Chapter 3. Transport noise

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1 Introduction n.1: stochastic heat transport

Let us oversimplify the fluid dynamics near the boundary. The following view is highly phenomenological and should be subject to much deeper research.

We assume that the fluid, in a region near the boundary, may be approximately described by the equations

$$\begin{aligned} \partial_t u + \nabla p &= \nu \Delta u - \frac{1}{\epsilon} u + \frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k \\ \operatorname{div} u &= 0 \\ u|_{\partial D} &= 0 \end{aligned}$$

This is Stokes model, strongly incorrect in itself for turbulent fluids, but complemented by the creation of eddies/vortices (the term $\frac{1}{\epsilon} \sum_{k \in K} \sigma_k \partial_t W^k$) and an extra-dissipation term of friction type $(-\frac{1}{\epsilon} u)$ to compensate the extra input of energy (in the average) due to the noise.

We have intentionally parametrized the problem by $\epsilon > 0$, in the very precise way written above, because we want to explore here a special scaling limit. Let us also, from now on, denote u by u^ϵ. The abstract semigroup formulation of this problem, with A given by the operator $\nu P \Delta$ as in the previous chapters, is

$$u^\epsilon (t) = e^{t\left(A - \frac{1}{\epsilon}\right)} u_0 + \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{(t-s)\left(A - \frac{1}{\epsilon}\right)} \sigma_k dW_s^k.$$

In Chapter 1, in order to avoid Itô integrals and cover rough noise sources of very different type, we have integrated by parts and used the following formulation:

$$u^\epsilon (t) = e^{t\left(A - \frac{1}{\epsilon}\right)} u_0 + \frac{1}{\epsilon} \sum_{k \in K} \sigma_k W_t^k + \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{(t-s)\left(A - \frac{1}{\epsilon}\right)} \left(A - \frac{1}{\epsilon}\right) \sigma_k W_s^k ds.$$

When W_s^k are independent Brownian motions, both formulations are meaningful and they are equivalent. In the next lines we shall apply a Fubini type theorem to the stochastic integral: one way to justify it rigorously is precisely to use the last formulation which involves only Lebesgue integrals.

Let us introduce two notations:

$$\begin{aligned} W^{\epsilon}(t,x) &= \int_0^t u^{\epsilon}(s,x)\,ds \\ W(t,x) &= \sum_{k\in K}\sigma_k(x)\,W_t^k. \end{aligned}$$

Then

$$\begin{aligned} W^{\epsilon}(t) &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\int_0^s e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k ds \\ &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\int_r^t e^{(s-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k ds dW_r^k \\ &= \frac{1}{\epsilon}\sum_{k\in K}\int_0^t\left(A-\frac{1}{\epsilon}\right)^{-1}\left[e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}-1\right]\sigma_k dW_r^k \\ &= \frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1}\sum_{k\in K}\int_0^t e^{(t-r)\left(A-\frac{1}{\epsilon}\right)}\sigma_k dW_r^k-\frac{1}{\epsilon}\left(A-\frac{1}{\epsilon}\right)^{-1}W(t). \end{aligned}$$

Now we use the fact (well known in the framework of Yosida approximations of semigroup theory) that

$$\lim_{\lambda\to\infty}\lambda(\lambda-A)^{-1}h=h$$

for all $h\in H$; being A^{-1} compact in our example, we can easily verify this property using the spectral decomposition. With minor additional arguments that we leave as exercise, it follows:

Lemma 1

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\|W^{\epsilon}(t)-W(t)\|_H^2\right]=0.$$

The result is also uniform in time, with supremum inside the expected value. The message of this lemma is that u converges in distribution to a white noise, the time derivative of the space-dependent Brownian motion W.

Why is this an interesting regime? Let us investigate this issue in the case of the evolution of an auxiliary quantity: heat. Assume the fluid has a variable temperature and is not strongly influenced by temperature, hence we do not change its equation of motion.

But temperature, next indicated by $\theta(t,x)$, evolves according to the diffusion-transport equation

$$\partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta + q$$

where $\kappa > 0$, typically small, is the heat diffusion constant and $u\cdot\nabla\theta$ is the transport due to the fluid motion; q is a heat source. If we take the limit $\epsilon \to 0$ in the model of fluid above and we apply the heuristics of Wong-Zakai result, we find the model

$$\partial_t\theta + \sum_{k\in K}(\sigma_k\cdot\nabla\theta)\circ\partial_t W^k = \kappa\Delta\theta + q$$

where the symbol $\circ$ stands for the Stratonovich operation. Below we explain why the correct Itô interpretation of this equation is

$$\partial_t\theta + \sum_{k\in K}(\sigma_k\cdot\nabla\theta)\,\partial_t W^k = (\kappa\Delta + \mathcal{L})\,\theta + q \tag{1}$$

where the stochastic term is now understood in the classical Itô sense and $\mathcal{L}$ is a suitable linear operator, precisely a second order elliptic differential operator, that we shall discover. The result of this modeling step is that we end-up with model (1) for the heat diffusion under a turbulent velocity field. Taking (heuristically at this stage) expectation of each term and introducing the mean temperature profile

$$\Theta(t,x) = \mathbb{E}[\theta(t,x)]$$

we get

$$\partial_t\Theta = (\kappa\Delta + \mathcal{L})\,\Theta + q.$$

If the noise has suitable properties, the elliptic operator $\mathcal{L}$ strongly increases the dissipation of the term $\kappa\Delta$. Moreover we shall prove that the random field $\theta(t,x)$ is close to its average $\Theta(t,x)$ under suitable assumptions. This will lead to the statement that *turbulent diffusion increases the original diffusion*, a fact that is observed in experiments. This model has the power to explain a well known experimental phenomenon, the so called *eddy diffusion*.

The results outlined in this introductory section will be developed below in some detail but additional informations can be found in the paper that initiated this research [29] and in subsequent references like [18], [46]; a different scaling can be seen in [21].

2 Introduction n.2: additional stochastic transport in the Navier-Stokes equations

Stochastic transport of passive scalars (the topic described in the previous section) is well known in the literature. On the contrary, this section introduces an analogous idea for the *internal modeling of a fluid*, which is less common and still debated. In some cases however it leads to results observed in the real world, hence it deserves to be investigated.

Fluids, in their complex regimes that we loosely name turbulent, show the activation of several scales: we observe large scale motions and small scale ones at the same time, with several intermediate scales; very small vortices, larger and larger ones, up to motion at the scale of the full domain. Oversimplifying this multiscale picture, let us think we want to split the fluid velocity in two components

$$u(t,x) = \overline{u}(t,x) + u'(t,x)$$

the first one containing most of the large scales, the second one mostly related to the small scales. A precise subdivision is impossibile, due to the multiscale nature of the problem.

An attempt to perform a precise subdivision is by means of projections. Let us mention two of them. One is by Fourier projections and was used already above as a technical tool for the rigorous investigation. If (e_n) is a complete orthonormal system of H as described in Chapter 2 and π_n are the associated finite dimensional projections, we may define

$$\overline{u}(t) = \pi_n u(t)$$

and thus $u'(t) = (I - \pi_n) u(t)$. The second approach is to take a smooth, possibly compact support, probability density θ, introduce the mollifiers $\theta_\epsilon(x) = \epsilon^{-d}\theta\left(\epsilon^{-1}x\right)$ (where d is the space dimension) and define

$$\overline{u}(t) = \theta_\epsilon * u(t)$$

with suitable corrections in bounded domains to cope with the problem that $\theta_\epsilon(\cdot - x_0)$ may not have the support in D.

With these definitions we guarantee a priori that $\overline{u}(t)$ is made only of "large scale structures". However, the equations for $\overline{u}(t)$ and $u'(t)$ are interlaced in a quite complex manner. An alternative approach is to consider the Navier-Stokes type system

$$\begin{aligned}
\partial_t\overline{u} + \left(\overline{u} + u'\right)\cdot\nabla\overline{u} + \nabla\overline{p} &= \nu\Delta\overline{u} + \overline{f} \\
\partial_t u' + \left(\overline{u} + u'\right)\cdot\nabla u' + \nabla p' &= \nu\Delta u' + f' \\
\operatorname{div}\overline{u} &= \operatorname{div} u' = 0, \qquad \overline{u}|_{\partial D} = u'|_{\partial D} = 0 \\
\overline{u}(0) &= \overline{u}_0, \qquad u'(0) = u_0'.
\end{aligned}$$

This system is equivalent to the original equation

$$\begin{aligned}
\partial_t u + u\cdot\nabla u + \nabla p &= \nu\Delta u + f \\
\operatorname{div} u &= 0, \qquad u|_{\partial D} = 0, \qquad u(0) = u_0
\end{aligned}$$

when

$$\begin{aligned}
f &= \overline{f} + f' \\
u_0 &= \overline{u}_0 + u_0'.
\end{aligned}$$

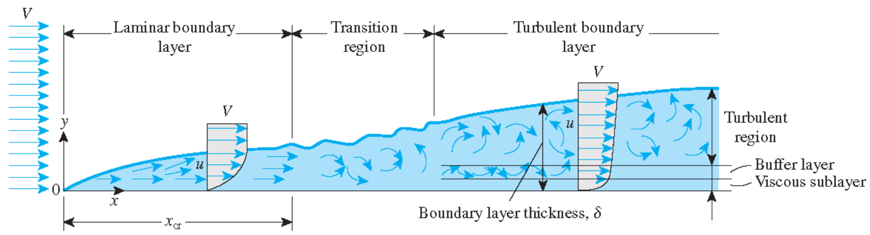


FIGURE 6–14
The development of the boundary layer for flow over a flat plate, and the different flow regimes.
Courtesy of University of Delaware.

Indeed, if $(\overline{u}, \overline{p}; u', p')$ is a solution of the system, then $u = \overline{u} + u'$, $p = \overline{p} + p'$ is a solution of the equations; viceversa, if (u, p) is a solution of the equations and $(\overline{u}, \overline{p})$ is a solution of

$$\partial_t \overline{u} + u \cdot \nabla \overline{u} + \nabla \overline{p} = \nu \Delta \overline{u} + \overline{f}$$

then $u' = u - \overline{u}$, $p' = p - \overline{p}$ is a solution of

$$\partial_t u' + \left(\overline{u} + u'\right) \cdot \nabla u' + \nabla p' = \nu \Delta u' + f'.$$

In the system we impose the small-large scale subdivision only on data: on the initial condition and on the forcing term. At least for a short time, this subdivision is expected to be maintained, approximately. How much it is maintained for longer times is a very difficult issue; certainly $\overline{u}$, for longer times is corrupted by small scales and u' by large scales; the open problem is how much.

Now let us come to stochastic modeling: looking at real situations with a boundary and the vortices produced near it, we suspect that the small scales are quite concentrated in a region near the boundary, the large scales are active everywhere.

Thus we replace the system above with the model

$$\begin{aligned}
\partial_t \overline{u} + \left(\overline{u} + u'\right) \cdot \nabla \overline{u} + \nabla \overline{p} &= \nu \Delta \overline{u} + \overline{f} \\
\partial_t u' + \nabla p' &= \nu \Delta u' - \frac{1}{\epsilon} u' + \frac{1}{\epsilon} \sum_k \sigma_k \partial_t W^k \\
\operatorname{div} \overline{u} &= \operatorname{div} u' = 0, \qquad \overline{u}|_{\partial D} = u'|_{\partial D} = 0 \\
\overline{u}\,(0) &= \overline{u}_0, \qquad u'\,(0) = u'_0
\end{aligned}$$

where both equations are considered in the full domain D but the second one is mostly active near the boundary thanks to the fact that the vector fields σ_k have small support near the boundary.

