Chapter 2. Stochastic Navier-Stokes equations and state dependent noise

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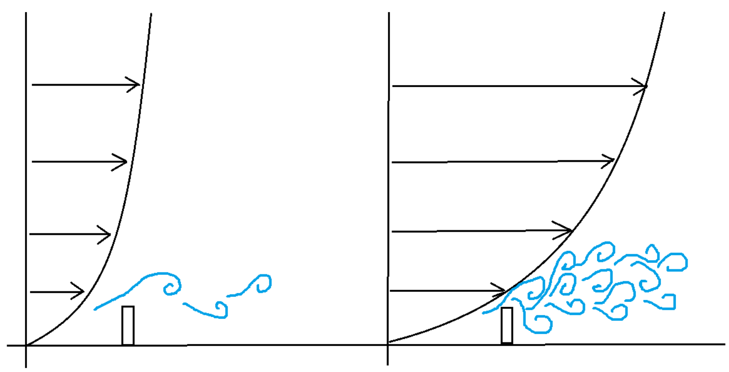
1 Introduction

Until now, although motivated by certain random input, we dealt with the Stochastic Navier-Stokes equations as if they were deterministic: given a single noise realization, we solve the equation.

This is possible in relatively few cases. The case treated above had the special feature that the random input was independent of the solution. But in real situations, as in the figure (discussed in a section below)



the noise may vary depending on the solution.

Mathematically speaking, in the previous chapter the noise entered the equation as an additive force; this was the key property which allowed us to study the linear Stokes problem first, independently of the solution of the nonlinear one. There are other cases (different from the additive case) which can be treated by similar ideas, but few.

If we have an equation of the form

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u + f + F(u) + \sigma(u) \partial_t W \\ \operatorname{div} u &= 0 \end{aligned}$$

where the distributional derivative $\partial_t W$ is multiplied by a function of the solution, we are in trouble. The problem is not just the fact that the Stokes problem

$$\begin{aligned}\partial_t z + \nabla q &= \nu \Delta z + \sigma(u)\,\partial_t W \\ \operatorname{div} z &= 0\end{aligned}$$

depends on u: this problem in principle could be solved by an iteration. The problem is that we cannot apply the trick of integration by parts in the mild formula for z:

$$\begin{aligned} z(t) &= e^{tA} z_0 + \int_0^t e^{(t-s)A} \sigma(u(s))\,\partial_s W(s)\,ds \\ &= e^{tA} z_0 + \left[e^{(t-s)A} \sigma(u(s))\, W(s) \right]_{s=0}^{s=t} - \int_0^t \frac{d}{ds} \left(e^{(t-s)A} \sigma(u(s)) \right) W(s)\,ds \\ &= e^{tA} z_0 + \sigma(u(t))\, W(t) - e^{tA} \sigma(u(0))\, W(0) \\ &\quad + \int_0^t A e^{(t-s)A} \sigma(u(s))\, W(s)\,ds + \int_0^t e^{(t-s)A} \frac{d}{ds} \sigma(u(s))\, W(s)\,ds \end{aligned}$$

and

$$\frac{d}{ds} \sigma(u(s)) = \langle D\sigma(u(s)), \partial_s u(s) \rangle$$

brings again into play the term $\partial_s W(s)$.

One way to escape this problem is using the theory of rough paths, which however is quite elaborated for our purposes. The most classical way is, when W is related to Brownian motions, to use stochastic calculus. The purpose of this chapter is illustrating the technique to study the Stochastic Navier-Stokes equations by stochastic calculus.

Remark 1 *The reader certainly noticed that we have introduced, in parallel to $\sigma(u)\,\partial_t W$, also a term $F(u)$. This is not for generality, which clearly is not our purpose in these notes. The reason is deep: if we introduce a term $\sigma(u)\,\partial_t W$, we also need to introduce a compensator $F(u)$, otherwise the Physics is wrong. This is Wong-Zakai principle: we shall describe it in two particular cases, in this and the next chapters.*

1.1 Filtered probability space

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration indexed by $t \geq 0$ is a family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras such that $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}$ for every $t_1 \leq t_2$. We call $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ a filtered probability space, and we abbreviate $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. A stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, taking values in a measurable space, is adapted if X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$. It is progressively measurable if the map $(s, \omega) \mapsto X_s(\omega)$ is measurable on $([0,t] \times \Omega, \mathcal{B}(0,t) \otimes \mathcal{F}_t)$ for every $t \geq 0$ ($\mathcal{B}(0,t)$ being the Borel σ-algebra on $[0,t]$). When the target space is metric with the Borel σ-algebra, and the process is continuous, the concepts of adapted and progressively measurable are equivalent. When we deal with processes such that, with respect to the time variable, are equivalence classes (with respect to zero sets for the Lebesgue measure on the time interval), like $L^2(0,T;V)$, we cannot use the concept of adapted process since X_t (given t) is not well defined. In this case we always use the concept of progressively measurable: for every t, the restriction on $[0,t]$ is a well defined equivalence class and the definition applies to it.

