# Logic and Game Theory

by Mamoru Kaneko, Waseda University, May 16, May 2017



## Why “logic and game theory”?

1. Similarities in targets and methods
2. “Symbolic” in Logic and Game Theory
3. What are missing in Logic and Game Theory

## 1. Similarities (or analogy)

- people are involved -- an ideal mathematician in Logic and players in GT

  **A**: an ideal mathematician - - logical inference about mathematical discourse

  **B**: players -- decision making and behavior by inference from his beliefs/knowledge

- **A** is the radical start of the modern logic (i.e., proof theory).
- von Neumann (‘28) got the idea of game theory from the above analogy between **A** and **B**.

## 2. "Symbols" in Symbolic Logics

- Use of symbols - - essential for a study of human thinking
- Separation between symbolic expressions and their intended contents

- This separation corresponds to the distinction between "syntax" (proof theory) and "semantics" (model theory).

## 3. What are missing in Logic and Game Theory

In game theory,

- explicit treatment reasoning (deductive and inductive)

In logic,

- Interactions between *ex ante* thoughts and *ex post* experiences
- Other people and their minds

# Three Research Projects

- Epistemic Logic
  - Logical inference to decision and prediction

    - - logic includes other aspects of human thinking such as “tense”, “deontic”, etc.
- Inductive Game Theory
  - To construct an individual view from **experiences**, e.g., prejudices and discrimination
- Social Justice
  - Evaluations (social welfare)
  - Design of social institution with its operational managements, e.g., progressive income taxation

Common features

- Use of Symbols

  symbolic expressions

  vs. their contents
- Constructive
- Operational

→ Bounded rationality

# Four fields in mathematical logic (foundations of mathematics)

## 1: Axiomatic Set Theory

- abstract and transcendental - - no suggestions for social science.

## 2: Model Theory (semantics)

- models (contents) → language expressions
- provides convenient tools, but not “radical” .

## 3: Proof Theory (syntax)

- language (symbolic) expressions → contents
- “radical” -- we may ignore “contents”
- new insights to human thinking.

## 4: Theory of computations (Turing machines, etc.)

- insightful in distinguishing between “finite” and “infinite”
- So far, it is still close to “infinite worlds”.

Fields 2 and 3 are connected by “soundness-completeness” theorem.

Bounded Rationality

- How ← logic
- Contents ← mathematics

# Agenda of this talk



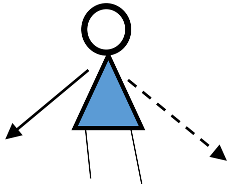
- I: Small introduction to game theory
- II: A difficulty in an existence proof
  - A proof is simple but not constructive
  - Related to Hilbert's 7th problem and a constructive proof needed 30 years
  - Implication to game theory?
- III: Mutual misunderstanding - - Comic story "Konnnyaku Mondo".
  - Mutual misunderstanding - -
    each person thinks they understand each other perfectly, but actually not.
  - A separation between "symbolic expressions" and "their contents".
- IV: Synthesizing
  - Synthesizes the above.

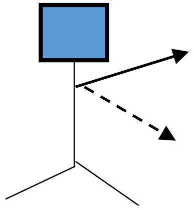
# Small introduction to game theory

- 2 players is facing a game:
  - each has 2 available actions C and N;
  - each makes independent choice of one action;
  - after their choices, their payoffs are given
  
  - - these are described in the bi-matrix.
- The above game is Prisoner's Dilemma.

  The second is Battle of the Sexes.

- Game theory makes a conjecture for the players to play the pair of actions "Nash equilibrium".





| | C | N |
|---|---|---|
| C | 5, 5 | 1, 6 |
| N | 6, 1 | 3, 3 Nash |

Prisoner's dilemma

| | M | B |
|---|---|---|
| M | 5, 10 Nash | 0, 0 |
| B | 2, 2 | 10, 5 Nash |

Battle of the Sexes

# Small introduction to game theory

- Some games have no “Nash equilibrium”.

- We have a Nash eq. when probabilistic choice is allowed.
  - Nash equilibrium - - each follows the probability distribution $\left(\frac{1}{2}, \frac{1}{2}\right)$ to choose an action.