Let us look only at the equation of large scales

$$\partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = \nu \Delta \overline{u} + \overline{f} - u' \cdot \nabla \overline{u}.$$

If we take the limit $\epsilon \to 0$ and argue as in the linear case of temperature diffusion, we get the equation

$$\partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = (\nu \Delta + \mathcal{L})\, \overline{u} + \overline{f} - \sum_{k \in K} (\sigma_k \cdot \nabla \overline{u})\, \partial_t W^k.$$

This is a *closed* model of large scales, influenced by turbulent small scales.

Is it useful and realistic? This difficult question is under investigation. Let us only mention one positive fact. Consider the associated deterministic equation

$$\begin{aligned} \partial_t U + U \cdot \nabla U + \nabla P &= (\nu \Delta + \mathcal{L})\, U + \overline{f} \\ \operatorname{div} U &= 0, \qquad U|_{\partial D} = 0, \qquad u'(0) = \overline{u}_0 \end{aligned}$$

(if $\overline{u}_0$ and $\overline{f}$ are deterministic, otherwise take their expectations). This equation has, for suitalbe $\mathcal{L}$, stronger dissipativity properties that the original one with just $\nu\Delta$. If we can prove that $\overline{u}$ is close to U, then we get that the large scale motion $\overline{u}$ reveals a stronger dissipativity, due to the presence of turbulent small scales. This is the observed phenomenon of *eddy viscosity*: turbulence improves the viscous properties. Mathematically, we can prove that $\overline{u}$ is close to U only in $d = 2$; in $d = 3$ there are essential obstructions. But at least for $d = 2$ we see that this model leads to realistic results.

The results outlined in the introductory section would require a chapter in themselves and will not be developed in this book. The reader may see some of the existing results in the following references: [16], [19], [25], [24].

3 The 3D Navier-Stokes equations with just transport

Preliminary to the concept described in this section, it is the concept of vorticity, mentioned several times in these lectures but never used explicitly, also because a rigorous use of vorticity in bounded domains lead to troubles.

Vorticity is defined as

$$\omega = \operatorname{curl} u$$

and in $d = 2$ it is a vector perpendicular to the plane of motion, hence it can be described by a scalar given by

$$\omega \overset{d=2}{=} \partial_2 u_1 - \partial_1 u_2.$$

From the Navier-Stokes equations, using some vector identities, we find the equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u + \operatorname{curl} f$$

which has the advantage that the pressure is disappeared; but the term $\omega \cdot \nabla u$, called *vortex stretching* term, provokes several troubles (it is responsible for the increase of intensity of the vorticity, which otherwise, for $\operatorname{curl} f = 0$, would be just transported by $u \cdot \nabla\omega$ and diffused by $\nu\Delta\omega$).

In $d = 2$ one can see that $\omega \cdot \nabla u = 0$ (indeed u lives in the plane of motion, hence also ∇u, but ω is perpendicular to such plane) and therefore the equation simplifies into the diffusion-transport equation

$$\partial_t \omega + u \cdot \nabla\omega \overset{d=2}{=} \nu\Delta\omega + \operatorname{curl} f$$

which is very useful in domains "without" boundary (the torus, the full space; when there is a boundary, the big problem is that the boundary conditions for ω are not a given datum but part of the solution). It leads to additional invariants and apriori estimates of great success.

Now, consider the topic discussed above of separating large and small scales and model the small scales bu a noise. We may perform this argument at the level of vorticity, instead of velocity. They are not equivalent, and which one is better for the Physics is still debated. Let us discuss here the application of such idea at the vorticity level.

In 2D, the procedure above leads to the stochastic equation (let us write it here in Stratonovich form for simplicity of notations)

$$\partial_t \overline{\omega} + \overline{u} \cdot \nabla\overline{\omega} \overset{d=2}{=} \nu\Delta\overline{\omega} - \sum_{k\in K} \sigma_k \cdot \nabla\overline{\omega} \circ \partial_t W^k + \overline{\operatorname{curl} f}.$$

This is an excellent equation, similar to the one of temperature diffusion and transport. In particular, one can discuss when $\overline{\omega}$ is close to the deterministic solution of an equation with increased dissipation of the form

$$\partial_t \Omega + U \cdot \nabla\Omega \overset{d=2}{=} (\nu\Delta + \mathcal{L})\,\Omega + \overline{\operatorname{curl} f}.$$

But let us discuss the 3D case. In this case we should find

$$\begin{aligned}\partial_t \overline{\omega} + (\overline{u} \cdot \nabla\overline{\omega} - \overline{\omega} \cdot \nabla\overline{u}) &\overset{d=3}{=} \nu\Delta\overline{\omega} + \overline{\operatorname{curl} f} \\ - \sum_{k\in K} (\sigma_k \cdot \nabla\overline{\omega} - \overline{\omega} \cdot \nabla\sigma_k) \circ \partial_t W^k\end{aligned}$$

Indeed, in the original vorticity equation there are two quadratic terms

$$u \cdot \nabla\omega - \omega \cdot \nabla u$$

and in both of them we have to replace u by $(\overline{u} + u')$, and then u' by noise. The previous stochastic equation has been investigated, at the level of local-in-time existence and uniqueness, but the link with an equation of the form

$$\partial_t \Omega + U \cdot \nabla\Omega \overset{d=3}{=} (\nu\Delta + \mathcal{L})\,\Omega + \Omega \cdot \nabla U + \overline{\operatorname{curl} f} \tag{2}$$

is not undestood until now. Maybe there are fluid regimes where there is a link, but this is still an open problem.

On the contrary, if we investigate the model, in 3D, with just transport noise,

$$\partial_t \overline{\omega} + \left(\overline{u} \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \overline{u}\right) \overset{d=3}{=} \nu \Delta \overline{\omega} + \overline{\operatorname{curl} f}$$
$$- \sum_{k \in K} P\left(\sigma_k \cdot \nabla \overline{\omega}\right) \circ \partial_t W^k$$

it is possible to prove a rigorous link with (2). Notice that we have introduced the projection $P : L^2 \to H$ in this equation: in general the term $\sigma_k \cdot \nabla \overline{\omega}$ is not divergence free, while the sum of all other terms is divergence free, hence without the projection there would be no solution in general. Moreover, notice that the previous model has been investigated only on the 3D torus, to avoid the problem of the boundary conditions for the vorticity.

One can prove that the solution $\overline{\omega}$ of the stochastic Navier-Stokes equations is close (in a suitable topology) to the solution Ω of the deterministic Navier-Stokes equations (2) with increased dissipation. This fact has a very important consequence: that *well-posedness is improved by noise*. In the deterministic case, the larger is the viscosity, the longer is the time interval of existence and uniqueness of smooth solutions; this interval becomes even infinite when the sizes of the initial condition and the viscosity (and the forcing term if it is not zero) satisfy a certain relation. Since the noise has the effect to introduce an extra-dissipation, it has the effect to increase the length of the time interval of existence and uniqueness of smooth solutions of the stochastic equation, length that again becomes infinite under certain conditions.

This is the first known regularization by noise result for 3D Navier-Stokes equations; it has been proved in [22]. It leaves open the very difficult question whether the same result holds when the noise affect also the stretching term. Other regularization by noise results along similar lines have been developed in [17].

4 The Wong-Zakai (Stratonovich) corrector

Key to the facts described in Section 1 is the emergence of the additional operator $\mathcal{L}$; we feel we need to justify it, at least heuristically. This is the reason for this intermediate section.

In this section we consider the heat transport equation

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \kappa \Delta \theta^\epsilon + q \tag{3}$$

where

$$u^\epsilon(t) = \frac{1}{\epsilon} \sum_{k \in K} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \sigma_k dW_s^k.$$

This is a simplified model with respect to the one of Section 1 (we drop the Stokes operator A, taking $u_0 = 0$ is only to simplify notations). We make this simplification in order to invoke the recent result of [46]; however, the result seems to be true in the case of Section 1.

Theorem 2 *If $\sigma_k \in D(A)$, $\phi \in C^\infty(D)$,*

$$\theta^\epsilon|_{t=0} = \theta_0 \in L^\infty(D)$$

then the weak solution θ^ϵ of equation (3) with initial condition θ_0 satisfies for every $t \geq 0$

$$\lim_{\epsilon \to 0} \langle \theta^\epsilon(t), \phi \rangle = \langle \theta(t), \phi \rangle$$

in probability, where $\theta(t)$ is the unique weak solution of equation (1), with

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sigma_k(x) \cdot \nabla(\sigma_k(x) \cdot \nabla\theta(x)).$$

The unique solvability of equation (1) is not a trivial task and will be postponed to a subsequent section. The unique solvability of equation (3) is classical, along with estimates of the form

$$\begin{gathered} \|\theta^\epsilon(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\nabla\theta^\epsilon(s)\|_{L^2}^2 \, ds = \|\theta_0\|_{L^2}^2 \\ \|\theta^\epsilon(t)\|_\infty \leq \|\theta_0\|_\infty . \end{gathered} \tag{4}$$

Let us give only the idea of proof of Theorem 2, subset of the results of [46]. Recall that, with the notations

$$\begin{aligned} W^\epsilon(t,x) &= \int_0^t u^\epsilon(s,x)\,ds \\ W(t,x) &= \sum_{k \in K} \sigma_k(x) W_t^k \end{aligned}$$

we have proved that

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\|W^\epsilon(t) - W(t)\|_H^2\right] = 0.$$

Let us introduce also some additional notations:

$$\begin{aligned} \xi_t^{k,\epsilon} &= \frac{1}{\epsilon} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} dW_s^k \\ W_t^{k,\epsilon} &= \int_0^t \xi_s^{k,\epsilon} ds \end{aligned}$$

so that $u^\epsilon(t,x) = \sum_{k \in K} \sigma_k(x) \xi_t^{k,\epsilon}$, $W^\epsilon(t,x) = \sum_{k \in K} \sigma_k(x) W_t^{k,\epsilon}$.