Denote by $L^2_{\mathcal{F}_t}(\Omega,H)$ the space of random variables (in fact equivalence classes) $X : \Omega \to H$ that are $\mathcal{F}_t$-measurable and square integrable. We denote by $C_{\mathcal{F}}([0,T];H)$ the space of continuous adapted processes $(X_t)_{t\in[0,T]}$ with values in H such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t\|_H^2\right] < \infty$$

and by $L^2_{\mathcal{F}}(0,T;V)$ the space of progressively measurable processes $(X_t)_{t\in[0,T]}$ with values in V such that

$$\mathbb{E}\left[\int_0^T \|X_t\|_V^2\,dt\right] < \infty.$$

Of course we may use similar notations also with different spaces in place of H and V; this is just the most common case in the sequel.

A (real valued) Brownian motion on $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ is a continuous adapted process $(W_t)_{t\geq 0}$ such that $\mathbb{P}(W_t = 0) = 1$, $W_t - W_s$ is independent of $\mathcal{F}_s$ for every $t \geq s \geq 0$, and $W_t - W_s$ is a centered Gaussian random variable with variance $t - s$ (we write $W_t - W_s \sim N(0,t-s)$). With probability one, paths are not only continuous but also locally Hölder continuous with any Hölder exponent $\alpha < \frac{1}{2}$.

The noise used in Chapter 1 had the form

$$W(t,x) := \sum_{k\in K}\sqrt{\lambda_k}\sigma_k(x)\,W_t^k \tag{1}$$

where K is a finite set, $\sigma_k \in D(A)$, $\left(W_t^k\right)_{t\geq 0}$ are independent Brownian motions on some filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. With probability one, the path $t \mapsto W(t,\cdot)$ is of class $C([0,T];D(A))$ (also $C^\alpha([0,T];D(A))$ for every $\alpha < \frac{1}{2}$).

In the previous chapter we have denoted by τ^k the average intertimes between creation of new eddies. Here we use the quantity

$$\lambda_k = \frac{1}{\tau^k}$$

which has the meaning of *rate* of eddy production. The reason is that, below, we modify the model with state-dependent rates and the notational analogy will be easier.

2 Additive noise under the view of stochastic calculus

Let us elaborate the result of Chapter 1 under the view of stochastic calculus. Consider the Itô type equation, in $d = 2$,

$$\begin{aligned} du + (u \cdot \nabla u + \nabla p)\, dt &= \nu \Delta u dt + \sum_{k \in K} \sqrt{\lambda_k} \sigma_k dW_t^k \\ \operatorname{div} u &= 0 \end{aligned} \tag{2}$$

with

$$\begin{aligned} u|_{\partial D} &= 0 \\ u(0) &= u_0. \end{aligned}$$

Definition 2 *Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and the noise $W(t,x)$ as in (1), given $u_0 : \Omega \to H$, $\mathcal{F}_0$-measurable, we say that a process u is a solution of equation (2), if its paths are of class*

$$u \in C([0,T]; H) \cap L^2(0,T;V)$$

with probability one, it is adapted as a process in H, progressively measurable in V, and

$$\begin{aligned} & \langle u(t), \phi \rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ = \; & \langle u_0, \phi \rangle + \int_0^t \langle u(s), A\phi \rangle\, ds + \sum_{k \in K} \sqrt{\lambda_k} \langle \sigma_k, \phi \rangle W_t^k \end{aligned}$$

for every $\phi \in D(A)$.

