A Solution Mapping and its Axiomatization in Two-Person Interval Games

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Abstract

An interval game is an extension of characteristic function form games in which players are assumed to face payoff uncertainty and thus the characteristic function assigns a closed interval, instead of a real number. In this study, we propose a new solution mapping of two-person interval games. We provide a collection of four axioms, consisting of Efficiency, Individual Rationality, and interval game versions of Shapley's Additivity and Nash's Independence of Irrelevant Alternatives, and show that the new solution mapping uniquely satisfies these axioms.

Keywords: Interval games, interval uncertainty, cooperative games, solution mapping, axiomatization

1   Introduction

This paper examines cooperative game theory in the situation where the players face uncertainty. One of the most familiar representations of cooperative game theory without uncertainties is coalitional form games with transferable utility (so-called coalition form or TU games) proposed by von Neumann and Morgenstern [17][1]. A coalition form game consists of the set of players N and a characteristic function v that associates with every subset S of N (coalition) a real number $v(S)$ (the worth of S). For each coalition S, $v(S)$ is the total payoff that S can obtain by itself and divide among its members in any possible way. A solution concept of coalition form games assigns to each game a set of outcomes. In the existing literature on coalitional form games, various types of solution concepts have been duly proposed, including imputation, core, von Neumann and Morgenstern stable set, Shapley value and nucleolus.

In reality, however, the payoffs that a coalition can obtain entail uncertainty. Therefore, introducing uncertainty into classical coalition form games is a natural extension. Here it should be noted that the existing literature on cooperative game theory with uncertainty has developed within two different groups. One group consists of models where uncertainty appears as a degree of cooperation in a coalition

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[1]Another standard specification is coalitional form games with nontransferable utilities (so-called NTU games) introduced by Aumann and Peleg [6].

or coalition formations (see fuzzy games in Aubin [1]). The other group considers models where player cooperation is as in the classical model, that is, using crisp coalitions, but uncertainty appears in the payoffs of coalitions (e.g., games in the stochastic characteristic function form of Charnes and Granot [9], stochastic payoffs of Suijs et al. [15], random payoffs of Timmer et al. [16], and interval payoffs of Branzei et al. [7] and Alparslan Gök et al. [5]). Of these, the interval games of [5][7] consider "interval uncertainty" in that the uncertainty regarding coalition payoffs is represented by an interval that a characteristic function assigns, rather than introducing the probability of stochastic components[2].

Similarly to coalition form games, an interval game consists of the set of players N and characteristic function w[3]. However, w uniquely associates with a coalition a closed interval of real numbers, called the worth set, rather than a real number as the characteristic function v in a classical coalition formation game. Therefore, we can regard interval games as an extension of coalition formation games. Note also that this specification is consistent with social situations in the real world where the potential rewards or costs are not known precisely, but we can estimate the intervals to which they belong.

The analysis of interval games then essentially examines, similarly to coalition form games, the issues of (i) coalition formation: or which coalition will be made and (ii) payoff allocation: or how to allocate the total payoff a coalition can obtain for each member in the coalition given coalition formation. Traditionally, coalition form games have mainly focused on (ii) and proposed various types of solution concepts regarding payoff allocation, and assuming the superadditivity under which the grand coalition is expected to be formed.

An interval game analysis can also impose superadditivity to focus on a desirable allocation under the grand coalition. However, we begin our analysis by arguing that interval game analyses have a unique subject to be carefully examined. Existing solution concepts for n-person interval games associate with each game (i) a (possibly empty or singleton) set of n-dimensional real-valued vectors ("selection-based solution concept") or (ii) a (possibly empty or singleton) set of n-dimensional interval payoff vectors ("interval solution concept"). However, if we return to the underlying social situations that interval game analyses address, such solution concepts in interval games do not seem necessarily appropriate in that this does not take into account the point that the uncertainty of outcomes represented by intervals that a characteristic functions assigns will be removed when an outcome is realized and allocated among players. Especially, if the grand coalition is assumed to be formed by imposing superadditivity, a solution concept for interval games should instruct how to allocate a realized outcome represented by a real number in the worth set of the grand coalition $w(N)$ among all players. Therefore, we should define a solution concept as a mapping from each realization to an n-dimensional real-valued vector, rather than directly providing a set of n-dimensional real-valued or interval vectors.

In fact, such mapping has been proposed in Alparslan Gök et al. [5] as ψ^{α}-value. However, the analysis of ψ^{α} is limited and, to our knowledge, no subsequent analysis has taken place since then, that is, all solution concepts for interval games proposed after [5] are either the "selection-based" or "interval" solution concepts referred to above. With this argument as our motivation, and after reviewing interval

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[2]This approach, from the viewpoint of microeconomics or noncooperative game theory, where the von Neumann and Morgenstern utility function is the building block, may not seem a standard way to address uncertainty. However, cooperative game theory has often employed this approach. For example, classical bankruptcy games essentially consider situations where uncertainty exists regarding the amount of an estate E and where they analyze those situations without the probability or distribution on E. It should also be noted that the results and implications derived from interval games in which any specific distributions are not imposed preserve high generality.

[3]Throughout our study w denotes a characteristic function in an interval game to distinguish it from a characteristic function v in a coalition form game.

games and the existing solution concepts, we propose a new solution mapping called σ^I focusing on a two-person interval game. We then show that the mapping satisfies the if and only if conditions of a collection of four axioms, including the interval game versions of Additivity in Shapley [14] and the Independence of Irrelevant Alternatives (IIA) in Nash [12].

2 Coalition form games and interval games

This section reviews classical coalition form games with transferable utility (TU games) and interval games.

2.1 Coalition form games with transferable utility

An n-person coalition form game with transferable utility (an n-person TU game) is a pair (N, v) where $N = \{1, 2, ..., n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that associates a real number $v(S) \in \mathbb{R}$ with each set $S \in 2^N$, such that $v(\phi) = 0$. The number $v(S)$ is called the worth of S. We refer to S and N as a coalition and the grand coalition, respectively. $v(S)$ is transferable utility (typically the amount of money) that the coalition S can obtain itself and divide among its members in any possible way. Let CG^N be the set of all n-person coalition form games.

