Review：Neural Network Structure Using Fractals

Sei-ichiro Kamata
Information, Production System Research Center
Waseda University

Fractal can be viewed as a type of image transform that exhibits self-similarity. Image transform essentially approximate a figure using a linear combination of simple curves (or surfaces) or smooth curves (or surfaces) known as kernels. However, images obtained from the real world are generally complex. In a sense, fractals can be seen as a tool for approximating objects with complex shapes in the real world. By introducing the concept of fractals, B. Mandelbrot demonstrated that complex shapes can be generated from simple operations. A key difference from conventional image representation is that images rendered using fractals possess infinite resolution. There are several concepts of Fractals such as Fractal dimensions in non-integer dimensions, fractals in iterated function systems via scaling maps, complex fractals according to Mandelbrot, fractals in chaos, In the following, we introduce self-similar set, Fractal dimensions, and iterated function Fractal systems, and fundamentals on deep learning.

■1 Overview of Fractals

Fractal is a term originated from *fractus*, the Latin adjective corresponding to the noun fraction (meaning fragment, part, fraction, etc.). Its verb form, frangere (to break, etc.), implies the creation of irregular fragments. In 1980, Mandelbrot published The Fractal Geometry of Nature [1] - a revised edition of his book Fractals- and demonstrated the existence of a self-similar set known as the Mandelbrot set. The Mandelbrot set is defined by a domain set of a parameter C such that if $n \to \infty$, two dimensional complex plane of $\{Z_n \mid n = 0,1,2,\cdots\}$ does not diverge using

$$Z_{n+1} = Z_n^2 + C \quad (C \text{ is a complex constant})$$

.

Following the publication of the Mandelbrot set, in the field of pattern recognition, A. Pentland [2] applied fractal dimension as a feature for texture analysis in 1984; meanwhile, in the field of image compression, M. Barnsley [3] proposed it as an accumulative compression method in the late 1980s, which attracted significant attention. In the following, we introduce self-similar sets via contraction mappings and fractals of iterated function systems via contraction mappings, which are fundamental concepts in fractal theory.

■ 2 Self-Similar Sets

In this section, contraction mapping and self-similar set, which are mathematically important fundamental concepts in fractals, are explained.

2.1 Contraction Mapping

Given two sets X, Y on plane, for two points $\mathbf{x}_1 \in X, \mathbf{x}_2 \in X$, a function $f: X \to Y$ is called a contraction mapping if there exists s satisfying

$$d\left(f(\mathbf{x}_1), f(\mathbf{x}_2)\right) \leq s \cdot d\left(\mathbf{x}_1, \mathbf{x}_2\right),$$

where s is called a contraction rate of a mapping function f, d is a distance. In addition to the Euclidean distance, various other distances can be considered. While the continuity of the contraction mapping clearly holds, refer to other literature for the proof.

2.2 Self-Similar Sets

Self-similar sets that are important in Fractals are defined by a set A satisfying

$$A = \bigcup_{n=1}^{N} F_n(A),$$

if a set A and the contraction mappings $F_1, F_2, \cdots, F_N : A \to A$ are given. For example. Cantor set is obtained by starting from initial set $A_0 = [0,1]$, and dividing the entire region A_n into three equal parts and applying two contraction mappings to the two end sections. Here the contraction mapping to the left section is F_1, and the contraction mapping to the right section is F_2. First, for $\forall x \in A_0$, if apply $F_1(x) = \frac{1}{3}x$, $F_2(x) = \frac{1}{3}(x+2)$, we obtain

$$A_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], A_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \cdots.$$

Furthermore, if applied an infinite number of times, $A_\infty = F_1(A_\infty) \cup F_2(A_\infty)$ hold.
Koch curve is given as a self-similar set as shown in Figure 2.

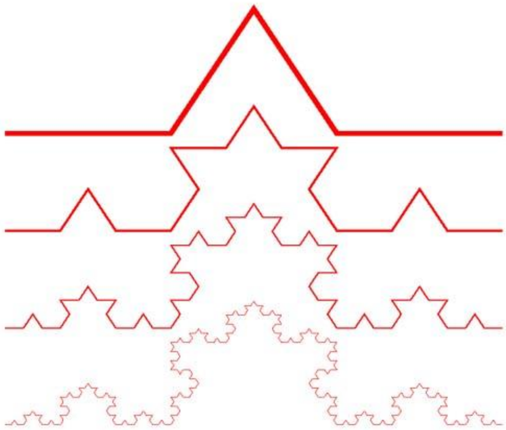

Figure 1　Koch curve

■3　Iterated Function Systems

A contraction map is defined using an affine mapping $F_n(\mathbf{x}) = s_n R_n \mathbf{x} + p_n$. This satisfies a length-preserving property. For initial set A_0, if we apply the following iteration function

$$A_{k+1} \equiv \mathbf{F}(A_k) = \bigcup_{n=1}^{N} F_n(A_k), k = 0,1,2,\cdots,$$

there exists a finite set (self-similar set) such that $\mathbf{F}(A) = A$. The set is called an attractor.