We use the weak formulation and try to pass to the limit term by term, taking great advantage of the fact that the equation is linear. In the weak formulation of equation (3), let us concentrate only on the difficult term

$$\int_0^t \langle u^\epsilon (s) \cdot \nabla \phi, \theta^\epsilon (s) \rangle \, ds$$

and split it on the partition π_ϵ:

$$\int_0^t \langle u^\epsilon (s) \cdot \nabla \phi, \theta^\epsilon (s) \rangle \, ds = \sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \langle u^\epsilon (s) \cdot \nabla \phi, \theta^\epsilon (s) \rangle \, ds.$$

Just for notational convenience (at the end we go back to the general case) assume $u^\epsilon (t)$ is made only of a single term

$$u^\epsilon (t, x) = \sigma (x) \, \xi_t^\epsilon$$

where

$$W_t^\epsilon := \int_0^t \xi^\epsilon (s) \, ds \to W_t.$$

Then

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \langle u^\epsilon (s) \cdot \nabla \phi, \theta^\epsilon (s) \rangle \, ds \\
&= \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla \phi, \theta^\epsilon (s) \rangle \, \xi_s^\epsilon ds \\
&= \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla \phi, \theta^\epsilon (t_i) \rangle \, \xi_s^\epsilon ds + \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla \phi, (\theta^\epsilon (s) - \theta^\epsilon (t_i)) \rangle \, \xi_s^\epsilon ds \\
&= \langle \sigma \cdot \nabla \phi, \theta^\epsilon (t_i) \rangle \left(W_{t_{i+1}}^\epsilon - W_{t_i}^\epsilon \right) + \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla \phi, (\theta^\epsilon (s) - \theta^\epsilon (t_i)) \rangle \, \xi_s^\epsilon ds.
\end{aligned}$$

The sum over the partition of the first term converge to the Itô integral $\int_0^t \langle \sigma \cdot \nabla \phi, \theta (s) \rangle \, dW_s$. More difficult is to understand the limit of

$$\sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \langle \sigma \cdot \nabla \phi, (\theta^\epsilon (s) - \theta^\epsilon (t_i)) \rangle \, \xi_s^\epsilon ds. \tag{5}$$

Notice first a potential mistake: one could think that, being $\theta^\epsilon (s) - \theta^\epsilon (t_i)$ small for $s \in [t_i, t_{i+1}]$, this sum will converge to zero. But ξ_s^ϵ, being related (in the limit) to the derivative of BM, is large, and the product $(\theta^\epsilon (s) - \theta^\epsilon (t_i)) \, \xi_s^\epsilon$ could have a non-zero compensation. Indeed, it has: roughly speaking $(\theta^\epsilon (s) - \theta^\epsilon (t_i))$ behaves like $\sqrt{t_{i+1} - t_i}$ and ξ_s^ϵ diverges like $\frac{1}{\sqrt{t_{i+1} - t_i}}$.

The way to capture the precise asymptotics is using again equation (**??**), written here for a generic test function ψ:

$$\left\langle \psi,\theta^{\epsilon}\left(s\right)-\theta^{\epsilon}\left(t_{i}\right)\right\rangle -\int_{t_{i}}^{s}\left\langle \sigma\cdot\nabla\psi,\theta^{\epsilon}\left(r\right)\right\rangle \xi_{r}^{\epsilon}dr=\int_{t_{i}}^{s}\left\langle \kappa\Delta\psi,\theta^{\epsilon}\left(r\right)\right\rangle dr. \tag{6}$$

Take $\psi=\sigma\cdot\nabla\phi$ to connect with the above term (5) to be investigated. We have now to deal with the two terms

$$\sum_{t_{i}\leq t}\int_{t_{i}}^{t_{i+1}}\int_{t_{i}}^{s}\left\langle \sigma\cdot\nabla\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(r\right)\right\rangle \xi_{r}^{\epsilon}\xi_{s}^{\epsilon}drds$$

and

$$\sum_{t_{i}\leq t}\int_{t_{i}}^{t_{i+1}}\left(\int_{t_{i}}^{s}\left\langle \kappa\Delta\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(r\right)\right\rangle dr\right)\xi_{s}^{\epsilon}ds. \tag{7}$$

Having assumed sufficient smoothness of σ and ϕ, we may use (4) to bound $\theta^{\epsilon}\left(r\right)$ uniformly and find (the inequality is even a.s., with a deterministic constant $C>0$)

$$\left|\int_{t_{i}}^{s}\left\langle \kappa\Delta\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(r\right)\right\rangle dr\right|\leq C\left(t_{i+1}-t_{i}\right).$$

Since $\int_{t_{i}}^{t_{i+1}}\left|\xi_{s}^{\epsilon}\right|ds$ is infinitesimal in a suitable probabilistic sense, it is easy to show that the term (7) goes to zero in probability. The difficult term is

$$\sum_{t_{i}\leq t}\int_{t_{i}}^{t_{i+1}}\int_{t_{i}}^{s}\left\langle \sigma\cdot\nabla\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(r\right)\right\rangle \xi_{r}^{\epsilon}\xi_{s}^{\epsilon}drds$$

But we start to see an auxiliary second order differential operator $\left(\sigma\cdot\nabla\sigma\cdot\nabla\right)$ arising here and this reinforces us to continue the computation. One has to play again the same trick above: rewrite the previous expression as

$$\begin{aligned}
&\sum_{t_{i}\leq t}\int_{t_{i}}^{t_{i+1}}\int_{t_{i}}^{s}\left\langle \sigma\cdot\nabla\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(t_{i}\right)\right\rangle \xi_{r}^{\epsilon}\xi_{s}^{\epsilon}drds\\
&=\sum_{t_{i}\leq t}\left\langle \sigma\cdot\nabla\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(t_{i}\right)\right\rangle \int_{t_{i}}^{t_{i+1}}\int_{t_{i}}^{s}\xi_{r}^{\epsilon}\xi_{s}^{\epsilon}drds
\end{aligned}$$

plus the remainder

$$R_{\epsilon}:=\sum_{t_{i}\leq t}\int_{t_{i}}^{t_{i+1}}\int_{t_{i}}^{s}\left\langle \sigma\cdot\nabla\left(\sigma\cdot\nabla\phi\right),\theta^{\epsilon}\left(r\right)-\theta^{\epsilon}\left(t_{i}\right)\right\rangle \xi_{r}^{\epsilon}\xi_{s}^{\epsilon}drds.$$

This time, one can show that the remainder is infinitesimal. The heuristic idea comes from the fact that it contains the product of three terms, all roughly speaking of order $\sqrt{t_{i+1} - t_i}$:

$$\theta^\epsilon(r) - \theta^\epsilon(t_i), \qquad W^\epsilon(t_{i+1}) - W^\epsilon(t_i), \qquad W^\epsilon(t_{i+1}) - W^\epsilon(t_i).$$

Again (6) and (4) are useful here.

Finally, we have to understand the limit of

$$\sum_{t_i \le t} \langle \sigma \cdot \nabla (\sigma \cdot \nabla \phi), \theta^\epsilon(t_i) \rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^\epsilon \xi_s^\epsilon dr ds.$$

In the case of general noise with several independent Brownian motions, we have to understand the limit of

$$\sum_{t_i \le t} \langle \sigma_k \cdot \nabla (\sigma_{k'} \cdot \nabla \phi), \theta^\epsilon(t_i) \rangle \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{k,\epsilon} \xi_s^{k',\epsilon} dr ds.$$

One can prove the following property on the joint quadratic variation:

$$\lim_{\epsilon \to 0} \sum_{t_i \le t} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \xi_r^{k,\epsilon} \xi_s^{k',\epsilon} dr ds \to \frac{1}{2} \delta_{k,k'} t$$

uniformly in time, in probability. From properties of Riemann-Stieltjes integrals, it follows that the previous sum converges to

$$\frac{\delta_{k,k'}}{2} \int_0^t \langle \sigma_k \cdot \nabla (\sigma_{k'} \cdot \nabla \phi), \theta(s) \rangle ds.$$

The final result is that, in the weak sense,

$$\begin{aligned} & \lim_{\epsilon \to 0} \int_0^t u^\epsilon(s) \cdot \nabla \theta^\epsilon(s) ds \\ = \quad & \sum_{k \in K} \int_0^t \sigma_k \cdot \nabla \theta dW_s^k + \frac{1}{2} \sum_{k \in K} \int_0^t (\sigma_k \cdot \nabla \sigma_k \cdot \nabla) \theta(s) ds. \end{aligned}$$

4.1 Divergence form of the operator

We have discovered that the additional term $\mathcal{L}\theta$ appearing in equation (1) has the form

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sigma_k(x) \cdot \nabla (\sigma_k(x) \cdot \nabla \theta(x)).$$

Componentwise we can write

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sum_{i,j=1}^{d} \sigma_k^i(x)\, \partial_i \left(\sigma_k^j(x)\, \partial_j \theta(x) \right).$$

Since $\sum_{i=1}^{d} \partial_i \sigma_k^i(x) = 0$, we deduce also

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{k \in K} \sum_{i,j=1}^{d} \partial_i \left(\sigma_k^i(x)\, \sigma_k^j(x)\, \partial_j \theta(x) \right).$$

Let us now introduce for the first time (but this doesn't mean it is a secondary concept) the covariance function of the noise, covariance with respect to the space variable. it is defined as

$$Q(x,y) = \mathbb{E}\left[W(t,x) \otimes W(t,y) \right] \qquad x, y \in D$$

and it is easily found to be

$$Q(x,y) = \sum_{k \in K} \sigma_k(x) \otimes \sigma_k(y).$$

Therefore we have found

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \left(Q_{ij}(x,x)\, \partial_j \theta(x) \right).$$

This is an elliptic operator in divergence form. Ellipticity comes from the property

$$\sum_{i,j=1}^{d} Q_{ij}(x,x)\, \xi_i \xi_j = \mathbb{E}\left[|W(t,x) \cdot \xi|^2 \right] \geq 0$$

for all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$.

5 Existence and uniqueness for the heat equation with transport noise

In this section we want to prove an existence and uniqueness result for the equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta)\, \partial_t W^k = (\kappa \Delta + \mathcal{L})\, \theta + q$$

in a bounded regular domain $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Other domains and boundary conditions can be studied as well.

We know two very efficient methods:

1. variational,
2. semigroups.

5.1 Variational method

We limit ourselves to the ideas.

- One has to introduce a sequence of approximating problems which have a unique solution by known results. We skip this step.
- On these approximations, one has to prove estimates independent of the approximating parameter.
- *We perform such step on the true equation, in the style of a priori estimates: we assume to have a smooth solution and see which estimates hold.*
- Such estimates provide the basis of application of the compactness method. We skip the details of this step.

5.1.1 A priori estimates using Stratonovich formulation

If we use Stratonovich formulation

$$\partial_t\theta + \sum_{k\in K} (\sigma_k \cdot \nabla\theta) \circ \partial_t W^k = \kappa\Delta\theta + q$$

and we accept that the rules of calculus (being the limit of smooth noise) are the classical ones, we get (recall $\operatorname{div}\sigma_k = 0$)

$$\begin{aligned} \frac{d}{dt}\|\theta(t)\|_{L^2}^2 &= -2\left\langle \theta, \sum_{k\in K} (\sigma_k \cdot \nabla\theta) \circ \partial_t W^k \right\rangle + 2\langle\theta, \kappa\Delta\theta\rangle \\ &= -2\kappa \|\nabla\theta(t)\|_{L^2}^2 + 2\langle\theta, q\rangle \end{aligned}$$

because

$$\begin{aligned} 2\int_D \langle\theta, \sigma_k \cdot \nabla\theta\rangle &= 2\int_D \theta(x)\,\sigma_k(x) \cdot \nabla\theta(x)\,dx \\ &= \int_D \sigma_k(x) \cdot \nabla\theta^2(x)\,dx = -\int_D \operatorname{div}\sigma_k(x)\,\theta^2(x)\,dx = 0. \end{aligned}$$

Therefore

$$\frac{d}{dt}\|\theta(t)\|_{L^2}^2 + 2\kappa\|\nabla\theta(t)\|_{L^2}^2 = 2\langle\theta(t), q(t)\rangle$$

leading to the a.s. (deterministic!) estimates. By easy classical steps one gets

$$\begin{aligned} \sup_{t\in[0,T]} \|\theta(t)\|_{L^2}^2 &\leq C \\ \int_0^T \|\nabla\theta(s)\|_{L^2}^2\,ds &\leq C \end{aligned}$$

with C depending only on κ, $\|\theta_0\|_{L^2}$, $\int_0^T \|q(s)\|_{L^2}^2\,ds$.

