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Abstract

Bramoullé et al. (2009) considered a linear social interaction model with network structures under complete information. However, their model is not appropriate for the case where the individual outcome is not completely observed or not precisely predictable by the other individuals in the same group. In this paper, we consider a linear social interaction model with network structures under incomplete information and derive the efficiency bound. The efficiency bound for the model considered in this paper had not been derived before. We also provide a sufficient condition for the existence of the efficiency bound.

1 Introduction

The social interaction model describes cases in which an individual's outcome is affected by others' outcomes. Manski (1993) showed that the social effects are not identified in a linear social interaction model, where each individual is affected by all the others in the group. In contrast, Bramoullé et al. (2009) showed that the social effects are identified through network structures. They considered a linear social interaction model under complete information, where the group members' outcomes are completely observable or precisely predictable by the individuals.

The efficiency bound is the lower bound of the asymptotic variance of an estimator. If the asymptotic variance of an estimator achieves the efficiency bound, the estimator results in the most precise estimation of a parameter. There are several results on the efficiency bound for the social interaction models. Aradillas-López (2019, 2021) derived the efficiency bound for the social interaction model, where the individual outcome is affected by the outcomes of all the other members in the group. Debarsy et al. (2024) derived the efficiency bound for the social interaction model, where individuals interact through a network under complete information. However, the efficiency bound for the social interaction model with network structures under incomplete information had not been derived before.

The purpose of this paper is to derive the efficiency bound for the social interaction model with network structures under incomplete information, using the computational method proposed by Severini and Tripathi (2001), which is based on the Riesz representer. Under

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incomplete information, individuals observe the expectations of others' outcomes rather than the outcomes themselves. This setting is more suitable for cases where the individual outcomes are not completely observed or not precisely predictable. We calculate the efficiency bound using the approach proposed by Severini and Tripathi (2001), which is simpler than the existing methods of van der Vaart (1991) and Newey (1990).

We provide a sufficient condition for the existence of the efficiency bound. Without any conditions guaranteeing the invertibility of the variance-covariance matrix, the efficiency bound may no longer exist. We reveal that the identification condition provided by Bramoullé et al. (2009) is sufficient to ensure the existence of the efficiency bound. Since their identification condition is easily verified based on the network structures, the sufficient condition for the existence of the efficiency bound is also easily verified.

The remainder of this paper is organized as follows. Section 2 presents the linear social interaction model with network structures under incomplete information. In Section 3, we derive the efficiency bound for the model presented in Section 2. Section 4 provides a sufficient condition for the existence of the efficiency bound. We prove that the identification condition provided by Bramoullé et al. (2009) is sufficient to guarantee the existence of the efficiency bound. Section 5 concludes the paper.

2 The Model

Suppose that there are L independent and identical networks. Each network $\ell = 1, \dots L$ has n individuals. The peer group $P_i \subset \{1, \dots, n\}$ is a set of n_i individuals who affect individual i. The adjacency matrix G is an $n \times n$ matrix whose elements are given by

$$(G)_{ij} = \begin{cases} \frac{1}{n_i} & j \in P_i \\ 0 & \text{otherwise.} \end{cases}$$

We set $(G)_{ij} = 0$ if $P_i = \emptyset$. We assume throughout this paper that the network structure G is exogenously given and non-stochastic.

The individual-level model is given by

$$y_{\ell i} = \alpha + \beta \sum_{j \in P_i} (G)_{ij} \mathbb{E}[y_{\ell j} | x_\ell] + \gamma x_{\ell i} + \delta \sum_{j \in P_i} (G)_{ij} x_{\ell j} + \varepsilon_{\ell i}, \quad i = 1, \dots, n, \ \ell = 1, \dots, L, \tag{2.1}$$

where $y_{\ell i} \in \mathbb{R}$ is an outcome variable for individual i in network ℓ, $x_{\ell i} \in \mathbb{R}$ is a scalar stochastic covariate for individual i in network ℓ, $x_\ell = (x_{\ell 1}, \dots, x_{\ell n})^\top$ is a covariate vector in network ℓ. The parameter of interest is $(\beta, \gamma, \delta) \in \mathbb{R}^3$ satisfying $|\beta| < 1$.

