A Common Notation System for Lambda Calculus and Combinatory Logic

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Today's Key Phrases

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Semantics of syntax

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What you see is (not) what you get

The syntax of Lambda Calculus and Combinatory Logic

$$
\begin{aligned}
\mathbb{X} &::= x, y, z, \cdots \\
M, N \in \Lambda &::= x \mid \lambda_x M \mid (M\ N)^0 \\
M, N \in \mathsf{CL} &::= x \mid \mathsf{I} \mid \mathsf{K} \mid \mathsf{S} \mid (M\ N)^0
\end{aligned}
$$

$(M\ N)^0$ stands for the *application* of the function M to its argument N. It is often written simply MN or $M(N)$, but we will always use the notation $(M\ N)^0$ for the application.

Lambda Calculus

$$M, N \in \Lambda \ ::= \ x \mid \lambda_x M \mid (M\ N)^0$$

$\lambda_x M$ stands for the function obtained from M by abstracting x in M.

$$(\lambda_x M\ N)^0 \to [x := N]M$$

Example

$$\begin{aligned}(\lambda_x x\ M)^0 &\to [x := M]x = M \\ ((\lambda_{xy} x\ M)^0\ N)^0 &\to ([x := M]\lambda_y x\ N)^0 = (\lambda_y M\ N)^0 \\ &\to [y := N]M = M\end{aligned}$$

Combinatory Logic

$$M, N \in \mathsf{CL} \ ::= \ x \mid \mathsf{I} \mid \mathsf{K} \mid \mathsf{S} \mid (M\ N)^0$$

$$(\mathsf{I}\ M)^0 \to M$$
$$((\mathsf{K}\ M)^0\ N)^0 \to M$$
$$((\mathsf{S}\ M)^0\ N)^0 P \to ((M\ P)^0\ (N\ P)^0)^0$$

These rules suggest the following identities.

$$\mathsf{I} = \lambda_x x$$
$$\mathsf{K} = \lambda_{xy} x$$
$$\mathsf{S} = \lambda_{xyz}((x\ z)^0\ (y\ z)^0)^0$$

By this identification, every combinatory term becomes a lambda term. Moreover, the above rewriting rules all hold in the lambda calculus.

Combinatory Logic

What about the converse direction? We can translate every lambda term to a combinatory term as follow.

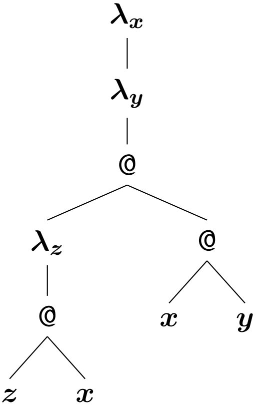
$$x^* = x$$
$$(\lambda_x M)^* = \lambda^*{}_x M^*$$
$$((M\ N)^0)^* = (M^*\ N^*)^0$$

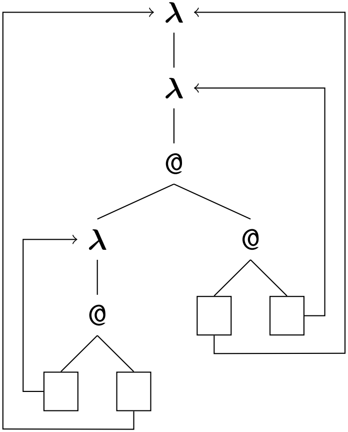
We used λ^* above, which is defined by:

$$\lambda^*{}_x x := \mathsf{I}$$
$$\lambda^*{}_x y := (\mathsf{K}\ y)^0 \text{ if } x \neq y$$
$$\lambda^*{}_x (M\ N)^0 := ((\mathsf{S}\ \lambda^*{}_x M)^0\ (\mathsf{S}\ \lambda^*{}_x N)^0)^0$$

Church's syntax and Quine-Bourbaki notation (1)

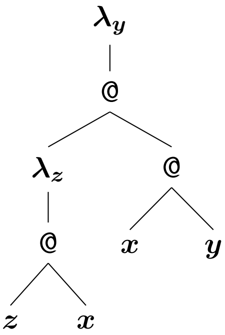
$$\lambda_x \lambda_y (\lambda_z (z\ x)^0\ (x\ y)^0)^0$$