5.1.2 Maximum Principle a priori estimates

Consider the Kolmogorov equation

$$\begin{aligned}\partial_t \theta + u \cdot \nabla \theta &= \kappa \Delta \theta + q \\ \theta|_{t=0} &= \theta_0\end{aligned}$$

on a time interval $[0, T]$. Introducing $\theta_T(t) = \theta(T-t)$, $u_T(t) = u(T-t)$, $q_T(t) = q(T-t)$, we get

$$\begin{aligned}\partial_t \theta_T - u_T \cdot \nabla \theta_T + \kappa \Delta \theta_T + q_T &= 0 \\ \theta_T|_{t=T} &= \theta_0.\end{aligned}$$

Denoting by $\varphi_{s,t}(x)$ the flow associated to the equation

$$\begin{aligned}d\varphi_{s,t}(x) &= -u_T\left(t, \varphi_{s,t}(x)\right) dt + \sqrt{2\kappa} dB_t \qquad t \in [s, T] \\ \varphi_{s,s}(x) &= x\end{aligned}$$

where B_t is an auxiliary Brownian motion, we have

$$\begin{aligned}d\theta_T\left(t, \varphi_{s,t}(x)\right) &= \partial_t \theta_T dt + \nabla \theta_T \cdot d\varphi_{s,t} + \kappa \Delta \theta_T dt \\ &= u_T \cdot \nabla \theta_T dt - \kappa \Delta \theta_T dt - q_T dt \\ &\quad - \nabla \theta_T \cdot u_T dt + \nabla \theta_T \cdot \sqrt{2\kappa} dB_t + \kappa \Delta \theta_T dt \\ &= -q_T dt + \nabla \theta_T \cdot \sqrt{2\kappa} dB_t\end{aligned}$$

and therefore

$$\mathbb{E}\left[\theta_0\left(\varphi_{s,T}(x)\right)\right] - \theta_T(s, x) = -\int_s^T \mathbb{E}\left[q_T\left(t, \varphi_{s,t}(x)\right)\right] dt.$$

Going back to the original variables we have

$$\mathbb{E}\left[\theta_0\left(\varphi_{s,T}(x)\right)\right] - \theta(T-s, x) = -\int_s^T \mathbb{E}\left[q\left(T-t, \varphi_{s,t}(x)\right)\right] dt$$

namely,

$$\theta(t, x) = \mathbb{E}\left[\theta_0\left(\varphi_{T-t,T}(x)\right)\right] + \int_{T-t}^T \mathbb{E}\left[q\left(T-r, \varphi_{T-t,r}(x)\right)\right] dr.$$

We deduce in particular

$$\|\theta(t)\|_\infty \leq \|\theta_0\|_\infty + \int_0^T \|q(r)\|_\infty dr. \tag{8}$$

The previous computation, performed here heuristically, can be made rigorous by convolution under very general assumptions. With due effort based on the theory of stochastic flows, it works also for the equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \circ \partial_t W^k = \kappa \Delta \theta + q$$

in Stratonovich form, being the limit of equations with regular coefficients. The final result is the same, a deterministic (a.s.) inequality in the supremum norm, a kind of Maximum Principle estimate.

5.1.3 A priori estimates using Itô formulation

Obviously the final result will be the same but let us see the computation when the equation contains the Itô-Stratonovich corrector; and the Itô formula is used to perform computations, with its correcting term. We use Itô formulation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \, \partial_t W^k = (\kappa \Delta + \mathcal{L}) \, \theta + q$$

and we apply Itô formula, we get

$$\begin{aligned} d \left\| \theta(t) \right\|_{L^2}^2 &= -2 \sum_{k \in K} \langle \theta, (\sigma_k \cdot \nabla \theta) \rangle \, dW^k + 2 \langle \theta, (\kappa \Delta + \mathcal{L}) \, \theta + q \rangle \, dt \\ &\quad + \sum_{k \in K} \left\| \sigma_k \cdot \nabla \theta \right\|_{L^2}^2 dt \\ &= -2\kappa \left\| \nabla \theta(t) \right\|_{L^2}^2 + 2 \langle \theta, q \rangle - 2 \frac{1}{2} \int_D \sum_{ij} Q(x,x) \, \partial_i \theta \partial_j \theta dx dt \\ &\quad + \sum_{k \in K} \int_D \sum_{ij} \sigma_k^i(x) \, \partial_i \theta \sigma_k^j(x) \, \partial_j \theta dx dt \end{aligned}$$

and get the same as above. *At the level of energy estimates, the Itô term and the corrector completely balance each other.*

5.2 Semigroup method

Consider the equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta) \, \partial_t W^k = (\kappa \Delta + \mathcal{L}) \, \theta + q. \tag{9}$$

Call: $H = L^2(D)$, $V = W_0^{1,2}(D)$, $D(A) = W^{2,2}(D) \cap V$, $A : D(A) \subset H \to H$

$$A\theta = (\kappa \Delta + \mathcal{L}) \, \theta$$

e^{tA}, $t \geq 0$, the analytic semigroup generated by A (under minimal regularity assumptions on $Q(x,x)$). Then

$$\theta(t) = e^{tA}\theta_0 - \sum_{k\in K}\int_0^t e^{(t-s)A}(\sigma_k \cdot \nabla\theta(s))\, dW_s^k.$$

We want to solve this equation by iterations. These equations are not trivial because there is a gradient of θ on the right-hand-side and thus iteration requires that also the left-hand-side accepts a gradient.

5.2.1 Notions of solution and main result

As already done in a previous chapter, let us denote by $L^2_{\mathcal{F}}(0,T;V)$ the space of progressively measurable process with values in V and by $C_{\mathcal{F}}([0,T];H)$ the space of continuous adapted square integrable processes. Assume σ_k smooth enough, $\theta_0 \in H$, $q \in L^2(0,T;H)$. A stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is assumed to be given (thus we deal with strong solutions).

Definition 3 *A stochastic process*

$$\theta \in C_{\mathcal{F}}([0,T];H) \cap L^2_{\mathcal{F}}(0,T;V)$$

is a weak solution if, for every $\phi \in C^2(D)$, we have

$$\begin{aligned}\langle\theta(t),\phi\rangle &= \langle\theta_0,\phi\rangle + \int_0^t \langle\theta(s),(\kappa\Delta+\mathcal{L})\phi\rangle\, ds \\ &\quad + \int_0^t \langle q(s),\phi\rangle\, ds + \sum_{k\in K}\int_0^t \langle\theta(s),\sigma_k\cdot\nabla\phi\rangle\, dW_t^k\end{aligned}$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s.

Notice that the stochastic integrals are well defined since $\sigma_k \cdot \nabla\phi \in H$, hence the integrand is a continuous adapted process; the deterministic integral is obviously well defined, since $s \mapsto \langle\theta(s),(\kappa\Delta+\mathcal{L})\phi\rangle$ is $\mathbb{P}$-a.s. continuous.

In the following alternative definition we use the heat semigroup e^{tA}.

Definition 4 *A stochastic process*

$$\theta \in C_{\mathcal{F}}([0,T];H) \cap L^2_{\mathcal{F}}(0,T;V)$$

is a mild solution if the following identity holds

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A}q(s)\, ds - \sum_{k\in K}\int_0^t e^{(t-s)A}\sigma_k\cdot\nabla\theta(s)\, dW_s^k$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s.

Proposition 5 *The two notions of solution coincide.*

The proof is not difficult and similar to one made in Chapter 1 for the Stokes problem. The main result proved below is:

Theorem 6 *For every $\theta_0 \in H$ and $q \in L^2(0,T;H)$, there exists one and only one (weak or mild) solution.*

5.2.2 General parabolic equations with Itô-type transport noise

In order to fully appreciate certain aspects of the previous result, consider the more general problem: the equation

$$\partial_t \theta + \sum_{k \in K} (\sigma_k \cdot \nabla \theta)\, \partial_t W^k = \sum_{i,j=1}^{d} \partial_j \left(a_{ij}(x)\, \partial_i \theta \right) + q \tag{10}$$

where $a_{i,j}$ is strongly elliptic and sufficiently regular so that the operator $A\theta = \sum_{i,j=1}^{d} \partial_j \left(a_{i,j}(x)\, \partial_i \theta \right)$ generates an analytic semigroup. The notions of solutions are the same.

Theorem 7 *Assume the exists $\eta < 1$ such that*

$$\frac{1}{2} \sum_{k \in K} (\sigma_k(x) \cdot \xi)^2 \leq \eta \sum_{i,j=1}^{d} a_{ij}(x)\, \xi_i \xi_j \tag{11}$$

for all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$. Then, for every $\theta_0 \in H$, there exists one and only one (weak or mild) solution.

5.2.3 Auxiliary variables and end of the proof

In order to study the equation

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds - \sum_{k \in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla \theta(s)\, dW_s^k$$

let us consider the auxiliary system

$$\begin{aligned} v_h(t) &= \sigma_h \cdot \nabla e^{tA}\theta_0 + \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} q(s)\, ds \\ &\quad - \sum_{k \in K} \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} v_k(s)\, dW_s^k \end{aligned}$$

for $h \in K$.These two problems are equivalent (specifying correctly the function spaces): if $\theta(t)$ is a solution of the first one then

$$\begin{aligned} v_k(t) &: = \sigma_k \cdot \nabla \theta(t) \\ v(t) &: = (v_k(t))_{k \in K} \end{aligned}$$

is a solution of the second one; and if $v(t) := (v_k(t))_{k \in K}$ is a solution of the second one, then $\theta(t)$ defined by

$$\theta(t) = e^{tA}\theta_0 + \int_0^t \sigma_h \cdot \nabla e^{(t-s)A} q(s)\, ds - \sum_{k \in K} \int_0^t e^{(t-s)A} v_k(s)\, dW_s^k$$

is a solution of the first one. Up to details related to continuity properties of stochastic convolutions, the key lemma to prove the theorem for the first equation, is the following result for the second one.

Consider the space X_T of vectors $(v_k(\cdot))_{k \in K}$ such that $v_k \in L^2_{\mathcal{F}}(0, T; H)$ and, in the case when K is countable,

$$\|v\|_T^2 := \sum_{h \in K} \mathbb{E} \int_0^T \|v_h(t)\|_H^2\, dt < \infty.$$

It is a Hilbert space and $\|v\|_T$ is the induced norm.

Proposition 8 *There exists a unique solution* $(v_k(\cdot))_{k \in K} \in X_T$.

Proof. Step 1 (preparation). Notice that, by assumption (11),

$$\begin{aligned} \sum_{k \in K} \|\sigma_k \cdot \nabla f\|_{L^2}^2 &= \int_D \sum_{k \in K} (\sigma_k(x) \cdot \nabla f(x))^2\, dx \\ &\leq 2\eta \int_D \sum_{i,j=1}^d a_{ij}(x)\, \partial_i f(x)\, \partial_j f(x)\, dx \\ &= -2\eta \int_D (Af)(x)\, f(x)\, dx = -2\eta \langle Af, f \rangle \end{aligned}$$

for every $f \in D(A)$. We use this fact in the inequalities below.

