Chapter 3. Transport noise

Franco Flandoli, Scuola Normale Superiore
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1 Introduction n.1: stochastic heat transport

Let us oversimplify the fluid dynamics near the boundary. The following view is highly phenomenological and should be subject to much deeper research.

We assume that the fluid, in a region near the boundary, may be approximately described by the equations

$$\begin{aligned}\partial_t u+\nabla p &= \nu\Delta u-\frac{1}{\epsilon}u+\frac{1}{\epsilon}\sum_{k\in K}\sigma_k\partial_t W^k\\ \operatorname{div} u &= 0\\ u|_{\partial D} &= 0\end{aligned}$$

This is Stokes model, strongly incorrect in itself for turbulent fluids, but complemented by the creation of eddies/vortices (the term $\frac{1}{\epsilon}\sum_{k\in K}\sigma_k\partial_t W^k$) and an extra-dissipation term of friction type ($-\frac{1}{\epsilon}u$) to compensate the extra input of energy (in the average) due to the noise. We have intentionally parametrized the problem by $\epsilon>0$, in the very precise way written above, because we want to explore here a special scaling limit. The abstract semigroup formulation of this problem, with A given by the operator $\nu P\Delta$ as in the previous chapters, is

$$u(t)=e^{t\left(A-\frac{1}{\epsilon}\right)}u_0+\frac{1}{\epsilon}\sum_{k\in K}\int_0^t e^{(t-s)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_s^k.$$

In Chapter 1, in order to avoid Itô integrals and cover rough noise sources of very different type, we have integrated by parts and used the following formulation:

$$u(t)=e^{t\left(A-\frac{1}{\epsilon}\right)}u_0+\frac{1}{\epsilon}\sum_{k\in K}\sigma_k W_t^k+\frac{1}{\epsilon}\sum_{k\in K}\int_0^t e^{(t-s)\left(A-\frac{1}{\epsilon}\right)}\left(A-\frac{1}{\epsilon}\right)\sigma_k W_s^k ds.$$

When W_s^k are independent Brownian motions, both formulations are meaningful and they are equivalent. In the next lines we shall apply a Fubini type theorem to the stochastic integral: one way to justify it rigorously is precisely to use the last formulation which involves only Lebesgue integrals.

Let us introduce two notations:

$$\begin{aligned} W^{\epsilon}(t,x) &:= \int_0^t u(s,x)\,ds \\ W(t,x) &= \sum_{k\in K} \sigma_k(x)\, W_t^k. \end{aligned}$$

Then

$$\begin{aligned} W^{\epsilon}(t) &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\int_0^s e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k ds \\ &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\int_r^t e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k ds dW_r^k \\ &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t \left(A-\frac{1}{\epsilon}\right)^{-1}\left[e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}-1\right]\sigma_k dW_r^k \\ &= \frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1}\sum_{k\in K}\int_0^t e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k - \frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1} W(t). \end{aligned}$$

Now we use the fact (well known in the framework of Yosida approximations of semigroup theory) that

$$\lim_{\lambda\to\infty}\lambda(\lambda-A)^{-1}h = h$$

for all $h\in H$; being A^{-1} compact in our example, we can easily verify this property using the spectral decomposition. With minor additional arguments that we leave as exercise, it follows:

Lemma 1

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\left\|W^{\epsilon}(t)-W(t)\right\|_H^2\right]=0.$$

The result is also uniform in time, with supremum inside the expected value. The message of this lemma is that u converges in distribution to a white noise, the time derivative of the space-dependent Brownian motion W.