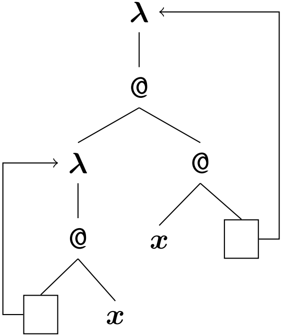




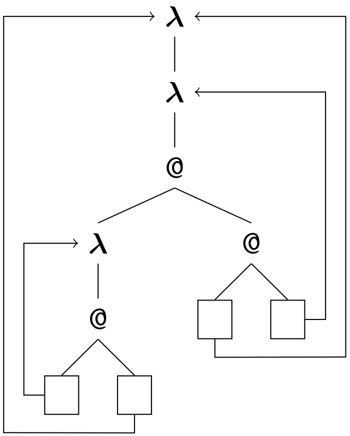
Church's syntax and Quine-Bourbaki notation (2)

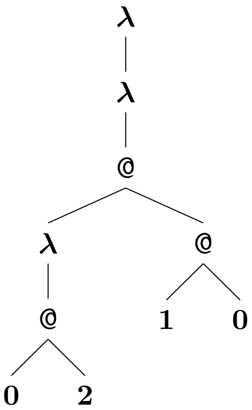
$$\lambda_y(\lambda_z(z\ x)^0\ (x\ y)^0)^0$$



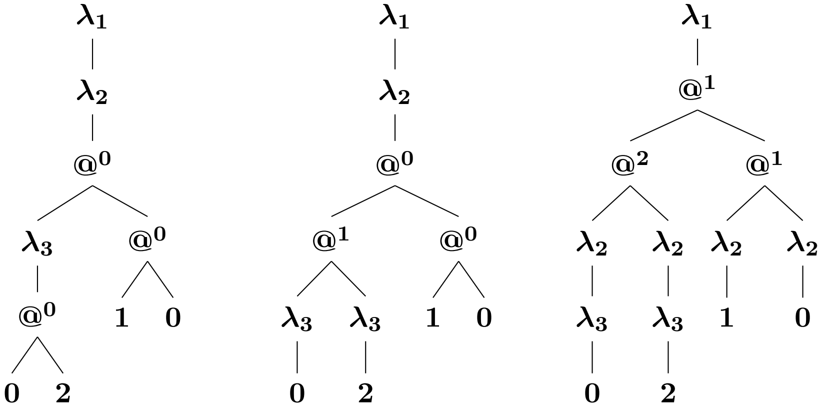


Quine-Bourbaki notation and de Bruijn notation

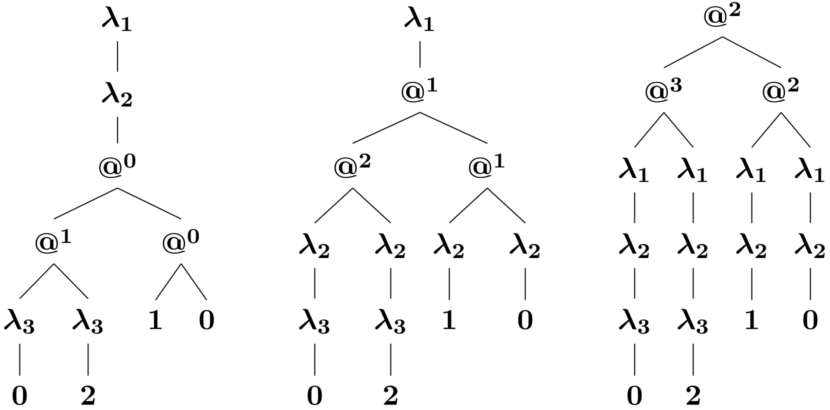




Generalized de Bruijn notation (1)



