Some variations of de Jongh’s theorem

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Workshop in Logic and Philosophy of Mathematics
Waseda University

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Motivation and abstract

- Some variations of de Jongh's theorem.

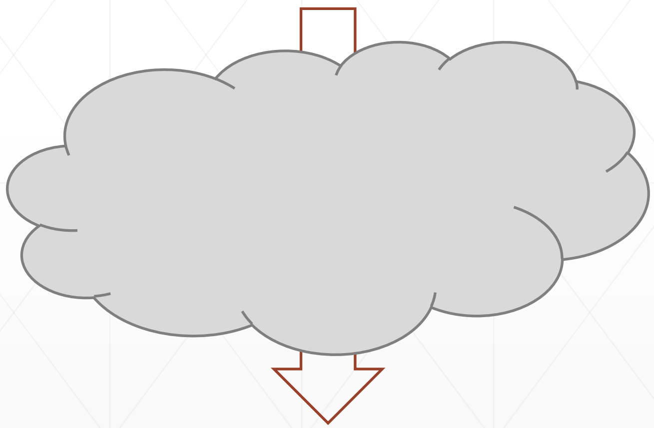
  [Smorinsky1973]

- Some variations of arithmetical completeness for logic of proofs.

  [Iwata-Kurahashi2017]

- Arithmetical completeness for intuitionistic Logic of Proofs.

  [Artemov-Iemhoff2007, Dashkov2009]



- Search for a modal logic which captures the provability predicate in Heyting Arithmetic. [Artemov-Beklemishev2005]

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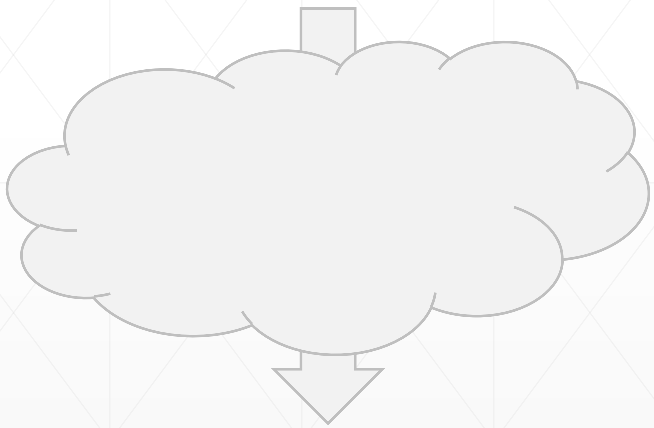

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Motivation and Abstract

- Kripke semantics
- Heyting arithmetic
- Notations

Preliminary

- AA : the set of Axioms of (Peano) Arithmetic
- Peano Arithmetic (PA) :
  - Classical Logic (CL) + AA
  - $\mathrm{PA} \vdash \alpha \Leftrightarrow \mathrm{AA} \vdash_{\mathrm{CL}} \alpha$
- Heyting Arithmetic (HA) :
  - Intuitionistic Logic (IL) + AA
  - $\mathrm{HA} \vdash \alpha \Leftrightarrow \mathrm{AA} \vdash_{\mathrm{IL}} \alpha$
- An arithmetical substitution is a mapping which assigns a arithmetical sentence to a propositional variable.

Heyting Arithmetic

- Kripke model $\mathcal{K} = \langle W, \leq, \{\mathcal{M}_w\}_{w \in W} \rangle$:
  - $\langle W, \leq \rangle$ is a partial order
  - each $\mathcal{M}_w$ is a (classical) model s.t.
    - $w \leq v \Rightarrow Dom(\mathcal{M}_w) \subseteq Dom(\mathcal{M}_v)$
    - $w \leq v$ and $\mathcal{M}_w \vDash \alpha \Rightarrow \mathcal{M}_v \vDash \alpha$ for each ($\mathcal{M}_w$-)sentence α
- $\mathcal{K}, w \Vdash \alpha$:
  - $\mathcal{K}, w \Vdash \alpha \Leftrightarrow \mathcal{M}_v \vDash \alpha$ if α is atomic ($\mathcal{M}_v$-)sectence
  - $\mathcal{K}, w \Vdash \alpha \wedge (\vee / \rightarrow) \beta \Leftrightarrow \mathcal{K}, v \Vdash \alpha$ and (or / implies) $\mathcal{K}, v \Vdash \beta$ for each $v \geq w$
  - $\mathcal{K}, w \Vdash \forall(\exists) x \alpha \Leftrightarrow \mathcal{K}, v \Vdash [x \coloneqq \bar{e}]\alpha$,