A TU game (N, v) is called superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \phi$. If a TU game is superadditive, it implies that the worth of the grand coalition $v(N)$ is at least as large as the sum of the worth of the members of any partition of N. This ensures that it is optimal that a coalition N of all players forms.

Representative solution concepts for TU games include the set of imputation, core, Shapley value and nucleolus[4]. A payoff vector $x \in \mathbb{R}^n$ is called an imputation for the TU game (N, v) if x satisfies (i) individual nationality: $x_i \geq v(\{i\})$ $\forall i \in N$ and (ii) efficiency: $\sum_{i \in N} x_i = v(N)$. The set of imputations of (N, v) is denoted by $I(v)$.

The core of a game (N, v), denoted by $C(v)$, is defined as follows:

$$C(v) = \left\{ x \in I(v) \;\middle|\; \sum_{i \in S} x_i \geq v(S) \quad \forall S \in 2^N \setminus \phi \right\}.$$

An imputation $x \in I(v)$ is in the core of (N, v) if and only if no coalition can improve on x. Thus, we interpret each member of the core as a highly stable allocation.

To define the Shapley value (Shapley [14]), let $\pi = (\pi(1), \pi(2), ..., \pi(n))$ be a permutation of the players in (N, v) and Π be the set of all permutations. Note that the number of the elements in Π is $n!$. For a permutation $\pi = (\pi(1), \pi(2), ..., \pi(n))$ and $i \in N$ with $i = \pi(k)$, let $\pi(j)$ be a predecessor of i for π is $\pi(j)$ satisfies $j < k$. Let $P^\pi(i)$ be the set of i's predecessors for a permutation π. Then, the n-dimensional real-valued vector $\phi(v) = (\phi_1(v), ..., \phi_n(v))$ that satisfies the following condition of (2.1) is called the Shapley value of (N, v):

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \left(v\left(P^\pi(i) \cup \{i\} \right) - v\left(P^\pi(i) \right) \right).$$

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[4]The imputation and the core relate to Subsection 2.3. We examine the coincidence of solution mappings in two-person interval games with Shapley value and the nucleolus in its equivalent coalition form game in the appendix.

The nucleolus proposed by Schmeidler [13] is defined by using the notion of excess. For an imputation $x \in I(v)$ and $S \in 2^N$ in a game (N, v), the excess of S for x is $e(S, x) = v(S) - \sum_{i \in S} x_i$. Let $\theta(x) = (\theta_1(x), ..., \theta_{2^n-2}(x))$ be a 2^n-2-dimensional vector that includes $e(S, x)$ for each $S \in 2^N \setminus \{N \cup \phi\}$ in descending order from the largest. For two different imputations $x, y \in I(v)$, we say x is more acceptable than y if there exists k with $1 \leq k \leq 2^n - 4$ satisfying the following:

$$\theta_l(x) = \theta_l(y) \quad \forall l = 1, ..., k, \quad \theta_{k+1}(x) < \theta_{k+1}(y).$$

For $x \in I(v)$, if there exists $y \in I(v)$ that is more acceptable than x, then we say x has a more acceptable imputation. The nucleolus is then the set of imputations that do not have a more acceptable imputation.

Other standard solution concepts in coalition form games are the von Neumann and Morgenstern stable set (von Neumann and Morgenstern [17]), bargaining sets (Aumann and Maschler [2]), and kernel (Davis and Maschler [10]). Related to our study, it should be noted that all of these solution concepts are defined as a (possibly empty or singleton) set of n-dimensional real-valued vectors.

2.2 Interval games

Similarly to an n-person coalition form game (N, v), an n-person interval game (with transferable utility) is a pair (N, w) where $N = \{1, 2, ..., n\}$ is a set of players and w is a characteristic function. An interval game differs from a coalition form game in that w assigns a closed interval to each coalition rather than a real number. Formally, letting $I(\mathbb{R})$ be the set of all closed and bounded intervals in $\mathbb{R}$, $w : 2^N \to I(\mathbb{R})$ is the characteristic function that assigns to each coalition $S \in 2^N$ a closed interval $w(S) \in I(\mathbb{R})$, satisfying $w(\phi) = [0, 0]$. $w(S)$ is called the worth set of S and the upper and lower bounds of $W(S)$ are denoted by $\underline{w}(S)$ and $\overline{w}(S)$ respectively, that is, $w(S) = [\underline{w}(S), \overline{w}(S)]$. An interval game (N, w) considers a situation in which players face "interval uncertainty" in that they know a coalition S could get $\underline{w}(S)$ as the minimal reward and $\overline{w}(S)$ as the maximal reward but do not know *ex ante* which reward between them would be realized *ex post*. Let IG^N be the set of all n-person interval games. We hereafter occasionally denote n-person coalition form games simply by w instead of an expression of (N, w). For an interval game w and a coalition $\{1, ..., k\} \in 2^N$, we write $w(i, ..., k)$ instead of $w(\{1, ..., k\})$.

Here we provide some interval calculus notations for the following analysis. Let $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ be two closed intervals. The $|I|$ is defined by $|I| = \overline{I} - \underline{I}$. The sum of I and J, denoted by $I + J$, is given as $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$. We say I is weakly better than J, denoted by $I \succeq J$ if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We say I is left to J if for each $a \in I$ and for each $b \in J$, $a \leq b$, or equivalently, $\overline{I} \leq \underline{J}$.

For two different interval games $w, w' \in IG^N$, the sum of the interval games $w + w' \in IG^N$ is also an interval game itself, defined by $(w + w')(S) = w(S) + w'(S)$ for every $S \in 2^N$.