Theorem 3 *There exists a unique solution.*

Proof. Given two solutions, with probability one their paths are two solutions in the sense of the theorem of the previous Chapter, hence they coincide. Path by path the existence of $u(\omega)$ is given by that theorem; since W is measurable, also u is measurable. But the measurability result can be applied on any subinterval $[0,t]$, the process u being always the same (namely the restriction to $[0,t]$ of the process on $[0,T]$), hence we have progressive measurability, which gives also adaptedness in H due to continuity. ■

We want now to apply Itô formula to compute

$$d\, \|u(t)\|_{L^2}^2 .$$

Let us recall, for comparison, that when X_t is a process in $\mathbb{R}^d$ satisfying the equation

$$dX_t^i = b_t^i dt + \sum_{k \in K} \sigma_t^{ik} dW_t^k$$

and f is a function of class $C^2(\mathbb{R}^d)$, then

$$df(X_t) = \sum_{i=1}^{d} \partial_i f(X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k\in K} \partial_i \partial_j f(X_t)\, \sigma_t^{ik} \sigma_t^{jk} dt$$

where we have to replace dX_t^i by the equation. Rigorously, all these identities have to be interpreted in integral form and the stochastic processes $X_t^i, b_t^i, \sigma_t^{ik}$ are assumed progressively measurable. In order to apply these facts we need a progressively measurable process (and this is provided by the previous theorem) and a finite dimensional reduction.

Theorem 4 *If* $\mathbb{E}\|u_0\|_{L^2}^2 < \infty$ *then*

$$u \in C_{\mathcal{F}}([0,T]; H) \cap L^2_{\mathcal{F}}(0,T;V)$$

and

$$\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] + 2\nu \int_0^t \mathbb{E}\|\nabla u(s)\|_{L^2}^2 ds = \mathbb{E}\left[\|u_0\|_{L^2}^2\right] + t\sum_{k\in K} \lambda_k \|\sigma_k\|_{L^2}^2$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_{L^2}^2\right] \le \mathbb{E}\left[\|u_0\|_{L^2}^2\right] + T\sum_{k\in K} \sqrt{\lambda_k}\, \|\sigma_k\|_{L^2}^2 + C\sum_{k\in K} \lambda_k \mathbb{E}\int_0^T \langle u(s), \sigma_k\rangle^2 ds.$$

Proof. Taken a complete orthonormal system in H, (e_i), made of eigenvectors of A, with eigenvalues $(-\lambda_i)$, called H_n the finite dimensional space generated by $e_1, .., e_n$ and π_n the projection onto H_n, called $u_n(t) = \pi_n u(t)$, called finally

$$b_n(u(s)) := \sum_{i=1}^{n} b(u(s), u(s), e_i)\, e_i$$

we have (from the weak formulation applied to each e_i)

$$u_n(t) + \int_0^t b_n(u(s))\, ds = \pi_n u_0 + \int_0^t A u_n(s)\, ds + \pi_n W(t).$$

Taken the function $f_n(x) = \sum_{i=1}^{n} \langle x, e_i\rangle^2$, which has the properties $\partial_i f_n(x) = 2\langle x, e_i\rangle$, $\partial_i \partial_j f_n(x) = 2\delta_{ij}$, using the fact that, with $\sigma_t^{ik} = \sqrt{\lambda_k}\langle \sigma_k, e_i\rangle$, one has $\sum_{i=1}^{\infty} \left(\sigma_t^{ik}\right)^2 = \lambda_k \|\sigma_k\|_{L^2}^2$, the classical Itô formula gives us

$$\begin{aligned} d\|u_n(t)\|_{L^2}^2 &= 2\langle u_n(t), du_n(t)\rangle + \sum_{k\in K} \lambda_k \|\pi_n \sigma_k\|_{L^2}^2 dt \\ &= -2\nu \|\nabla u_n(t)\|_{L^2}^2 dt + \sum_{k\in K} \lambda_k \|\pi_n \sigma_k\|_{L^2}^2 dt \\ &\quad +2\sum_{k\in K} \sqrt{\lambda_k}\langle u_n(t), \pi_n \sigma_k\rangle dW_t^k + b(u(s), u(s), u_n(s))\, dt \end{aligned}$$

where we have used

$$\langle u_n(s), b_n(u(s))\rangle = b(u(s), u(s), u_n(s)).$$

This identity has to be interpreted in integral form. Using the convergence properties of π_n and the regularity of u, it is not difficult to pass to the limit and obtain

$$\begin{aligned}\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds &= \|u_0\|_{L^2}^2 + t\sum_{k\in K}\lambda_k\|\sigma_k\|_{L^2}^2 \\ &\quad + 2\sum_{k\in K}\sqrt{\lambda_k}\int_0^t \langle u(s), \sigma_k\rangle\, dW_s^k\end{aligned} \tag{3}$$

where the last term is an Itô-integral. In order to take expected values we have to use a localization argument that we explain here forever, namely we omit the repetition below when it is used several times. For sake of simplicity of notations assume that u is a solution defined for all $t \geq 0$ (we can do this, T is arbitrary).For every $R > 0$, let τ_R be the stopping time defined as