    for each (some) $e \in Dom(\mathcal{M}_v)$, for each $v \geq w$
- $\mathcal{K} \Vdash \alpha \Leftrightarrow \mathcal{K}, w \Vdash \alpha$ for each $w \in W$
- $\Gamma \Vdash \alpha \Leftrightarrow \mathcal{K} \Vdash \alpha$ for each $\mathcal{K}$ s.t. $\mathcal{K} \Vdash \gamma$ for each $\gamma \in \Gamma$

Thm [Kripke 1965]

$\Gamma \vdash_{IL} \alpha$ iff $\Gamma \Vdash \alpha$.

Kripke semantics

φ : propositional formula.

CPL : Classical Propositional Logic.

IPL : Intuitionistic Propositional Logic.

$\mathbb{N}$: standard PA model.

Notations

- CPL vs PA
- IPL vs HA

Introduction to de Jongh's theorem

Thm1 If $\vdash_{CPL} \varphi$, then $PA \vdash \sigma(\varphi)$ for each arithmetical substitution σ.

$\rightarrow$ very easy.

Thm2 If $\nvdash_{CPL} \varphi$, then $PA \nvdash \sigma(\varphi)$ for some arithmetical substitution σ.

Proof If $\nvdash_{CPL} \varphi$, then $\mathcal{M} \nvDash \varphi$ for some model $\mathcal{M}$.

Let σ be the following arithmetical substitution:

$$\sigma(p) = \begin{cases} 0 = 0 & \text{if } \mathcal{M} \vDash p \\ 0 \neq 0 & \text{if } \mathcal{M} \nvDash p \end{cases},$$

then we have $\mathbb{N} \nvDash \varphi$.

$\mathcal{M}$

$\mathcal{M} \vDash p_1, p_2, p_4$

$\mathcal{M} \nvDash p_3$

$\mathbb{N}$

$\mathbb{N} \vDash 0 = 0$

$\mathbb{N} \nvDash 0 \neq 0$

CPL vs PA

Thm2 If $\nvdash_{CPL} \varphi$, then $PA \nvdash \sigma(\varphi)$ for some arithmetical substitution σ.

Thm2+ There exists an arithmetical substitution σ s.t. if $\nvdash_{CPL} \varphi$, then $PA \nvdash \sigma(\varphi)$.

Σ_1

↑

Lemma [Myhill1972]

There are Σ_1-sentences $S_1, S_2, \ldots$ s.t.

if $S_i' \equiv S_i$ or $\neg S_i$ then $AA \cup \{S_1', S_2', \ldots\}$ has a model.

$\sigma(p_i) = S_i$:

$\mathcal{M} \nvDash \varphi(p_1, p_2, p_3, p_4)$

$\mathcal{M} \vDash p_1, p_2, p_4$

$\mathcal{M} \nvDash p_3$

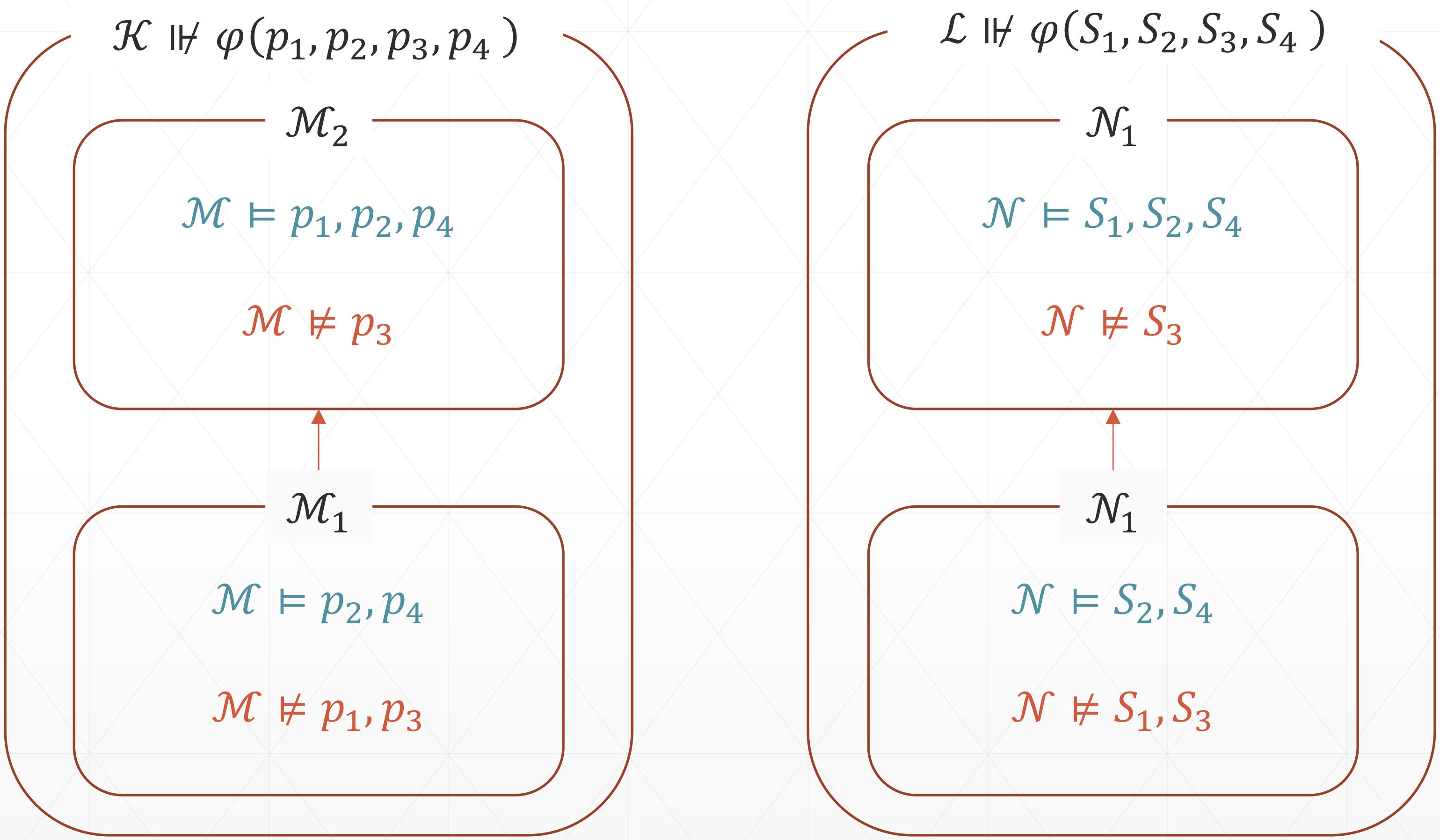
$\mathcal{N} \nvDash \varphi(S_1, S_2, S_3, S_4)$

$\mathcal{N} \vDash S_1, S_2, S_4$

$\mathcal{N} \nvDash S_3$

Q If $\nvdash_{IPL} \varphi$, then does $HA \nvdash \sigma(\varphi)$ hold for some σ ?

$\leftrightarrow$ Can we find the following $S_1, S_2, \ldots$ and $\mathcal{L}$?



Q+ Is there an arithmetical substitution σ s.t. if $\nvdash_{IPL} \varphi$, then $HA \nvdash \sigma(\varphi)$?

CPL vs PA

Q If $\nvdash_{IPL} \varphi$, then does $HA \nvdash \sigma(\varphi)$ hold for some σ ?

A Yes! There is a substitution.

(Weak version of de Jongh's theorem / Arithmetical completeness)

Q+ Is there an arithmetical substitution σ s.t. if $\nvdash_{IPL} \varphi$, then $HA \nvdash \sigma(\varphi)$?