Why is this an interesting regime? Let us investigate this issue in the case of the evolution of an auxiliary quantity: heat. Assume the fluid has a variable temperature and is not strongly influenced by temperature, hence we do not change its equation of motion. But temperature, next indicated by $\theta(t,x)$, evolves according to the diffusion-transport equation

$$\partial_t\theta = \kappa\Delta\theta + u\cdot\nabla\theta$$

where $\kappa > 0$, typically small, is the heat diffusion constant and $u \cdot \nabla\theta$ is the transport due to the fluid motion. If we take the limit $\epsilon \to 0$ in the model of fluid above and we apply the heuristics of Wong-Zakai result, we find the model

$$\partial_t \theta = \kappa\Delta\theta + \sum_{k\in K} (\sigma_k \cdot \nabla\theta) \circ \partial_t W^k$$

where the symbol $\circ$ stands for the Stratonovich operation. Below we explain why the correct Itô interpretation of this equation is

$$\partial_t \theta = (\kappa\Delta + \mathcal{L})\,\theta + \sum_{k\in K} (\sigma_k \cdot \nabla\theta)\,\partial_t W^k \tag{1}$$

where the stochastic term is now understood in the classical Itô sense and $\mathcal{L}$ is a suitable linear operator, precisely a second order elliptic differential operator, that we shall discover. The result of this modeling step is that we end-up with model (1) for the heat diffusion under a turbulent velocity field. Taking (heuristically at this stage) expectation of each term and introducing the mean temperature profile

$$\Theta(t,x) = \mathbb{E}[\theta(t,x)]$$

we get

$$\partial_t \Theta = (\kappa\Delta + \mathcal{L})\,\Theta.$$

If the noise has suitable properties, the elliptic operator $\mathcal{L}$ strongly increases the dissipation of the term $\kappa\Delta$. Moreover we shall prove that the random field $\theta(t,x)$ is close to its average $\Theta(t,x)$ under suitable assumptions. This will lead to the statement that *turbulent diffusion increases the original diffusion*, a fact that is observed in experiments. This model has the power to explain a well known experimental phenomenon, the so called *eddy diffusion*.

2 Introduction n.2: additional stochastic transport in the Navier-Stokes equations

Stochastic transport of passive scalars (the topic described in the previous section) is well known in the literature. On the contrary, this section introduces an analogous idea for the *internal modeling of a fluid*, which is less common and still debated. In some cases however it leads to results observed in the real world, hence it deserves to be investigated.

Fluids, in their complex regimes that we loosely name turbulent, show the activation of several scales: we observe large scale motions and small scale ones at the same time, with several intermediate scales; very small vortices, larger and larger ones, up to motion at the scale of the full domain. Oversimplifying this multiscale picture, let us think we want to split the fluid velocity in two components

$$u(t,x) = \overline{u}(t,x) + u'(t,x)$$

the first one containing most of the large scales, the second one mostly related to the small scales. A precise subdivision is impossibile, due to the multiscale nature of the problem.

An attempt to perform a precise subdivision is by means of projections. Let us mention two of them. One is by Fourier projections and was used already above as a technical tool for the rigorous investigation. If (e_n) is a complete orthonormal system of H as described in Chapter 2 and π_n are the associated finite dimensional projections, we may define

$$\overline{u}(t) = \pi_n u(t)$$

and thus $u'(t) = (I - \pi_n) u(t)$. The second approach is to take a smooth, possibly compact support, probability density θ, introduce the mollifiers $\theta_\epsilon(x) = \epsilon^{-d}\theta\left(\epsilon^{-1}x\right)$ (where d is the space dimension) and define

$$\overline{u}(t) = \theta_\epsilon * u(t)$$

with suitable corrections in bounded domains to cope with the problem that $\theta_\epsilon(\cdot - x_0)$ may not have the support in D.

With these definitions we guarantee a priori that $\overline{u}(t)$ is made only of "large scale structures". However, the equations for $\overline{u}(t)$ and $u'(t)$ are interlaced in a quite complex manner. An alternative approach is to consider the Navier-Stokes type system

$$\begin{aligned} \partial_t \overline{u} + \left(\overline{u} + u'\right) \cdot \nabla \overline{u} + \nabla \overline{p} &= \nu \Delta \overline{u} + \overline{f} \\ \partial_t u' + \left(\overline{u} + u'\right) \cdot \nabla u' + \nabla p' &= \nu \Delta u' + f' \\ \operatorname{div} \overline{u} &= \operatorname{div} u' = 0, \qquad \overline{u}|_{\partial D} = u'|_{\partial D} = 0 \\ \overline{u}(0) &= \overline{u}_0, \qquad u'(0) = u'_0. \end{aligned}$$