$$\tau_R = \inf\{t > 0 : \|u(t)\|_{L^2} > R\}$$

or equal to $+\infty$ if the set is empty. Compute the previous identity at time $t\wedge\tau_R$ (it helps the fact that the process u is continuous in H):

$$\begin{aligned}\|u(t\wedge\tau_R)\|_{L^2}^2 + 2\nu\int_0^t 1_{s\leq\tau_R}\|\nabla u(s)\|_{L^2}^2\, ds &= \|u_0\|_{L^2}^2 + (t\wedge\tau_R)\sum_{k\in K}\lambda_k\|\sigma_k\|_{L^2}^2 \\ &\quad + 2\sum_{k\in K}\sqrt{\lambda_k}\int_0^t 1_{s\leq\tau_R}\langle u(s), \sigma_k\rangle\, dW_s^k.\end{aligned}$$

Now $\mathbb{E}\int_0^T 1_{s\leq\tau_R}\langle u(s), \sigma_k\rangle^2\, ds < \infty$ hence the Itô integrals of this identity are true martingales; their expected values are thus equal to zero. Moreover, the other terms on the right-hand-side have finite expected value, hence the same is true for the sum of the two terms on the left-hand-side, and then also individually for each of them, being non-negative. We get

$$\begin{aligned}&\mathbb{E}\left[\|u(t\wedge\tau_R)\|_{L^2}^2\right] + 2\nu\mathbb{E}\int_0^t 1_{s\leq\tau_R}\|\nabla u(s)\|_{L^2}^2\, ds \\ = &\ \mathbb{E}\left[\|u_0\|_{L^2}^2\right] + \mathbb{E}[t\wedge\tau_R]\sum_{k\in K}\lambda_k\|\sigma_k\|_{L^2}^2.\end{aligned}$$

Since $\lim_{R\to\infty}\tau_R = +\infty$, and u is continuous in H, we deduce as $R\to\infty$

$$\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] + 2\nu\mathbb{E}\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds = \mathbb{E}\left[\|u_0\|_{L^2}^2\right] + t\sum_{k\in K}\lambda_k\|\sigma_k\|_{L^2}^2.$$

From this result, which is already part of the thesis, we deduce $u \in L^2_{\mathcal{F}}(0,T;V)$. In order to prove $u \in C_{\mathcal{F}}([0,T];H)$ we restart from (3) where now, as a consequence of the estimates just proved, we know that the Itô integrals are square integrable martingales. Let us simplify (3) into

$$\|u(t)\|^2_{L^2} \leq \|u_0\|^2_{L^2} + t\sum_{k\in K}\lambda_k \|\sigma_k\|^2_{L^2} + 2\sum_{k\in K}\sqrt{\lambda_k}\int_0^t \langle u(s), \sigma_k\rangle dW^k_s.$$

By Doob's inequality,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\|u(t)\|^2_{L^2}\right] &\leq \mathbb{E}\left[\|u_0\|^2_{L^2}\right] + T\sum_{k\in K}\lambda_k\|\sigma_k\|^2_{L^2} \\ &\quad + C\sum_{k\in K}\lambda_k\mathbb{E}\int_0^T \langle u(s), \sigma_k\rangle^2 ds\end{aligned}$$

and the right-hand-side is bounded as in the statement of the theorem. Hence in particular $u \in C_{\mathcal{F}}([0,T];H)$. ■

2.1 Consequences

The message we get from this theorem is manifold.

- The solution has integrability properties in ω reflecting analogous properties assumed on the data.
- In the modeling of emergence of vortices developed in the previous section we have made a mistake: creating vortices from nothing we introduce energy into the system. Therefore we have to include an extra dissipation mechanism. There is a loss of energy due to the impact of the flow with the obstacle (which, let us remember, is not included into the boundary conditions); part of this energy is given back in the form of emerging vortices. We do not have a sufficiently good solution to this mistake, which then we leave as an open problem. A possible proposal is adding a friction term $-\lambda(x)u$

  $$du + (u\cdot\nabla u + \nabla p)\,dt = (\nu\Delta u - \lambda(x)u)\,dt + \sum_{k\in K}\sqrt{\lambda_k}\sigma_k dW^k_t$$

  with a friction coefficient possibly depending on x and localized near the boundary: in this way the Physical idea is that energy of large scales is subtracted near the boundary; and re-injected through the vortices σ_k. The energy balance is now

  $$\begin{aligned}&\mathbb{E}\left[\|u(t)\|^2_{L^2}\right] + 2\nu\int_0^t \mathbb{E}\|\nabla u(s)\|^2_{L^2} ds + 2\mathbb{E}\int_0^t\int_D \lambda(x)|u(s,x)|^2 dxds \\ =\ &\mathbb{E}\left[\|u_0\|^2_{L^2}\right] + t\sum_{k\in K}\lambda_k\|\sigma_k\|^2_{L^2}.\end{aligned}$$