By stacking the individual-level model (2.1) for all individuals in the network, we get

$$y_\ell = \alpha \iota + \beta G \mathbb{E}[y_\ell | x_\ell] + \gamma x_\ell + \delta G x_\ell + \varepsilon_\ell, \tag{2.2}$$

where

$$y_\ell = \begin{bmatrix} y_{\ell 1} \\ \vdots \\ y_{\ell n} \end{bmatrix}, \ \iota = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \ x_\ell = \begin{bmatrix} x_{\ell 1} \\ \vdots \\ x_{\ell n} \end{bmatrix}, \ \varepsilon_\ell = \begin{bmatrix} \varepsilon_{\ell 1} \\ \vdots \\ \varepsilon_{\ell n} \end{bmatrix}.$$

We assume that the error term satisfies $x_\ell \perp\!\!\!\perp \varepsilon_\ell$ and $\mathbb{E}[\varepsilon_\ell] = 0$. By taking the conditional expectation of (2.2) with respect to x_ℓ, we have

$$\mathbb{E}[y_\ell|x_\ell] = \alpha\iota + \beta G\mathbb{E}[y_\ell|x_\ell] + \gamma x_\ell + \delta G x_\ell. \tag{2.3}$$

By Proposition A.2 in the Appendix, the operator norm $\|G\|^*$ satisfies $\|G\|^* \leq 1$. Since $\|G\|^* \leq 1$ and $|\beta| < 1$ hold, $(I - \beta G)^{-1}$ exists. By solving for $\mathbb{E}[y_\ell|x_\ell]$ in (2.3), we have

$$\mathbb{E}[y_\ell|x_\ell] = (I - \beta G)^{-1}(\alpha\iota + \gamma x_\ell + \delta G x_\ell). \tag{2.4}$$

By substituting (2.4) into (2.2), we obtain the reduced-form equation given by

$$y_\ell = (I - \beta G)^{-1}(\alpha\iota + \gamma x_\ell + \delta G x_\ell) + \varepsilon_\ell.$$

Suppose that v_0^2 is the true density generating x_ℓ, and φ_0^2 is the true density generating ε_ℓ. Suppose also that $p(y_\ell, x_\ell; \lambda_0)$, $\lambda_0 = (\alpha_0, \beta_0, \gamma_0, \delta_0, v_0, \varphi_0)$ is the true density generating (y_ℓ, x_ℓ).

Assumption 1. (1) The support of the distribution of x_ℓ is not contained in any proper linear subspaces of $\mathbb{R}^n$.

(2) We assume

$$\int \cdots \int \left| \frac{\partial}{\partial \varepsilon_{\ell i}} \left[\frac{1}{2} \varphi_0(\varepsilon_\ell)^2 \right] \right| d\varepsilon_{\ell 1} \cdots d\varepsilon_{\ell n} < \infty \tag{2.5}$$

and

$$\lim_{\|\varepsilon_\ell\| \to \infty} \varphi_0(\varepsilon_\ell) = 0, \tag{2.6}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

3 Efficiency Bound

In this section, we derive the efficiency bound of $\theta_0 = (\beta_0, \gamma_0, \delta_0) \in \mathbb{R}^3$ based on the computational method proposed by Severini and Tripathi (2001).

Theorem 3.1. Let X_ℓ be the $n \times 3$ matrix given by

$$X_\ell = \begin{bmatrix} (I - \beta_0 G)^{-1} G (I - \beta_0 G)^{-1}(\alpha_0 \iota + \gamma_0 x_\ell + \delta_0 G x_\ell) \\ (I - \beta_0 G)^{-1} x_\ell \\ (I - \beta_0 G)^{-1} G x_\ell \end{bmatrix}^\top,$$

let $\nabla \varphi_0(\varepsilon_\ell)$ be the gradient vector of φ_0 given by

$$\nabla \varphi_0(\varepsilon_\ell) = \begin{bmatrix} \frac{\partial \varphi_0}{\partial \varepsilon_{\ell 1}} \\ \vdots \\ \frac{\partial \varphi_0}{\partial \varepsilon_{\ell n}} \end{bmatrix},$$

and let Σ be the 3×3 matrix given by

$$\Sigma = 4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}\mathrm{Var}(X_\ell^\top \nabla \varphi_0(\varepsilon_\ell))\right].$$

