Maximal regularity and the Newton polygon approach

Maximal Regularity Theorems and Mathematical Fluid Dynamics

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Konstanz

A map of Germany

Konstanz (population $\sim$ 80 000)

University of Konstanz

11,000 students, 190 full professors

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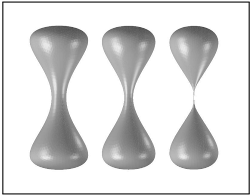
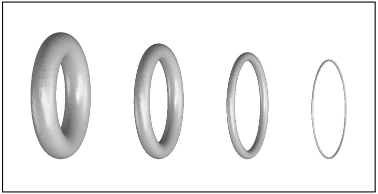
An example: mean curvature flow

The mean curvature flow is described by the equation $V = -H$. Here V is the velocity of the surface in normal direction, and H is the mean curvature.





One aim of parabolic theory is to show (local) well-posedness of the equation:

Theorem (what we want to show)

For every initial surface, the mean curvature equation has a unique solution with maximal existence interval. The solution is infinitely smooth in time and space.

An example: mean curvature flow

In local coordinates, the mean curvature flow is given by

$$\begin{aligned} \partial_t u - \Delta u &= -\sum_{i=1}^{n} \frac{\partial_i u \partial_j u}{1 + |\nabla u|^2} \, \partial_i \partial_j u \quad (t \in (0, T)), \\ u(0) &= u_0 \end{aligned} \tag{1}$$

with $T \in (0, \infty)$. Here $\partial_i = \frac{\partial}{\partial x_i}$, and $\Delta := \partial_1^2 + \cdots + \partial_n^2$ is the Laplace operator. Equation (2) is an example of a quasilinear parabolic partial differential equation. General form:

$$\begin{aligned} \partial_t u - Au &= G(u), \\ u(0) &= u_0. \end{aligned}$$

Here A is a linear operator (e.g., differential operator in space) and G is a nonlinear operator with $G(0) = G'(0) = 0$.

Linearization

We want to solve the quasilinear equation

$$\partial_t u - Au = G(u), \\ u(0) = u_0.$$

For this, we linearize the equation. So consider for fixed v the linear equation

$$\partial_t u - Au = G(v) \quad (t \in (0, T)), \\ u(0) = u_0.$$

Idea of maximal regularity:

If the solution of the linear problem is smooth enough, we can apply Banach's fixed point theorem (contraction mapping principle) to get a unique local solution of the nonlinear problem.

Maximal regularity

The linearized problem has the form

$$\begin{aligned} \partial_t u - Au &= f \quad (t \in (0,T)), \\ u(0) &= u_0 \end{aligned} \tag{2}$$

with $f := G(v)$.

Function spaces:

We are looking for spaces

$$u \in \mathbb{E},\ f \in \mathbb{F},\ u_0 \in \gamma_t\mathbb{E}$$

such that

$$u \mapsto (f, u_0),\ \mathbb{E} \to \mathbb{F} \times \gamma_t\mathbb{E}$$

is an isomorphism. Typical choices are:

- Hölder spaces C^α,
- L^p-Sobolev spaces.

Maximal L^p-regularity

Let X be a complex Banach space and $A\colon X \supset D(A) \to X$ be a closed operator. We consider the abstract Cauchy problem

$$\begin{aligned} \partial_t u - Au &= f \quad (t > 0), \\ u(0) &= u_0. \end{aligned}$$

In the L^p-setting, the natural space for f is

$$f \in \mathbb{F} := L^p((0,T);X).$$

For maximal regularity we want to have $\partial_t u \in \mathbb{F}$ and $Au \in \mathbb{F}$, so the natural space for u is

$$u \in \mathbb{E} := W_p^1((0,T);X) \cap L^p((0,T);D(A)).$$

Here, $D(A)$ is endowed with the graph norm $\|\cdot\|_A$.

Function spaces

The Cauchy problem has the form

$$\partial_t u - Au = f \quad (t > 0),$$
$$u(0) = u_0.$$

Let $\gamma_t \colon u \mapsto u|_{t=0}$ be the time trace. The natural trace space is given by

$$\gamma_t \mathbb{E} := \left\{ u_0 \in X : \exists\, u \in \mathbb{E} : \gamma_t u = u_0 \right\}$$

with norm

$$\|u_0\|_{\gamma_t \mathbb{E}} := \inf \left\{ \|u\|_{\mathbb{E}} : u \in \mathbb{E},\, \gamma_t u = u_0 \right\}.$$

Remark: If $D(A) = W_p^k(\mathbb{R}^n)$, we know that $\gamma_t \mathbb{E} = B_{pp}^{k-k/p}(\mathbb{R}^n)$.

Maximal L^p-regularity

Let X be a Banach space and $A\colon X \supset D(A) \to X$ be a closed operator. Let $p \in (1,\infty)$ and $T \in (0,\infty)$.

Definition

The operator A has maximal L^p-regularity in $(0,T)$ if

$$\begin{pmatrix} \partial_t - A \\ \gamma_t \end{pmatrix} : \mathbb{E} \to \mathbb{F} \times \gamma_t \mathbb{E}, \quad u \mapsto \begin{pmatrix} f \\ u_0 \end{pmatrix} := \begin{pmatrix} \partial_t u - Au \\ u\big|_{t=0} \end{pmatrix}$$

is an isomorphism.

In this case we have a continuous solution operator

$$S = \begin{pmatrix} \partial_t - A \\ \gamma_t \end{pmatrix}^{-1} : (f, u_0) \mapsto u,$$

i.e., $u = S(f, u_0)$ is the unique solution of

$$\begin{aligned} \partial_t u - Au &= f \quad (t > 0), \\ u\big|_{t=0} &= u_0. \end{aligned}$$

Remarks on maximal regularity

- To show maximal regularity, we may assume $u_0 = 0$.
- The nonlinear problem

$$\begin{aligned} \partial_t u + Au &= G(u) \quad (t \in (0, T)), \\ \gamma_t u &= u_0 \end{aligned}$$

is equivalent to the fixed-point equation

$$u = S(G(u), u_0).$$

Maximal regularity

The linearization approach gives:

Theorem

If A has maximal regularity and if

$$u \mapsto S(G(u), u_0)$$

is a contraction then the nonlinear equation has a unique maximal solution, i.e. a unique solution (in L^p-sense) defined on the maximal interval of existence.

To obtain a contraction, in application we usually have

- a condition on p to control the nonlinearity $G(u)$ by Sobolev imbedding results,
- a condition on the smallness of T or of u_0.

Application to mean curvature flow

The graphical mean curvature flow equation is given by

$$\begin{aligned} \partial_t u - \Delta u &= -\sum_{i,j=1}^{n} \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\, \partial_i\partial_j u \quad (t \in (0,T)), \\ u\big|_{t=0} &= u_0. \end{aligned} \tag{3}$$

Theorem

Let $p \in (n+2, \infty)$. Then for all initial values $u_0 \in B_{pp}^{2-2/p}(\mathbb{R}^n)$ there exists a time interval $(0,T)$ with $T > 0$ such that (3) has a unique solution

$$u \in \mathbb{E} = W_p^1((0,T); L^p(\mathbb{R}^n)) \cap L^p((0,T); W_p^2(\mathbb{R}^n)).$$

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Maximal regularity and Fourier transform

We want to prove maximal L^p-regularity for the problem

$$\begin{aligned} \partial_t u - Au &= f \quad (t \in (0,\infty)), \\ \gamma_t u &= u_0. \end{aligned}$$

- We may assume $u_0 = 0$ (see above).
- We extend f and u to the whole line $t \in \mathbb{R}$ by zero.

We will apply Fourier transform with respect to time

$$(\mathscr{F}_t u)(\tau) := (2\pi)^{-1/2} \int_{\mathbb{R}} u(t) e^{-it\tau}\, dt.$$

Note that

$$\big[\mathscr{F}_t(\partial_t u)\big](\tau) = i\tau(\mathscr{F}_t u)(\tau).$$

(There is a close connection to the Laplace transform.)

Fourier transform and maximal regularity

Taking Fourier transform $\mathscr{F}_t$ with respect to t, we get

$$(i\tau - A)(\mathscr{F}_t u)(\tau) = (\mathscr{F}_t f)(\tau).$$

For maximal regularity we need

$$\partial_t u = \mathscr{F}_t^{-1} i\tau (i\tau - A)^{-1} \mathscr{F}_t f \in L^p((0,T);X).$$

Theorem

The operator A has maximal L^p-regularity if and only if

$$\mathscr{F}_t^{-1}\, i\tau (i\tau - A)^{-1} \mathscr{F}_t$$

defines a continuous operator in $L^p(\mathbb{R};X)$.

Fourier multipliers

Definition

Let $m \in L^\infty(\mathbb{R}^n; L(X))$ be an operator-valued symbol. The m is called an L^p-Fourier multiplier if for every $f \in \mathscr{S}(\mathbb{R}^n; X)$ we have $\operatorname{op}[m]f \in L^p(\mathbb{R}^n; X)$ and

$$\|\operatorname{op}[m]f\|_{L^p(\mathbb{R}^n;X)} \leq C\|f\|_{L^p(\mathbb{R}^n;X)}.$$

In this case, we can extend $\operatorname{op}[m]$ to a bounded linear operator

$$\operatorname{op}[m] \in L(L^p(\mathbb{R}^n; X)).$$

- To show maximal regularity for A, we have to show that the

$$m(\tau) := i\tau(i\tau - A)^{-1}$$

is a Fourier multiplier in $L^p(\mathbb{R}; X)$.