But we should be able to choose $\lambda(x)$ in such a way that

$$2\mathbb{E}\int_0^t \int_D \lambda(x) |u(s,x)|^2 \, dxds \sim t \sum_{k\in K} \lambda_k \|\sigma_k\|_{L^2}^2 .$$

We do not know how to reach this target.

- Assume $u(t)$ is a statistically stationary solution; this implies that $\mathbb{E}\|u(t)\|_{L^2}^2 = \mathbb{E}\|u_0\|_{L^2}^2$ and $\mathbb{E}\|\nabla u(s)\|_{L^2}^2$ is independent s, which then we denote by $\mathbb{E}\|\nabla u\|_{L^2}^2$. Then, stressing the dependence of u on ν,

$$\epsilon := \nu\mathbb{E}\|\nabla u_\nu\|_{L^2}^2 \, ds = \frac{1}{2}\sum_{k\in K} \lambda_k \|\sigma_k\|_{L^2}^2 .$$

  The dissipation ϵ of energy due to viscosity remains constant in the inviscid limit $\epsilon \to 0$ (it is a statement of K41 theory), if the energy injection is constant.

- We may use a small variant of the previous result to study state-dependent noise by iterations, see below.

2.2 Example of state-dependent noise

In Chapter 1 we have introduced a noise modeling the emergence of vortices at a boundary due to instability. However, when the fluid is at rest, certainly no vortex is created; similarly, we do not expect frequent creations if the velocity of the flow is very small. The rate of creation of vortices hence should depend on some feature of the flow itself. This doesn't mean that the model of the previous Chapter is useless: it is reasonable when the mean flow is roughly constant, and the rates τ^k should be taken appropriately with respect to the constant mean flow value.

When the state $u(t,\cdot)$ affects the rate of creation, we may introduce (corresponding to each k) an instantaneous rate $\lambda_k(u(t))$ depending on an average intensity of $u(t,\cdot)$, e.g.

$$\lambda_k(u(t)) = \chi^2\left(\frac{1}{|B(x_k,r)|}\int_{B(x_k,r)} |u(t,y)| \, dy\right)$$

where χ^2 is a nondecreasing non-negative function, equal to zero in zero and $r > 0$ is a length scale relevant to the problem. Then we introduce the cumulative rate

$$\Lambda_k(t) = \int_0^t \lambda_k(u(s)) \, ds$$

and finally we modify the Poisson process N_t^k by this rate, namely we consider the process

$$N_{\Lambda_k(t)}^k .$$

The case previously considered was simply

$$\lambda_k(u(t)) = \lambda_k, \qquad \Lambda_k(t) = \lambda_k, \qquad N^k_{\lambda_k t}.$$

The jump times of the noise in the equation will be the jump times of this processes, which are delayed or accelerated depending on the average intensity of $u(t)$:

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \sigma_k \partial_t N^k_{\Lambda_k(t)} \tag{4}$$

or

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \frac{1}{\sqrt{2}} \sigma_k \partial_t \left(N^{k,1}_{\Lambda_k(t)} - N^{k,2}_{\Lambda_k(t)} \right) \tag{5}$$

depending whether we assume that both vortices $\sigma_k(x)$ and $-\sigma_k(x)$ appear and are equally likely.

This is already a very interesting model which could deserve investigation. Otherwise, in the case of (5), we may rescale the noise as

$$\sum_{k \in K} \frac{1}{n\sqrt{2}} \sigma_k(x) \left(N^{k,1}_{n^2 \Lambda_k(t)} - N^{k,2}_{n^2 \Lambda_k(t)} \right). \tag{6}$$

Notice that, in order to increase the rate at time t, we have to use the instantaneous rate $n^2 \lambda_k(t)$, whence the expression $n^2 \Lambda_k(t)$ (instead of $\Lambda_k(n^2 t)$ which has a completely different and wrong meaning).