A Yes! There is a uniform substitution. Π_2

(de Jongh's theorem / Uniform version of arithmetical completeness)

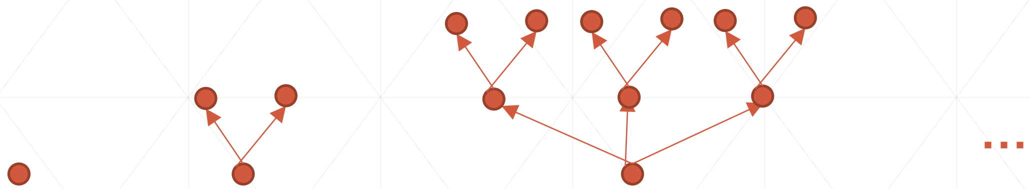
- Proof of (weak) de Jongh’s theorem
- Strong verision
- de Jongh’s theorem for one propositional variable

Some variants of de Jongh’s theorem

Thm If $\nvdash_{IPL} \varphi$, then $HA \nvdash \sigma(\varphi)$ for some arithmetical substitution σ.

Lemma [Jaskowski1936,Smorinski1973]

If $\nvdash_{IPL} \varphi$, then $\mathcal{K} \nVdash \varphi$ for some $\mathcal{K}$ based on either of the following tree:



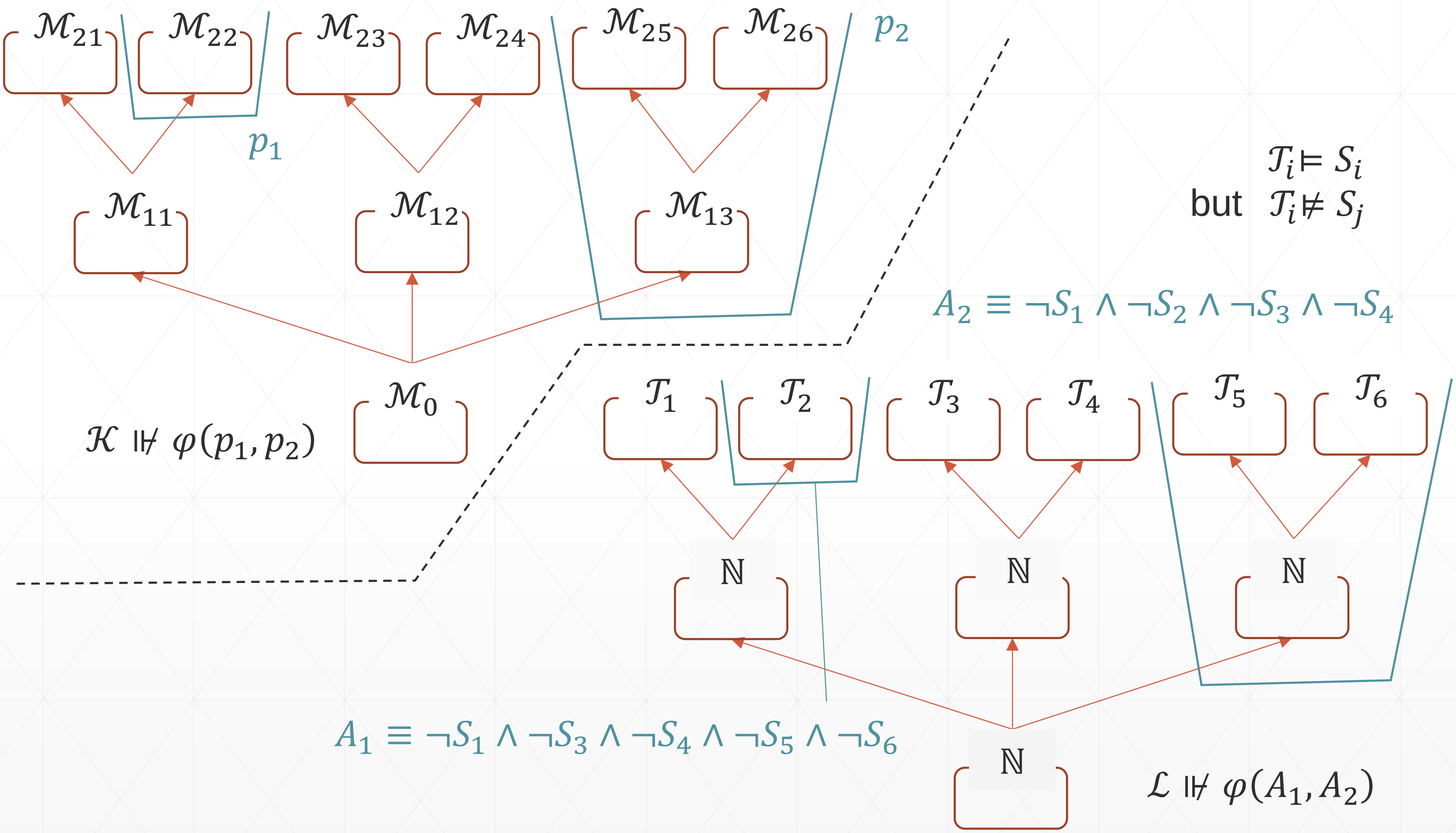
Lemma [Myhill1972]