This system is equivalent to the original equation

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f \\ \operatorname{div} u &= 0, \qquad u|_{\partial D} = 0, \qquad u(0) = u_0 \end{aligned}$$

when

$$\begin{aligned} f &= \overline{f} + f' \\ u_0 &= \overline{u}_0 + u'_0. \end{aligned}$$

Indeed, if $(\overline{u}, \overline{p}; u', p')$ is a solution of the system, then $u = \overline{u} + u'$, $p = \overline{p} + p'$ is a solution of the equations; viceversa, if (u, p) is a solution of the equations and $(\overline{u}, \overline{p})$ is a solution of

$$\partial_t \overline{u} + u \cdot \nabla \overline{u} + \nabla \overline{p} = \nu \Delta \overline{u} + \overline{f}$$

then $u' = u - \overline{u}$, $p' = p - \overline{p}$ is a solution of

$$\partial_t u' + \left(\overline{u} + u'\right) \cdot \nabla u' + \nabla p' = \nu \Delta u' + f'.$$

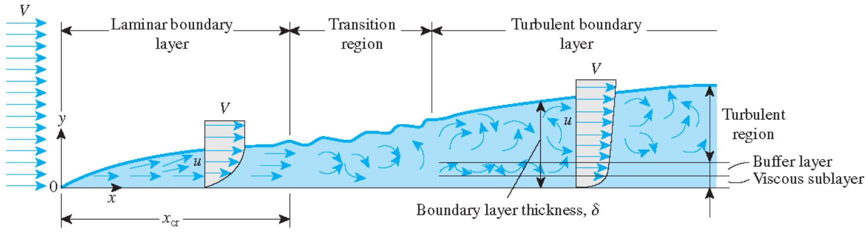

FIGURE 6–14
The development of the boundary layer for flow over a flat plate, and the different flow regimes.
Courtesy of University of Delaware.

In the system we impose the small-large scale subdivision only on data: on the initial condition and on the forcing term. At least for a short time, this subdivision is expected to be maintained, approximately. How much it is maintained for longer times is a very difficult issue; certainly $\overline{u}$, for longer times is corrupted by small scales and u' by large scales; the open problem is how much.

Now let us come to stochastic modeling: looking at real situations with a boundary and the vortices produced near it, we suspect that the small scales are quite concentrated in a region near the boundary, the large scales are active everywhere.

Thus we replace the system above with the model

$$\begin{aligned}
\partial_t \overline{u} + \left(\overline{u} + u'\right) \cdot \nabla \overline{u} + \nabla \overline{p} &= \nu \Delta \overline{u} + \overline{f} \\
\partial_t u' + \nabla p' &= \nu \Delta u' - \frac{1}{\epsilon} u' + \frac{1}{\epsilon} \sum_k \sigma_k \partial_t W^k \\
\operatorname{div} \overline{u} &= \operatorname{div} u' = 0, \qquad \overline{u}|_{\partial D} = u'|_{\partial D} = 0 \\
\overline{u}(0) &= \overline{u}_0, \qquad u'(0) = u'_0
\end{aligned}$$

where both equations are considered in the full domain D but the second one is mostly active near the boundary thanks to the fact that the vector fields σ_k have small support near the boundary.

Let us look only at the equation of large scales

$$\partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = \nu \Delta \overline{u} + \overline{f} - u' \cdot \nabla \overline{u}.$$

If we take the limit $\epsilon \to 0$ and argue as in the linear case of temperature diffusion, we get the equation

$$\partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = (\nu \Delta + \mathcal{L}) \overline{u} + \overline{f} - \sum_{k \in K} (\sigma_k \cdot \nabla \overline{u}) \partial_t W^k.$$

This is a *closed* model of large scales, influenced by turbulent small scales.