If Σ is a non-singular matrix, the efficiency bound for the parameter of interest $\theta_0 = (\beta_0, \gamma_0, \delta_0)^\top$ is given by

$$\text{l.b.}(\theta_0) = \Sigma^{-1}.$$

Proof. The parameter set is given by $\Lambda = A \times \Theta \times V \times \Phi$, where

$$A = \mathbb{R}$$

$$\Theta = (-1, 1) \times \mathbb{R} \times \mathbb{R}$$

$$V = \left\{ v \in L_2(\mathbb{R}^n) : v(x_\ell) > 0, \int v(x_\ell)^2 dx_\ell = 1 \right\}$$

$$\Phi = \left\{ \varphi \in L_2(\mathbb{R}^n) : \varphi(\varepsilon_\ell) > 0, \ \int \varphi(\varepsilon_\ell)^2 d\varepsilon_\ell = 1, \ \int \varepsilon_\ell \varphi(\varepsilon_\ell)^2 d\varepsilon_\ell = 0 \right\}.$$

For sufficiently small $\eta > 0$, let

$$(-\eta, \eta) \ni t \mapsto \lambda_t := (\alpha_t, \beta_t, \gamma_t, \delta_t, v_t, \varphi_t) \in \Lambda$$

be a curve passing through $\lambda_0 = (\alpha_0, \beta_0, \gamma_0, \delta_0, v_0, \varphi_0)$ at $t = 0$. We derive the efficiency bound of $\rho(\lambda_0) = c^\top \theta_0$ for any constant vector $c \in \mathbb{R}^3$.

The tangent set is given by $T(\Lambda, \lambda_0) = T(A, \alpha_0) \times T(\Theta, \theta_0) \times T(V, v_0) \times T(\Phi, \varphi_0)$, where

$$T(A, \alpha_0) = \mathbb{R}$$

$$T(\Theta, \theta_0) = \mathbb{R}^3$$

$$T(V, v_0) = \left\{ \dot{v} \in L_2(\mathbb{R}^n) : \int \dot{v}(x_\ell) v_0(x_\ell) dx_\ell = 0 \right\}$$

$$T(\Phi, \varphi) = \left\{ \dot{\varphi} \in L_2(\mathbb{R}^n) : \int \dot{\varphi}(\varepsilon_\ell)\varphi_0(\varepsilon_\ell) d\varepsilon_\ell = 0, \ \int \varepsilon_\ell \dot{\varphi}(\varepsilon_\ell)\varphi_0(\varepsilon_\ell) d\varepsilon_\ell = 0 \right\}.$$

The tangent space is

$$\overline{T(\Lambda, \lambda_0)} = T(A, \alpha_0) \times T(\Theta, \theta_0) \times \overline{T(V, v_0)} \times \overline{T(\Phi, \varphi_0)},$$

where the closure of $\overline{T(V, v_0)}$ and $\overline{T(\Phi, \varphi_0)}$ is taken with respect to the $L_2(\mathbb{R}^n)$-norm.

The density of (y_ℓ, x_ℓ) is given by

$$p(y_\ell, x_\ell; \lambda) = \varphi(y_\ell - (I - \beta G)^{-1}(\alpha\iota + \gamma x_\ell + \delta G x_\ell))^2 v(x_\ell)^2.$$

By differentiating $(I - \beta_t G)^{-1}$ with respect to t using Proposition A.1 in the Appendix, we get the score function $U = \left(\frac{d}{dt}\right)_{t=0} \log p(y_\ell, x_\ell; \lambda_t)$ as

$$\begin{aligned} U &= \left(\frac{d}{dt}\right)_{t=0} \left\{ 2\log\varphi_t(y_\ell - (I - \beta_t G)^{-1}(\alpha_t\iota + \gamma_t x_\ell + \delta_t G x_\ell)) + 2\log v_t(x_\ell) \right\} \\ &= \frac{2\{\dot{\varphi}(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I - \beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})\}}{\varphi_0(\varepsilon_\ell)} + \frac{2\dot{v}(x_\ell)}{v_0(x_\ell)}. \end{aligned}$$

The Fisher information for estimating t is

$$
\begin{aligned}
i_{\mathrm{F}}(t) &= \mathbb{E}\left[\left\{\frac{2\{\dot{\varphi}(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})\}}{\varphi_0(\varepsilon_\ell)} + \frac{2\dot{v}(x_\ell)}{v_0(x_\ell)}\right\}^2\right] \\
&= 4\mathbb{E}\left[\frac{\{\dot{\varphi}(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})\}^2}{\varphi_0(\varepsilon_\ell)^2}\right] + 4\mathbb{E}\left[\frac{\dot{v}(x_\ell)^2}{v_0(x_\ell)^2}\right] \\
&\quad + 8\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})}{\varphi_0(\varepsilon_\ell)}\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right].
\end{aligned}
$$