Iterated function systems, IFS) defined by M. Barnsley can generate several images if we consider pairs consisting of a contraction mapping $F_1, F_2, \cdots, F_N : A \to A$ and its corresponding probability $p_1, p_2, \cdots, p_N$, and apply them repeatedly to an initial set A_0 in the plane. Although Barnsley discusses the concept of iterated functional systems on a Hausdorff space which is an extension of Euclidean space, our discussion is limited below to a general overview in two-dimensional Euclidean space. Since the mathematical proofs have also been omitted, please refer to his book [3]. For example, attractor generation by four contraction mappings starts from initial set $A_0 = [0,1]^2$ and is obtained by

$$A_{k+1} \equiv \mathbf{F}(A_k) = \bigcup_{n=1}^{4} F_n(A_k), k = 0,1,2,\cdots.$$

If we repeat this an infinite number of times, we obtain an attractor usng four contraction mappings. Looking at it another way, we can say that “if a given image is identical to the attractor of four contraction mappings, then this image is represented by the parameters of those four contraction mappings.” This is the concept behind fractal image representation. Therefore, it follows that this image has infinite resolution due to the continuity of the contraction mapping.

The problem here is how to construct an iterative function system that has attractors close to the given image set. Although this can be solved using the Collage Theorem, we will omit the details here; please refer to Barnesley's book [3].

■4　Neural Network Structure：A New Model　FractalNEXT [4]

There has been research applying self-similar fractal structures to neural networks. FractalNet, one of the conventional neural network models used in the field of image recognition, was presented at the 2017 International Conference on Learning Representations (ICLR)—a leading international conference—in a paper titled "FractalNet: Ultra-Deep Neural Networks Without Residuals." This approach offers advantages in terms of constructing ultra-deep networks without using residual connections, such as those found in ResNet. However, it has been confirmed that when this model is implemented, it exhibits recognition performance inferior to that of ResNet due to factors such as the loss of spatial information during image tokenization and the inability to sufficiently preserve frequency information without residual connections. While FractalNet is an effective approach for avoiding residual blocks, achieving higher recognition accuracy requires determining how to construct fractal blocks—that is, a new design for fractal blocks characterized by self-similarity is needed.

The conventional model, FractalNet, overlooks the problem of recognition performance degradation due to the limit of residual connections while maintaining the stability of information propagation over frequency. To address this problem, we considered Multiple Wavelet Patch Partition (MWPP), a method that extracts wavelet patch tokens while preserving spatial information within each patch. In addition, we introduced a selective wavelet connection (SWC) that is aware of frequency information to strengthen the residual connection, thereby increasing the stability of information propagation on frequency and compensating for the information loss caused by the token mixer. We designed a new scalable fractal architecture FractalNEXT, which integrates both convolution and self-attention as a token mixer, based on MWPP and SWC. This is generated by structuring a Fractal block starting from initial mapping function $F_1(\mathbf{x}) = \left(Mix \circ Mix\right)(\mathbf{x}) + SWC(\mathbf{x})$ for arbitrary tensor $\mathbf{x}$, and applying the following equation

$$F_{n+1}\left(\mathbf{x}\right) = \left(F_n \circ F_n\right)(\mathbf{x}) + SWC(\mathbf{x}), \quad n = 1, 2, 3, \cdots.$$

However, regarding modules related to wavelet transform, we will explain the manuscript again.

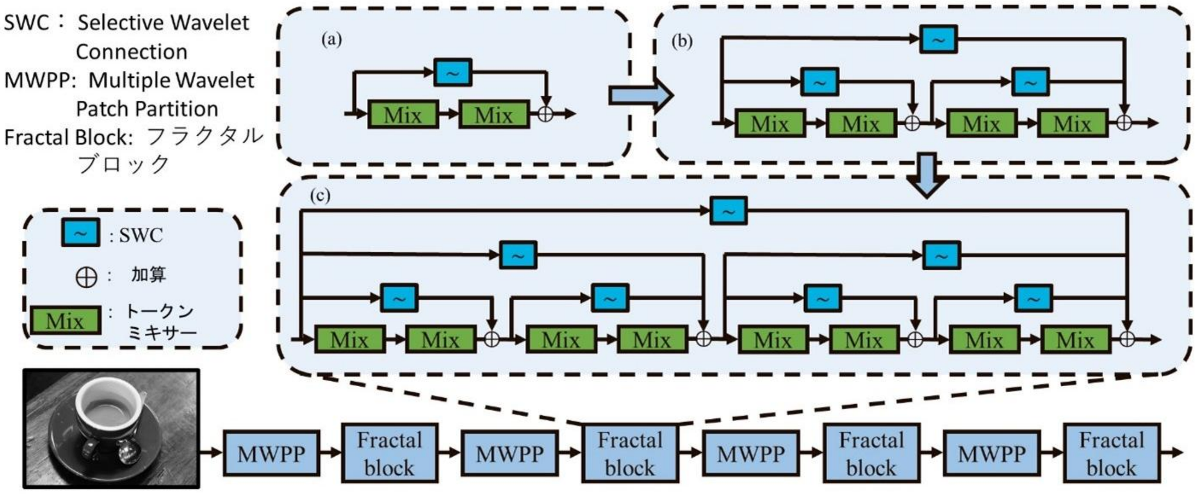

Figure 2　　Overview of FractalNEXT

References

[1] B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman and Co., San Francisco, 1982.

[2] A. Pentland, Fractal-Based Description of Natural Scenes, IEEE Trans. PAMI, vol.PAMI-15, pp.850-863, 1993.

[3] N,F, Barnsley, Fractal everywhere, Academic Press, 1988.

[4] P. Lu, S. Kamata, M. Zhang, FracNeXt: Enhancing visual representation learning in sequence models with fractal wavelets, Neural Networks, June 2026.