Is it useful and realistic? This difficult question is under investigation. Let us only mention one positive fact. Consider the associated deterministic equation

$$\begin{aligned} \partial_t U + U \cdot \nabla U + \nabla P &= (\nu\Delta + \mathcal{L}) U + \overline{f} \\ \operatorname{div} U &= 0, \qquad U|_{\partial D} = 0, \qquad u'(0) = \overline{u}_0 \end{aligned}$$

(if $\overline{u}_0$ and $\overline{f}$ are deterministic, otherwise take their expectations). This equation has, for suitalbe $\mathcal{L}$, stronger dissipativity properties that the original one with just $\nu\Delta$. If we can prove that $\overline{u}$ is close to U, then we get that the large scale motion $\overline{u}$ reveals a stronger dissipativity, due to the presence of turbulent small scales. This is the observed phenomenon of *eddy viscosity*: turbulence improves the viscous properties. Mathematically, we can prove that $\overline{u}$ is close to U only in $d = 2$; in $d = 3$ there are essential obstructions. But at least for $d = 2$ we see that this model leads to realistic results.

3 The 3D Navier-Stokes equations with just transport

Preliminary to the concept described in this section, it is the concept of vorticity, mentioned several times in these lectures but never used explicitly, also because a rigorous use of vorticity in bounded domains lead to troubles.

Vorticity is defined as

$$\omega = \operatorname{curl} u$$

and in $d = 2$ it is a vector perpendicular to the plane of motion, hence it can be described by a scalar given by

$$\omega \overset{d=2}{=} \partial_2 u_1 - \partial_1 u_2.$$

From the Navier-Stokes equations, using some vector identities, we find the equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu\Delta\omega + \omega \cdot \nabla u + \operatorname{curl} f$$

which has the advantage that the pressure is disappeared; but the term $\omega \cdot \nabla u$, called *vortex stretching* term, provokes several troubles (it is responsible for the increase of intensity of the vorticity, which otherwise, for $\operatorname{curl} f = 0$, would be just transported by $u \cdot \nabla \omega$ and diffused by $\nu\Delta\omega$).

In $d = 2$ one can see that $\omega \cdot \nabla u = 0$ (indeed u lives in the plane of motion, hence also ∇u, but ω is perpendicular to such plane) and therefore the equation simplifies into the diffusion-transport equation

$$\partial_t \omega + u \cdot \nabla \omega \overset{d=2}{=} \nu\Delta\omega + \operatorname{curl} f$$

which is very useful in domains "without" boundary (the torus, the full space; when there is a boundary, the big problem is that the boundary conditions for ω are not a given datum but part of the solution). It leads to additional invariants and apriori estimates of great success.

Now, consider the topic discussed above of separating large and small scales and model the small scales bu a noise. We may perform this argument at the level of vorticity, instead of velocity. They are not equivalent, and which one is better for the Physics is still debated. Let us discuss here the application of such idea at the vorticity level.

In 2D, the procedure above leads to the stochastic equation (let us write it here in Stratonovich form for simplicity of notations)

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla \overline{\omega} \overset{d=2}{=} \nu \Delta \overline{\omega} - \sum_{k \in K} \sigma_k \cdot \nabla \overline{\omega} \circ \partial_t W^k + \overline{\operatorname{curl} f}.$$

This is an excellent equation, similar to the one of temperature diffusion and transport. In particular, one can discuss when $\overline{\omega}$ is close to the deterministic solution of an equation with increased dissipation of the form

$$\partial_t \Omega + U \cdot \nabla \Omega \overset{d=2}{=} (\nu \Delta + \mathcal{L}) \Omega + \overline{\operatorname{curl} f}.$$

But let us discuss the 3D case. In this case we should find

$$\begin{aligned} &\partial_t \overline{\omega} + (\overline{u} \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \overline{u}) \overset{d=3}{=} \nu \Delta \overline{\omega} + \overline{\operatorname{curl} f} \\ &- \sum_{k \in K} (\sigma_k \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \sigma_k) \circ \partial_t W^k \end{aligned}$$

Indeed, in the original vorticity equation there are two quadratic terms

$$u \cdot \nabla \omega - \omega \cdot \nabla u$$

and in both of them we have to replace u by $(\overline{u} + u')$, and then u' by noise. The previous stochastic equation has been investigated, at the level of local-in-time existence and uniqueness, but the link with an equation of the form

$$\partial_t \Omega + U \cdot \nabla \Omega \overset{d=3}{=} (\nu \Delta + \mathcal{L}) \Omega + \Omega \cdot \nabla U + \overline{\operatorname{curl} f} \tag{2}$$

is not undestood until now. Maybe there are fluid regimes where there is a link, but this is still an open problem.