➡ How to prove that a symbol is a Fourier multiplier?

$\mathcal{R}$-boundedness

Definition

A family $\mathscr{T} \subset L(X)$ of bounded linear operators is $\mathcal{R}$-bounded if there exists a constant $C > 0$ with

$$\sum_{\varepsilon_1,\dots,\varepsilon_N=\pm 1} \Big\| \sum_{j=1}^N \varepsilon_j T_j x_j \Big\|_X \le C \sum_{\varepsilon_1,\dots,\varepsilon_N=\pm 1} \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_X$$

for all $x_j \in X$, $T_j \in \mathscr{T}$ and $N \in \mathbb{N}$. The smallest possible C is called the $\mathcal{R}$-bound $\mathcal{R}(\mathscr{T})$.

- Setting $N = 1$ in the definition, we get

$$\|Tx\|_X \le C\|x\|_X \quad (x \in X,\ T \in \mathscr{T}),$$

  i.e., $\mathcal{R}$-bounded implies bounded.
- If X is a Hilbert space, $\mathcal{R}$-bounded is equivalent to bounded.

Vector-valued version of Mikhlin's theorem

The following variant of Mikhlin's theorem was crucial for maximal L^p-regularity:

Theorem (Weis 2001)

Let $p \in (1, \infty)$, X be a Banach space of class HT, and let $m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X))$ with

$$\mathcal{R}\big(\{\xi^\beta \partial_\xi^\beta m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\, \beta \in \{0,1\}^n\}\big) < \infty.$$

Then m is a Fourier multiplier, i.e., $\operatorname{op}[m] \in L(L^p(\mathbb{R}^n; X))$.

This can be seen as

- $\mathcal{R}$-bounded symbols lead to bounded operators.

Fourier multipliers and $\mathcal{R}$-boundedness

The following result gives an equivalent condition for maximal regularity.

Theorem (Weis 2001)

Let $p \in (1, \infty)$, let X be a Banach space of class HT, and let A be a sectorial operator. Then the following statements are equivalent:

1. *A has maximal L^p-regularity ,*
2. *the $L(X)$-valued function $m(\tau) := i\tau(i\tau - A)^{-1}$ is an L^p-Fourier multiplier ,*
3. *the set $\{i\tau(i\tau - A)^{-1} : \tau \in \mathbb{R}\}$ is $\mathcal{R}$-bounded .*

- The equivalence of (i) and (ii) has been shown above.
- For the equivalence of (ii) and (iii), one needs the vector-valued version of Mikhlin's theorem in one dimension.

Vector-valued version of Mikhlin's theorem

The following result makes an iteration possible:

Theorem (Girardi-Weis 2003)

Let $1 < p < \infty$, X be a Banach space of class HT with property (α), and let $\{m_\lambda : \lambda \in \Lambda\} \subset C^n(\mathbb{R}^n \setminus \{0\}, L(X))$ with

$$\mathcal{R}\Big(\{\xi^\beta \partial_\xi^\beta m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0,1\}^n,\ \lambda \in \Lambda\}\Big) < \infty.$$

Then the set of associated Fourier multipliers $\{\mathscr{F}^{-1} m_\lambda \mathscr{F} : \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded in $L(L^p(\mathbb{R}^n; X))$.

- $\mathcal{R}$-bounded symbols lead to $\mathcal{R}$-bounded operators.
- If X is a Hilbert space (e.g., $X = \mathbb{C}$ or $X = \mathbb{C}^N$), then bounded symbols lead to $\mathcal{R}$-bounded operators.

Properties of operators

For an operator in a Banach space of class HT, we have the following implications:

$$A \text{ is sectorial, i.e. } \|\lambda(\lambda - A)^{-1}\| \leq C \text{ for } \operatorname{Re}\lambda \geq 0$$

$$\Updownarrow$$

$$A \text{ generates an analytic semigroup}$$

$$\Uparrow$$

$$A \text{ is } \mathcal{R}\text{-sectorial, i.e. } \mathcal{R}\big(\{\lambda(\lambda - A)^{-1} : \operatorname{Re}\lambda \geq 0\}\big) < \infty$$

$$\Updownarrow$$

$$A \text{ has maximal } L^p\text{-regularity for all } p \in (1, \infty)$$

$$\Uparrow$$

$$A \text{ has bounded imaginary powers}$$

$$\Uparrow$$

$$A \text{ admits a bounded } H^\infty\text{-calculus}$$

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Solving boundary value problems

Let $p \in (1, \infty)$, $G \subset \mathbb{R}^n$ be a bounded sufficiently smooth domain. Consider a general linear partial differential operator

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

with $m \in \mathbb{N}$, $a_\alpha \colon \overline{G} \to \mathbb{C}$, $D^\alpha := (-i)^{|\alpha|} \partial^\alpha$.

Let $B_1, \dots, B_m$ be boundary operators of the form

$$B_j(x, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x') \gamma_0 D^\beta$$

with $m_j < 2m$, $b_{j\beta} \colon \partial G \to \mathbb{C}$ and $\gamma_0 u = u|_{\partial G}$.

We always assume the coefficients $a_\alpha, b_{j\beta}$ to be sufficiently smooth.

Parabolic differential operators

Consider the operator

$$A(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha.$$

Definition

The principal symbol of $A(x, D)$ is defined by

$$a(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \quad (x \in \overline{G},\ \xi \in \mathbb{R}^n).$$

Definition

The operator $\partial_t - A(x, D)$ is called parabolic if

$$\lambda - a(x, \xi) \neq 0 \quad (x \in \overline{G},\ (\xi, \lambda) \in (\mathbb{R}^n \times \overline{\mathbb{C}_+}) \setminus \{0\}).$$

Here, $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$.

The Shapiro–Lopatinskii condition

We define the principal symbol of the boundary operators

$$b_j(x',\xi) := \sum_{|\beta|=m_j} b_{j\beta}(x')\xi^\beta.$$

Fix $x' \in \partial G$ and choose a coordinate system associated to x' (i.e., $x' = 0$ and the positive x_n-axis is the direction of the inner normal). In these coordinates, apply partial Fourier transform $\mathscr{F}'$ in tangential direction and obtain an ODE:

$$\begin{aligned}(\lambda - a(x',\xi',D_n))v(x_n) &= 0 \quad (x_n > 0),\\ b_j(x',\xi',D_n)v(0) &= h_j \quad (j = 1,\dots,m).\end{aligned}$$

The Shapiro–Lopatinskii condition

Key observations:

- The stable solutions of the homogeneous equation

$$(\lambda - a(x', \xi', D_n))v(x_n) = 0 \quad (x_n > 0)$$

  are given by $e^{i\tau x_n}$ with $\lambda - a(x', \xi', \tau) = 0$, $\operatorname{Im}\tau > 0$ (modification for non-simple zeros).

- ➡ m-dimensional space of stable solutions.

- Let $\tau_1, \dots, \tau_m$, $\tau_j = \tau_j(x', \xi', \lambda)$, be the zeros with positive imaginary part and set

$$a_+(x', \xi', \tau, \lambda) := \prod_{j=1}^{m} (\tau - \tau_j).$$

- The initial value problem is uniquely solvable if and only if

$$b_1(x', \xi', \cdot), \dots, b_m(x', \xi', \cdot)$$

are linearly independent modulo $a_+(x', \xi', \cdot, \lambda)$.

The Lopatinskii matrix

For $j = 1, \dots, m$ write

$$b_j(x', \xi', \tau) \equiv c_{j1} + c_{j2}\tau + \dots + c_{jm}\tau^{m-1} \mod a_+(x', \xi', \tau, \lambda).$$

with $c_{jk} = c_{jk}(x', \xi', \lambda)$.

Then $b_1(x', \xi', \cdot), \dots, b_m(x', \xi', \cdot)$ are linearly independent modulo a_+ if and only if the matrix

$$L := \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mm} \end{pmatrix}$$

is non-singular.

The matrix $L = L(x', \xi', \lambda)$ is called the Lopatinskii matrix of the boundary value problem.

The Shapiro–Lopatinskii condition

Definition (Shapiro-Lopatinskii condition)

Let $\partial_t - A(x, D)$ be parabolic. Then the boundary value problem $(\partial_t - A, B)$ is called parabolic if for all $x' \in \partial G$, all $\xi' \in \mathbb{R}^{n-1}$ and $\operatorname{Re}\lambda \geq 0$, $(\xi', \lambda) \neq 0$, the ODE (in local coordinates)

$$\begin{aligned} (\lambda - a(x', \xi', D_n))v(x_n) &= 0 \quad (x_n > 0), \\ b_j(x', \xi', D_n)v(0) &= 0 \quad (j = 1, \dots, m) \end{aligned}$$

has only the trivial stable solution $v = 0$.