Recalling the convergence of rescaled Poisson processes to Brownian motion discussed in Chapter 1, it can be proved that the limit process of (6), in law, is

$$\sum_{k \in K} \sigma_k(x) B^k_{\Lambda_k(t)}$$

where B^k_t are independent Brownian motions. Then, by a deep theorem on martingales, there exists (possibly on a larger probability space) independent Brownian motions W^k_t such that, in law

$$B^k_{\Lambda_k(t)} = \int_0^t \sqrt{\lambda_k(u(s))} dW^k_s$$

(jointly in k). This result in undoubtedly advanced and not trivial even at the heuristic level but notice at least the analogy with the coefficients $\sqrt{\lambda_k}$ in the case of constant rate: when $\lambda_k(u(s)) = \lambda_k$, $\Lambda_k(t) = \lambda_k$, the previous identity reads

$$B^k_{\lambda_k t} = \int_0^t \sqrt{\lambda_k} dW^k_s = \sqrt{\lambda_k} W^k_t$$

and it is well known that $\lambda_k^{-1/2} B^k_{\lambda_k t}$ is a new Brownian motion.

The final equation is

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \sigma_k \sqrt{\lambda_k(u)} \partial_t W_t^k.$$

We write it in the form

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f + F(u) + \sum_{k \in K} \sigma_k(u) \partial_t W_t^k \tag{7}$$

by introducing the maps $\sigma_k : H \to H$ given by

$$\sigma_k(u)(x) = \sigma_k(x) \sqrt{\lambda_k(u)}.$$

2.3 The Wong-Zakai corrector

Equations (4)-(5) are mathematically correct (whether they are physically relevant, it should be investigated more deeply). On the contrary, equation (7) requires a special choice of $F(u)$ to be the right one:

$$F(u) = \frac{1}{2} \sum_{k \in K} D\sigma_k(u) \sigma_k(u).$$

Here by $D\sigma_k(u)$ we mean the Frechét Jacobian of $\sigma_k(u)$, which is a linear bounded operator from H to H, under suitable assumptions, and $D\sigma_k(u)\sigma_k(u)$ is the application of the linear map $D\sigma_k(u)$ to the element $\sigma_k(u)$ of H. We do not know whether a full proof of this fact has been given and under which assumptions. We assume this is the correct result by heuristic extension of a known argument for finite dimensional equations. Let us explain it in the simple case of a one-dimensional equation.

Consider the one dimensional equation, with $\sigma(x) \geq \nu > 0$,

$$\frac{dX_t^\epsilon}{dt} = \sigma(X_t^\epsilon) \frac{dW_t^\epsilon}{dt}$$

where W_t^ϵ is an approximation of a Brownian motion W_t. It is an equation with separated variables. Then

$$\frac{\frac{dX_t^\epsilon}{dt}}{\sigma(X_t^\epsilon)} = \frac{dW_t^\epsilon}{dt}$$

$$\int_0^T \frac{\frac{dX_t^\epsilon}{dt}}{\sigma(X_t^\epsilon)} dt = \int_0^T \frac{dW_t^\epsilon}{dt} dt$$

$$\Phi(X_T^\epsilon) - \Phi(x_0) = W_T^\epsilon, \qquad \Phi'(x) = \frac{1}{\sigma(x)}$$

$$X_t^\epsilon = \Phi^{-1}(\Phi(x_0) + W_t^\epsilon)$$

Hence $X^{\epsilon}_{\cdot}$ converges weakly to $X_{\cdot}$ given by

$$X_t = \Phi^{-1}\left(\Phi\left(x_0\right) + W_t\right).$$

From Ito formula, since

$$\begin{aligned} D\Phi^{-1}(x) &= \frac{1}{\Phi'\left(\Phi^{-1}(x)\right)} = \sigma\left(\Phi^{-1}(x)\right) \\ D^2\Phi^{-1}(x) &= D\left[\sigma\left(\Phi^{-1}(x)\right)\right] = \sigma'\left(\Phi^{-1}(x)\right) D\Phi^{-1}(x) \\ &= \sigma'\left(\Phi^{-1}(x)\right)\sigma\left(\Phi^{-1}(x)\right) \end{aligned}$$

$$\begin{aligned} dX_t &= \sigma\left(\Phi^{-1}\left(\Phi\left(x_0\right)+W_t\right)\right) dW_t + \frac{1}{2}\sigma'\left(\Phi^{-1}\left(\Phi\left(x_0\right)+W_t\right)\right)\sigma\left(\Phi^{-1}\left(\Phi\left(x_0\right)+W_t\right)\right) dt \\ &= \sigma\left(X_t\right) dW_t + \frac{1}{2}\sigma'\left(X_t\right)\sigma\left(X_t\right) dt. \end{aligned}$$

We have found the corrector above.