On the contrary, if we investigate the model, in 3D, with just transport noise,

$$\begin{aligned} &\partial_t \overline{\omega} + (\overline{u} \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \overline{u}) \overset{d=3}{=} \nu \Delta \overline{\omega} + \overline{\operatorname{curl} f} \\ &- \sum_{k \in K} P (\sigma_k \cdot \nabla \overline{\omega}) \circ \partial_t W^k \end{aligned}$$

it is possible to prove a rigorous link with (2). Notice that we have introduced the projection $P : L^2 \to H$ in this equation: in general the term $\sigma_k \cdot \nabla \overline{\omega}$ is not divergence free, while the sum of all other terms is divergence free, hence without the projection there would be no solution in general. Moreover, notice that the previous model has been investigated only on the 3D torus, to avoid the problem of the boundary conditions for the vorticity.

One can prove that the solution $\overline{\omega}$ of the stochastic Navier-Stokes equations is close (in a suitable topology) to the solution Ω of the deterministic Navier-Stokes equations (2) with increased dissipation. This fact has a very important consequence: that *well-posedness is improved by noise*. In the deterministic case, the larger is the viscosity, the longer is the time interval of existence and uniqueness of smooth solutions; this interval becomes even infinite when the sizes of the initial condition and the viscosity (and the forcing term if it is not zero) satisfy a certain relation. Since the noise has the effect to introduce an extra-dissipation, it has the effect to increase the length of the time interval of existence and uniqueness of smooth solutions of the stochastic equation, length that again becomes infinite under certain conditions.

This is the first known regularization by noise result for 3D Navier-Stokes equations; it has been proved by Dejun Luo and F.F. in a recent work. It leaves open the very difficult question whether the same result holds when the noise affect also the stretching term.

4 The Wong-Zakai (Stratonovich) corrector

In this section we consider the heat transport equation

$$\partial_t \theta^\epsilon = \kappa \Delta \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon$$

where u^ϵ is an approximation, in the senso of distributions in time, of the white noise $\sum_{k \in K} \sigma_k \partial_t W^k$. We want to show the convergence to Stratonovich noise. There are rigorous results in the literature on this issue, but we limit ourselves to an heuristic description, for shortness.

Certainly if u^ϵ is smooth, we can say that θ^ϵ is smooth. Hence the we could perform the following computations in a very classical way. However, somewhere in the argument it is necessary to have good estimates on θ^ϵ, uniform in ϵ. These estimates hold only at low level or regularity. For this reason we develop the next computations in a weak sense, close to a rigorous proof, in spite of the fact that we do not give the details. Concerning which uniform-in-ϵ estimates hold, at least two of them: energy estimates and maximum principle hold

$$\left\| \theta^\epsilon(t) \right\|_{L^2}^2 + 2\kappa \int_0^t \left\| \nabla \theta^\epsilon(s) \right\|_{L^2}^2 ds = \left\| \theta_0 \right\|_{L^2}^2$$

$$\left\| \theta^\epsilon(t) \right\|_\infty \leq \left\| \theta_0 \right\|_\infty .$$

In the weak formulation of the equation, let us consider the difficult term and split it on a partition of the time interval:

$$\int_0^t \langle u^\epsilon(s)\cdot\nabla\phi, \theta^\epsilon(s)\rangle\, ds = \sum_i \int_{t_i}^{t_{i+1}} \langle u^\epsilon(s)\cdot\nabla\phi, \theta^\epsilon(s)\rangle\, ds.$$