Equivalent condition:

$$\det L(x', \xi', \lambda) \neq 0 \quad (x' \in \partial G,\ \xi' \in \mathbb{R}^{n-1},\ \operatorname{Re}\lambda \geq 0,\ (\xi', \lambda) \neq 0).$$

How to solve a boundary value problem

We want to solve

$$\partial_t u - A(x,D)u = f \quad \text{in } G,$$
$$B_j(x,D)u = g_j \quad (j = 1,\dots,m) \quad \text{on } \partial G.$$

Standard steps of reduction:

- Laplace transform $t \rightsquigarrow \lambda = i\tau$,
- localization and freezing the coefficients $x \rightsquigarrow x_0$

  ➡ model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$,

- solve $(\lambda - A(x_0,D))u_1 = e_+ f$ in $\mathbb{R}^n$ ➡ solution $u_1 = R(\lambda)e_+ f$,
- consider $u - r_+ u_1$

  ➡ reduction to $f = 0$, with $g_j \rightsquigarrow h_j := g_j - B_j(x_0,D)r_+R(\lambda)e_+ f$.

The fundamental solution

We have to solve

$$\begin{aligned} (\lambda - A(x_0, D))u &= 0 \quad \text{in } \mathbb{R}^n_+, \\ B_j(x_0, D)u &= h_j \quad (j = 1, \dots, m) \text{ on } \mathbb{R}^{n-1}. \end{aligned} \tag{4}$$

Define the fundamental solution $w_k = w_k(x_0, \xi', \cdot)$ by

$$\begin{aligned} (\lambda - a(x_0, \xi', D_n))w_k(x_n) &= 0 \quad (x_n > 0), \\ b_j(x_0, \xi', D_n)w_k(0) &= \delta_{kj} \quad (j = 1, \dots, m). \end{aligned}$$

Then the solution of (4) is given by

$$u = \sum_{j=1}^{m} (\mathscr{F}')^{-1} w_j(x_0, \xi', x_n)(\mathscr{F}' h_j)(\xi', 0).$$

Solving the boundary value problem in $\mathbb{R}^n_+$

Theorem (Solution operators)

The unique solution of the model boundary value problem in $\mathbb{R}^n_+$ is given by

$$u = r_+ R(\lambda) e_+ f + \sum_{j=1}^{m} K_j \Big(g_j - B_j(x_0, D) r_+ R(\lambda) e_+ f \Big).$$

Here $R(\lambda) = \operatorname{op}[(\lambda - a(x_0, \xi))^{-1}]$ *is the whole-space resolvent, and the operators* K_j *are defined by*

$$(K_j \varphi)(x', x_n) := (\mathscr{F}')^{-1} w_j(x', \xi', x_n)(\mathscr{F}' \varphi)(\xi', 0),$$

where $w_1, \ldots, w_m$ *are the fundamental solutions defined above.*

$\mathcal{R}$-boundedness of the solution operators

In the solution, we have the following operators:

- whole-space resolvent $R(\lambda) = \operatorname{op}[(\lambda - a(x_0,\xi))^{-1}]$
- operators in $\mathbb{R}^n_+$ of the form

$$
\begin{aligned}
(K\varphi)(x',x_n) &= (\mathscr{F}')^{-1} w_j(x',\xi',x_n)(\mathscr{F}'\varphi)(\xi',0) \\
&= -\int_0^\infty (\mathscr{F}')^{-1}(\partial_n w_j)(x',\xi',x_n+y_n)(\mathscr{F}'\varphi)(\xi',y_n)dy_n \\
&\quad - \int_0^\infty (\mathscr{F}')^{-1} w_j(x',\xi',x_n+y_n)(\mathscr{F}'\partial_n\varphi)(\xi',y_n)dy_n.
\end{aligned}
$$

(Poisson operators)

All these operators are $\mathcal{R}$-bounded!

Maximal regularity for boundary value problems

Let $(\partial_t - A(x, D), B_1(x, D), \ldots, B_m(x, D))$ be parabolic. Define A_B by

$$D(A_B) := \{u \in W_p^{2m}(G) : B_1(x, D)u = \ldots = B_m(x, D)u = 0\}$$

and $A_B u := A(x, D)u$.

Theorem

The L^p-realization A_B has maximal L^p-regularity. Therefore, for every $f \in \mathbb{F} := L^p((0, T) \times G)$ and every $u_0 \in \gamma_t \mathbb{E} := B_{pp}^{2m-2m/p}(G)$, there exists a unique solution

$$u \in \mathbb{E} := W_p^1((0, T); L^p(G)) \cap L^p((0, T); W_p^{2m}(G))$$

of the initial boundary value problem

$$\begin{aligned} \partial_t u - A(x, D)u &= f \quad \text{in } (0, T) \times G, \\ B_j(x, D)u &= 0 \quad \text{on } (0, T) \times \partial G, \\ u\big|_{t=0} &= u_0 \quad \text{in } G. \end{aligned}$$

Maximal regularity for boundary value problems

Remarks:

- We have even found a solution operator for inhomogeneous boundary conditions:

$$B_j(x, D)u = g_j \quad (j = 1, \dots, m) \text{ on } (0, T) \times \partial G.$$

Here g_j belongs to the boundary trace space

$$g_j \in B_{pp}^{(2m-m_j-1/p)/(2m)}((0, T); L^p(\partial G)) \cap L^p((0, T); B_{pp}^{2m-m_j-1/p}(\partial G)).$$

- Analog results are possible for $f \in L^p((0, T); L^q(G))$ with $p \neq q$.

Contents

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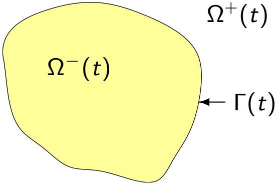
An non-standard example: Stefan problem

Consider the Stefan problem with Gibbs-Thomson correction (free boundary problem)

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in } \Omega^\pm(t), \\ u &= \kappa \quad \text{on } \Gamma(t), \\ V &= [\partial_\nu u] \quad \text{on } \Gamma(t), \\ u(0) &= u_0 \quad \text{in } \Omega^\pm(0), \\ \Gamma(0) &= \Gamma_0. \end{aligned}$$



κ: sum of principal curvatures of $\Gamma(t)$,
V: normal velocity of $\Gamma(t)$,
$[\partial_\nu u]$: jump of normal derivatives.

Stefan problem with Gibbs-Thomson correction

The above Stefan problem leads to the linearized model problem (Escher-Prüss-Simonett 2003)

$$\begin{aligned} (\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ u\big|_{\mathbb{R}^{n-1}} + \Delta' \, \sigma &= g \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ -\partial_n u\big|_{\mathbb{R}^{n-1}} + \partial_t \, \sigma &= h \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ u\big|_{t=0} &= u_0 \quad \text{in } \mathbb{R}^n_+, \\ \sigma\big|_{t=0} &= \sigma_0 \quad \text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{1}$$

Here, $\Delta' := \partial_1^2 + \cdots + \partial_{n-1}^2$. The unknowns are u describing the temperature and σ describing (locally) the boundary as a graph. Note that

- σ is defined only on the boundary $\mathbb{R}^{n-1}$,
- there is a time derivative with respect to σ (dynamic boundary condition),
- this problem cannot be solved with $\mathcal{R}$-sectoriality.

Stefan problem with Gibbs-Thomson correction

The above Stefan problem leads to the linearized model problem (Escher-Prüss-Simonett 2003)

$$(\partial_t - \Delta)u = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \tag{2}$$

$$u\big|_{\mathbb{R}^{n-1}} + \Delta' \sigma = g \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \tag{3}$$

$$-\partial_{x_n} u\big|_{\mathbb{R}^{n-1}} + \partial_t \sigma = h \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \tag{4}$$

What is the space for σ?

We have $u \in W^1_p(\mathbb{R}_+; L^p(\mathbb{R}^n_+)) \cap L^p(\mathbb{R}_+; W^2_p(\mathbb{R}^n_+))$ and therefore $u|_{\mathbb{R}^{n-1}} \in B^{1-1/(2p)}_{pp}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1})) \cap L^p(\mathbb{R}_+; B^{2-1/p}_{pp}(\mathbb{R}^{n-1}))$.

- From (3): $\sigma \in B^{1-1/(2p)}_{pp}(\mathbb{R}_+; W^2_p(\mathbb{R}^{n-1})) \cap L^p(\mathbb{R}_+; B^{4-1/p}_{pp}(\mathbb{R}^{n-1}))$
- From (4): $\sigma \in B^{3/2-1/(2p)}_{pp}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1})) \cap W^1_p(\mathbb{R}_+; B^{1-1/p}_{pp}(\mathbb{R}^{n-1}))$

The Lopatinskii matrix of the Stefan problem

$$
\begin{aligned}
(\partial_t - \Delta) u = 0 \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\
u\big|_{\mathbb{R}^{n-1}} + \Delta' \,\sigma = g \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
-\partial_{x_n} u\big|_{\mathbb{R}^{n-1}} + \partial_t \,\sigma = h \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1},
\end{aligned}
$$

We apply Laplace transform $\mathscr{L}_{t\to\lambda}$ and partial Fourier transform $\mathscr{F}'_{x'\to\xi'}$ and obtain

$$(\lambda + |\xi'|^2 - \partial_n^2)\hat{u}(x_n) = 0 \quad (x_n > 0).$$

The stable solution of this ODE is $\hat{u}(x_n) = \hat{u}(0) \exp(-\sqrt{|\xi'|^2 + \lambda}\, x_n)$ which yields

$$\begin{pmatrix} 1 & -|\xi'|^2 \\ \sqrt{|\xi'|^2 + \lambda} & \lambda \end{pmatrix} \begin{pmatrix} \hat{u}(0) \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \hat{g} \\ \hat{h} \end{pmatrix}.$$

This matrix is the (generalized) Lopatinskii matrix of the problem.

