Stochastic Partial Differential Equations in Fluid Mechanics
Lecture 1: The Navier-Stokes equations with rough force

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The Navier-Stokes equations

Assume D is a regular bounded connected open domain.
In D we have a fluid described by its velocity $u = u(t, x)$ (a vector field) and pressure $p = p(t, x)$ (a scalar field). The equations are

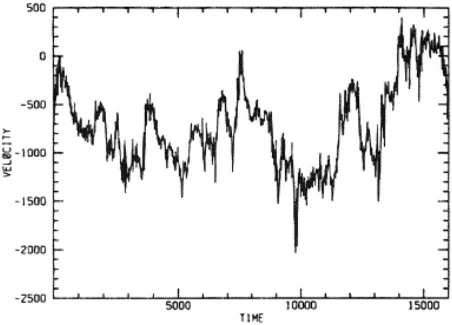
$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f \\ \operatorname{div} u &= 0 \end{aligned}$$

supplemented by boundary and initial condition

$$\begin{aligned} u|_{\partial D} &= 0 \\ u|_{t=0} &= u_0. \end{aligned}$$

The density field is assumed to be constant.

Fluid dynamics is therefore part of continuum mechanics and its laws are deterministic. However, some observations look random:



Turbulent velocity signal, from Sreenivasan, Ann. Rev. Fluid Mech. 23, 1991.

The theory of deterministic dynamical systems has developed outstanding ideas to understand how a random signal may arise from a deterministic motion. But difficult to apply to the Navier-Stokes equations.

Statistical hydrodynamics approaches the question from a statistical viewpoint, with little use of the Navier-Stokes equations.

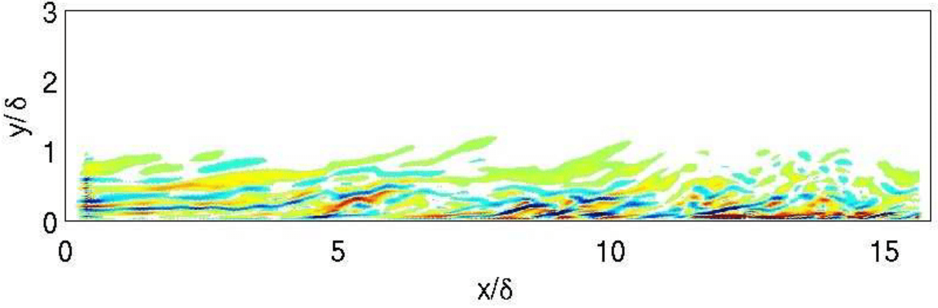
Stochastic fluid dynamics, the theory described in these lectures, is somewhat in between, based on classical equations of continuum mechanics, but enriched by means of random elements. But, **where does the noise come from**?

Where the noise comes from

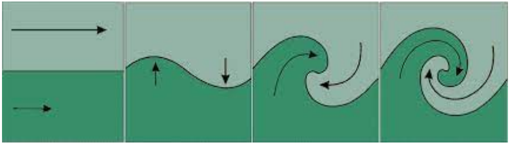
1. **vorticity production at boundaries**
2. perturbations at the interaction between different fluids or fluid/structure
3. vorticity production in shear flows
4. the dance of the vortex structures
5. **how small scales affect large ones**.

Our lectures will take into account 1 and 5, but just here in the introduction let us mention also 2, 3 and 4.

1. *Vorticity production at boundaries*. Most physical boundaries have some degree of irregularity: think of the irregularities of the hearth surface like mountains, hills, trees and houses. Mathematical models cannot take them into account.
This is what we want to replace by noise: the irregularities of the boundary.

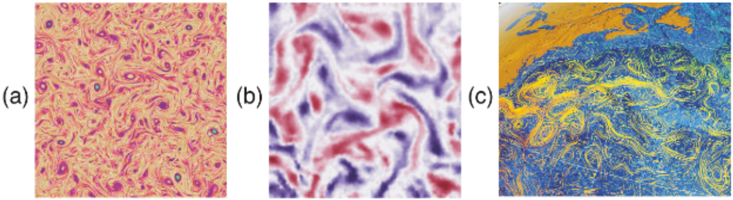


3. *Vorticity production in shear flows*. Instability due to shear lead to creation of vortices, complex motion and ultimately "noise".