Just for notational convenience (at the end we go back to the general case) assume $u^\epsilon(t)$ is made of a single term

$$u^\epsilon(t,x) = \sigma(x)\,\xi^\epsilon(t)$$

where

$$W^\epsilon(t) := \int_0^t \xi^\epsilon(s)\, ds \to W_t.$$

Then

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \langle u^\epsilon(s)\cdot\nabla\phi, \theta^\epsilon(s)\rangle\, ds \\
&= \int_{t_i}^{t_{i+1}} \langle \sigma\cdot\nabla\phi, \theta^\epsilon(s)\rangle\, \xi^\epsilon(s)\, ds \\
&= \int_{t_i}^{t_{i+1}} \langle \sigma\cdot\nabla\phi, \theta^\epsilon(t_i)\rangle\, \xi^\epsilon(s)\, ds + \int_{t_i}^{t_{i+1}} \langle \sigma\cdot\nabla\phi, (\theta^\epsilon(s)-\theta^\epsilon(t_i))\rangle\, \xi^\epsilon(s)\, ds \\
&= \langle \sigma\cdot\nabla\phi, \theta^\epsilon(t_i)\rangle\, (W^\epsilon(t_{i+1}) - W^\epsilon(t_i)) + \int_{t_i}^{t_{i+1}} \langle \sigma\cdot\nabla\phi, (\theta^\epsilon(s)-\theta^\epsilon(t_i))\rangle\, \xi^\epsilon(s)\, ds.
\end{aligned}$$

If the form of the approximations ξ^ϵ is not bad (and those described above with the Stokes problem are good), one can show that

$$\sum_i \langle \sigma\cdot\nabla\phi, \theta^\epsilon(t_i)\rangle\, (W^\epsilon(t_{i+1}) - W^\epsilon(t_i)) \to \int_0^t \langle \sigma\cdot\nabla\phi, \theta\rangle\, dW$$

the limit object understood as an Itô integral. We have thus to understand the limit of

$$\sum_i \int_{t_i}^{t_{i+1}} \langle \sigma\cdot\nabla\phi, (\theta^\epsilon(s)-\theta^\epsilon(t_i))\rangle\, \xi^\epsilon(s)\, ds.$$

Notice first a potential mistake: one could think that, being $\theta^\epsilon(s) - \theta^\epsilon(t_i)$ small for $s \in [t_i, t_{i+1}]$, this sum will converge to zero. But $\xi^\epsilon(s)$, being related (in the limit) to the derivative of BM, is large, and the product $(\theta^\epsilon(s) - \theta^\epsilon(t_i))\,\xi^\epsilon(s)$ could have a non-zero compensation. Indeed, it has: roughly speaking $(\theta^\epsilon(s) - \theta^\epsilon(t_i))$ behaves like $\sqrt{t_{i+1}-t_i}$ and $\xi^\epsilon(s)$ diverges like $\frac{1}{\sqrt{t_{i+1}-t_i}}$.

The way to capture the precise asymptotics is using again equation (**??**):

$$\langle \psi, \theta^\epsilon(s) - \theta^\epsilon(t_i)\rangle - \int_{t_i}^s \langle \sigma\cdot\nabla\psi, \theta^\epsilon(r)\rangle\, \xi^\epsilon(r)\, dr = \int_{t_i}^s \langle \kappa\Delta\psi, \theta^\epsilon(r)\rangle\, dr.$$

We have now to deal with the two terms

$$\sum_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon(r) \rangle \xi^\epsilon(r) \xi^\epsilon(s) \, drds$$

and

$$\sum_i \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{s} \langle \kappa \Delta (\sigma \cdot \nabla \phi), \theta^\epsilon(r) \rangle \, dr \right) \xi^\epsilon(s) \, ds.$$

Assuming sufficient smoothness of σ and θ^ϵ,

$$\left| \int_{t_i}^{s} \langle \kappa \Delta (\sigma \cdot \nabla \phi), \theta^\epsilon(r) \rangle \, dr \right| \leq C (t_{i+1} - t_i).$$

Since $\int_{t_i}^{t_{i+1}} \xi^\epsilon(s) \, ds = W^\epsilon(t_{i+1}) - W^\epsilon(t_i)$, with some work that we do not illustrate now, one can show that the last term goes to zero. The difficult term is

$$\sum_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon(r) \rangle \xi^\epsilon(r) \xi^\epsilon(s) \, drds$$