Moreover, we use the following fact:

$$\begin{aligned} -2 \int_0^T \left\langle A e^{tA}\theta_0, e^{tA}\theta_0 \right\rangle dt &= -\int_0^T \frac{d}{dt} \left\langle e^{tA}\theta_0, e^{tA}\theta_0 \right\rangle dt \\ &= -\left(\left\| e^{TA}\theta_0 \right\|_H^2 - \|\theta_0\|_H^2 \right) \leq \|\theta_0\|_H^2 . \end{aligned}$$

Similarly, one has

$$\begin{aligned}
& -2\int_0^T\int_s^T \left\langle Ae^{(t-s)A}v_k\left(s\right),e^{(t-s)A}v_k\left(s\right)\right\rangle dtds \\
= & -\int_0^T\int_s^T \frac{d}{dt}\left\langle e^{(t-s)A}v_k\left(s\right),e^{(t-s)A}v_k\left(s\right)\right\rangle dtds \\
= & -\int_0^T\left(\left\|e^{(T-s)A}v_k\left(s\right)\right\|_H^2-\left\|v_k\left(s\right)\right\|_H^2\right)ds \\
\leq & \int_0^T\left\|v_k\left(s\right)\right\|_H^2 ds.
\end{aligned}$$

Step 2 (fixed point). Choose a number $\epsilon>0$ so small that $\eta\left(1+\epsilon\right)<1$. Consider the map Γ defined on X_T as

$$\left(\Gamma v\right)_h\left(t\right):=w_h\left(t\right)+\sum_{k\in K}\int_0^t\sigma_h\cdot\nabla e^{(t-s)A}v_k\left(s\right)dW_s^k$$

$h\in K$, where we have set

$$w_h\left(t\right):=\sigma_h\cdot\nabla e^{tA}\theta_0+\int_0^t\sigma_h\cdot\nabla e^{(t-s)A}q\left(s\right)ds.$$

We prove it takes values in X_T and it is a contraction; thus it has a unique fixed point. Notice that, opposite to many other applications of contraction mapping principle, we do not need to take T small.

Using $\left(a+b\right)^2\leq\left(1+\epsilon\right)a^2+\left(1+\frac{4}{\epsilon}\right)b^2$ we have

$$\begin{aligned}
\left\|\Gamma v\right\|_T^2 \leq & \left(1+\frac{4}{\epsilon}\right)\sum_{h\in K}\int_0^T\mathbb{E}\left[\left\|w_h\left(t\right)\right\|_{L^2}^2\right]dt \\
& +\left(1+\epsilon\right)\sum_{h\in K}\int_0^T\mathbb{E}\left[\left\|\sum_{k\in K}\int_0^t\sigma_h\cdot\nabla e^{(t-s)A}v_k\left(s\right)dW_s^k\right\|_{L^2}^2\right]dt.
\end{aligned}$$

Using a result of the first step and similar estimates for the convolution integral, we get

$$\left(1+\frac{4}{\epsilon}\right)\sum_{h\in K}\int_0^T\mathbb{E}\left[\left\|w_h\left(t\right)\right\|_{L^2}^2\right]dt\leq C_1<\infty$$

Therefore

$$\left\|\Gamma v\right\|_T^2\leq C_1+\left(1+\epsilon\right)\sum_{h\in K}\int_0^T\int_s^T\mathbb{E}\left[\sum_{h\in K}\left\|\sigma_h\cdot\nabla e^{(t-s)A}v_k\left(s\right)\right\|_{L^2}^2\right]dtds$$

$$\begin{aligned} &\le C_1 - 2\eta(1+\epsilon)\sum_{k\in K}\int_0^T\int_s^T \left\langle Ae^{(t-s)A}v_k(s), e^{(t-s)A}v_k(s)\right\rangle dtds \\ &\le C_1 + \eta(1+\epsilon)\|v\|_T^2 \end{aligned}$$

from another fact proved in Step 1. The previous computation shows that $\Gamma v \in X_T$. Then by the same computation we have

$$\left\|\Gamma v' - \Gamma v''\right\|_T^2 \le \eta(1+\epsilon)\left\|v' - v''\right\|_T^2$$

and $\eta(1+\epsilon) < 1$, hence Γ is a contraction. ■

5.2.4 Super-parabolicity condition and Stratonovich formulation

We have solved the general parabolic equation (10) under assumption (11), sometimes called super-parabolicity condition, very famous in the theory of nonlinear filtering and Zakai equations. The parabolic equation

$$\partial_t\theta = \sum_{i,j=1}^d \partial_j\left(a_{ij}(x)\partial_i\theta\right)$$

is well posed when a_{ij} is strongly parabolic, namely when there exists $\nu > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \ge \nu\|\xi\|^2$$

for all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$. The condition of the stochastic case is therefore much more restrictive. However, when the problem (10) comes from a Stratonovich equation of the form (9), we have

$$a_{ij}(x) = \kappa\delta_{ij} + \frac{1}{2}Q_{ij}(x,x)$$

with

$$Q_{ij}(x,x) = \sum_{k\in K}\sigma_k^i(x)\sigma_k^j(x).$$

The super-parabolicity condition in this case requires to find $\eta \in (0,1)$ such that

$$\begin{aligned} \frac{1}{2}\sum_{k\in K}(\sigma_k(x)\cdot\xi)^2 &\le \eta\sum_{i,j=1}^d\left(\kappa\delta_{ij} + \frac{1}{2}\sum_{k\in K}\sigma_k^i(x)\sigma_k^j(x)\right)\xi_i\xi_j \\ &= \eta\kappa\|\xi\|^2 + \frac{\eta}{2}\sum_{k\in K}(\sigma_k(x)\cdot\xi)^2 \end{aligned}$$

namely such that

$$\sum_{k\in K}\left(\sigma_k\left(x\right)\cdot\xi\right)^2\leq\frac{2\eta\kappa}{1-\eta}\left\|\xi\right\|^2.$$

Under the summability conditions which guarantee to have $Q\left(x,y\right)$ well defined and bounded, such η exists, sufficiently close to 1. Therefore the Stratonovich equation is always well posed.

5.3 The equation for the average

We have immediately a result if we take the average, called as above

$$\Theta\left(t,x\right):=\mathbb{E}\left[\theta\left(t,x\right)\right].$$

We assume here that $\theta_0\in H$ is deterministic.

Proposition 9 *If $\theta\left(t,x\right)$ is the solution given by Corollary* **??**, *then $\Theta\left(t,x\right)$ is a (weak or mild) solution of the deterministic equation*

$$\begin{aligned}\partial_t\Theta &= \left(\kappa\Delta+\mathcal{L}\right)\Theta+q\\ \Theta|_{t=0} &= \theta_0.\end{aligned}$$

Proof. We take $q=0$ for shortness. Take for instance the weak formulation, for $\phi\in D\left(A\right)$:

$$\left\langle\theta\left(t\right),\phi\right\rangle=\left\langle\theta_0,\phi\right\rangle+\int_0^t\left\langle\theta\left(s\right),\left(\kappa\Delta+\mathcal{L}\right)\phi\right\rangle ds+\sum_{k\in K}\int_0^t\left\langle\theta\left(s\right),\sigma_k\cdot\nabla\phi\right\rangle dW_t^k.$$

The stochastic integral $\int_0^t\left\langle\theta\left(s\right),\sigma_k\cdot\nabla\phi\right\rangle dW_t^k$ is a martingale because $\theta\in L^2_{\mathcal{F}}\left(0,T;H\right)$ (it is much more than this). Therefore

$$\left\langle\Theta\left(t\right),\phi\right\rangle=\left\langle\theta_0,\phi\right\rangle+\int_0^t\left\langle\Theta\left(s\right),\left(\kappa\Delta+\mathcal{L}\right)\phi\right\rangle ds.$$

Moreover,

$$\Theta\in C\left(\left[0,T\right];H\right)\cap L^2\left(0,T;V\right)$$

as a consequence of the property

$$\theta\in C_{\mathcal{F}}\left(\left[0,T\right];H\right)\cap L^2_{\mathcal{F}}\left(0,T;V\right).$$

Therefore it is a weak solution. The proof that it is a mild solution is similar, or it follows from the equivalence between the two concepts, under our regularity, in the deterministic case. ■

6 When θ is close to Θ

6.1 Main assumption and result

Define $\varepsilon_{Q,\kappa} \geq 0$ as the smallest number such that

$$\int\int v(x)^T Q(x,y) v(y)\, dxdy \leq \varepsilon_{Q,\kappa} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x) v(x) \right) dx \tag{12}$$

for all $v \in L^2(D, \mathbb{R}^d)$. When $v(x) = f(x) \nabla w(x)$, it gives

$$\begin{aligned} & \int\int \nabla w(x)^T Q(x,y) \nabla w(y)\, dxdy \\ \leq\ & \varepsilon_{Q,\kappa} \int |f(x)|^2 \left(\kappa |\nabla w(x)|^2 + \frac{1}{2} \nabla w(x)^T Q(x,x) \nabla w(x) \right) dx \\ \leq\ & -\varepsilon_{Q,\kappa} \|f\|_\infty^2 \langle Aw, w \rangle . \end{aligned}$$

In the next theorem we assume $\theta_0 \in L^\infty(D)$, $q \in L^\infty([0,T] \times D)$. Call $C_\infty(T, \theta_0, q) > 0$ a constant such that

$$\sup_{s \in [0,T]} \mathbb{E} \|\theta(s)\|_\infty^2 \leq C_\infty(T, \theta_0, q).$$

In Section 5.1.2 above we have outlined one method to prove a bound of this form, in that case even an a.s. bound:

$$\|\theta(t)\|_\infty \leq \|\theta_0\|_\infty + T \|q\|_\infty .$$

However there are other bounds available, in the average, using regularity theory for $\theta(t)$, see [23], which improve the depedence on T.