Contents

Parabolicity for mixed order systems

Let $A = (a_{ij}(D_{x'}, \partial_t))_{i,j=1,\dots,N}$ be a mixed order system with

$$\operatorname{ord} a_{ij} \le l_i + m_j \quad (i,j = 1,\dots,N).$$

Then the principal symbol is defined by $A_0(\xi',\lambda) = (a_{ij}^0(\xi',\lambda))_{i,j=1,\dots,N}$ with

$$a_{ij}^0(\xi',\lambda) := \begin{cases} a_{ij,0}(\xi',\lambda) & \text{if } \operatorname{ord} a_{ij} = l_i + m_j, \\ 0 & \text{if } \operatorname{ord} a_{ij} < l_i + m_j. \end{cases}$$

Definition (first attempt)

The mixed order system $A(D_{x'}, \partial_t)$ is called parabolic if

$$\det A_0(\xi',\lambda) \ne 0 \quad (\xi' \in \mathbb{R}^{n-1},\, \operatorname{Re}\lambda \ge 0, (\xi',\lambda) \ne (0,0)).$$

Parabolicity for mixed order systems

The Lopatinskii matrix of the Stefan problem is given by

$$L(\xi', \lambda) = \begin{pmatrix} \frac{1}{\sqrt{|\xi'|^2 + \lambda}} & -|\xi'|^2 \\ & \lambda \end{pmatrix}.$$

We obtain the following order structure and principal part:

	order	principal symbol
no scaling, $\lvert\lambda\rvert \approx \lvert\xi'\rvert$	$\begin{array}{c\|cc} & 2 & 2 \\ \hline 0 & 0 & 2 \\ -1 & 1 & 1 \end{array}$	$\begin{pmatrix} 0 & -\lvert\xi'\rvert^2 \\ \sqrt{\lvert\xi'\rvert^2} & \lambda \end{pmatrix}$
parabolic scaling, $\lvert\lambda\rvert \approx \lvert\xi'\rvert^2$	$\begin{array}{c\|cc} & 1 & 2 \\ \hline 0 & 0 & 2 \\ 0 & 1 & 2 \end{array}$	$\begin{pmatrix} 0 & -\lvert\xi'\rvert^2 \\ \sqrt{\lvert\xi'\rvert^2 + \lambda} & \lambda \end{pmatrix}$

Parabolicity for mixed-order systems

The determinant of the principal part (with parabolic scaling) is given by

$$\det L_0(\xi', \lambda) = |\xi'|^2 \sqrt{|\xi'|^2 + \lambda}.$$

For $\xi' = 0$ and $\lambda \neq 0$ we have $\det L_0(\xi', \lambda) = 0$, so the Stefan problem is not parabolic in the classical sense.

The first definition is not appropriate because

- there is no fixed relation between the co-variables λ and ξ' (i.e., time and space derivatives),
- there is no principal symbol of the Lopatinskii determinant.

The Newton polygon

The Lopatinskii determinant for the Stefan problem was given by

$$\det L(\xi',\lambda) = \lambda + |\xi'|^2\sqrt{|\xi'|^2+\lambda}.$$

Compare with the symbol of the heat equation: $A(\xi',\lambda) = \lambda + |\xi'|^2$.

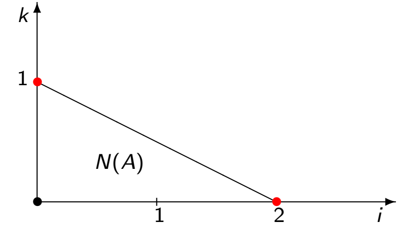
Definition

Let $A(\xi',\lambda) = \sum_{\alpha,k} a_{\alpha k}\lambda^k(\xi')^\alpha$. Then the Newton polygon is defined as the convex hull of all points

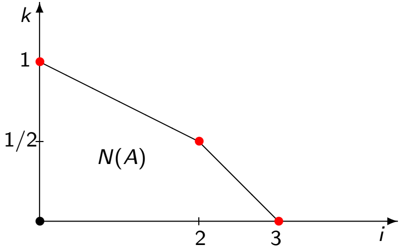
$$(|\alpha|, k) \quad \text{with} \;\; a_{\alpha k} \neq 0$$

and their projections onto the axes.

(a) Heat equation: $A(\xi', \lambda) = \lambda + |\xi'|^2$.



(b) Stefan problem: $A(\xi', \lambda) = \lambda + |\xi'|^2 \sqrt{\lambda + |\xi'|^2}$.



Definition of parabolicity for mixed-order systems

Definition

The scalar operator $A(D_{x'}, \partial_t)$ is called N-parabolic if

- the Newton polygon $N(A)$ is regular, i.e. it has no edge parallel to the axes,
- the estimate

$$|A(\xi', \lambda)| \geq C \sum_{(i,k)} |\lambda|^k |\xi'|^i$$

holds for $\operatorname{Re} \lambda \geq 0$. The sum runs over all vertices of $N(A)$.

Definition

A mixed-order system is called N-parabolic if its determinant is N-parabolic.

(Gindikin-Volevich 1992), (Mennicken-Volevich-D. 1998)

A family of principal symbols

In the Stefan problem we have the inhomogeneous symbol

$$A(\xi', \lambda) = \det L(\xi', \lambda) = \lambda + |\xi'|^2 \sqrt{|\xi'|^2 + \lambda}.$$

What is the principal symbol?

Idea: For every $\gamma > 0$ we set

$$|\lambda| \approx |\xi'|^\gamma$$

and get a family of principal symbols $(\pi_\gamma A(\xi', \lambda))_{\gamma > 0}$:

$$\begin{array}{ll} 0 < \gamma < 2: & \pi_\gamma A = |\xi'|^3, \\ \gamma = 2: & \pi_\gamma A = |\xi'|^2 \sqrt{\lambda + |\xi'|^2}, \\ 2 < \gamma < 4: & \pi_\gamma A = |\xi'|^2 \sqrt{\lambda}, \\ \gamma = 4: & \pi_\gamma A = \lambda + |\xi'|^2 \sqrt{\lambda}, \\ \gamma > 4: & \pi_\gamma A = \lambda. \end{array}$$

A family of principal symbols

Theorem

Let $A(x', D_{x'}, \partial_t)$ be a scalar operator. Then the following statements are equivalent:

- *A is parabolic in the sense of the Newton polygon.*
- *For every $\gamma > 0$ we have*

$$\pi_\gamma A(x', \xi', \lambda) \neq 0 \quad (\operatorname{Re}\lambda \geq 0,\ \xi' \neq 0,\ \lambda \neq 0).$$

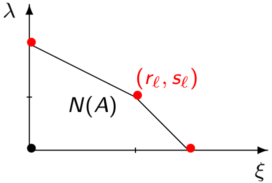
(Gindikin-Volevich 1992, D.-Saal-Seiler 2008, D.-Kaip 2013)

Idea of proof:

- partition of unity in the covariable space determined by the geometry of the Newton polygon,
- in each subset the full symbol is a perturbation of the γ-principal part for some γ.

Contents

Spaces related to the Newton polygon



For each vertex (r_ℓ, s_ℓ) of the Newton polygon, we consider the space

$$ {}_0\mathcal{F}_\ell^{s_\ell}((0,T), \mathcal{K}_\ell^{r_\ell}(\mathbb{R}^n)) $$

with $\mathcal{F}_\ell \in \{B_{p_0q_0}, H_{p_0}, F_{p_0q_0}\}$, $\mathcal{K}_\ell \in \{B_{p_1q_1}, H_{p_1}, F_{p_1q_1}\}$, $p_i, q_i \in (1,\infty)$.

The Sobolev space related to the Newton polygon $N(A)$ is the intersection of these spaces:

$$ \mathbb{H} := \bigcap_\ell {}_0\mathcal{F}_\ell^{s_\ell}((0,T), \mathcal{K}_\ell^{r_\ell}(\mathbb{R}^n)). $$

- mixture of scales can be chosen

N-parabolic equations

Main results (Gindikin-Volevich 1992, D.-Saal-Seiler 2008, D.-Kaip 2013):

Theorem

Let $A(\xi', \lambda)$ be N-parabolic, i.e. assume that

$$\pi_\gamma A(\xi', \lambda) \neq 0 \quad (\operatorname{Re} \lambda \geq 0, \lambda \neq 0, \xi' \neq 0, \gamma > 0).$$

Then $A(D_{x'}, \partial_t)$ is an isomorphism in the spaces related to the Newton polygon $N(A)$.

- The operator $A(D_{x'}, \partial_t)$ can be defined as a Fourier multiplier or by a joint H^∞-calculus of the sectorial and bisectorial operators $\partial_t, \partial_{x_1}, \dots, \partial_{x_n}$ (Dore-Venni 2005).