There are Σ_1-sentences $S_1, S_2, \ldots$ s.t.

if $S_i' \equiv S_i$ or $\neg S_i$ then $AA \cup \{S_1', S_2', \ldots\}$ has a PA model.

Proof of (weak) de Jongh's theorem

Thm If $\nvdash_{IPL} \varphi$, then $HA \nvdash \sigma(\varphi)$ for some arithmetical substitution σ.



Proof of (weak) de Jongh's theorem

Thm If $\nvdash_{IPL} \varphi$, then $HA \nvdash \sigma(\varphi)$ for some arithmetical substitution σ.

Thm+ If $\nvdash_{IPL} \varphi$, then $HA \nvdash \sigma(\varphi)$ for some arithmetical Σ_1-substitution σ.

Thm++ There exists an arithmetical Π_2-substitution σ s.t. $\nvdash_{IPL} \varphi$ implies $HA \vdash \sigma(\varphi)$.

Thm+++ Let $S_1, S_2, \ldots$ be arbitrary Π_2-sentences, independent over $PA + \{\text{true } \Pi_1\text{-sentences}\}$, and let $\sigma(p_i) = S_i$. Then $\nvdash_{IPL} \varphi$ implies $HA \vdash \sigma(\varphi)$.

Strong version

Thm

- $\varphi(p)$: propositional formula consisting of only one propositional variable p.
- S : arbitrary Σ_1-sentence independent over PA.

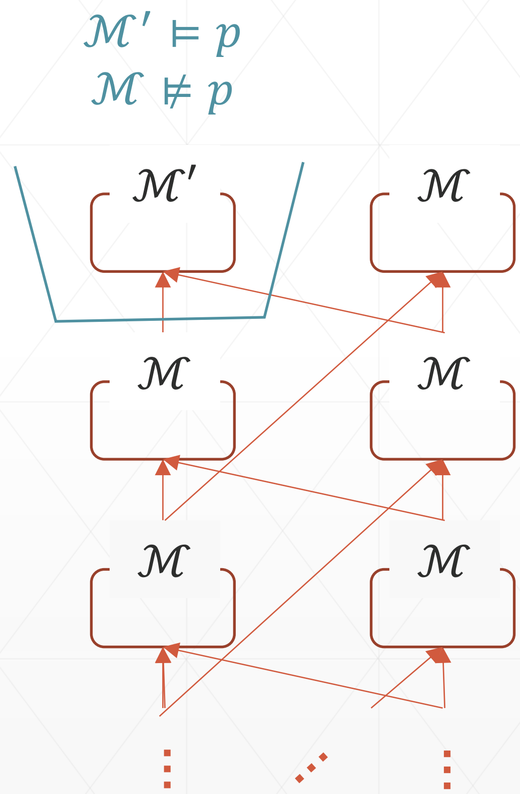
$\Rightarrow\ \nvdash_{IPL} \varphi$ implies $HA \nvdash \sigma(S)$.

Lemma [Nishimura1960]

- $\varphi(p)$: propositional formula consisting of only one propositional variable p.
- $\nvdash_{IPL} \varphi(p)$.

$\Rightarrow \mathcal{N} \Vdash\!\!\!\!/\ \varphi(p)$

$\mathcal{N}$:



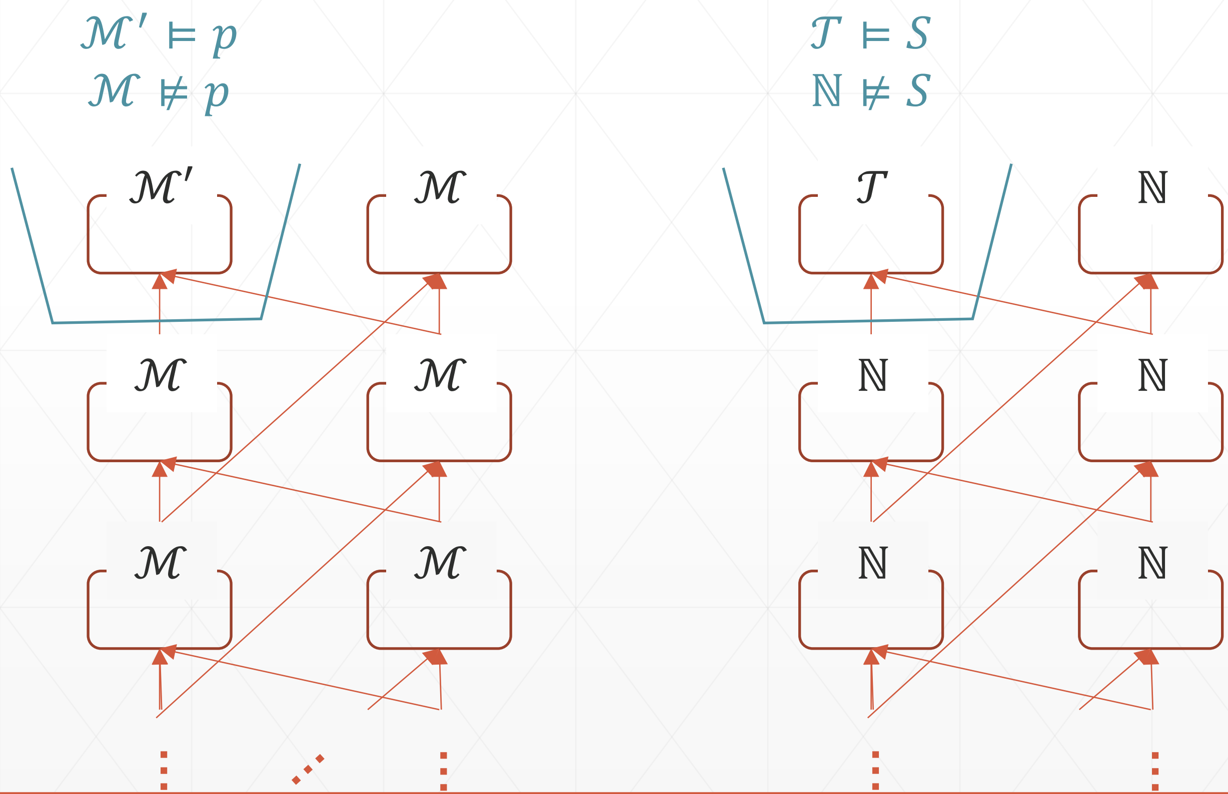
de Jongh's theorem for one propositional variable

Thm

- $\varphi(p)$: propositional formula consisting of only one propositional variable p.
- S : arbitrary Σ_1-sentence independent over PA.

$\Rightarrow\ \nvdash_{IPL} \varphi$ implies $HA \nvdash \sigma(S)$.

Proof



de Jongh's theorem for one propositional variable

Summary

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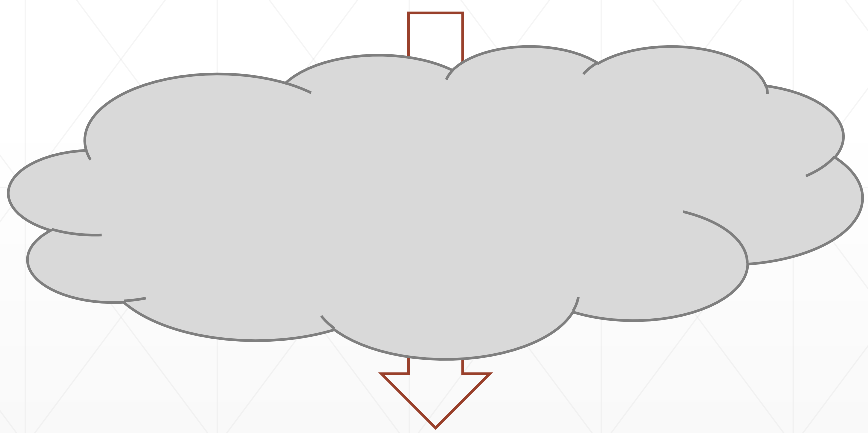

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Summary

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