Theorem 10 *For every $\phi \in L^2(D)$,*

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] \leq \varepsilon_{Q,\kappa} \|\phi\|_{L^2}^2 C_\infty(T, \theta_0, q).$$

Proof. Recall the identity

$$\theta(t) = e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds - \sum_{k \in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla\theta(s)\, dW_s^k.$$

Here $e^{tA}\theta_0 + \int_0^t e^{(t-s)A} q(s)\, ds$ is precisely $\Theta(t)$, hence

$$\theta(t) - \Theta(t) = -\sum_{k \in K} \int_0^t e^{(t-s)A} \sigma_k \cdot \nabla\theta(s)\, dW_s^k.$$

If $\phi \in H$,

$$\langle \theta(t) - \Theta(t), \phi \rangle = \sum_{k \in K} \int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla \theta e^{(t-s)A} \phi \right\rangle dW_s^k.$$

Then (here we take advantage of the cancellations of Itô integrals)

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] = \sum_{k \in K} \mathbb{E} \int_0^t \left\langle \theta(s), \sigma_k \cdot \nabla e^{(t-s)A} \phi \right\rangle^2 ds.$$

Write $\phi_{t,s} := e^{(t-s)A}\phi$. Then

$$\begin{aligned}
& \sum_{k \in K} \left\langle \theta(s), \sigma_k \cdot \nabla \phi_{t,s} \right\rangle^2 \\
= \; & \sum_{k \in K} \int \int \theta(s,x) \, \theta(s,y) \, \sigma_k(x) \cdot \nabla \phi_{t,s}(x) \, \sigma_k(y) \cdot \nabla \phi_{t,s}(y) \, dx dy \\
= \; & \int \int \theta(s,y) \, \nabla \phi_{t,s}(y)^T \, Q(x,y) \, \nabla \phi_{t,s}(x) \, \theta(s,x) \, dx dy \\
\leq \; & -\varepsilon_{Q,\kappa} \left\| \theta(s) \right\|_\infty^2 \left\langle A e^{(t-s)A}\phi, e^{(t-s)A}\phi \right\rangle.
\end{aligned}$$

Therefore, with the notation $C_\infty(T, \theta_0, q)$,

$$\begin{aligned}
& \mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] \\
\leq \; & \varepsilon_{Q,\kappa} C_\infty(T, \theta_0, q) \int_0^t \left\langle (-A) \, e^{(t-s)A}\phi, e^{(t-s)A}\phi \right\rangle ds \\
= \; & \varepsilon_{Q,\kappa} C_\infty(T, \theta_0, q) \int_0^t \frac{d}{ds} \left\| e^{(t-s)A}\phi \right\|_{L^2}^2 ds \\
\leq \; & \varepsilon_{Q,\kappa} C_\infty(T, \theta_0, q) \left\| \phi \right\|_{L^2}^2
\end{aligned}$$

after a computation made already above for $\int_0^t \frac{d}{ds} \left\| e^{(t-s)A}\phi \right\|_{L^2}^2 ds$. ■

6.2 When $\varepsilon_{Q,\kappa}$ is small (and $\mathcal{L}$ is not small)

Inequality (12) is not immediately transparent. Let us discuss it in two cases, which however do not exhaust all opportunities.

6.2.1 The case when $Q(x,x)$ is degenerate

The first one neglects the second term on the right-hand-side, the term with $Q(x,x)$, because in very relevant cases it is degenerate. This happens precisely in the case considered everywhere in this lectures, namely the case of a viscous fluid in a bounded domain D, satisfying the no-slip boundary condition $u|_{\partial D} = 0$. In this case $Q(x,x) = 0$ for $x \in \partial D$. We do not exclude that, in spite of this degeneracy, $Q(x,x)$ may help on the right-hand-side of (12). But a priori it is difficult to use it.

In this case we look for the smallest constant $\epsilon_Q \geq 0$ such that

$$\int\int v(x)^T Q(x,y) v(y)\, dxdy \leq \epsilon_Q \int |v(x)|^2 dx \tag{13}$$

for all $v \in L^2(D, \mathbb{R}^d)$. Then

$$\varepsilon_{Q,\kappa} \leq \frac{\epsilon_Q}{\kappa}$$

because, if (13) holds, being

$$\epsilon_Q \int |v(x)|^2 dx \leq \frac{\epsilon_Q}{\kappa} \int \left(\kappa |v(x)|^2 + \frac{1}{2} v(x)^T Q(x,x) v(x) \right) dx$$

we have that $\frac{\epsilon_Q}{\kappa}$ is a constant fulfilling (12), hence the smallest one is less or equal to $\frac{\epsilon_Q}{\kappa}$. We thus have:

Corollary 11

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] \leq \frac{\epsilon_Q}{\kappa} \|\phi\|_{L^2}^2 \left(\|\theta_0\|_\infty + T \|q\|_\infty \right)^2 .$$

Therefore, one way to have $\theta(t)$ close to $\Theta(t)$ is to have a very small ϵ_Q. However, any small noise realizes this target but then also the additional operator $\mathcal{L}$ is small. Thus the true question is: are there noises such that ϵ_Q is small and the operator $\mathcal{L}$ is *substantial*?

The name "substantial" may refer to different properties. We have in mind two of them:

- improve the decay rate κ (eddy diffusion)
- produce a significantly modified profile (turbulent boundary layer heat profile).

In [18] we have constructed a noise, made of vortex structures, in simple 2D domains, with the following properties: given $\epsilon, \delta > 0$ (small) and $\sigma^2 > 0$ (large) we have

$$\begin{aligned} \epsilon_Q &\leq \epsilon \\ Q(x,x) &\geq \sigma^2 I \qquad \text{for all } x \in D \text{ such that } d(x, \partial D) \geq \delta. \end{aligned}$$

The first conditon guarantees that the profile of $\theta(t)$ (smoothed by the scalar product $\langle \theta(t), \phi \rangle$) is close to the profile of $\Theta(t)$. The second condition implies that the deterministic equation of $\Theta(t)$ has an enhanced diffusion, still effective in spite of the vanishing-diffusion boundary layer. In [18] we have proved the following dissipativity property:

Theorem 12 *Assume $D = B(0,1) \subset \mathbb{R}^d$. Call $\lambda_{D,\kappa,Q}$ the first eigenvalue of $-A$ (it measures the rate of decay of $\Theta(t)$). Then there exists a constant $C_{D,d} > 0$ such that*

$$\lambda_{D,\kappa,Q} \geq C_{D,d} \min\left(\sigma^2, \frac{\kappa}{\delta}\right).$$

asymptotically as $\delta \to 0$ one can take $C_{D,d} = d/2$ and one also has $\lambda_{D,\kappa,Q} \geq \frac{\kappa d}{\kappa+\delta\sigma^2}\sigma^2$.

This result corresponds to the improvement of the decay rate κ (eddy diffusion) mentioned above. Considering the other sentence, namely producing a significantly modified profile (diffusion boundary layer), we have the following result, in a modified geometry with respect to the one of these lectures. The domain now is the infinite channel

$$D = \mathbb{R} \times [-1,1]$$

with Dirichlet boundary condition for both temperature and fluid at the upper and bottom boundaries:

$$\theta(x_1, \pm 1) = \sigma_k(x_1, \pm 1) = 0 \qquad \text{for every } x_1 \in \mathbb{R},\ k \in K.$$

The theoretical results are similar to those above. In addition, let us consider the stationary deterministic profile for a given $q = q(x)$, element of H: we have to solve

$$A\Theta_{st} + q = 0$$

namely

$$\Theta_{st} = -A^{-1}q.$$

In practice, assume that in a region $x \in [-L,L] \times [-1,1]$ the function $q(x)$ is equal to a constant q, and both the stationary solution $\Theta_{st}(x)$ and $Q(x,x)$ depend only on the vertical direction $z \in [-1,1]$ and they are symmetric with respect to $z=0$. The equation

$$\operatorname{div}\left(\left(\kappa I + \frac{1}{2}Q(x,x)\right)\nabla\Theta_{st}(x)\right) = -q(x)$$

becomes

$$\partial_z\left(\left(\kappa + Q_{22}(z)\right)\partial_z\Theta_{st}(z)\right) = -q.$$

It gives us

$$\left(\kappa + Q_{22}(z)\right)\partial_z\Theta_{st}(z) = -qz$$

without constants, since both sides of the identity should vanish at $z=0$ (the function Θ_{st} is symmetric with respect to $z=0$ and smooth, hence $\partial_z\Theta_{st}(0) = 0$). Therefore we have to solve

$$\begin{aligned} \partial_z\Theta_{st}(z) &= -\frac{qz}{\kappa + Q_{22}(z)} \\ \Theta_{st}(1) &= 0. \end{aligned}$$

The solution of the previous equation is

$$\Theta_{st}(z) = -\int_{-1}^{z} \frac{qs}{\kappa + Q_{22}(s)} ds.$$

Without noise the solution is

$$\Theta_{st}^{Q=0}(z) = \frac{q}{\kappa}\frac{1-z^2}{2} = \frac{q}{2\kappa} - \frac{q}{2\kappa}z^2$$

so the curvature $\frac{q}{\kappa}$ is large (for κ small) and also the maximum is large:

$$\max \Theta_{st}^{Q=0} = \frac{q}{2\kappa}.$$

Assume

$$c_2\sigma^2 1_{[-1+\delta,1-\delta]} \leq Q_{22}(z) \leq c_2\sigma^2$$

with large σ^2 and small δ. Then

$$\frac{q}{\kappa + c_2\sigma^2}\frac{1-z^2}{2} \leq \Theta_{st}(z)(z) \leq -\int_{-1}^{z} \frac{qs}{\kappa + c_1\sigma^2 1_{[-1+\delta,1-\delta]}(s)} ds.$$

If $z \in [-1, -1+\delta]$ we have

$$\Theta_{st}(z)(z) \leq \frac{q}{\kappa}\frac{1-z^2}{2}$$

like in the case without noise but, for $z \in [-1+\delta, 0]$ we have

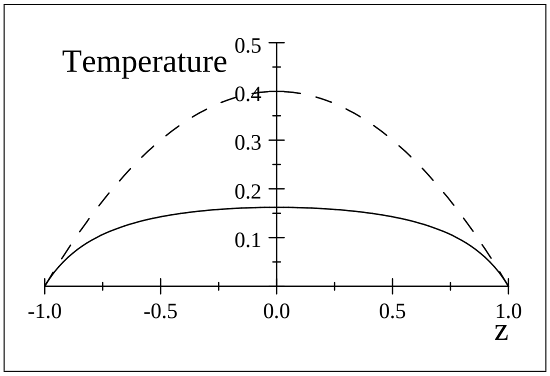
$$\begin{aligned}\Theta_{st}(z)(z) &\leq \frac{q}{\kappa}\frac{1-(1-\delta)^2}{2} + \frac{q}{\kappa + c_1\sigma^2}\frac{(1-\delta)^2 - z^2}{2} \\ &= C\left(\kappa, q, \delta, \sigma^2\right) - \frac{q}{\kappa + c_1\sigma^2}\frac{z^2}{2}.\end{aligned}$$

The curvature $\frac{q}{\kappa + c_1\sigma^2}$ is much smaller than $\frac{q}{\kappa}$ and the maximum

$$\max \Theta_{st}(z) = C\left(\kappa, q, \delta, \sigma^2\right) \geq \frac{q}{\kappa + c_1\sigma^2}\frac{(1-\delta)^2}{2}$$

is very small for large σ^2 and small δ.

The Figure 1 illustrates the modification of profile, from the standard parabolic one of free diffusion in a steady medium, to the case of turbulent decay. The reduction in heat content can be dramatic, due to turbulence, creating a fundamental engineering problem.



The dashed profile is the classical parabolic profile with $Q = 0$. The solid-line profile is the one obtained by a large σ^2 and small δ.

6.2.2 The case when $Q(x, x)$ is non-degenerate

In bounded domains with no-slip boundary conditions for the fluid, $Q(x, x)$ is always degenerate. However, in other geometries, like the torus or the full space, we may have non-degenerate $Q(x, x)$.