N-parabolic systems

Theorem (D.-Kaip 2013)

Let $\mathscr{L} = \big(\mathscr{L}_{jk}(\xi', \lambda)\big)_{j,k=1,\dots,N}$ *be a mixed-order matrix of symbols. Assume that* $\det \mathscr{L}$ *is N-parabolic. Then* $\mathscr{L}(D_{x'}, \partial_t)$ *is an isomorphism*

$$\mathscr{L}(D_{x'}, \partial_t) \in L_{Isom}\Big(\prod_{j=1}^N \mathbb{H}_j, \prod_{j=1}^N \mathbb{F}_j\Big),$$

where the spaces are defined by the Newton polygon structure of the matrix.

- In each component, we have a Newton polygon space.
- The description of the spaces depends on the Douglis-Nirenberg structure of the system

$$\operatorname{ord}_\gamma(\mathscr{L}_{ij}) \le l_i(\gamma) + m_j(\gamma)$$

  (order functions).

Maximal regularity
and the Newton polygon approach

Maximal Regularity Theorems and Mathematical Fluid Dynamics

Waseda University, Tokyo

Robert Denk

University of Konstanz

March 9–12, 2021

Contents

Contents

Spaces for the Stefan problem

We want to prove maximal regularity for the Stefan problem:

$$\begin{aligned}(\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+,\\ u\big|_{\mathbb{R}^{n-1}} + \Delta' \sigma &= g \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1},\\ -\partial_n u\big|_{\mathbb{R}^{n-1}} + \partial_t \sigma &= h \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}\end{aligned}$$

(plus zero initial conditions).

(i) Space for f: For L^p-maximal regularity, we choose

$$f \in \mathbb{F} := L^p((0,T); L^p(\mathbb{R}^n_+)).$$

(ii) Space for u: The natural solution space for u is

$$u \in \mathbb{E} := {}_0H^1_p((0,T); L^p(\mathbb{R}^n_+)) \cap L^p((0,T); H^2_p(\mathbb{R}^n_+)).$$

Spaces for the Stefan problem

We want to prove maximal regularity for the Stefan problem:

$$\begin{aligned} (\partial_t - \Delta)u = f \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ u\big|_{\mathbb{R}^{n-1}} + \Delta'\,\sigma = g \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ -\partial_n u\big|_{\mathbb{R}^{n-1}} + \partial_t\,\sigma = h \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}. \end{aligned}$$

(iii) Spaces for g and h: The spaces for g and h are the boundary trace spaces:

$$\begin{aligned} g \in \mathbb{G} := \gamma_0 \mathbb{E} := {}_0B_{pp}^{1-1/(2p)}((0,T); L^p(\mathbb{R}^{n-1})) \cap L^p((0,T); B_{pp}^{2-1/p}(\mathbb{R}^{n-1})), \\ h \in \mathbb{H} := {}_0B_{pp}^{1/2-1/(2p)}((0,T); L^p(\mathbb{R}^{n-1})) \cap L^p((0,T); B_{pp}^{1-1/p}(\mathbb{R}^{n-1})). \end{aligned}$$

The space for σ can be determined by the Newton polygon method.

N-parabolicity

The determinant of the Lopatinskii matrix was given by

$$A(\xi',\lambda) := \det L(\xi',\lambda) = \lambda + |\xi'|^2\sqrt{|\xi'|^2+\lambda}.$$

This gives the family of principal symbols $(\pi_\gamma A(\xi',\lambda))_{\gamma>0}$:

$0<\gamma<2$:	$\pi_\gamma A = |\xi'|^3$,
$\gamma = 2$:	$\pi_\gamma A = |\xi'|^2\sqrt{\lambda+|\xi'|^2}$,
$2<\gamma<4$:	$\pi_\gamma A = |\xi'|^2\sqrt{\lambda}$,
$\gamma = 4$:	$\pi_\gamma A = \lambda + |\xi'|^2\sqrt{\lambda}$,
$\gamma > 4$:	$\pi_\gamma A = \lambda$.

We immediately see

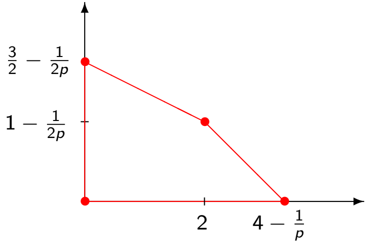
$$\pi_\gamma A(\xi',\lambda) \neq 0 \quad (\xi' \in \mathbb{R}^{n-1}\setminus\{0\},\ \operatorname{Re}\lambda \geq 0,\ \lambda\neq 0).$$

Therefore, L is N-parabolic.

Spaces for the Stefan problem

The Newton polygon method gives the space for σ:

$$\begin{aligned}\sigma \in \mathbb{S} := \; & B_{pp}^{3/2-1/(2p)}((0,T); L^p(\mathbb{R}^{n-1})) \\ & \cap B_{pp}^{1-1/(2p)}((0,T); H_p^2(\mathbb{R}^{n-1})) \\ & \cap L^p((0,T); B_{pp}^{4-1/p}(\mathbb{R}^{n-1})).\end{aligned}$$



Maximal L^p-regularity for the Stefan problem

The Lopatinskii matrix of the Stefan problem is given by

$$L(\xi', \lambda) = \begin{pmatrix} \dfrac{1}{\sqrt{|\xi'|^2+\lambda}} & -|\xi'|^2 \\ & \lambda \end{pmatrix}.$$

Theorem

a) For $p \in (1,\infty)$ and $T \in (0,\infty)$, L induces an isomorphism

$$L(D_{x'}, \partial_t)\colon \gamma_0\mathbb{E} \times \mathbb{S} \to \mathbb{G} \times \mathbb{H},\ (\gamma_0 u, \sigma) \mapsto (g, h).$$

b) For every $f \in \mathbb{F}$, $g \in \mathbb{G}$ and $h \in \mathbb{H}$, the Stefan problem has a unique solution

$$\begin{aligned} u \in \mathbb{E} &= {}_0H_p^1((0,T); L^p(\mathbb{R}^n_+)) \cap L^p((0,T); H_p^2(\mathbb{R}^n_+)), \\ \sigma \in \mathbb{S} &= {}_0B_{pp}^{3/2-1/(2p)}((0,T), L^p(\mathbb{R}^{n-1})) \cap {}_0B_{pp}^{1-1/(2p)}((0,T), H_p^2(\mathbb{R}^{n-1})) \\ &\qquad \cap L^p(J; B_{pp}^{4-1/p}(\mathbb{R}^{n-1})). \end{aligned}$$

(see Escher-Prüss-Simonett 2003)

The Stefan problem in the L^p-L^q-setting

Now we consider the Stefan problem in the L^p-L^q-setting with $p, q \in (1, \infty)$.

Space for f: We choose

$$f \in \mathbb{F} := L^p((0,T); L^q(\mathbb{R}^n_+)).$$

Space for u: Then the space for u is

$$u \in \mathbb{E} := {}_0H^1_p((0,T); L^q(\mathbb{R}^n_+)) \cap L^p((0,T); H^2_q(\mathbb{R}^n_+)).$$

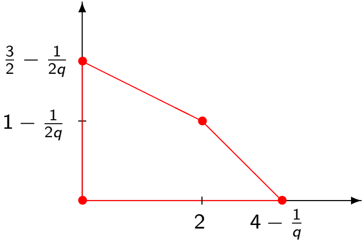
Spaces for f and g: they are given as boundary trace spaces

$$\begin{aligned} \gamma_0 u \in \gamma_0\mathbb{E} &:= {}_0F^{1-1/(2q)}_{pq}(J, L^q(\mathbb{R}^{n-1})) \cap L^p(J, B^{2-1/q}_{qq}(\mathbb{R}^{n-1})), \\ g \in \mathbb{G} &:= \gamma_0\mathbb{E}, \\ h \in \mathbb{H} &:= {}_0F^{1/2-1/(2q)}_{pq}(J, L^q(\mathbb{R}^{n-1})) \cap L^p(J, B^{1-1/q}_{qq}(\mathbb{R}^{n-1})). \end{aligned}$$

The Stefan problem in the L^p-L^q-setting

The space for σ is given by the Newton polygon:

$$\sigma \in \mathbb{S} := {}_0F_{pq}^{3/2-1/(2q)}(J, L^q(\mathbb{R}^{n-1})) \cap {}_0F_{pq}^{1-1/(2q)}(J, H_q^2(\mathbb{R}^{n-1})) \\ \cap L^p(J; B_{qq}^{4-1/q}(\mathbb{R}^{n-1})).$$



The Stefan problem in the L^p-L^q-setting

The Lopatinskii matrix of the Stefan problem is given by

$$L(\xi', \lambda) = \begin{pmatrix} \frac{1}{\sqrt{|\xi'|^2 + \lambda}} & -|\xi'|^2 \\ & \lambda \end{pmatrix}.$$

Theorem (Kaip 2012, Meyries-Veraar 2014)

a) For $p, q \in (1, \infty)$ and $J = (0, T)$ with $T < \infty$, L induces an isomorphism

$$L(\partial_t, D_{x'})\colon \gamma_0 \mathbb{E} \times \mathbb{S} \to \mathbb{G} \times \mathbb{H}$$