We start to see an auxiliary second order differential operator $(\sigma \cdot \nabla \sigma \cdot \nabla)$ arising here. One has to play again the same trick above: rewrite the previous expression as

$$\begin{aligned} &\sum_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon(t_i) \rangle \xi^\epsilon(r) \xi^\epsilon(s) \, drds \\ &= \sum_i \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon(t_i) \rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi^\epsilon(r) \xi^\epsilon(s) \, drds \end{aligned}$$

plus the remainder. This time, one can show that the remainder is infinitesimal, since rougly speaking it contains the product of three terms, all roughly speaking of order $\sqrt{t_{i+1} - t_i}$:

$$\theta^\epsilon(r) - \theta^\epsilon(t_i), \qquad W^\epsilon(t_{i+1}) - W^\epsilon(t_i), \qquad W^\epsilon(t_{i+1}) - W^\epsilon(t_i).$$

Finally, we have to understand the limit of

$$\sum_i \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon(t_i) \rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi^\epsilon(r) \xi^\epsilon(s) \, drds.$$

For reasonable approximations of white noise one has, as a limit in probability,

$$\sum_i \int_{t_i}^{t_{i+1}} \int_{\widetilde{t}_i}^{s} \xi^\epsilon(r) \xi^\epsilon(s) \, drds \to \frac{1}{2} t.$$

The factor $\frac{1}{2}$ comes from the fact that we integrate over a triangle. For instance, if $W^{\epsilon}(t)$ is the piecewise linear interpolation of $W(t)$ on the grid (t_i), so that $\xi^{\epsilon}(t) = \frac{W(t_{i+1})-W(t_i)}{t_{i+1}-t_i}$ when $t \in (t_i, t_{i+1})$, then

$$\begin{aligned}
& \sum_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi^{\epsilon}(r) \xi^{\epsilon}(s) \, dr ds \\
= & \sum_i \left(\frac{W(t_{i+1}) - W(t_i)}{t_{i+1} - t_i} \right)^2 \frac{(t_{i+1} - t_i)^2}{2} \\
= & \frac{1}{2} \sum_i (W(t_{i+1}) - W(t_i))^2 \to \frac{1}{2} t
\end{aligned}$$

by the famous theorem on the quadratic variation. Then one can prove that

$$\sum_i (\sigma \cdot \nabla \sigma \cdot \nabla) \theta^{\epsilon}(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi^{\epsilon}(r) \xi^{\epsilon}(s) \, dr ds \to \frac{1}{2} \int_0^t (\sigma \cdot \nabla \sigma \cdot \nabla) \theta(s) \, ds.$$

The result, under suitable assumptions, extends to u^{ϵ} of the form

$$u^{\epsilon}(t, x) = \sum_{k \in K} \sigma_k(x) \xi_k^{\epsilon}(t)$$

also in the case of a countable sum, and give rise to the result that

$$\int_0^t u^{\epsilon}(s) \cdot \nabla \theta^{\epsilon}(s) \, ds \to \sum_{k \in K} \int_0^t \sigma_k \cdot \nabla \theta dW_s^k + \frac{1}{2} \sum_{k \in K} \int_0^t (\sigma_k \cdot \nabla \sigma_k \cdot \nabla) \theta(s) \, ds.$$

The fact that the limit sum reduces to a single index k is due to the fact that

$$\sum_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_k^{\epsilon}(r) \xi_{k'}^{\epsilon}(s) \, dr ds \to 0$$

when $k \neq k'$; for instance, in the case of piecewise linear interpolation of $W(t)$ on the grid (t_i), we have

$$\begin{aligned}
& \sum_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_k^{\epsilon}(r) \xi_{k'}^{\epsilon}(s) \, dr ds \\
= & \sum_i \frac{W_k(t_{i+1}) - W_k(t_i)}{t_{i+1} - t_i} \frac{W_{k'}(t_{i+1}) - W_{k'}(t_i)}{t_{i+1} - t_i} \frac{(t_{i+1} - t_i)^2}{2} \\
= & \frac{1}{2} \sum_i (W_k(t_{i+1}) - W_k(t_i)) (W_{k'}(t_{i+1}) - W_{k'}(t_i)) \to 0.
\end{aligned}$$