Assume, for some $\sigma^2 > 0$ (large), we have

$$Q(x, x) \geq \sigma^2 I \qquad \text{for all } x \in D \ .$$

Then

$$\begin{aligned} & \int \left(\kappa \left| v(x) \right|^2 + \frac{1}{2} v(x)^T Q(x, x) v(x) \right) dx \\ \geq \quad & \left(\kappa + \frac{\sigma^2}{2} \right) \int \left| v(x) \right|^2 dx. \end{aligned}$$

If (13) holds, being

$$\epsilon_Q \int \left| v(x) \right|^2 dx \leq \frac{\epsilon_Q}{\kappa + \frac{\sigma^2}{2}} \int \left(\kappa \left| v(x) \right|^2 + \frac{1}{2} v(x)^T Q(x, x) v(x) \right) dx$$

we deduce (as above)

$$\varepsilon_{Q,\kappa} \leq \frac{\epsilon_Q}{\kappa + \frac{\sigma^2}{2}}.$$

We thus have:

Corollary 13

$$\mathbb{E}\left[\langle \theta(t) - \Theta(t), \phi \rangle^2 \right] \leq \frac{\epsilon_Q}{\kappa + \frac{\sigma^2}{2}} \left\| \phi \right\|_{L^2}^2 \left(\left\| \theta_0 \right\|_\infty + T \left\| q \right\|_\infty \right)^2 .$$

Therefore, another way to have $\theta(t)$ close to $\Theta(t)$, different from ϵ_Q small (or concurring with it) is to have σ^2 large.

Assume we are in full space $\mathbb{R}^d$. A famous noise satisfying the previous conditions (for suitable values of its parameters) is R. Kraichnan noise, [39], [38]. Id is space-homogeneous, $Q(x,y) = Q(x-y)$, with the form

$$Q(z) = \sigma^2 k_0^\zeta \int_{k_0 \leq |k| < k_1} \frac{1}{|k|^{d+\zeta}} e^{ik\cdot z} \left(I - \frac{k \otimes k}{|k|^2}\right) dk.$$

This model has a meaning and an interest for both positive and negative ζ. Assume $\zeta > 0$ (the so-called Kolmogorov 41 case is $\zeta = 4/3$). In this case, take $k_1 = +\infty$. Assume

$$k_0 = k_0^N$$

and take $k_0^N \to \infty$. Then

$$\begin{aligned} Q(x,x) &= Q(0) = \sigma^2 k_0^\zeta \int_{k_0 \leq |k| < \infty} \frac{1}{|k|^{d+\zeta}} \left(I - \frac{k \otimes k}{|k|^2}\right) dk \\ &\overset{k'=k/k_0}{=} \sigma^2 k_0^\zeta \int_{1 \leq |k'| < \infty} \frac{1}{k_0^{d+\zeta} |k'|^{d+\zeta}} \left(I - \frac{k' \otimes k'}{|k'|^2}\right) k_0^d dk' \\ &= \sigma^2 \int_{1 \leq |k| < \infty} \frac{1}{|k|^{d+\zeta}} \left(I - \frac{k \otimes k}{|k|^2}\right) dk \end{aligned}$$

is independent of k_0 and therefore of N. This is the matrix appearing in the limit parabolic equation. But, concerning ϵ_Q, we have

$$\begin{aligned} &\int \int v(x)^T Q(x,y) v(y) \, dx dy \\ &\leq \sigma^2 k_0^\varsigma \int_{k_0 \leq |k| < \infty} \frac{1}{|k|^{d+\varsigma}} |\widehat{v}(k)|^2 \, dk \\ &\leq \sigma^2 k_0^{-d} \int_{k_0 \leq |k| < \infty} |\widehat{v}(k)|^2 \, dk \leq \sigma^2 k_0^{-d} \|v\|_{L^2}^2 . \end{aligned}$$

Thus ϵ_Q is small if $\sigma^2 k_0^{-d}$ is small, hence if $k_0^N \to \infty$.

Remark 14 *If $-d \leq \zeta \leq 0$, $k_0 = 1$, σ^2 small, and k_1 is so large that $\sigma^2 \int_{1 \leq k \leq k_1} \frac{1}{k^{\zeta+1}} dk$ is large, then $Q(x,x)$ is large and ϵ_Q is small.*

Remark 15 *We have seen that, in order to fulfill our conditions, the noise has to activate very small scales (large k) with high energy.*

7 The 3D case

Similarly to the 2D case, it is meaningful and useful to investigate first the linear case, then the nonlinear one. Thus we start with the equation of a passive vector field, typically a magnetic field in applications.

7.1 Passive magnetic field

The equations for a magnetic field M in a fluid u are

$$\partial_t M + u \cdot \nabla M = \eta \Delta M + M \cdot \nabla u.$$

Similarly to the scalar case, we model u by a white noise, with the Stratonovich interpretation:

$$dM + \sum_{k \in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt + \sum_{k \in K} M \cdot \nabla \sigma_k \circ dW_t^k.$$

The equation can be written as

$$dM = (\eta \Delta + \mathcal{L}) M dt + \text{Itô terms}$$

for a suitable second order differential operator $\mathcal{L}$. And $\overline{M}(t) := \mathbb{E}[M]$ satisfies

$$\partial_t \overline{M} = (\eta \Delta + \mathcal{L}) \overline{M}.$$

Thus, as above, the question arises whether $\mathbb{E}\left[\left\langle M(t) - \overline{M}(t), \phi \right\rangle^2\right]$ is small.

This question is open. We shall see below that in the case of special noise (space-homogeneous and mirror symmetric) the operator $\mathcal{L}$ is the same as the one of the scalar case. In this situation there exists the following conjecture:

F. Krause, K.-H. Rädler, Mean Field Magnetohydrodynamics, 1980, page 12: "homogeneous isotropic mirror symmetric turbulence only influences the decay rate of the mean magnetic fields, which is enhanced in almost all cases of physical interest."

7.1.1 The corrector

If we define

$$B_k M = M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M$$

then the corrector is $\frac{1}{2} \sum_{k \in K} B_k B_k M$. Thus let us compute $B_k B_k M$. We have

$$\begin{aligned} B_k B_k M &= (B_k M) \cdot \nabla \sigma_k - \sigma_k \cdot \nabla (B_k M) \\ &= (M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M) \cdot \nabla \sigma_k - \sigma_k \cdot \nabla (M \cdot \nabla \sigma_k - \sigma_k \cdot \nabla M) \end{aligned}$$

$$= (M \cdot \nabla \sigma_k) \cdot \nabla \sigma_k - (\sigma_k \cdot \nabla M) \cdot \nabla \sigma_k - \sigma_k \cdot \nabla (M \cdot \nabla \sigma_k) + \sigma_k \cdot \nabla (\sigma_k \cdot \nabla M).$$

Lemma 16

$$\frac{1}{2}\sum_{k\in K} B_k B_k M = \mathcal{L}M - \sum_{k\in K}\sum_{i,j}\sigma_k^i \partial_i M_j \partial_j \sigma_k$$
$$+\frac{1}{2}\sum_{k\in K}\sum_{i,j}\left(\partial_j \sigma_k^i \partial_i \sigma_k - \sigma_k^i \partial_i \partial_j \sigma_k\right) M_j.$$

Proof. The term

$$\frac{1}{2}\sum_{k\in K}\sigma_k \cdot \nabla\left(\sigma_k \cdot \nabla M\right)$$

is equal to $\mathcal{L}M$, as in the previous sections. The term $\sigma_k \cdot \nabla\left(M \cdot \nabla \sigma_k\right)$ is equal to

$$\left(\sigma_k \cdot \nabla M\right) \cdot \nabla \sigma_k + \sum_{i,j}\left(\sigma_k^i \partial_i \partial_j \sigma_k\right) M_j$$

hence its first addendum, $\left(\sigma_k \cdot \nabla M\right) \cdot \nabla \sigma_k$, adds to another equal term in the total sum; they form the term

$$-\sum_{k\in K}\sum_{i,j}\sigma_k^i \partial_i M_j \partial_j \sigma_k$$

in the final result. The zero order term is thus the remainder of this computation. ■

Lemma 17 *Assume the noise is space-homogeneous:*

$$Q\left(x,y\right) = Q\left(x-y\right)$$

and $Q\left(x,x\right) = Q\left(0\right)$*, a constant matrix. Then*

$$\frac{1}{2}\sum_{k\in K}\sum_{i,j}\left(\partial_j \sigma_k^i \partial_i \sigma_k - \sigma_k^i \partial_i \partial_j \sigma_k\right) M_j = 0.$$

Proof. Step 1. The sum $\sum_{k\in K}\sigma_k^i\left(x\right)\sigma_k^\alpha\left(x\right)$ is constant, equal to $Q_{i,\alpha}\left(0\right)$, for every $i,\alpha = 1,2,3$. Thus their derivatives are equal to zero. It follows that

$$\sum_{k\in K}\left(\partial_j \sigma_k^i\right)\left(x\right)\sigma_k^\alpha\left(x\right) = -\sum_{k\in K}\sigma_k^i\left(x\right)\left(\partial_j \sigma_k^\alpha\right)\left(x\right).$$

Moreover, it follows also

$$\sum_i \partial_i \sum_{k\in K}\sigma_k^i\left(x\right)\sigma_k^\alpha\left(x\right) = 0$$

which implies

$$\sum_{k\in K}\sum_i \sigma_k^i\left(x\right)\partial_i \sigma_k^\alpha\left(x\right) = 0$$

because $\operatorname{div}\sigma_k = 0$.

Step 2. Not only the sum $\sum_{k\in K}\sigma_k^i(x)\,\sigma_k^\alpha(x)$ is constant, but also $\sum_{k\in K}\left(\partial_j\sigma_k^i\right)(x)\,\sigma_k^\alpha(x)$. Indeed, we have

$$\begin{aligned}\sum_{k\in K}\left(\partial_j\sigma_k^i\right)(x)\,\sigma_k^\alpha(y) &= \partial_{x_j}\sum_{k\in K}\sigma_k^i(x)\,\sigma_k^\alpha(y)\\ &= \partial_{x_j}Q_{i,\alpha}(x-y) = \left(\partial_j Q_{i,\alpha}\right)(x-y)\end{aligned}$$

which implies

$$\sum_{k\in K}\left(\partial_j\sigma_k^i\right)(x)\,\sigma_k^\alpha(x) = \left(\partial_j Q_{i,\alpha}\right)(0).$$

This implies

$$\partial_i\sum_{k\in K}\left(\partial_j\sigma_k^i\right)(x)\,\sigma_k^\alpha(x) = 0.$$

Step 3. Now, first the two terms we have to investigate are opposite one to the other:

$$\begin{aligned}\sum_{k\in K}\sum_i \partial_j\sigma_k^i\partial_i\sigma_k &= \partial_j\sum_{k\in K}\sum_i \sigma_k^i\partial_i\sigma_k - \sum_{k\in K}\sum_i \sigma_k^i\partial_i\partial_j\sigma_k\\ &= -\sum_{k\in K}\sum_i \sigma_k^i\partial_i\partial_j\sigma_k\end{aligned}$$

where we have used the fact that $\sum_{k\in K}\sum_i \sigma_k^i\partial_i\sigma_k$ is equal to zero (Step 1). Therefore it is sufficient to prove that

$$\sum_{k\in K}\sum_i \partial_j\sigma_k^i\partial_i\sigma_k = 0.$$

But this term can be written as

$$\sum_i \partial_i\sum_{k\in K}\partial_j\sigma_k^i\sigma_k$$

which is zero, because of Step 2. The identity between the previous two terms is due to the fact that $\sum_i \partial_i\partial_j\sigma_k^i = 0$, being $\operatorname{div}\sigma_k = 0$. ■

Corollary 18 *If the noise is space-homogeneous, then*

$$\frac{1}{2}\sum_{k\in K}B_kB_kM = \mathcal{L}M - \sum_j \partial_j Q(0)\cdot\nabla M_j$$

where $\partial_j Q(0)$ is the matrix with entries $\left(\partial_j Q_{\alpha,i}\right)(0)$. In the particular case when

$$Q(-x) = Q(x)$$

(mirror symmetry) then $\partial_j Q(0) = 0$ and thus

$$\frac{1}{2}\sum_{k\in K}B_kB_kM = \mathcal{L}M.$$

Proof. For the first identity it remains to show that

$$\begin{aligned}\sum_{k\in K}\sum_{i,j}\sigma_k^i\partial_i M_j\partial_j\sigma_k^\alpha &= \sum_{i,j}\left(\sum_{k\in K}\sigma_k^i\partial_j\sigma_k^\alpha\right)\partial_i M_j \\ &= \sum_j \left(\partial_j Q_{\alpha,i}\right)(0)\,\partial_i M_j\end{aligned}$$

where we have used an indentity proved in Step 2 of the previous proof.