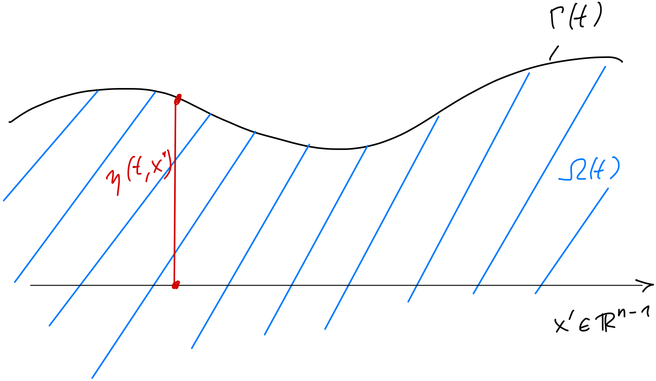
b) For every $f \in \mathbb{F} = L^p(J; L^q(\mathbb{R}^n_+))$ and every $g \in \mathbb{G}$ and $h \in \mathbb{H}$, the Stefan problem has a unique solution

$$\begin{aligned} u &\in \mathbb{E} = {}_0H^1_p(J; L^q(\mathbb{R}^n_+)) \cap L^p(J; H^2_q(\mathbb{R}^n_+)), \\ \sigma &\in \mathbb{S} = {}_0F^{3/2-1/(2q)}_{pq}(J, L^q(\mathbb{R}^{n-1})) \cap {}_0F^{1-1/(2q)}_{pq}(J, H^2_q(\mathbb{R}^{n-1})) \\ &\qquad \cap L^p(J; B^{4-1/q}_{qq}(\mathbb{R}^{n-1})). \end{aligned}$$

Contents

A fluid-structure interaction model

jointly with J. Saal (2020)



The model

We consider the following one-phase fluid-structure interaction model: (Grandmont-Hillairet 2016; Badra-Takahashi 2017)

$$\begin{aligned}
\rho(\partial_t u + (u\cdot\nabla)u)) - \operatorname{div} T(u,\, q) &= 0, & t>0,\ x\in\Omega(t),\\
\operatorname{div} u &= 0, & t>0,\ x\in\Omega(t),\\
u &= V_\Gamma, & t\ge 0,\ x\in\Gamma(t),\\
\tfrac{1}{\nu\cdot e_n} e_n^\tau T(u,\, q)\nu &= \phi_\Gamma, & t\ge 0,\ x\in\Gamma(t),\\
\Gamma(0)=\Gamma_0,\quad V_\Gamma(0)=V_0,\quad u(0) &= u_0, & x\in\Omega(0),
\end{aligned}$$

The unknowns in the model are the velocity u, the pressure q and the interface $\Gamma(t) = \partial\Omega(t)$.

- We assume the fluid to be incompressible and the stress to be given as

$$T(u,\, q) = 2\mu D(u) - qI, \qquad D(u) = \tfrac{1}{2}(\nabla u + (\nabla u)^\tau).$$

One-phase fluid-structure interaction model

$$\begin{aligned}
\rho(\partial_t u + (u\cdot\nabla)u)) - \operatorname{div} T(u,\,q) &= 0, && t>0,\ x\in\Omega(t),\\
\operatorname{div} u &= 0, && t>0,\ x\in\Omega(t),\\
u &= V_\Gamma, && t\geq 0,\ x\in\Gamma(t),\\
\tfrac{1}{\nu\cdot e_n} e_n^\tau T(u,\,q)\nu &= \phi_\Gamma, && t\geq 0,\ x\in\Gamma(t),\\
\Gamma(0)=\Gamma_0,\quad V_\Gamma(0)=V_0,\quad u(0) &= u_0, && x\in\Omega(0),
\end{aligned}$$

- Here, ν is the exterior unit normal at Γ, and V_Γ is the velocity of Γ, where we assume that $\Gamma(t)$ is the graph of a function:

$$\Gamma(t) = \{(x',\eta(t,x')) : x'\in\mathbb{R}^{n-1}\},$$

- The elastic response is of damped Kirchhoff type :

$$\phi_\Gamma = m(\partial_t,\partial')\eta := \partial_t^2\eta + \alpha(\Delta')^2\eta - \beta\Delta'\eta - \gamma\partial_t\Delta'\eta$$

with $\alpha,\beta,\gamma>0$. Here, Δ' is the Laplacian in $\mathbb{R}^{n-1}$.

Some references

- Quarteroni–Tuveri-Veneziani (2000):
  This model with application to cardiovascular systems
- Badra–Takahashi (2017):
  2D, generation of an analytic semigroup in L^2-setting
- Grandmont–Hillairet (2016):
  Global strong solution in L^2
- Chambolle–Desjardins–Esteban–Grandmont (2005), Grandmont (2008), Lengeler (2014), Lengeler–Růžička (2014), . . . :
  Weak solutions in L^2, also for parabolic-hyperbolic setting
- Beirão da Veiga (2004), Coutand–Shkoller (2006), Lequeurre (2011, 2013), Galdi–Kyed (2009), Muha–Canic (2015), . . . :
  Strong solutions in L^2
- Maity–Takahashi (2020), Kyed (this workshop) :
  maximal L^p-regularity

Main result: The spaces for the solution

We have the unknowns u (velocity), q (pressure), and η describing the boundary.

- For u and q, we have the standard spaces (in variable domains):

$$\begin{aligned} &u \in H_p^1(J; L^p(\Omega(t))) \cap L^p(J; H_p^2(\Omega(t))), \\ &q \in L^p(J; \dot{H}_p^1(\Omega(t))), \end{aligned}$$

- For η, we have a non-standard space including a dominating mixed derivative (Newton polygon space):

$$\begin{aligned} \eta \in \mathbb{E}_\eta := B_{pp}^{9/4-1/(4p)}(J; L^p(\mathbb{R}^{n-1})) \cap H_p^2(J; B_{pp}^{1-1/p}(\mathbb{R}^{n-1})) \\ \cap L^p(J; B_{pp}^{5-1/p}(\mathbb{R}^{n-1})), \end{aligned}$$

H_p^k: classical Sobolev space, $\dot{H}_p^1$: homogeneous Sobolev space,
B_{pp}^s: Besov space

Main result: The spaces for the initial values

We have the following initial values at time $t = 0$:

- $u(0) = u_0 \in B_{pp}^{2-2/p}(\Omega(0))$
- $\Gamma(0) = \Gamma_0$ which is the graph of the function

$$\eta_0 \in B_{pp}^{5-3/p}(\mathbb{R}^{n-1})$$

- $V_\Gamma(0) = V_0$ with $V_0(x') = (0, \eta_1(x'))$ $(x' \in \mathbb{R}^{n-1})$ with

$$\eta_1 \in B_{pp}^{3-3/p}(\mathbb{R}^{n-1})$$

For η_0 and η_1, we need results on the traces of Newton polygon spaces (D.-Saal-Seiler 2008).

Main result

Theorem

Let $n \geq 2$, $p \geq (n+2)/3$, $T > 0$, and $J = (0, T)$. Then there exists some $\kappa = \kappa(T) > 0$ such that for all initial values u_0, η_0 and η_1 satisfying the compatibility conditions and

$$\|u_0\|_{B_{pp}^{2-2/p}(\Omega(0))} + \|\eta_0\|_{B_{pp}^{5-3/p}(\mathbb{R}^{n-1})} + \|\eta_1\|_{B_{pp}^{3-3/p}(\mathbb{R}^{n-1})} < \kappa,$$

there exists a unique solution (u, q, Γ) of the fluid-structure interaction system such that $\Gamma = \text{graph}(\eta)$ in the solution spaces above. The solution depends continuously on the data.

- One can also get short-time solution for arbitrary data.
- For the physically relevant cases $n = 2$ and $n = 3$, the case $p = 2$ is included. This could help for considering the singular limit $\gamma \to 0$ (undamped plate model).

Transformation and linearization

- By re-scaling, we may assume $\rho = \mu = 1$.
- Transformation to the half-space $\mathbb{R}^n_+$:

$$\theta : J \times \mathbb{R}^n_+ \to \bigcup_{t\in J} \{t\} \times \Omega(t), \; (t, x', y) := \theta(t, x', x_n) := (t, x', x_n + \eta(t, x')).$$

Here, $J := (0, T)$ and $(x', x_n) \in \mathbb{R}^n_+$ with $x' \in \mathbb{R}^{n-1}$.

- New unknowns $v := \theta^* u$, $p := \theta^* q$.