Note that for an interval game (N, w), if all the worth sets are singleton, that is, $\underline{w}(S) = \overline{w}(S)$ for every $S \in 2^N$, then (N, w) corresponds to the coalition form game (N, v) defined as $v(S) = \underline{w}(S) = \overline{w}(S)$. We say in this case (N, v) and (N, w) are equivalent or (N, w) is equivalent to (N, v). This means that a coalition form game can be a special case of an interval game. For different interval games $w, w' \in IG^N$ and $S \in 2^N$, if $\underline{w'}(S) \leq \underline{w}(S)$ and $\overline{w}(S) \leq \overline{w'}(S)$, then we denote $w(S) \subset w'(S)$.

2.3 Solution concepts in interval games

For the interval games defined above, several solution concepts have been proposed, all of which are categorized either into "selection-based" or "interval" solution concepts.

Selection-based solution concepts are based on the notion of the "selection" of an interval game. For an interval game of (N, w), a function $v : 2^N \to \mathbb{R}$ is called a selection of w if $v(S) \in w(S)$ for each $S \in 2^N$. Note that the set (N, v) constitutes a coalition form game. Let $Sel(w)$ be the set of all selections of w. The following two representative selection-based solution concepts of an interval game (N, v), the imputation set $I(w)$ and the core set $C(w)$, are given by (Alparslan Gök et al. [5]):

$$
\begin{aligned}
I(w) &= \cup \{I(v) | v \in sel(w)\} \\
C(w) &= \cup \{C(v) | v \in sel(w)\}
\end{aligned}
$$

where $I(v)$ and $C(v)$ are the sets of imputation and core, respectively, in the coalition form game (N, v). Note that both $I(v)$ and $C(v)$ are (possibly empty or singleton) sets of n-dimensional real-valued vectors.

Interval solution concepts are defined as a (possibly empty or singleton) set of n-dimensional interval vectors. Formally, letting $I_i \in I(\mathbb{R})$ be the interval payoff of player i and $I = (I_1, ..., I_n) \in I(\mathbb{R})^N$ be an n-dimensional interval (payoff) vector, an interval solution concept in (N, w) assigns a set of n-dimensional interval vectors whose components belong to $I(\mathbb{R})$. The interval imputation set $II(w)$ and the interval core $IC(w)$ of an interval game (N, w) are given by

$$
\begin{aligned}
II(w) &= \left\{ I = (I_1, ..., I_n) \in I(\mathbb{R})^N \,\middle|\, \sum_{i \in N} I_i = w(N),\ w(i) \preceq I_i,\ \forall\, i \in N \right\} \\
IC(w) &= \left\{ I = (I_1, ..., I_n) \in I(\mathbb{R})^N \,\middle|\, \sum_{i \in N} I_i = w(N),\ w(S) \preceq \sum_{i \in S} I_i,\ \forall\, S \in 2^N \setminus \{\phi\} \right\}.
\end{aligned}
$$

Other solution concepts of interval games such as the mini-core set, interval Shapley value, and interval stable set (Alparslan Gök et al. [4][5]) are also defined as a (possibly empty or singleton) set of n-dimensional real-valued vectors or a set of n-dimensional interval payoff vectors.

We consider the following account that an interval game analysis essentially assumes. However, an alternative approach may be appropriate in constructing a solution concept. That is, the account assumes first that players who face payoff uncertainties represented by worth sets negotiate *ex ante* a desirable or acceptable payoff allocation before the uncertainties are removed. Then, one of the outcomes in a worth set is realized, the uncertainties are removed, and the outcome is allocated *ex post* among players based on a specific "rule" or "protocol." Especially, when the grand coalition forms, the realized outcome in $w(N)$ is allocated. Based on this interpretation, a solution concept in this account is required to specify both (i) possible agreement in the *ex ante* negotiation and (ii) *ex post* allocation after removing uncertainty. However, both selection-based and interval solution concepts seem to address only the issue of (i) and do not provide a clear answer to (ii) by themselves. As a natural consequence, another framework to tackle the second issue and to examine desirable "protocols" handling interval-solution concepts and giving *ex post* allocation among players is additionally needed, as in Branzei et al. [8]. However, a clear instruction to provide an agreeable *ex post* allocation does not seem to have been derived.

An alternative approach that we argue for in this study is to employ a mapping as a solution concept,

rather than a set of *n*-dimensional real-valued vectors (selection-based solution concepts) or a set of *n*-dimensional interval payoff vectors (interval solution concepts), that is, the mapping that assigns *n*-dimensional real-valued vectors to each realized outcome in the worth set of the grand coalition $w(N)$. Indeed, Alparslan Gök et al. [5] proposed such a solution mapping called ψ^α-value in line with this approach by focusing on two-person interval games. However, that mapping analysis limited and, to our knowledge, no subsequent analysis has taken place since then. The next section reviews the existing solution mapping ψ^α, highlights some elements to be modified, and proposes a new solution mapping.

3 The solution mappings ψ^α and σ^I in two-person interval games: a heuristic approach

Hereafter, we focus on two-person interval games, following Alparslan Gök et al. [5]. We impose this restriction mainly because there has been no existing analysis or progress on solution mappings since [5] focusing on two-person interval games. An extension to an *n*-person game should then be proposed as a topic for further research.

Let $IG^{\{1,2\}}$ be the set of all two-person interval games. Following Alparslan Gök et al. [5], a two-person interval game $w \in IG^{\{1,2\}}$ is called superadditive if $w(1) + w(2)$ is left to $w(1,2)$, that is,

$$\overline{w}(1) + \overline{w}(2) \le \underline{w}(1,2). \tag{1}$$

A two-person interval game $w \in IG^{\{1,2\}}$ is superadditive if and only if w is strongly balanced (Alparslan Gök et al. [5]). Let $SBIG^{\{1,2\}}$ be the set of all strongly balanced two-person interval games.