4.0.1 Divergence form of the operator

We have discovered that the additional term $\mathcal{L}\theta$ appearing in equation (1) has the form

$$(\mathcal{L}\theta)(x) = \sum_{k\in K} \sigma_k(x) \cdot \nabla (\sigma_k(x) \cdot \nabla\theta(x)).$$

Componentwise we can write

$$(\mathcal{L}\theta)(x) = \sum_{k\in K} \sum_{i,j=1}^{d} \sigma_k^i(x)\, \partial_i \left(\sigma_k^j(x)\, \partial_j \theta(x)\right).$$

Since $\sum_{i=1}^{d} \partial_i \sigma_k^i(x) = 0$, we deduce also

$$(\mathcal{L}\theta)(x) = \sum_{k\in K} \sum_{i,j=1}^{d} \partial_i \left(\sigma_k^i(x)\, \sigma_k^j(x)\, \partial_j \theta(x)\right).$$

Let us now introduce for the first time (but this doesn't mean it is a secondary concept) the covariance function of the noise, covariance with respect to the space variable. it is defined as

$$Q(x,y) = \mathbb{E}\left[W(t,x) \otimes W(t,y)\right] \qquad x,y \in D$$

and it is easily found to be

$$Q(x,y) = \sum_{k\in K} \sigma_k(x) \otimes \sigma_k(y).$$

Therefore we have found

$$(\mathcal{L}\theta)(x) = \sum_{i,j=1}^{d} \partial_i \left(Q_{ij}(x,x)\, \partial_j \theta(x)\right).$$

This is an elliptic operator in divergence form. Ellipticity comes from the property

$$\sum_{i,j=1}^{d} Q_{ij}(x,x)\, \xi_i \xi_j = \mathbb{E}\left[\left|W(t,x) \cdot \xi\right|^2\right] \geq 0$$

for all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$.

5 Summary

In this chapter we have discussed a second class of noise: the one of transport type. There is a third class, variant of the second one, namely noise of transport-stretching type in 3D, which is only mentioned but should receive due attention.

Noise of transport type in the equations for auxiliary quantities, like heat, have been investigated by several authors. Here we have introduced them as a Wong-Zakai limit to emphasize the presence of a correcting term, essential to preserve the Physics and to get useful informations. In the case of heat transport our investigation culminates in the proof of a property of eddy dissipation.

But similar ideas may be applied to the internal structure of the fluid itself when we introduce the subdivision in large and small scales. Here the noise is used to summarize the dynamics of small scales and affects the closed equation for the large scales. This is the motivation for considering stochastic Navier-Stokes equations with transport type noise (and, as mentioned above, also with transport-stretching noise in 3D). The 2D case starts to be well understood and, in particular, similarly to the case of heat transfer, one can prove a result of eddy viscosity: turbulence enhances the viscosity of the fluid itself. This fact, clearly observed in real situations, is perhpas the main confirmation that the heuristic discussion made here about stochastic modeling of small scales and consequent transport noise in the large ones may have a deep physical meaning, in spite of poor justifycation at the level of continuum mechanics that we can provide at present.

Moving these ideas to the 3D case but with the limitation of a transport type noise, we may show that noise improves the theory of 3D Navier-Stokes equations. This was a long standing project in the case of additive noise, frustrated however by several technical difficulties. The case of transport noise reveald to be more promising. However, for future research, the understanding of case of transport-stretching noise must be considered the most important open problem.

Let us also add the following very heurisitc remark. In these lectures we started from additive perturbations motivated by the roughness of boundaries. Additive noise, as just mentioned, have not been shown to improve so much the theory of 3D Navier-Stokes equations. But additive noise in the small scales, as shown in the present chapter, may lead to a multiplicative transport noise in the large scales. And transport noise has a better regularizing power. At the end it seems, then, that *it is the additive noise at small scales which regularizes*! Presumably the long-standing conjecture that additive noise regularizes could be correct but the path to reveal its power is very complex. Until now the efforts to prove that additive noise regularizes were based on the similarity with the finite dimensional case, where additive noise is so succeessful. But this is probably a too abstract viewpoint for the Navier-Stokes equations. The deep reason of regularization stands inside the links between scales, a fact proper of fluid dynamics and not of general evolution equations.