- One central theorem in game theory is: the existence of a Nash equilibrium.
  - The existence ensures each player to choose an action following the Nash equilibrium.
  - Really?

| | H | T |
|---|---|---|
| H | 1, -1 | -1, 1 |
| T | -1, 1 | 1, -1 |

Matching Pennies

# Part I: A Non-constructive Existence Theorem

Theorem: There exist (possibly identical) irrational (real) numbers $\alpha$ and $\beta$ s.t $\alpha^{\beta}$ is rational.

**Proof**: Consider $\sqrt{2}$, which is known to be irrational. Consider the two cases.

(1): Suppose $\sqrt{2}^{\sqrt{2}}$ be rational. Then, we have two irrationals $\alpha = \beta = \sqrt{2}$.

(2): Suppose $\sqrt{2}^{\sqrt{2}}$ be irrational. Let $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$. Then,

$$\alpha^{\beta} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^{2} = 2.//$$

- The law of excluded middle, $A \vee (\neg A)$, is used:

  (1) and (2) share "existence of irrational $\alpha$ and $\beta$ with rational $\alpha^{\beta}$" in common.

- However, it does not indicate which of (1) and (2) is the case.

  The above proof is accepted by classical mathematics but not by constructive mathematics.

- Hilbert's 7th problem (1900): Prove that $2^{\sqrt{2}}$ is irrational and moreover transcendental. Around 1934, Gelfand solved this affirmatively, which implies (2) is the case.

# Implications game theory



Game theory is not targeting problems in mathematics but human behavior in society.

- An existence proof suggests the *existence* of a target which a player should play, but it may not tell what it is.
- Actually, "complexity" of logical inference is a related issue.
  - The standard notion of "complexity" is about an algorithm.
  - The above theorem talks about a particular choice, rather than an algorithm.

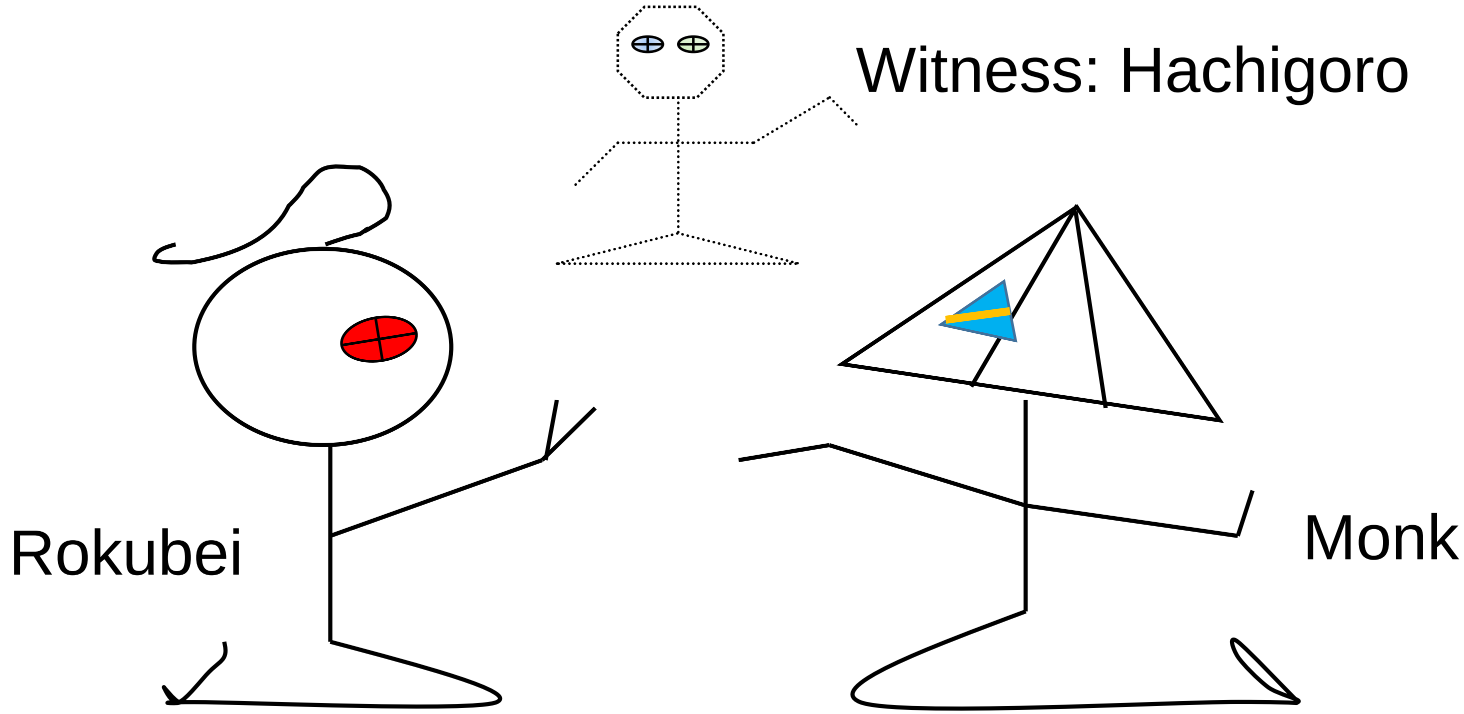
- Concepts required for playing a game here
  - "Transcendental number"?
  - How does the player generate an irrational number probability $\frac{\sqrt{2}}{2}$ ?
  - How does he play it?