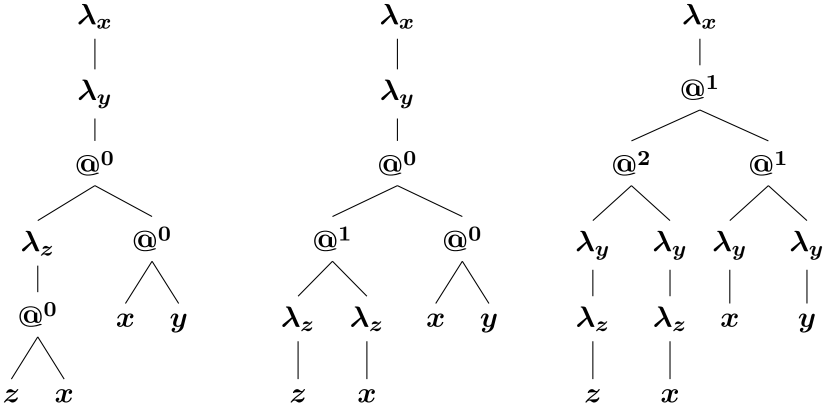
Generalized de Bruijn notation (2)



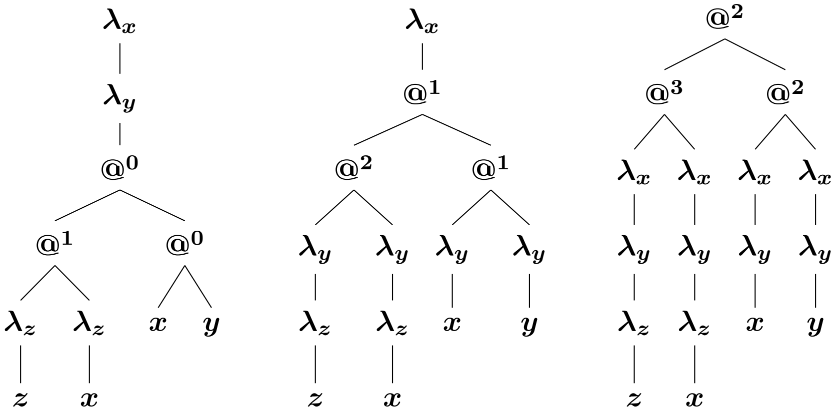
Nameless binder and distributive law

$$\lambda(D\ E)^n = (\lambda D\ \lambda E)^{n+1}$$

Generalized Church's syntax (1)

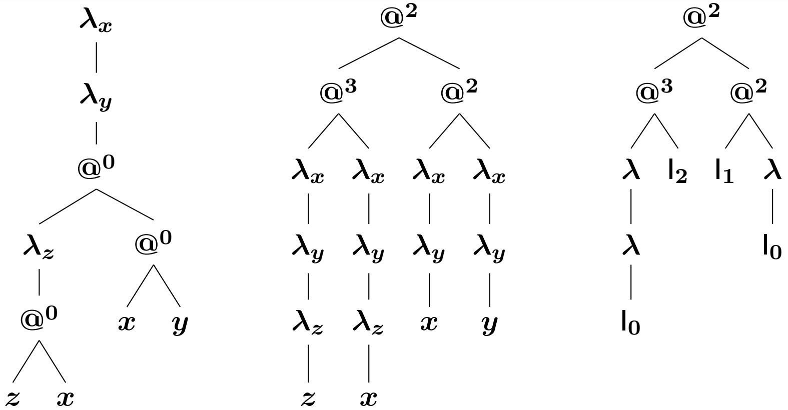


Generalized Church's syntax (2)



Distributive Law: $\lambda_x(D\ E)^n = (\lambda_x D\ \lambda_x E)^{n+1}$.

α-reduction



$$\lambda_x x \to_\alpha \mathsf{I}_0,\ \lambda_x\lambda_y x \to_\alpha \mathsf{I}_1,\ \lambda_x\lambda_y\lambda_z x \to_\alpha \mathsf{I}_2, \ldots$$

$$\lambda_x \mathsf{I}_k \to_\alpha \lambda\mathsf{I}_k,\ \lambda_x\lambda\mathsf{I}_k \to_\alpha \lambda\lambda\mathsf{I}_k,\ \lambda_x\lambda\lambda\mathsf{I}_k \to_\alpha \lambda\lambda\lambda\mathsf{I}_k, \ldots$$

α-reduction rules can compute α normal form.
To achieve this, we must extend Church's syntax!