Bu this is already into the deterministic equations, we should not introduce it by force.

4. *The dance of the vortex structures.*



From R. E. Ecke, J. Fluid Mech. 828, 2017.

2D vortex structures dance one around the other, merging from smaller to larger ones, in a very complex manner. Emergence of stochastic features in the motion of interacting particles is a classical research topic.

The Newtonian equations

Assume D is a regular bounded connected open domain.
In D we have a fluid described by its velocity $u = u(t, x)$ (a vector field) and pressure $p = p(t, x)$ (a scalar field). The equations are

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f \\ \mathrm{div}\, u &= 0 \end{aligned} \tag{1}$$

supplemented by boundary and initial condition

$$\begin{aligned} u|_{\partial D} &= 0 \\ u|_{t=0} &= u_0. \end{aligned}$$

The density field is assumed to be constant.

Energy balance

Assume (u, p) is a smooth pair satisfying

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f.$$

Then

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_D |u(t,x)|^2 dx &= \int_D u(t,x) \cdot \partial_t u(t,x) dx \\ &= -\int_D u \cdot (u \cdot \nabla u) dx - \int_D u \cdot \nabla p dx \\ &\quad + \nu \int_D u \cdot \Delta u dx + \int_D u \cdot f dx. \end{aligned}$$

Energy balance

Now

$$\int_D u \cdot (u \cdot \nabla u)\, dx = \frac{1}{2} \int_D u \cdot \nabla |u|^2\, dx = -\frac{1}{2} \int_D \operatorname{div} u \cdot |u|^2\, dx = 0$$

(we have used also $u|_{\partial D} = 0$); similarly

$$\int_D u \cdot \nabla p dx = -\int_D p \operatorname{div} u dx = 0$$

$$\int_D u \cdot \Delta u dx = -\int_D |\nabla u|^2\, dx.$$

Therefore we get

$$\frac{d}{dt} \frac{1}{2} \int_D |u(t,x)|^2\, dx + \nu \int_D |\nabla u|^2\, dx = \int_D u \cdot f dx.$$

Functional analysis

Assume D is a regular bounded connected open domain.
Denote by $H^k\left(D,\mathbb{R}^2\right)$, $k=1,2,\ldots$the classical Sobolev spaces or vector fields.
Denote by $H_0^k\left(D,\mathbb{R}^2\right)$ the subspace of those which are zero at the boundary.
Denote by H (resp. V, $D\left(A\right)$) the closure in $L^2\left(D;\mathbb{R}^2\right)$ (resp. $H^1\left(D,\mathbb{R}^2\right)$, $H^2\left(D,\mathbb{R}^2\right)$) of smooth compact support fields $v\in C_c^\infty\left(D;\mathbb{R}^2\right)$ such that $\operatorname{div} v=0$.
H is the space of $L^2\left(D;\mathbb{R}^2\right)$-vector fields v, divergence free, such that $v\cdot n|_{\partial D}=0$ where n is the normal to ∂D. Denote by P the projection of $L^2\left(D;\mathbb{R}^2\right)$ on H.
V (resp. $D\left(A\right)$) is the space of all $v\in H_0^1\left(D,\mathbb{R}^2\right)$ (resp. $v\in H^2\left(D,\mathbb{R}^2\right)\cap H_0^1\left(D,\mathbb{R}^2\right)$) such that $\operatorname{div} v=0$.

Functional analysis

Define $A : D(A) \subset H \to H$ by the identity

$$\langle Av, w \rangle = \nu \langle \Delta v, w \rangle$$

for all $v \in D(A)$ and $w \in H$, or as

$$Av = \nu P \Delta v.$$

Denote by $\mathbb{L}^4$ the space $L^4(D, \mathbb{R}^2) \cap H$, with the usual topology of $L^4(D, \mathbb{R}^2)$. Define the trilinear form $b : \mathbb{L}^4 \times V \times \mathbb{L}^4 \to \mathbb{R}$ as

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_D u_i(x) \partial_i v_j(x) w_j(x) \, dx = \int_D (u \cdot \nabla v) \cdot w dx$$

(it is well defined and continuous on $\mathbb{L}^4 \times V \times \mathbb{L}^4$ by Hölder inequality.