So far, I have not talked about problems involving multiple players.

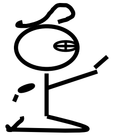
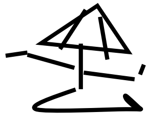
Any other problems due to interactions between players?

**Konnyaku Mondo (蒟蒻問答) (Konnyaku - - food product (devil's tongue jelly)) - - a Japanese comic story performed by a Rakugo teller.**

- ○ The other's mind and false beliefs:
- ○ CK as shared information, but
- ○ Mutual misunderstanding in interpretations of exchanged gestures



# Konnyaku Mondo 1

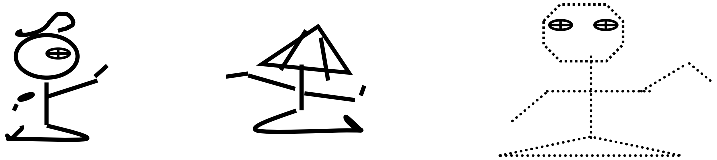
There was a temple where no monks were living any longer.
A devil's tongue jelly maker, named Rokubei, lived next door.
He moved into the temple and started pretending to be a monk.

One day, a traveling Zen Buddhist monk passed by and challenged Rokubei to a debate on Buddhism.
Rokubei had no knowledge on Buddhism and was not able to have a debate.
He tried to refuse, but he couldn't escape and finally agreed.

The Buddhist dialogue started but Rokubei didn't know how to perform and he kept silent.
The Buddhist monk tried to communicate to Rokubei in many ways.

After some time, Rokubei started responding with gestures to the body movements the monk made.

# Konnyaku Mondo 2





The monk took this as a style of dialogue and tried to answer in gestures, too. They exchanged gestures, and after some time, the monk told Rokubei,

"your thoughts are profound and mine are of no comparison.

I am very sorry to have bothered you".

After saying this, he left the temple.

Hachigoro, a neighbor of Rokubei, witnessed the whole debate, and followed the monk to ask what had happened.

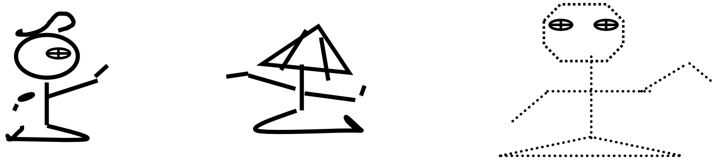
The monk answered,

"I'm not trained enough in Buddhist thoughts to compete with that master.

Please convey to him my earnest apology for having left so abruptly".

Almost as quickly as the words had left his lips, he ran away.