Common extension of lambda calculus and combinatory logic

Definition (The datatypes $\mathbb{M}$, $\boldsymbol{\Lambda}$ and CL)

$$M, N \in \mathbb{M} \ ::= \ x \mid \mathsf{I}_k \mid \lambda_x M \mid \lambda M \mid (M\ N)^i$$
$$M, N \in \boldsymbol{\Lambda} \ ::= \ x \mid \lambda_x M \mid (M\ N)^0$$
$$M, N \in \mathsf{CL} \ ::= \ x \mid \mathsf{I} \mid \mathsf{K} \mid \mathsf{S} \mid (M\ N)^0$$

Combinators I, K and S are definable in $\mathbb{M}$ as abbreviations:

$$\mathsf{I} := \mathsf{I}_0$$
$$\mathsf{K} := \mathsf{I}_1$$
$$\mathsf{S} := ((\mathsf{I}_2\ \lambda\lambda\mathsf{I}_0)^3\ (\lambda\mathsf{I}_1\ \lambda\lambda\mathsf{I}_0)^3)^3$$, or, Atsushi Igarashi remarked,
$$\mathsf{S} := (\mathsf{I}_1\ \lambda\lambda\mathsf{I}_0)^3$$

Definition (One step α-reduction on $\mathbb{M}$ and α-nf)

$$\frac{}{\lambda_x \lambda^i \mathsf{I}_k \to_{1\alpha} \lambda^{i+1} \mathsf{I}_k}\ \mathsf{E_1}$$

$$\frac{}{\lambda_x \lambda^i x \to_{1\alpha} \mathsf{I}_i}\ \mathsf{E_2} \qquad \frac{x \neq y}{\lambda_x \lambda^i y \to_{1\alpha} \lambda^{i+1} y}\ \mathsf{E_3}$$

$$\frac{}{\lambda_*(M\ N)^i \to_{1\alpha} (\lambda_* M\ \lambda_* N)^{i+1}}\ \mathsf{D} \qquad \frac{M \to_{1\alpha} M'}{\lambda_* M \to_{1\alpha} \lambda_* M'}\ \mathsf{C_1}$$

$$\frac{M \to_{1\alpha} M'}{(M\ N)^i \to_{1\alpha} (M'\ N)^i}\ \mathsf{C_2} \qquad \frac{N \to_{1\alpha} N'}{(M\ N)^i \to_{1\alpha} (M\ N')^i}\ \mathsf{C_3}$$

Example

This example shows how the variable-binders $\boldsymbol{\lambda}_x$ and $\boldsymbol{\lambda}_y$ are eliminated by one step α-reductions.

$$
\begin{aligned}
\boldsymbol{\lambda}_x\boldsymbol{\lambda}_y(y\ x)^0 &\rightarrow_{1\alpha} \boldsymbol{\lambda}_x(\boldsymbol{\lambda}_y y\ \boldsymbol{\lambda}_y x)^1 \\
&\rightarrow_{1\alpha} \boldsymbol{\lambda}_x(\mathsf{I}\ \boldsymbol{\lambda}_y x)^1 \\
&\rightarrow_{1\alpha} \boldsymbol{\lambda}_x(\mathsf{I}\ \boldsymbol{\lambda} x)^1 \\
&\rightarrow_{1\alpha} (\boldsymbol{\lambda}_x\mathsf{I}\ \boldsymbol{\lambda}_x\boldsymbol{\lambda} x)^2 \\
&\rightarrow_{1\alpha} (\boldsymbol{\lambda}\mathsf{I}\ \boldsymbol{\lambda}_x\boldsymbol{\lambda} x)^2 \\
&\rightarrow_{1\alpha} (\boldsymbol{\lambda}\mathsf{I}\ \mathsf{K})^2 \quad \square
\end{aligned}
$$

The datatype $\mathbb{L}$

Definition (The datatypes $\mathbb{T}$ and $\mathbb{L}$)

$$
\begin{aligned}
t \in \mathbb{T} &::= \lambda^i \mathsf{I}_k \mid \lambda^i x \\
M, N \in \mathbb{L} &::= t \mid (M\ N)^i
\end{aligned}
$$

Elements of $\mathbb{T}$ are called *threads*.

Theorem

An $\mathbb{M}$-term M is an α-nf if and only if M is an $\mathbb{L}$-term.