Define the operator

$$B : \mathbb{L}^4 \times \mathbb{L}^4 \to V'$$

$$\langle B(u,v), \phi \rangle = -b(u,\phi,v) = -\int_D (u \cdot \nabla \phi) \cdot v dx$$

for all $\phi \in V$. It is explicitly given by

$$B(u,v) = P(u \cdot \nabla v)$$

when u, v are more regular: we have

$$\langle B(u,v), \phi \rangle = \int_D (u \cdot \nabla v) \cdot \phi dx = -\int_D (u \cdot \nabla \phi) \cdot v dx = -b(u,\phi,v).$$

Let V' be the dual of V; $D(A) \subset V \subset H \equiv H' \subset V'$.
$\langle \cdot, \cdot \rangle$ will be the scalar product in H and also the dual pairing between V and V'.

Definition

Given $u_0 \in H$ and $f \in L^2(0, T; V')$, we say that

$$u \in C([0, T]; H) \cap L^2(0, T; V)$$

is a weak solution of equation (1) if

$$\begin{aligned} & \langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ = \; & \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle\, ds + \int_0^t \langle f(s), \phi \rangle\, ds \end{aligned}$$

for every $\phi \in D(A)$.

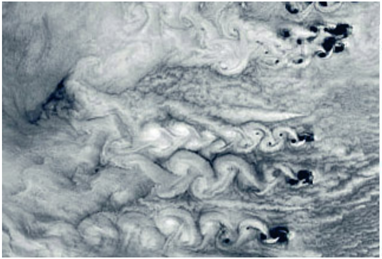
Theorem

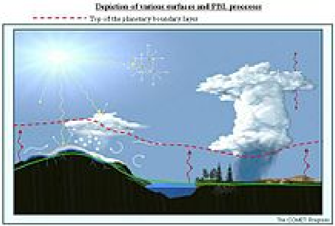
For every $u_0 \in H$ and $f \in L^2(0,T;V')$ there exists a unique weak solution of equation (1). It satisfies

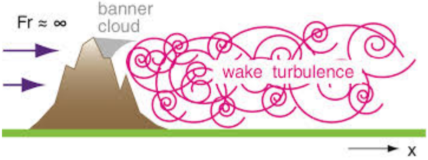
$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds = \|u_0\|_{L^2}^2 + 2\int_0^t \langle u(s), f(s) \rangle \, ds.$$

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\omega \mapsto (u_0(\omega), f(\omega))$ is a measurable map from $(\Omega, \mathcal{F})$ to $H \times L^2(0,T;V')$ (endowed with the Borel σ-algebra) then, called $u(\omega)$ the weak solution corresponding to $(u_0(\omega), f(\omega))$, we have that $\omega \mapsto u(\omega)$ is measurable from $(\Omega, \mathcal{F})$ to $C([0,T];H) \cap L^2(0,T;V)$.

Example of noise: generation of vortices near obstacles







- If vortices are produced by instability of a flat boundary, that is continuum mechanics, not noise.
- If vortices are produced by irregularities of the boundary which are not included in the mathematical model, noise may be the way to include them.
- This is the viewpoint adopted here.

Proposal: jump times t_i, where new "eddies" $\sigma(x)$ appear

$$u\left(t_i^+, x\right) = u\left(t_i^-, x\right) + \sigma(x).$$

Several obstacles with locations x_k, $k \in K$, several type of eddies σ_k:

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \sum_{k \in K} \sum_i \delta\left(t - t_i^k\right) \sigma_k.$$

- Family $\left\{ \left(N_t^k\right)_{t \geq 0} ; k \in K \right\}$ of independent standard (rate 1) Poisson processes
- mean inter-times τ^k
- $t_1^k < t_2^k < \ldots$ are the random times when the Poisson process N_{t/τ^k}^k jumps

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \sum_{k \in K} \sigma_k \frac{dN_{t/\tau^k}^k}{dt}.$$

Introduce the function

$$W(t,x) = \sum_{k \in K} \sigma_k(x) N^k_{t/\tau^k} = \sum_{k \in K} \sum_{i \in \mathbb{N}: t^k_i \leq t} \sigma_k(x)$$

and write the equation in the form

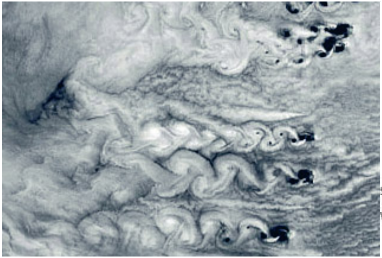
$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \partial_t W.$$