As is evident from the expression (1), when an interval game $w \in IG^{\{1,2\}}$ is superadditive, the sum of the maximum amount of payoff when each player acts individually is never larger than the minimum amount of payoff when players jointly act. Therefore, superadditivity is considered a sufficient condition that the grand coalition forms in an interval game [5].

With this setup, Alparslan Gök et al. [5] proposed ψ^α-values as a solution concept in two-person interval games. ψ^α-values are a solution mapping that assigns two-dimensional payoff vectors to each realization of the outcome in $w(1,2)$ for a given interval game $w \in IG^N$ [6]. Specifically, letting $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$ be an optimism vector, they first defined $s_1^{\alpha_1}(w)$ and $s_2^{\alpha_2}(w)$ as follows:

$$s_1^{\alpha_1}(w) = \alpha_1 \overline{w}(1) + (1 - \alpha_1)\underline{w}(1), \quad s_2^{\alpha_2}(w) = \alpha_2 \overline{w}(2) + (1 - \alpha_2)\underline{w}(2). \tag{2}$$

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[5]It should be noted that there is another (weaker) version of superadditivity in interval games that includes the condition of (i) $\overline{w}(1) + \overline{w}(2) \le \overline{w}(1,2)$ *and* (ii) $\underline{w}(1) + \underline{w}(2) \le \underline{w}(1,2)$ (see Alparslan Gök [3]). When an interval game satisfies this weaker version, but does not satisfy the (stronger) superadditivity of (1), then the grand coalition is less likely to form. For example, suppose $\underline{w}(1) + \underline{w}(2) < \underline{w}(1,2) < \overline{w}(1) + \overline{w}(2)$, satisfying only the weaker version. In this case, if each player *conservatively* forecasts that $\underline{w}(i)$ would be realized when acting individually, then the grand coalition is expected to form. On the other hand, if each player *optimistically* forecasts that $\overline{w}(i)$ would be realized, then the grand coalition is expected not to form. Therefore, under weak superadditivity, the likelihood of the grand coalition critically depends on players' risk attitudes. In contrast, under the strong superadditivity of (1), the grand coalition is expected to form independent from players' risk attitudes.

Following Alparslan Gök et al. [5], our study employs the strong superadditivity of (1). In particular, as we will see in Section 4, this will be used when we restrict the range of interval games such that the axiom of individual rationality should be satisfied. In other words, we do not explicitly assume that all interval games to be studied satisfy strong superadditivity. However, it should be noted that our analysis as well as Alparslan Gök et al. [5] implicitly assumes that the grand coalition forms because the solution concepts adopted by these studies specify a payoff allocation in the grand coalition. An examination of the condition under which the grand coalition forms in interval games when violating strong superadditivity should be a topic for further research.

[6]The following description of ψ^α-values is based on Alparslan Gök et al. [5].

Consider a mapping $\kappa : [a,b] \to \mathbb{R}^2$ that assigns two-dimensional payoff vectors to each realization in the closed interval of $[a,b]$. Let the set of all κ be $K(\mathbb{R}^2)$ and $F : IG^{\{1,2\}} \to K(\mathbb{R}^2)$ be a map that assigns to each interval game w a unique mapping $F(w) \in K(\mathbb{R}^2)$. When the domain of $F(w)$ is the worth set of the grand coalition $w(1,2) = [\underline{w}(1,2), \overline{w}(1,2)]$, that is, $F(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathbb{R}^2$, we call F a solution or a solution mapping in two-person interval games. $F(\cdot)(\cdot)$ assigns two-dimensional payoff vectors to each realization $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ for each interval game $w \in IG^N$.

Alparslan Gök et al. [5] proposed the following $\psi^\alpha : IG^{\{1,2\}} \to K(\mathbb{R}^2)$ as a solution mapping. For each $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, ψ^α is defined as :

$$\psi^\alpha(w)(t) = (s_1^{\alpha_1}(w) + \beta, \; s_2^{\alpha_2}(w) + \beta), \tag{3}$$

where $s_1^{\alpha_1}(w)$ and $s_2^{\alpha_2}(w)$ are defined in (2) and $\beta = \frac{1}{2}(t - s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w))$. Note that (3) can be rearranged as follows:

$$\psi^\alpha(w)(t) = \left(\frac{1}{2}(t + s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w)), \; \frac{1}{2}(t - s_1^{\alpha_1}(w) + s_2^{\alpha_2}(w)) \right). \tag{4}$$

$\psi^\alpha(w)(t)$ gives a two-dimensional payoff vector to each realization $t \in w(1,2)$ in a given two-person interval game $w \in IG^{\{1,2\}}$. ψ^α is taken as a solution concept that specifies both (i) possible agreement in the *ex ante* negotiation among players and (ii) *ex post* allocation after uncertainties are removed. Namely, in the *ex ante* bargaining, players agree on ψ^α. Then, after payoff uncertainties are removed, t is allocated among players based on ψ^α. ψ^α satisfies some axioms including efficiency and a variant of symmetry (Alparslan Gök et al. [5]). Furthermore, when a two-person interval game $w \in IG^{\{1,2\}}$ is equivalent to a two-person coalition form game $v \in CG^{\{1,2\}}$, ψ^α in w coincides with the Shapley value and nucleolus in w (see the appendix for the proof of this statement[7]).

It should be noted that, however, the solution mapping ψ^α seems to entail some elements that need to be carefully examined and altered.

The first element relates to how the notion of the optimism vector can be interpreted particularly in connection with the value sets of $w(1)$ and $w(2)$. The allocation rule of ψ^α expressed in (3) seems to be based on the following idea. First, $s_i^{\alpha_i}(w)$ is determined according to the optimism vector (α_1, α_2) and allocated to each player. Then, the residual of $t - (s_1^{\alpha_1}(w) + s_2^{\alpha_2}(w))$ is equally divided. In this account, (α_1, α_2) seems to reflect each player's bargaining power in the *ex ante* negotiation before payoff uncertainties are removed. However, taking the underlying situations and construction of interval games into consideration, such negotiation powers have already been reflected in $w(i)$, that is, the outcome that i can obtain by itself when the negotiation would fail. In the framework ψ^α-value, the optimism vector α_i is exogenously added and assumed independent from $w(i)$. However, as both represent player bargaining power, there is redundancy in applying ψ^α-value to an interval game. Furthermore, $\psi^\alpha(w)(t)$ may not necessarily work as an appropriate "protocol" as it cannot provide a specific payoff vector unless (α_1, α_2) is determined.