⇒ quasilinear system for (v, p, η)

$$\begin{array}{rcll}
\partial_t v - \Delta v + \nabla p & = & F(v, p, \eta) & \text{in} \quad J \times \mathbb{R}^n_+, \\
\operatorname{div} v & = & G(v, \eta) & \text{in} \quad J \times \mathbb{R}^n_+, \\
v' & = & 0 & \text{on} \quad J \times \mathbb{R}^{n-1}, \\
\partial_t \eta - v^n & = & 0 & \text{on} \quad J \times \mathbb{R}^{n-1}, \\
-2\partial_n v^n + p - m(\partial_t, \partial')\eta & = & H(v, \eta) & \text{on} \quad J \times \mathbb{R}^{n-1}, \\
v\big|_{t=0} & = & v_0 & \text{in} \quad \mathbb{R}^n_+, \\
\eta\big|_{t=0} & = & \eta_0 & \text{in} \quad \mathbb{R}^{n-1}, \\
\partial_t \eta\big|_{t=0} & = & \eta_1 & \text{in} \quad \mathbb{R}^{n-1}.
\end{array}$$

Transformation and linearization

After transformation to the fixed domain $\mathbb{R}^n_+ := \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$, we obtain a quasilinear system for the transformed unknowns v, p, and η in the time interval $J = (0, T)$:

$$\begin{aligned}
\partial_t v - \Delta v + \nabla p &= F(v,p,\eta) && \text{in} \quad J \times \mathbb{R}^n_+,\\
\operatorname{div} v &= G(v,\eta) && \text{in} \quad J \times \mathbb{R}^n_+,\\
v' &= 0 && \text{on} \quad J \times \mathbb{R}^{n-1},\\
\partial_t \eta - v^n &= 0 && \text{on} \quad J \times \mathbb{R}^{n-1},\\
-2\partial_n v^n + p - m(\partial_t, \partial')\eta &= H(v,\eta) && \text{on} \quad J \times \mathbb{R}^{n-1},
\end{aligned}$$

The non-linear right-hand sides are given as

$$\begin{aligned}
F(v,p,\eta) &= (\partial_t \eta - \Delta'\eta)\partial_n v - 2(\nabla'\eta \cdot \nabla')\partial_n v + |\nabla'\eta|^2 \partial_n^2 v\\
&\quad -(v \cdot \nabla)v + (v' \cdot \nabla'\eta)\partial_n v + (\nabla'\eta, 0)^\tau \partial_n p,\\
G(v,\eta) &= \nabla'\eta \cdot \partial_n v',\\
H(v,\eta) &= -\nabla'\eta \cdot \partial_n v' - \nabla'\eta \cdot \nabla' v^n.
\end{aligned}$$

The linearized system

The linearized problem is given by

$$
\begin{array}{rcccl}
\partial_t v - \Delta v + \nabla p & = & f & \text{in} & J \times \mathbb{R}^n_+, \\
\operatorname{div} v & = & g & \text{in} & J \times \mathbb{R}^n_+, \\
v' & = & 0 & \text{on} & J \times \mathbb{R}^{n-1}, \\
\partial_t \eta - v^n & = & 0 & \text{on} & J \times \mathbb{R}^{n-1}, \\
-2\partial_n v^n + p - m(\partial_t, \partial')\eta & = & h & \text{on} & J \times \mathbb{R}^{n-1}
\end{array}
\qquad \text{(LP)}
$$

with

$$
\begin{aligned}
f &\in \mathbb{F}_f := L^p(J; L^p(\mathbb{R}^n_+)), \\
g &\in \mathbb{F}_g := H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^n_+)) \cap L^p(J; H^1_p(\mathbb{R}^n_+)), \\
h &\in \mathbb{F}_h := L^p(J; \dot{B}^{1-1/p}_{pp}(\mathbb{R}^{n-1})).
\end{aligned}
$$

and with initial values $u(0) = u_0 \in B^{2-2/p}_{pp}(\mathbb{R}^n_+)$, $\eta(0) = \eta_0 \in B^{5-3/p}_{pp}(\mathbb{R}^{n-1})$, and $\partial_t \eta(0) = \eta_1 \in B^{3-3/p}_{pp}(\mathbb{R}^{n-1})$.

Maximal regularity for the linearized system

Theorem (Maximal regularity)

The linearized system (LP) has a solution

$$
\begin{aligned}
v \in \mathbb{E}_v &:= H^1_p(J; L^p(\Omega(t))) \cap L^p(J; H^2_p(\Omega(t))),\\
p \in \mathbb{E}_p &:= L^p(J; \dot{H}^1_p(\Omega(t))),\\
\eta \in \mathbb{E}_\eta &:= B^{9/4-1/(4p)}_{pp}(J; L^p(\mathbb{R}^{n-1})) \cap H^2_p(J; B^{1-1/p}_{pp}(\mathbb{R}^{n-1}))\\
&\qquad \cap L^p(J; B^{5-1/p}_{pp}(\mathbb{R}^{n-1})),
\end{aligned}
$$

if and only if the data $(f, g, h, u_0, \eta_0, \eta_1)$ *belong to the spaces above and satisfy the compatibility conditions.*

Some words on the proof

After some calculations, we obtain (on symbol level) the relation

$$\hat{\eta}(\lambda, \xi') = -\frac{|\xi'|^2}{N_L(\lambda, |\xi'|)} \hat{h}(\lambda, \xi')$$

with

$$\begin{aligned} N_L(\lambda, |\xi'|) &:= |\xi'|^2 m(\lambda, \xi') + \lambda\omega^2(\omega + |\xi'|), \\ m(\lambda, \xi') &:= \lambda^2 + \alpha|\xi'|^4 + \gamma\lambda|\xi'|^2 + \beta|\xi'|^2, \\ \omega &:= \sqrt{\lambda + |\xi'|^2}. \end{aligned}$$

➡ We need mapping properties of the operator $N_L(\partial_t, \sqrt{-\Delta'})$

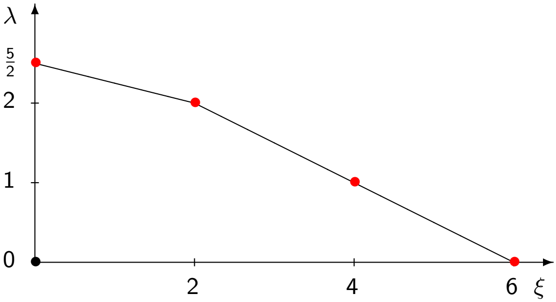
- This operator can be defined by joint H^∞-calculus.
- The mapping properties are given by the Newton polygon theory.

Application of the Newton polygon method

We want to study mapping properties of $N(\partial_t, \sqrt{-\Delta'})$ for

$$N_L(\lambda, z) := z^2 m(\lambda, z) + \lambda(\lambda + z^2)(\sqrt{\lambda + z^2} + z),$$

where $m(\lambda, z) := \lambda^2 + \alpha z^4 + \gamma\lambda z^2 + \beta z^2$ and $z := |\xi'|$.



The Newton polygon method: principal parts

We want to study mapping properties of $N(\partial_t, \sqrt{-\Delta'})$ for

$$N_L(\lambda, z) := z^2 m(\lambda, z) + \lambda(\lambda + z^2)(\sqrt{\lambda + z^2} + z),$$

where $m(\lambda, z) := \lambda^2 + \alpha z^4 + \gamma\lambda z^2 + \beta z^2$ and $z := |\xi'|$.

We obtain the following principal parts:

$$\pi_\gamma(N_L(\lambda, z)) = \begin{cases} \alpha z^6, & 0 < \gamma < 2, \\ (\lambda^2 + \alpha z^4 + \gamma\lambda z^2) z^2, & \gamma = 2, \\ \lambda^2 z^2, & 2 < \gamma < 4, \\ \lambda^2 z^2 + \lambda^{5/2}, & \gamma = 4, \\ \lambda^{5/2}, & \gamma > 4. \end{cases}$$

- Note $\pi_\gamma(N_L(\lambda, z)) \neq 0$ for all $\gamma > 0$, all $z \neq 0$ and all $\lambda \neq 0$ with $\operatorname{Re}\lambda \geq 0$.

➡ $N_L(\lambda, z)$ is N-parabolic.

Application of the Newton polygon method

For the linearized model, we had

$$\hat{\eta}(\lambda, \xi') = -\frac{|\xi'|^2}{N_L(\lambda, |\xi'|)} \hat{h}(\lambda, \xi').$$

Corollary

a) The operator $N_L(\partial_t, \sqrt{-\Delta'}) \colon \mathbb{E}_N \to L^p(J; B^0_{pp}(\mathbb{R}^{n-1}))$ is an isomorphism, where

$$\mathbb{E}_N := {}_0H_p^{5/2}(J; B^0_{pp}(\mathbb{R}^{n-1})) \cap {}_0H_p^2(J; B^2_{pp}(\mathbb{R}^{n-1})) \cap L^p(J; B^6_{pp}(\mathbb{R}^{n-1})).$$

b) For every $h \in \mathbb{F}_h = L^p(J; \dot{B}_{pp}^{1-1/p}(\mathbb{R}^{n-1}))$, we have

$$\eta := \Delta' \big[N_L(\partial_t, \sqrt{-\Delta'}) \big]^{-1} h \in \mathbb{E}_\eta.$$

This is the key step in the proof of the maximal regularity for the linear system.

Proof of the main result for the nonlinear system

The nonlinear system was given by

$$
\begin{aligned}
\partial_t v - \Delta v + \nabla p &= F(v,p,\eta) && \text{in} && J \times \mathbb{R}^n_+,\\
\operatorname{div} v &= G(v,\eta) && \text{in} && J \times \mathbb{R}^n_+,\\
v' &= 0 && \text{on} && J \times \mathbb{R}^{n-1},\\
\partial_t \eta - v^n &= 0 && \text{on} && J \times \mathbb{R}^{n-1},\\
-2\partial_n v^n + p - m(\partial_t, \partial')\eta &= H(v,\eta) && \text{on} && J \times \mathbb{R}^{n-1},
\end{aligned}
$$

- We can write this in an abstract way as $\mathscr{L}(v,p,\eta) = \mathscr{N}(v,p,\eta)$ with $\mathscr{L}$ being the linear part and $\mathscr{N}$ the nonlinear part.
- By maximal regularity, we know that $\mathscr{L}$ is invertible.
- By construction, we have $\mathscr{N}(0) = 0$ and $D\mathscr{N}(0) = 0$ for the Fréchet derivative.
- The main point to show is that $\mathscr{N}$ maps into the correct spaces.