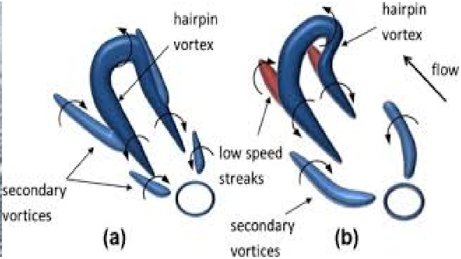
It is a Navier-Stokes equation with rough forcing term (recall the title of the Lecture).

Generation of pairs

Assume the generated vortices come in pairs:

$$W(t,x) = \sum_{k \in K} \frac{1}{\sqrt{2}} \left(\sigma_k(x) \frac{dN^{k,1}_{t/\tau^k}}{dt} - \sigma_k(x) \frac{dN^{k,2}_{t/\tau^k}}{dt} \right).$$





Rescale by n:

$$W_n(t,x) = \sum_{k\in K} \frac{1}{n}\sigma_k(x) \frac{N^{k,1}_{n^2 t/\tau^k} - N^{k,2}_{n^2 t/\tau^k}}{\sqrt{2}}$$

The heuristics is that we make much more jumps but of smaller size. The precise rescaling has been chosen in order to have a non-zero finite limit:

$$\mathbb{E}[W_n(t,x)] = 0$$

$$\mathbb{E}\left[|W_n(t,x)|^2\right] = t\sum_{k\in K} \frac{|\sigma_k(x)|^2}{\tau^k}.$$

Proof is based on independence and $\mathbb{E}\left[N^{k,j}_{n^2 t/\tau^k}\right] = \frac{n^2 t}{\tau^k}$, $Var\left[N^{k,j}_{n^2 t/\tau^k}\right] = \frac{n^2 t}{\tau^k}$.

Donsker invariance principle:

$$\frac{1}{n}\left(N_{n^2 t} - n^2 t\right) \to W_t \text{ (Brownian motion).}$$

One can prove that

$$W_n\left(t, x\right) = \sum_{k \in K} \frac{1}{n} \sigma_k\left(x\right) \frac{N^{k,1}_{n^2 t/\tau^k} - N^{k,2}_{n^2 t/\tau^k}}{\sqrt{2}}$$

converges in law to

$$W\left(t, x\right) := \sum_{k \in K} \frac{1}{\sqrt{\tau^k}} \sigma_k\left(x\right) W^k_t$$

where $\left(W^k_t\right)_{t \geq 0}$ are independent Brownian motions.

We get the stochastic Navier-Stokes equations

$$du + (u \cdot \nabla u + \nabla p)\, dt = \nu \Delta u dt + \sum_{k \in K} \frac{1}{\sqrt{\tau^k}} \sigma_k dW_t^k .$$

Summarizing, we have at least two examples of non-differentiable force. Recall that, with probability one, a trajectory of Brownian motion is nowhere differentiable, not of bounded variation, not Hölder of exponent $\alpha \geq \frac{1}{2}$ on any interval, but it is locally Hölder of any exponent $\alpha < \frac{1}{2}$.

Summary

- Our proposal, to justify *stochastic Navier-Stokes equations*, is to model the effect of the irregularity of a real boundary by means of a rough force.
- One restriction of the previous example is the need of *pairs of vortices*.
- Indeed, $\frac{1}{n}\left(N_{n^2 t} - n^2 t\right) \rightarrow W_t$. Just $\frac{1}{n} N_{n^2 t}$ behaves like $W_t + nt$, diverging as $n \rightarrow \infty$.
- A mistake is that we have introduced energy. We do not know how to subtract it (see Lecture 2).

Rigorous analysis of rough force

Consider the equation

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + \partial_t W \\ \operatorname{div} u &= 0 \end{aligned}$$

when W is not differentiable, with

$$\begin{aligned} u\big|_{\partial D} &= 0 \\ u\big|_{t=0} &= u_0 . \end{aligned}$$

Our aim is to giving a rigorous definition of solution and proving, in 2D, existence and uniqueness.