Our conclusion, supported by the previous heuristic evidences, is that the right stochastic equations is

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f + \frac{1}{2}\sum_{k\in K} D\sigma_k(u)\,\sigma_k(u) + \sum_{k\in K}\sigma_k(u)\,\partial_t W_t^k.$$

Remark 5 *There is a notion of stochastic integral, different from the Itô one, called Stratonovich integral and denoted by* $\int_0^t \sigma_k(u(s))\circ dW_s^k$*, such that*

$$\int_0^t \sigma_k(u(s))\circ dW_s^k = \int_0^t \sigma_k(u(s))\,dW_s^k + \frac{1}{2}\int_0^t D\sigma_k(u(s))\,\sigma_k(u(s))\,ds$$

when u *solves equation above. Therefore, with such notion, the equation has the form*

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f + \sum_{k\in K}\sigma_k(u)\circ\partial_t W_t^k.$$

3 2D Stochastic Navier-Stokes equations

Consider now the equations

$$\begin{aligned} \partial_t u + u\cdot\nabla u + \nabla p &= \nu\Delta u + f + F(u) + \sum_{k\in K}\sigma_k(u)\,\partial_t W_t^k \qquad (8) \\ \operatorname{div} u &= 0 \end{aligned}$$

with

$$\begin{aligned} u|_{\partial D} &= 0 \\ u(0) &= u_0. \end{aligned}$$

Assume

$$\begin{aligned} F &\in Lip(H,H) \\ \sigma_k &\in Lip(H,H) \cap C(H, D(A)), \text{ bounded in } H, \qquad k \in K. \end{aligned}$$

With some additional elements of stochastic analysis (Itô formula for $\|u(t)\|_{L^2}^p$ and Burkholder-Davis-Gundy inequality) one can drop the assumption that σ_k are bounded, so it is made here only for simplicity of exposition. The assumption $C(H, D(A))$ is also made just for simplicity, but it is clear from the estimates below that it is absolutely unessential.

Definition 6 *Given* $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ *and* $f \in L^2_{\mathcal{F}}(0,T;V')$*, we say that*

$$u \in C_{\mathcal{F}}([0,T]; H) \cap L^2_{\mathcal{F}}(0,T;V)$$

is a weak solution of equation (8) if

$$\begin{aligned} &\langle u(t), \phi\rangle - \int_0^t b(u(s), \phi, u(s))\, ds \\ =\ &\langle u_0, \phi\rangle + \int_0^t \langle u(s), A\phi\rangle\, ds + \int_0^t \langle f(s), \phi\rangle\, ds \\ &+ \int_0^t \langle F(u(s)), \phi\rangle\, ds + \sum_{k\in K} \int_0^t \langle \sigma_k(u(s)), \phi\rangle\, dW_s^k \end{aligned}$$

for every $\phi \in D(A)$.

Theorem 7 *For every* $u_0 \in L^2_{\mathcal{F}_0}(\Omega, H)$ *and* $f \in L^2_{\mathcal{F}}(0,T;V')$*, there exists a unique weak solution of equation (8). It satisfies*

$$\begin{aligned} &\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] + 2\nu\mathbb{E}\int_0^t \|\nabla u(s)\|_{L^2}^2\, ds \\ =\ &\mathbb{E}\left[\|u_0\|_{L^2}^2\right] + 2\mathbb{E}\int_0^t \langle u(s), f(s) + F(u(s))\rangle\, ds \\ &+ \sum_{k\in K} \mathbb{E}\int_0^t \|\sigma_k(u(s))\|_{L^2}^2\, ds. \end{aligned}$$

3.1 Proof of uniqueness

Let $u^{(i)}$ be two solutions. Then $w = u^{(1)} - u^{(2)}$ satisfies

$$\begin{aligned} & \langle w(t), \phi \rangle - \int_0^t \left(b\left(u^{(1)}, \phi, u^{(1)}\right) - b\left(u^{(2)}, \phi, u^{(2)}\right) \right)(s)\, ds \\ = & \int_0^t \langle w(s), A\phi \rangle\, ds + \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), \phi \right\rangle ds \\ & + \sum_{k \in K} \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), \phi \right\rangle dW_s^k \end{aligned}$$

and since

$$\begin{aligned} & b\left(u^{(1)}, \phi, u^{(1)}\right) - b\left(u^{(2)}, \phi, u^{(2)}\right) - b(w, \phi, w) \\ = & b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right) \end{aligned}$$

we get

$$\begin{aligned} & \langle w(t), \phi \rangle - \int_0^t (b(w(s), \phi, w(s)))\, ds \\ = & \int_0^t \langle w(s), A\phi \rangle\, ds + \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), \phi \right\rangle ds \\ & + \sum_k \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), \phi \right\rangle dW_s^k \\ & - \int_0^t \left(b\left(u^{(2)}, \phi, w\right) + b\left(w, \phi, u^{(2)}\right) \right)(s)\, ds. \end{aligned}$$