The second issue to be examined is individual rationality when $w \in IG^{\{1,2\}}$ is superadditive. Here individual rationality means that, letting $x^* = (x_1^*, x_2^*)$ be an allocated payoff vector, $x_i^* \geq \overline{w}(i)$. If $w \in IG^{\{1,2\}}$ is superadditive, that is, if (1) holds, then the following is true with respect to player 1 (and the same for player 2). Starting from (4), we obtain:

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[7]The appendix provides the proof of this statement as Alparslan Gök et al. [5] did not explicitly deal with this proposition.

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$$\frac{1}{2}(t + s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w)) \geq \frac{1}{2}(\underline{w}(1,2) + s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w)) \geq \frac{1}{2}(\overline{w}(1) + \overline{w}(2) + s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w))$$

$$\geq \frac{1}{2}(\overline{w}(1) + \overline{w}(2) + \underline{w}(1) - \overline{w}(2)) = \frac{1}{2}(\overline{w}(1) + \underline{w}(1)) \geq \underline{w}(1). \qquad (5)$$

The second inequality holds from superadditivity. (5) implies that, if $w \in IG^{\{1,2\}}$ is superadditive, ψ^α always gives at least $\underline{w}(i)$ to each player. However, ψ^α does not guarantee that both players obtain at least $\overline{w}(i)$. If so, it is not clear that the grand coalition would be formed even with superadditivity[8].

To examine these issues further, consider the following example.

Example 3.1

$$w(\phi) = [0,0], \ w(1) = [10,30], \ w(2) = [30,50], \ w(1,2) = [90,130]$$

$w \in IG^{\{1,2\}}$ is superadditive since $\overline{w}(1) + \overline{w}(2) = 30 + 50 \leq 90 = \underline{w}(1,2)$. When $\alpha = (\alpha_1, \alpha_2) = (0,1)$ is assumed, then we obtain

$$\psi^\alpha(w)(t) = (10 + \beta, \ 50 + \beta) \ where \ \beta = \frac{1}{2}(t - 60)$$

given $s_1^{\alpha_1}(w) = \underline{w}(1) = 10, s_2^{\alpha_2}(w) = \overline{w}(2) = 50$. If $t = 90$, $\psi^\alpha(w)(90) = (25,65)$.

As discussed, α is considered to reflect the relative relationship of bargaining powers between players 1 and 2. Thus, the assumption of $\alpha = (0,1)$ corresponds to a situation where player 1's bargaining power is smaller relative to player 2. In this sense, the resulting allocation $(25,65)$ giving a small number to player 1 might seem to be reasonable. However, the assumption of superadditivity corresponds to a situation that even the minimum payoff by forming the grand coalition cannot be attained if each player acts individually and succeeds in getting its maximum payoff. In such a case where the "synergy effect" stemming from the grand coalition are large, it is clear that the grand coalition is formed, but if so, a desirable allocation rule might be based on the following idea. First, the maximum payoff when acting individually is distributed to each player. Then, the amount of payoff generated by acting jointly (i.e., the synergy effect) is equally divided among players. If this idea is employed, the payoff of $(25,65)$ allocated by $\psi^\alpha(w)(90)$ seems to lack validity given $25 < 30 = \overline{w}(1)$. Therefore, when superadditivity holds, the allocated payoff of $x^* = (x_1^*, x_2^*)$ should not only satisfy $x_i^* \geq \underline{w}(i)$ as in (5) but also the condition of $x_i^* \geq \overline{w}(i)$.

Based on the above argument, consider the following allocation rule.

- **STEP 1** give player 1 $\overline{w}(1) = 30$ and player 2 $\overline{w}(2) = 50$.

- **STEP 2** equally divide the "synergy effect" of $t - (50 + 30)$ and add to the allocation in **STEP 1**.

When $t = 90$, this allocation rule gives $x^* = (x_1^*, x_2^*) = (35,55)$.

Here, it should be noted that in this allocation rule, the issues underlying the notion of ψ^α discussed above are appropriately addressed. First, this rule specifies the two-dimensional payoff vector $(35,55)$ for a realized payoff $t = 90$ in $w(1,2) = [90,130]$ and for given $w \in IG^{\{1,2\}}$ by NOT using parameters as α in ψ^α. The asymmetry of $x^* = (35,55)$ simply comes from the ratio between $x(1)$ and $x(2)$, both

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[8]Recall we employ the "stronger" superadditivity, a sufficient condition for the grand coalition to form. See footnote 5.

of which are considered as sources of players' bargaining power. Second, by giving $\overline{w}(i)$ to each player in **STEP 1**, this allocation rule always satisfies individual rationality of $x_i^* \geq \overline{w}(i)$ as long as an interval game is superadditive as $t - (\overline{w}(1) + \overline{w}(2)) \geq 0$ always holds.

Based on the above discussion, we formally define the following solution mapping σ^I as an alternative to ψ^α.