Definition (Height (Ht) of $\mathbb{L}$-terms)

$$
\begin{aligned}
\mathsf{Ht}(\lambda^i \mathsf{I}_k) &:= i + k + 1 \\
\mathsf{Ht}(\lambda^i x) &:= i \\
\mathsf{Ht}((M\ N)^i) &:= \min\{i, \mathsf{Ht}(M), \mathsf{Ht}(N)\}
\end{aligned}
$$

α-reduction

Definition (α-reduction on $\mathbb{M}$ and α-equality)

$$\frac{M_0 \to_{1\alpha} M_1 \quad M_1 \to_{1\alpha} M_2 \quad \cdots \quad M_{n-1} \to_{1\alpha} M_n}{M_0 \to_\alpha M_n}$$

When we have $M_0 \to_\alpha M_n$ by this rule, we say that M_0 *α-reduces to M_n in n steps*.

$$\frac{M \to_\alpha P \quad N \to_\alpha P}{M =_\alpha N}$$

$=_\alpha$ is a decidable equivalence relation

Theorem

Given any $\mathbb{M}$-term M, there uniquely exists an N such that $M \to_\alpha N$ and N is an α-nf.

Remark

1. $(-)_{\alpha} : \mathbb{M} \to \mathbb{M}$ is idempotent, i.e., $(M_{\alpha})_{\alpha} = M_{\alpha}$ and image of $(-)_{\alpha}$ is $\mathbb{L}$.
2. For any $M \in \mathbb{M}$, $M =_{\alpha} M_{\alpha}$.
3. For any $M \in \mathbb{M}$, $M = M_{\alpha}$ iff $M \in \mathbb{L}$.
4. $M =_{\alpha} N$ iff $M_{\alpha} = N_{\alpha}$.

Thus M_{α} is a natural representative of the equivalence class $\{N \in \mathbb{M} \mid N =_{\alpha} M\}$ containing M.

Instantiation

Definition (Instantiation of threads at level n)

If $t \in T^{n+1}$ and $u \in T^n$, then $\langle t\ u \rangle^n$ can be computed by the following equations.

$$\langle \lambda^i \mathsf{l}_k\ \lambda^j \mathsf{l}_\ell \rangle^n := \begin{cases} \lambda^{i-1}\mathsf{l}_k & \text{if } n < i, \\ \lambda^{j+k}\mathsf{l}_\ell & \text{if } n = i \leq j, \\ \lambda^{j}\mathsf{l}_{\ell+k} & \text{if } n = i > j, \\ \lambda^{i}\mathsf{l}_{k-1} & \text{if } n > i. \end{cases}$$

$$\langle \lambda^i \mathsf{l}_k\ \lambda^j x \rangle^n := \begin{cases} \lambda^{i-1}\mathsf{l}_k & \text{if } n < i, \\ \lambda^{j+k}x & \text{if } n = i, \\ \lambda^{i}\mathsf{l}_{k-1} & \text{if } n > i. \end{cases}$$

$$\langle \lambda^i x\ t \rangle^n := \lambda^{i-1}x$$

Instantiation at level n

Define lift $\uparrow_n^k : \mathbb{L}^n \to \mathbb{L}^{n+k}$ by

$$\uparrow_n^k \lambda^j \mathsf{I}_\ell := \begin{cases} \lambda^{j+k}\mathsf{I}_\ell & \text{if } n \leq j, \\ \lambda^j \mathsf{I}_{\ell+k} & \text{if } n > j. \end{cases}$$

$$\uparrow_n^k \lambda^j x := \lambda^{j+k} x$$

$$\uparrow_n^k (M\ N)^j := (\uparrow_n^k M\ \ \uparrow_n^k N)^{j+k}.$$

Definition (Instantiation at level n)

If $M \in \mathbb{L}^{n+1}$ and $P \in \mathbb{L}^n$, then $\langle M\ P\rangle^n$ is defined by the following equations.

$$\langle \lambda^i \mathsf{I}_k\ P\rangle^n := \begin{cases} \lambda^{i-1}\mathsf{I}_k & \text{if } n < i, \\ \uparrow_n^k P & \text{if } n = i, \\ \lambda^i \mathsf{I}_{k-1} & \text{if } n > i. \end{cases}$$

$$\langle \lambda^i x\ P\rangle^n := \lambda^{i-1} x.$$

$$\langle (M\ N)^{i+1}\ P\rangle^n := (\langle M\ P\rangle^n\ \langle N\ P\rangle^n)^i.$$

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