We need the Itô formula to continue; it can be proved similarly to Theorem 4. It gives us

$$\begin{aligned} \|w(t)\|_H^2 + 2\nu \int_0^t \|\nabla w(s)\|_H^2\, ds = & \; 2 \int_0^t \left\langle F\left(u^{(1)}(s)\right) - F\left(u^{(2)}(s)\right), w(s) \right\rangle ds \\ & -2 \int_0^t \left(b\left(u^{(2)}, w, w\right) + b\left(w, w, u^{(2)}\right) \right)(s)\, ds \\ & + \sum_{k \in K} \int_0^t \left\| \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right) \right\|_{L^2}^2 ds \\ & + M_t \end{aligned}$$

where

$$M_t := \sum_k \int_0^t \left\langle \sigma_k\left(u^{(1)}(s)\right) - \sigma_k\left(u^{(2)}(s)\right), w(s) \right\rangle dW_s^k.$$

Therefore, if L_F and L_k are the Lipschitz constants of F and σ_k respectively, using estimates of Chapter 1 we get

$$\begin{aligned}\|w(t)\|_H^2+\nu\int_0^t\|\nabla w(s)\|_H^2\,ds &\leq \left(2L_F+\sum_{k\in K}L_k^2\right)\int_0^t\|w(s)\|_H^2\,ds\\ &+C\int_0^t\|w(s)\|_H^2\left(1+\left\|u^{(2)}(s)\right\|_{\mathbb{L}^4}^2\right)ds\\ &+M_t.\end{aligned}$$

We need now a very interesting trick that we have learned from Bjorn Schmalfuss: introduced

$$\rho_t=\exp\left(-C\int_0^t\left(1+\left\|u^{(2)}(s)\right\|_{\mathbb{L}^4}^2\right)ds\right)$$

we have, from Itô formula again,

$$\|w(t)\|_H^2\rho_t+\nu\int_0^t\|\nabla w(s)\|_H^2\rho_s ds\leq\left(2L_F+\sum_{k\in K}L_k^2\right)\int_0^t\|w(s)\|_H^2\rho_s ds+\widetilde{M}_t$$

where

$$\widetilde{M}_t:=\sum_{k\in K}\int_0^t\left\langle\sigma_k\left(u^{(1)}(s)\right)-\sigma_k\left(u^{(2)}(s)\right),w(s)\right\rangle\rho_s dW_s^k.$$

Omitting the necessary localization argument entirely similar to the one used in Theorem 4, we get

$$\begin{aligned}&\mathbb{E}\left[\|w(t)\|_H^2\rho_t\right]+\nu\mathbb{E}\int_0^t\|\nabla w(s)\|_H^2\rho_s ds\\ \leq&\left(2L_F+\sum_{k\in K}L_k^2\right)\int_0^t\mathbb{E}\left[\|w(s)\|_H^2\rho_s\right]ds\end{aligned}$$

which leads to $\mathbb{E}\left[\|w(t)\|_H^2\rho_t\right]=0$ by Gronwall lemma. But, thanks to the regularity of $u^{(2)}$, $\mathbb{P}(\rho_t>0)=1$. Hence $\mathbb{P}(w(t)=0)=1$. Since this is true for all t, the processes $u^{(1)}$ and $u^{(2)}$ are modifications; but they are continuous, hence they are indistinguishable.

4 Proof of existence

(to be continued)

5 Summary

The main open problem outlined in this Chapter is the continuation of the one posed in the previous chapter, namely the link between a real irregular boundary and stochastic models of fluids; here the problem is enriched of the dependence on the flow intensity, a very realistic feature, which poses a new technical issue, namely the presence of the Wong-Zakai corrector in the limit equation. We have also seen that noise introduces energy, in the average, hence the model should be corrected by an energy loss.

The main techniques illustrated in this Chapter are the use of Itô formula and an interesting idea for uniqueness, until now (then comment on existence).