Definition 3.1 *For a two-person interval game* $w \in IG^{\{1,2\}}$ *and a realization of the value set of the grand coalition* $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, σ^I *is defined as follows:*

$$\sigma^I(w)(t) = (\overline{w}(1) + \beta, \overline{w}(2) + \beta) \quad where \quad \beta = \frac{1}{2}\left(t - \overline{w}(1) - \overline{w}(2)\right). \tag{6}$$

σ^I is a solution mapping that assigns a two-dimensional payoff vector to each realization t in a given $w \in IG^{\{1,2\}}$. Note that σ^I always exists. Similarly to (4), we note that $\sigma^I(w)(t)$ can be rearranged as follows:

$$\sigma^I(w)(t) = \left(\frac{1}{2}\left(t + \overline{w}(1) - \overline{w}(2)\right), \frac{1}{2}\left(t - \overline{w}(1) + \overline{w}(2)\right)\right). \tag{7}$$

As we show in Subsection 4, σ^I satisfies efficiency and individual rationality when an interval game is superadditive. We point out here the following property regarding the relation between σ^I and ψ^α.

Property 3.1 $\sigma^I(w)(t) = \psi^{(1,1)}(w)(t).$

Proof Suppose $\alpha = (\alpha_1, \alpha_2) = (1, 1)$, implying that $s_1^{\alpha_1}(w) = \overline{w}(1)$ and $s_2^{\alpha_2}(w) = \overline{w}(2)$. By inserting these into (4), we obtain:

$$\psi^{(1,1)}(w)(t) = \left(\frac{1}{2}\left(t + \overline{w}(1) - \overline{w}(2)\right), \frac{1}{2}\left(t - \overline{w}(1) + \overline{w}(2)\right)\right) = \sigma^I(w)(t). \quad \blacksquare$$

Property 3.1 says σ^I is mathematically identical to ψ^α when $\alpha = (1, 1)$. However, it should be noted that we have heuristically derived the solution mapping σ^I defined in (6) by directly using information on $w(i)$, not by introducing the notion of α.

4   An axiomatization of the solution mapping σ^I

Let $\sigma : IG^{\{1,2\}} \rightarrow K(R^2)$ be an solution mapping in $IG^{\{1,2\}}$. This subsection proposes a collection of axioms and shows that σ^I satisfies the collection and that σ^I is the unique solution mapping to satisfy them. In particular, we consider the following axioms.

- Axiom 1: Efficiency [EF]

$$\left(\sigma_1(w)(t) + \sigma_2(w)(t) = t\right) \left(\forall w \in IG^{\{1,2\}}\right) \left(\forall t \in w(1,2)\right).$$

- Axiom 2: Individual Rationality [IR]

$$\left(\sigma_i(w)(t) \geq \overline{w}(i)\right) \left(\forall w \in SBIG^{\{1,2\}}\right) \left(\forall t \in w(1,2)\right) \left(\forall i \in \{1,2\}\right).$$

- Axiom 3: Symmetry [SYM]

$$\Big(\sigma_1(w)(t) = \sigma_2(w)(t)\Big) \Big(\forall w \in IG^{\{1,2\}} \text{ with } w(1) = w(2)\Big) \Big(\forall t \in w(1,2)\Big).$$

- Axiom 4: Additivity [AD]

$$\Big(\sigma_i(w + w')(t + t') = \sigma_i(w)(t) + \sigma_i(w')(t')\Big)$$
$$\Big(\forall w, w' \in IG^{\{1,2\}}\Big) \Big(\forall t \in w(1,2)\Big) \Big(\forall t' \in w'(1,2)\Big) \Big(\forall i \in \{1,2\}\Big).$$

Axiom EF asserts that all of the $t \in w(1,2)$ is allocated to either player 1 or player 2 and no residual exists. Axiom IR asserts that, as we discussed previously, when an interval game is superadditive, no player should get less in the solution than the maximum value $\overline{w}(i)$ they could get on their own. Axiom SYM argues that only what a player can get themselves in the game should matter, not their specific name or label in the set N (Myerson [11]). Axiom AD comes from Shapley [14], which considers the interval game $(w + w') \in IG^{\{1,2\}}$ defined as $(w + w')(S) = w(S) + w'(S) \ \forall S \in 2^N$ for two different games $w, w' \in IG^{\{1,2\}}$ and asserts that, when σ gives $\sigma_i(w)(t)$ to player i for a realization t in w and $\sigma_i(w')(t')$ to i for a realization t' in w', then in the newly created game $(w + w') \in IG^{\{1,2\}}$ σ should give $\sigma_i(w)(t) + \sigma_i(w')(t')$ to player i for realization $t + t'$.

We show that σ^I defined by (6) is the unique solution mapping satisfying these axioms. First, Theorem 4.1 shows that σ^I satisfies all the axioms.

Theorem 4.1 *In every two-person interval game $w \in IG^{\{1,2\}}$, σ^I satisfies Axioms 1 to 4.*

Proof Suppose an interval game $w \in IG^{\{1,2\}}$ and σ^I defined as (6) in w. Then, we show that σ^I satisfies Axioms 1 to 4.

Axiom 1 $\sigma_1^I(w)(t) + \sigma_2^I(w)(t) = \overline{w}(1) + \overline{w}(2) + 2\beta = \overline{w}(1) + \overline{w}(2) + t - \overline{w}(1) - \overline{w}(2) = t.$

Axiom 2 $\sigma_i^I(w)(t) - \overline{w}(i) = \frac{1}{2}\big(t + \overline{w}(i) - \overline{w}(j)\big) - \overline{w}(i) = \frac{1}{2}\big(t - [\overline{w}(i) + \overline{w}(j)]\big) \geq \frac{1}{2}\big(t - \underline{w}(1,2)\big) \geq 0.$
The first inequality comes from superadditivity and the second from $t \in [\underline{w}(1,2), \overline{w}(1,2)]$.

Axiom 3 when $w(1) = w(2)$, $\sigma_1^I(w)(t) = \frac{1}{2}t = \sigma_2^I(w)(t).$

Axiom 4 for simplicity, focus on player 1 (the same argument holds for player 2). For any $w, w' \in IG^{\{1,2\}}$, any $t \in w(1,2)$ and any $t' \in w'(1,2)$, the following holds:
$\sigma_1^I(w + w')(t + t') = \frac{1}{2}\big(t + t' + [\overline{w}(1) + \overline{w}'(1)] - [\overline{w}(2) + \overline{w}'(2)]\big) = \frac{1}{2}\big(t + \overline{w}(1) - \overline{w}(2)\big) + \frac{1}{2}(t' + \overline{w}'(1) - \overline{w}'(2)) = \sigma_1^I(w)(t) + \sigma_1^I(w')(t').$ ∎

Then, Theorem 4.2 shows the uniqueness of σ^I.