# Konnyaku Mondo 3



Hachigoro returned to the temple and
asked Rokubei if he knew anything about Buddhism thoughts.

Rokubei answered,
"No, I have no idea about Buddhism, but
the guy talked badly about my jelly products.
That's why I gave him a lesson".

# Implications



1. Separation between symbolic expressions and associated meanings
    - Symbolic expressions, gestures → syntax (proof theory)
    - Associated meanings, interpretations → semantics (model theory)

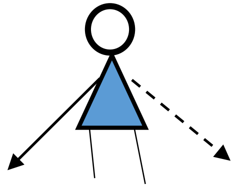
2. Subjectivities involved in two (or more) people
    - Different beliefs for them;
    - False beliefs without the true referential fact.

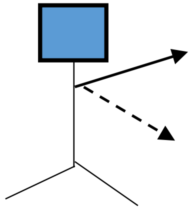
3. Each person believes “common knowledge”.
    - It is symbolic, but probabilistic.
    - It is not “knowledge (justified true belief)”.

4. Interactions between *ex ante* decisions and *ex post* experiences.

# Konnyaky Mondo Phenomena in Society

- Players 1 and 2 are playing game G repeatedly.
  - they have reached a stationary state $(a_2, b_2)$ ;
  - after each play of the game, each player observes
    - the pair of actions played
    - his (her) own payoff value
    - with low frequencies, each makes trials-errors of the other action.





| | $b_1$ | $b_2$ |
|---|---|---|
| $a_1$ | 5, 5 | 0, 6 |
| $a_2$ | 2, 1 | 10, 3 NE played |

G: Game played

- From these experiences, each constructs his (her) understanding of the game adding the missing payoffs;
  - player 1 thinks they are playing Battle of the Sexes;
  - player 2 thinks they are playing Prisoner's dilemma.

| | M | B |
|---|---|---|
| M | 5, 10 | 0, 0 |
| B | 2, 2 | 10, 5 |

| | C | N |
|---|---|---|
| C | 5, 5 | 1, 6 |
| N | 6, 1 | 3, 3 |

- These views contain false believes about the other's payoff
- **However, they cannot correct them with experiences.**

## Elements and Issues to be considered;

- External objectivity vs. internal subjectivity
  - Inductive game theory - - interactions between prediction making about the other's decision; and his own decision making - - *ex ante*
  - Experiences (observation) - - *ex post*
- Other's mind?
  - How can a person learn (guess) the other's mind?
  - Social roles

- One important application:
  - Discrimination and prejudices (Kaneko-Matsui (1999)).

# Symbolic Expressions vs. their Contents

Common

Syntax: Language - formula generation rules

## Epistemic Logic and Game Theory

*ex ante* prediction/decision with explicitly formulated logical abilities for each player

## Inductive Game Theory

Interactions between *ex ante* prediction/decision and *ex post* observations (experiences)

# Summary



- Logic and Game theory are related in many entangled manners.
  - Objectivity vs. Subjectivity
  - *Ex ante* decisions/prediction making vs. *ex post* observations
- These lead us to better understandings of societies and ourselves.

# References


- Fagin, R., J. Y. Halpern, Y. Moses and M. Y. Verdi, (1995), Reasoning about Knowledge, The MIT Press, Cambridge.
- Kaneko, M., and T. Nagashima, (1996), Game Logic and its Applications, Studia Logica 57, 325--354.
- Kaneko, M., (2002), Epistemic logics and their game theoretical applications: Introduction. Economic Theory 19, 7-62.
- Kaneko, M., (2004), Game Theory and Mutual Misunderstanding,  Springer.
- Kaneko, M., and A. Matsui, (1999), Inductive game theory: discrimination and prejudices, (with A. Matsui). Journal of Public Economic Theory 1, 101-137.
- Kaneko, M., and J. J. Kline (2008), Inductive Game Theory: A Basic Scenario,  Journal of Mathematical Economics 44, (2008), 1332-1363.
- 金子守、石川竜一郎, (2013), 制度と認識の経済学の構築に向けて、 経済セミナー 2013年、10-11月号、69-75