Under mirror symmetry, $Q_{\alpha,i}(x)$ is a smooth even function, hence its derivatives at zero are equal to zero. ∎

7.1.2 The difficulty

We have shown that in the particular case of space-homogeneous noise with mirror symmetry the Itô form of the equation is

$$dM+\sum_{k\in K}\sigma_k\cdot\nabla M dW_t^k=(\eta\Delta+\mathcal{L})\,M dt+\sum_{k\in K}M\cdot\nabla\sigma_k dW_t^k$$

similarly to the passive scalar case. Without mirror symmetry we would have an additional first-order differential operator, related to the so called α-effect in the theory of dynamo.

The problem is that, in spite of the positive sentence of F. Krause, K.-H. Rädler, recalled above, we cannot prove that the stochastic process M is close to its average $\overline{M}$, solution of

$$\partial_t\overline{M}=(\eta\Delta+\mathcal{L})\,\overline{M}.$$

We cannot extend the theory of eddy viscosity to the 3D case.

The reason stands in the estimates on M. We do not have anymore the energy conservation estimate, because

$$\langle\sigma_k\cdot\nabla M,M\rangle=0$$

hence

$$d\,\|M(t)\|_{L^2}^2+2\eta\,\|\nabla M(t)\|_{L^2}^2\,dt=2\sum_{k\in K}\langle M\cdot\nabla\sigma_k,M\rangle\circ dW_t^k$$

but $\langle M\cdot\nabla\sigma_k,M\rangle$ is not zero and contributes a lot, at least a priori.

Similarly, the Lagrangian property should be reformulated here as

$$M(t,x)=D\varphi_{-t}(x)\,M_0\left(\varphi_{-t}(x)\right)$$

and the Lagrangian deformation tensor $D\varphi_{-t}(x)$ may have, a priori, an enormous effect of stretching on $M_0\left(\varphi_{-t}(x)\right)$. Thus, even if we may start the computation as in the scalar case

$$\begin{aligned} \langle M(t), \phi \rangle - \langle \overline{M}(t), \phi \rangle &= +\sum_{k \in K} \int_0^t \left\langle M(s), e^{(t-s)A} \sigma_k \cdot \nabla \phi \right\rangle dW_t^k \\ &\quad + \sum_{k \in K} \int_0^t \left\langle M(s) \nabla \sigma_k, e^{(t-s)A} \phi \right\rangle dW_t^k \end{aligned}$$

then we do not have good estimates on $M(s)$ to control in mean square the stochastic terms.

7.1.3 The purely transport case

If we consider the ideal model

$$dM + \sum_{k \in K} \sigma_k \cdot \nabla M \circ dW_t^k = \eta \Delta M dt$$

where the noise acts only on the transport term, we get the equation

$$dM + \sum_{k \in K} \sigma_k \cdot \nabla M dW_t^k = (\eta \Delta + \mathcal{L}) M dt$$

which satisfies the estimates

$$\|M(t)\|_{L^2}^2 + 2\eta \int_0^t \|\nabla M(s)\|_{L^2}^2 ds = \|M_0\|_{L^2}^2$$

$$\|M(t)\|_\infty \leq \|M_0\|_\infty .$$

and therefore we may control the difference

$$\langle M(t), \phi \rangle - \langle \overline{M}(t), \phi \rangle = +\sum_{k \in K} \int_0^t \left\langle M(s), e^{(t-s)A} \sigma_k \cdot \nabla \phi \right\rangle dW_t^k$$

exactly as in the scalar scase.

From the physical viewpoint the stretching term $\sum_{k \in K} M \cdot \nabla \sigma_k \circ dW_t^k$ cannot be neglected. However, it is possible that there are regimes where its effect is small.

Remark 19 *In this model we should not assume* $\operatorname{div} M = 0$ *otherwise the model is incorrect, because* $\sigma_k \cdot \nabla M$ *is not divergence free in general, while the other terms of the equation would be divergence free* ($\sigma_k \cdot \nabla M - M \cdot \nabla \sigma_k$ *is divergence free, on the contrary). If we want the additional property that* M *is divergence free, then we have to consider the more difficult model*

$$dM + \sum_{k \in K} P(\sigma_k \cdot \nabla M) \circ dW_t^k = \eta \Delta M dt$$

where P is the projector introduced in the previous chapters. The Itô-Stratonovich corrector now is much more complex. This difficulty is necessary in the case below of the Navier-Stokes equations, where the role of M is taken by the vorticity ω, which is divergence free. Hence the simple ideas described in this subsection are more complex, for the 3D Navier-Stokes equations, in two respects: the problem is nonlinear, hence it is not sufficient to control $\langle M(t), \phi\rangle - \langle \overline{M}(t), \phi\rangle$, and the corrector is non-local, since it contains P.

7.2 The 3D navier-Stokes equations with transport noise

Consider, on the 3D torus, the vorticity equation with noise only in the transport component:

$$\partial_t \omega + u \cdot \nabla \omega + P\left(u' \circ \nabla \omega\right) = \Delta \omega + \omega \cdot \nabla u.$$

with noise u' of the form

$$u'(t,x) = \sum_k \sigma_k(x) \circ \partial_t W_t^k$$

Notice the presence of the projection in the term $P(u' \circ \nabla \omega)$, necessary for compatibility, but source of great technical difficulties (the Itô-Stratonovich corrector is a nonlocal differential operator).

Call ω the unique local solution, for $\omega_0 \in H$ (the space L^2 with usual conditions).

Theorem 20 *Given $T, R_0, \epsilon > 0$ there exists $(\sigma_k)_{k\in K}$ with the following property: for every initial condition $\omega_0 \in H$ with $\|\omega_0\|_H \leq R_0$, the 3D Navier-Stokes equations with transport noise (and viscosity = 1) has a global unique solution on $[0,T]$, up to probability ϵ.*

The full proof requires too many details, see [22]. Let us mention only one fact. The norm $\|\omega(t)\|_H^2$ can be controlled *locally* from

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega$$

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_H^2 + \|\nabla \omega(t)\|_H^2 = \langle \omega \cdot \nabla u, \omega\rangle.$$

The term $\langle \omega \cdot \nabla u, \omega\rangle$ describes the *stretching* of vorticity ω produced by the deformation tensor ∇u. This is the potential source of unboundedness of $\|\omega(t)\|_H^2$.

Sobolev and interpolation inequalities:

$$\langle \omega \cdot \nabla u, \omega\rangle \leq \|\omega\|_{L^3}^3 \leq \|\omega\|_{W^{1/2,2}}^3 \leq \|\omega\|_{L^2}^{3/2}\|\omega\|_{W^{1,2}}^{3/2} \leq \|\omega\|_{W^{1,2}}^2 + \|\omega\|_{L^2}^6$$

lead to

$$\frac{d}{dt}\|\omega(t)\|_H^2 \leq C\|\omega\|_H^6$$

which provides only a local control.

However the interval of existence depends on the *viscosity coefficient* ν:

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \nu \Delta \omega$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_H^2 + \nu\|\nabla\omega(t)\|_H^2 &= \langle \omega\cdot\nabla u, \omega\rangle \\
&\leq \|\omega\|_{L^2}^{3/2}\|\omega\|_{W^{1,2}}^{3/2} \\
&\leq \nu\|\nabla\omega(t)\|_H^2 + \frac{C}{\nu^3}\|\omega\|_H^6
\end{aligned}$$

$$\frac{d}{dt}\|\omega(t)\|_H^2 \leq \frac{C}{\nu^3}\|\omega\|_H^6$$

The explosion is delayed for large ν. Not only: beyond a threshold the solution is global.

This is the key for a regularization by noise: transport noise improves dissipation, hence it delays blow-up.

8 Summary

In this chapter we have discussed a second class of noise: the one of transport type. There is a third class, variant of the second one, namely noise of transport-stretching type in 3D, which is only mentioned but should receive due attention.

Noise of transport type in the equations for auxiliary quantities, like heat, have been investigated by several authors. Here we have introduced them as a Wong-Zakai limit to emphasize the presence of a correcting term, essential to preserve the Physics and to get useful informations. In the case of heat transport our investigation culminates in the proof of a property of eddy dissipation.

But similar ideas may be applied to the internal structure of the fluid itself when we introduce the subdivision in large and small scales. Here the noise is used to summarize the dynamics of small scales and affects the closed equation for the large scales. This is the motivation for considering stochastic Navier-Stokes equations with transport type noise (and, as mentioned above, also with transport-stretching noise in 3D). The 2D case starts to be well understood and, in particular, similarly to the case of heat transfer, one can prove a result of eddy viscosity: turbulence enhances the viscosity of the fluid itself. This fact, clearly observed in real situations, is perhpas the main confirmation that the heuristic discussion made here about stochastic modeling of small scales and consequent transport noise in the large ones may have a deep physical meaning, in spite of poor justifycation at the level of continuum mechanics that we can provide at present.

Moving these ideas to the 3D case but with the limitation of a transport type noise, we may show that noise improves the theory of 3D Navier-Stokes equations. This was a long standing project in the case of additive noise, frustrated however by several technical difficulties. The case of transport noise reveald to be more promising. However, for future research, the understanding of case of transport-stretching noise must be considered the most important open problem.

Let us also add the following very heurisitc remark. In these lectures we started from additive perturbations motivated by the roughness of boundaries. Additive noise, as just mentioned, have not been shown to improve so much the theory of 3D Navier-Stokes equations. But additive noise in the small scales, as shown in the present chapter, may lead to a multiplicative transport noise in the large scales. And transport noise has a better regularizing power. At the end it seems, then, that *it is the additive noise at small scales which regularizes*! Presumably the long-standing conjecture that additive noise regularizes could be correct but the path to reveal its power is very complex. Until now the efforts to prove that additive noise regularizes were based on the similarity with the finite dimensional case, where additive noise is so succeessful. But this is probably a too abstract viewpoint for the Navier-Stokes equations. The deep reason of regularization stands inside the links between scales, a fact proper of fluid dynamics and not of general evolution equations.

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