Theorem 4.2 *In every two-person interval game $w \in IG^{\{1,2\}}$, σ^I is the unique solution mapping satisfying Axioms 1 to 4.*

Proof In a two-person interval game $w \in IG^{\{1,2\}}$, suppose a solution mapping σ satisfies Axioms 1 to 4. Then, we show σ must be σ^I.

First, for $w \in IG^{\{1,2\}}$ we define the following two-person interval games $w_1, w_2 \in IG^{\{1,2\}}$:

$$w_1(1) = w(1), \ w_1(2) = w(2), \ w_1(1,2) = [\overline{w}(1) + \overline{w}(2), \overline{w}(1) + \overline{w}(2)]$$

$$w_2(1) = [0,0], \ w_2(2) = [0,0], \ w_2(1,2) = [\underline{w}(1,2) - \overline{w}(1) - \overline{w}(2), \overline{w}(1,2) - \overline{w}(1) - \overline{w}(2)].$$

Note that $w_1 + w_2 = w$ and w_1 is superadditive, or equivalently, strongly balanced ($w_1 \in SBIG^{\{1,2\}}$) given $\overline{w_1}(1) + \overline{w_1}(2) = \overline{w}(1) + \overline{w}(2) \leq \underline{w_1}(1,2)$ holds with equality.

Then, we show the following (8) holds.

$$\sigma(w_1) \left(\overline{w}(1) + \overline{w}(2) \right) = \left(\overline{w}(1), \ \overline{w}(2) \right) \tag{8}$$

(**Proof of (8)**) Note first that $\overline{w}(1) + \overline{w}(2) \in w_1(1,2)$. Given $w_1 \in SBIG^{\{1,2\}}$, σ holds $\sigma_1(w_1) \left(\overline{w}(1) + \overline{w}(2) \right)$ $\geq \overline{w_1}(1) = \overline{w}(1)$ and $\sigma_2(w_1) \left(\overline{w}(1) + \overline{w}(2) \right) \geq \overline{w_1}(2) = \overline{w}(2)$ from IR. From EF, on the other hand, $\sigma_1(w_1)(\overline{w}(1) + \overline{w}(2)) + \sigma_2(w_1) \left(\overline{w}(1) + \overline{w}(2) \right) = \overline{w}(1) + \overline{w}(2)$. Those conditions are satisfied only when (8) holds. (**End of Proof of (8)**)

Next we show the following (9) holds.

$$\sigma(w_2) \left(t - [\overline{w}(1) + \overline{w}(2)] \right) = \left(\frac{t - [\overline{w}(1) + \overline{w}(2)]}{2}, \frac{t - [\overline{w}(1) + \overline{w}(2)]}{2} \right) \ \forall t \in w(1,2) \tag{9}$$

(**Proof of (9)**) First, as $t \in w(1,2)$, $\underline{w}(1,2) \leq t \leq \overline{w}(1,2) \Leftrightarrow \underline{w}(1,2) - [\overline{w}(1) + \overline{w}(2)] \leq t - [\overline{w}(1) + \overline{w}(2)] \leq \overline{w}(1,2) - [\overline{w}(1) + \overline{w}(2)]$. Given $\underline{w}(1,2) - [\overline{w}(1) + \overline{w}(2)] = \underline{w_2}(1,2)$ and $\overline{w}(1,2) - [\overline{w}(1) + \overline{w}(2)] = \overline{w_2}(1,2)$, it follows that $t - [\overline{w}(1) + \overline{w}(2)] \in w_2(1,2)$.

Given $w_2(1) = w_2(2)$ and σ is assumed to satisfy SYM, $\sigma_1(w_2) \left(t - [\overline{w}(1) + \overline{w}(2)] \right) = \sigma_2(w_2)$ $\left(t - [\overline{w}(1) + \overline{w}(2)] \right)$. Furthermore, $\sigma_1(w_2) \left(t - [\overline{w}(1) + \overline{w}(2)] \right) + \sigma_2(w_2) \left(t - [\overline{w}(1) + \overline{w}(2)] \right) = t - [\overline{w}(1) + \overline{w}(2)]$ holds from EF. These two conditions are satisfied only when (9) holds. (**End of Proof of (9)**)

From (8) and (9), $w_1 + w_2 = w$ and SYM of σ, the following holds:

$$\sigma(w)(t) = \sigma(w_1 + w_2) \left(\{\overline{w}(1) + \overline{w}(2)\} + \{t - [\overline{w}(1) + \overline{w}(2)]\} \right)$$

$$= \sigma(w_1) \left(\overline{w}(1) + \overline{w}(2) \right) + \sigma(w_2) \left(t - [\overline{w}(1) + \overline{w}(2)] \right) \quad (\because \ additivity \ of \ \sigma)$$

$$= \left(\overline{w}(1) + \frac{t - [\overline{w}(1) + \overline{w}(2)]}{2}, \ \overline{w}(2) + \frac{t - [\overline{w}(1) + \overline{w}(2)]}{2} \right) \quad ((8) \ and \ (9))$$

$$= \left(\frac{1}{2} \left(t + \overline{w}(1) - \overline{w}(2) \right), \ \frac{1}{2} \left(t - \overline{w}(1) + \overline{w}(2) \right) \right) \ = \ \sigma^I(w)(t). \ \blacksquare$$

We next consider the following axiom originally proposed by Nash [12] in the Nash bargaining game.