Mapping properties of the nonlinearities (1)

In order to show that the nonlinearities map into the data spaces, we need embedding results.

Example: For $v \in \mathbb{E}_v$ and $\eta \in \mathbb{E}_\eta$, we have to show that

$$\partial_t \eta \, \partial_n v \in \mathbb{F}_f := L^p(J; L^p(\mathbb{R}^n_+)).$$

Step 1: Embedding for $\partial_t \eta$

- By definition of $\mathbb{E}_\eta$ and the mixed derivative theorem, we have

$$\eta \in H^2_p(J, B^{1-1/p}_{pp}(\mathbb{R}^{n-1})) \cap L^p(J, B^{5-1/p}_{pp}(\mathbb{R}^{n-1})) \subset H^1(J, B^{3-1/p}_{pp}(\mathbb{R}^{n-1})).$$

- For $\partial_t \eta$, we obtain again by the mixed derivative theorem

$$\begin{aligned} \partial_t \eta &\in B^{1-1/2p}_{pp}(J, L^p(\mathbb{R}^{n-1})) \cap L^p(J, B^{2-1/p}_{pp}(\mathbb{R}^{n-1})) \\ &=: B^{2-1/p,(2,1)}_{pp}(J \times \mathbb{R}^{n-1}) \end{aligned}$$

- This is an anisotropic Sobolev space.

Mapping properties of the nonlinearities (2)

Step 2: Embedding for anisotropic Sobolev spaces

Define the anisotropic Sobolev space for $s \geq 0$ by

$$H_p^{s,(2,1)}(J \times \mathbb{R}_+^n) := H_p^{s/2}(J; L^p(\mathbb{R}_+^n)) \cap L^p(J; H_p^s(\mathbb{R}_+^n)).$$

Then $v \in H_p^{2,(2,1)}(J \times \mathbb{R}_+^n)$. So we have

$$\partial_t \eta \in B_{pp}^{2-1/p,(2,1)}(J \times \mathbb{R}^{n-1}),$$
$$\partial_n v \in H_p^{1,(2,1)}(J \times \mathbb{R}_+^n).$$

Now, $\partial_t \eta \, \partial_n v \in L^p(J; L^p(\mathbb{R}_+^n))$ follows from the anisotropic embedding

$$B_{pp}^{2-1/p,(2,1)}(J \times \mathbb{R}^{n-1}) \cdot H_p^{1,(2,1)}(J \times \mathbb{R}_+^n) \subset H_p^{0,(1,2)}(J \times \mathbb{R}_+^n) \quad (p \geq \tfrac{n+2}{3}).$$

(Köhne–Saal 2018)

Main result

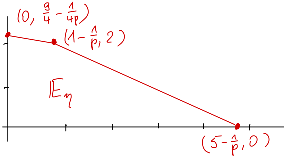
This finishes the proof of the main result.

Theorem

Let $n \geq 2$, $p \geq (n+2)/3$, $T > 0$, and $J = (0, T)$. Then for all sufficiently small data satisfying the compatibility conditions, there exists a unique solution (u, q, Γ) of the fluid-structure interaction system such that $\Gamma = \text{graph}(\eta)$ in the spaces

$$
\begin{aligned}
u &\in H_p^1(J; L^p(\Omega(t))) \cap L^p(J; H_p^2(\Omega(t))),\\
q &\in L^p(J; \dot{H}_p^1(\Omega(t))),\\
\eta &\in B_{pp}^{9/4-1/(4p)}(J; L^p(\mathbb{R}^{n-1})) \cap H_p^2(J; B_{pp}^{1-1/p}(\mathbb{R}^{n-1})) \cap L^p(J; B_{pp}^{5-1/p}(\mathbb{R}^{n-1})).
\end{aligned}
$$

- Newton polygon method for the linearization
- Sobolev embeddings for the nonlinear part

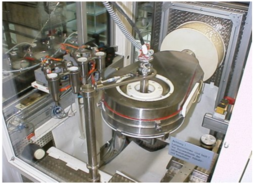


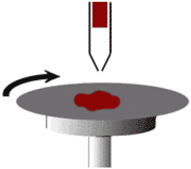
Contents

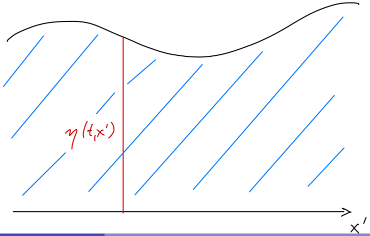
Spin-coating

jointly with Geissert, Hieber, Saal, Sawada (2011)

Spin-coating processes are described by a Navier-Stokes equation with additional centrifugal force terms and Coriolis force terms.







Spin-coating: the equations

One model for the spin-coating is given by

$$\begin{aligned}
\rho(\partial_t u + (u\cdot\nabla)u) &= \mu\Delta u - \nabla q - \rho\big[2\tilde\omega \times u + \tilde\omega \times (\tilde\omega \times \chi_R x)\big] \quad \text{in } \Omega(t),\\
\operatorname{div} u &= 0 \quad \text{in } \Omega(t),\\
T(u,q) &= \sigma\kappa\nu \quad \text{on } \Gamma^+(t),\\
V &= u\cdot\nu \quad \text{on } \Gamma^+(t)
\end{aligned}$$

(+ Navier slip condition on $\Gamma^-(t)$ + initial values)

Spin-coating

The Lopatinskii matrix for the top layer boundary has the symbol

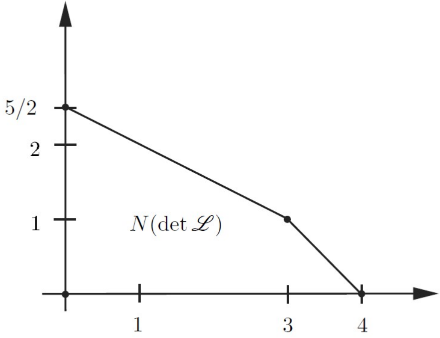
$$L(\xi,\tau) = \begin{pmatrix} i\xi_1 & i\xi_2 & -\omega & 0 & 0 \\ 0 & 0 & 1 & \frac{|\xi'|}{\omega(\omega+|\xi'|)} & \lambda \\ \omega & 0 & -i\xi_1 & -\frac{i\xi_1(\omega-|\xi'|)}{\omega(\omega+|\xi'|)} & 0 \\ 0 & \omega & -i\xi_2 & -\frac{i\xi_2(\omega-|\xi'|)}{\omega(\omega+|\xi'|)} & 0 \\ 0 & 0 & -2\omega & -1 & \sigma|\xi|^2 \end{pmatrix}.$$

Here $\omega := \sqrt{\lambda + |\xi'|^2}$ and $\lambda = i\tau$.

This matrix is N-parabolic.

The Newton polygon

The Newton polygon of $\det L(\xi, \tau)$ has the following form:

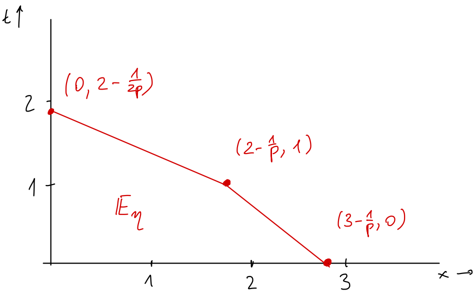


Spin-coating: maximal regularity

Theorem

For sufficiently small time, we obtain for appropriate initial values (satisfying the compatibility conditions) a unique solution

$$u \in W^1_p(J; L^p(\Omega(t))) \cap L^p(J; W^2_p(\Omega(t))),$$
$$q \in L^p(J; \dot{H}^1_p(\Omega(t))),$$
$$\eta \in B^{2-1/(2p)}_{pp}(J; L^p(\mathbb{R}^{n-1})) \cap W^1_p(J; B^{2-1/p}_{pp}(\mathbb{R}^{n-1})) \cap L^p(J; B^{3-1/p}_{pp}(\mathbb{R}^{n-1})).$$



Further applications

The following problems are covered by the N-parabolic theory:

- Generalized L_p-L_q thermoelastic plate equation in $\mathbb{R}^n$ (D.-Racke 2006),
- Bi-Laplacian with Wentzell boundary conditions (D.-Kunze-Ploss 2021),
- Cahn-Hilliard equations (Prüss-Racke-Zheng 2006), (Wilke 2007),
- Generalized L_p-L_q Stokes problem in $\mathbb{R}^n$ (Bothe-Prüss 2007),
- Two-phase Navier-Stokes equation with surface tension and gravity (Prüss-Simonett 2009-2011), (Shibata-Shimizu 2011)
- Two-phase Navier Stokes equation with Boussinesq-Scriven surface fluid (Prüss-Bothe 2010),