- Axiom 5: Independence of Irrelevant Alternatives [IIA]

$$\left(\sigma_i(w)(t) = \sigma_i(w')(t) \right)$$

$$\left(\forall w, w' \in IG^{\{1,2\}} \text{ with } w(1,2) \subset w'(1,2) \text{ and } w(i) = w'(i), \forall i \in \{1,2\} \right) \left(\forall t \in w(1,2) \right) \left(\forall i \in \{1,2\} \right).$$

Axiom IIA asserts that eliminating feasible alternatives that would not have been otherwise chosen should not affect the solution. As for IIA, we establish the following property.

Property 4.1 *In every two-person interval game $w \in IG^{\{1,2\}}$, σ^I satisfies Axiom 5.*

Proof This is because equation (6) does not depend on $w(1,2)$. Formally, for any $t \in w(1,2)$ and for any $w, w' \in IG^{\{1,2\}}$ satisfying (i) $w(i) = w'(i) \; \forall i \in \{1,2\}$ and (ii) $w(1,2) \subset w'(1,2)$, it holds that $t \in w'(1,2)$. Therefore, $\sigma_i^I(w')(t)$ exists and it follows that $\sigma_i^I(w')(t) = \dfrac{1}{2}\left(t + \overline{w}'(i) - \overline{w}'(j)\right) = \dfrac{1}{2}\left(t + \overline{w}(i) - \overline{w}(j)\right) = \sigma_i^I(w)(t)$. ∎

5   Concluding remarks

An interval game is an extension of classical coalition form games that addresses the situation in which players face payoff uncertainty. In this study, we propose a new solution mapping called σ^I that assigns a payoff allocation to each realization of the outcome. Furthermore, we construct a collection of axioms including interval game versions of Shapley's Additivity and Nash's Independence of Irrelevant Alternatives and show that the new solution mapping uniquely satisfies these axioms.

We conclude the analysis by pointing out some topics for further research. First, it would be interesting to extend our results to n-person interval games. Second, we could examine other solution mappings as an alternative to σ^I. As discussed in Section 3, σ^I is constructed based on the following idea. First, the maximum payoff when acting individually is distributed to each player. Then, a "synergy effect" generated by acting jointly is equally divided among the players. An alternative solution mapping may be constructed, as an example, based on the following idea. First, the synergy effect is computed in some way and distributed equally to each player. Then, the residual is allocated proportionally to $w(i)$. Of course, such alternative solution mappings should be justified in terms of axiomatization or equivalency with standard solution concepts in coalition form games, as argued here. Finally, as referred to in footnote 5, conditions under which the grand coalition forms in interval games when strong superadditivity is violated should also be considered for further research.

Acknowledgements

Appendix : Coincidence of the payoff vector allocated by ψ^α in a two-person interval game with Shapley value and nucleolus in its equivalent coalition form game

This appendix examines the relationship between a payoff vector supported by the interval solution mapping ψ^α in an interval game $w \in IG^{\{1,2\}}$ and solution concepts in an coalition form game $v \in CG^{\{1,2\}}$ when v and w are equivalent, that is, $w(S) = \underline{w}(S) = \overline{w}(S) = v(S) \; \forall S \in 2^N$.

We first refer to the following without proof.

Property 5.1 *Define the egalitarian nonseparable contribution solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ in a two-person*

coalition form game $v \in CG^{\{1,2\}}$ *as follows:*

$$(\tilde{x}_1, \tilde{x}_2) = \left(v(1) + \frac{v(1,2) - v(1) - v(2)}{2}, \ v(2) + \frac{v(1,2) - v(1) - v(2)}{2} \right). \tag{10}$$

Then, $\tilde{x}$ coincides with the Shapley value and nucleolus in v.

Then, we show the following lemma.

Lemma 5.1 *In a two-person interval game $w \in IG^{\{1,2\}}$, suppose $w(S) = \underline{w}(S) = \overline{w}(S) \ \forall S \in 2^N$. Furthermore, let $v \in CG^{\{1,2\}}$ be the two-person coalition form game equivalent to w such that $v(S) = w(S) \ \forall S \in 2^N$ and let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ be the egalitarian nonseparable contribution solution defined in (10) in v. Then, for every optimism vector $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, the solution mapping ψ^α satisfies the following:*

$$\psi^\alpha(w)(t) = (\tilde{x}_1, \tilde{x}_2).$$

Proof From the expression (4), construction of w and $t = w(1,2) = v(1,2)$, the following holds for every $\alpha = (\alpha_1, \alpha_2)$:

$$\begin{aligned}
\psi^\alpha(w)(t) &= \left(\frac{1}{2}(t + s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w)), \ \frac{1}{2}(t - s_1^{\alpha_1}(w) + s_2^{\alpha_2}(w)) \right) \\
&= \left(\frac{1}{2}(v(1,2) + v(1) - v(2)), \ \frac{1}{2}(v(1,2) - v(1) + v(2)) \right) \\
&= \left(v(1) + \frac{v(1,2) - v(1) - v(2)}{2}, \ v(2) + \frac{v(1,2) - v(1) - v(2)}{2} \right) = (\tilde{x}_1, \tilde{x}_2). \quad \blacksquare
\end{aligned}$$

From Property 5.1 and Lemma 5.1, we obtain the following.

Theorem 5.1 *In a two-person interval game $w \in IG^{\{1,2\}}$, suppose $w(S) = \underline{w}(S) = \overline{w}(S)$ for every $S \in 2^N$. Then, $\psi^\alpha(w)(t) \in \mathbb{R}^2$ is identical to to the Shapley value and nucleolus in the coalition form game $v \in CG^{\{1,2\}}$ equivalent to $w \in IG^{\{1,2\}}$*

As Theorem 5.1 holds for every α, it also holds for σ^I (see Property 3.1).

As discussed, interval games can be regarded as an extension of coalition form games. Therefore, the proved coincidence of solution mappings ψ^α and σ^I in interval games with standard solution concepts in coalition form games yields a justification for employing ψ^α and σ^I as solution concepts in interval games.

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