Note that we have

$$
\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell)}{\varphi_0(\varepsilon_\ell)}\right] = \int \frac{\dot{\varphi}(\varepsilon_\ell)}{\varphi_0(\varepsilon_\ell)}\varphi_0(\varepsilon_\ell)^2 d\varepsilon_\ell = \int \dot{\varphi}(\varepsilon_\ell)\varphi_0(\varepsilon_\ell) d\varepsilon_\ell = 0 \tag{3.1}
$$

and

$$
\mathbb{E}\left[\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right] = \int \frac{\dot{v}(x_\ell)}{v_0(x_\ell)} v_0(x_\ell)^2 dx_\ell = \int \dot{v}(x_\ell) v_0(x_\ell) dx_\ell = 0. \tag{3.2}
$$

Each element of

$$
\mathbb{E}\left[\frac{\nabla\varphi_0(\varepsilon_\ell)^\top}{\varphi_0(\varepsilon_\ell)}\right] = \mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)}\begin{bmatrix}\frac{\partial\varphi_0}{\partial\varepsilon_{\ell 1}} & \cdots & \frac{\partial\varphi_0}{\partial\varepsilon_{\ell n}}\end{bmatrix}\right]
$$

is calculated as

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)}\frac{\partial\varphi_0(\varepsilon_\ell)}{\partial\varepsilon_{\ell i}}\right] &= \int_{\varepsilon_{\ell 1}=-\infty}^{\infty}\cdots\int_{\varepsilon_{\ell i}=-\infty}^{\infty}\cdots\int_{\varepsilon_{\ell n}=-\infty}^{\infty}\frac{\partial}{\partial\varepsilon_{\ell i}}\left[\frac{1}{2}\varphi_0(\varepsilon_{\ell 1},\cdots,\varepsilon_{\ell n})^2\right] d\varepsilon_{\ell 1}\cdots d\varepsilon_{\ell n} \\
&= \int_{\varepsilon_{\ell i}=-\infty}^{\infty}\left\{\left[\frac{1}{2}\varphi_0(\varepsilon_\ell)^2\right]_{\varepsilon_{\ell i}=-\infty}^{\infty}\right\} d\varepsilon_{-\ell i} \\
&= 0,
\end{aligned} \tag{3.3}
$$

where $\int_{\varepsilon_{\ell i}=-\infty}^{\infty}\left\{\left[\frac{1}{2}\varphi_0(\varepsilon_\ell)^2\right]_{\varepsilon_{\ell i}=-\infty}^{\infty}\right\} d\varepsilon_{-\ell i}$ denotes the integral of $(\varepsilon_{\ell 1},\ldots,\varepsilon_{\ell n})$ except for $\varepsilon_{\ell i}$, the second equality is due to (2.5) in Assumption 1 (2), and the last equality comes from (2.6) in Assumption 1 (2). Using (3.1), (3.2), and (3.3), we can calculate the cross term of the Fisher information as

$$
\begin{aligned}
& 8\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})}{\varphi_0(\varepsilon_\ell)}\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right] \\
&= 8\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell)}{\varphi_0(\varepsilon_\ell)}\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right] - 8\mathbb{E}\left[\frac{\nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})}{\varphi_0(\varepsilon_\ell)}\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right] \\
&= 8\underbrace{\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell)}{\varphi_0(\varepsilon_\ell)}\right]}_{=0}\mathbb{E}\left[\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right] - 8\underbrace{\mathbb{E}\left[\frac{\nabla\varphi_0(\varepsilon_\ell)^\top}{\varphi_0(\varepsilon_\ell)}\right]}_{=0}\mathbb{E}\left[((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})\frac{\dot{v}(x_\ell)}{v_0(x_\ell)}\right] \\
&= 0.
\end{aligned}
$$

Thus, the Fisher information is given by

$$i_{\mathrm{F}}(t) = 4\mathbb{E}\left[\frac{\{\dot{\varphi}(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}\iota + X_\ell\dot{\theta})\}^2}{\varphi_0(\varepsilon_\ell)^2}\right] + 4\mathbb{E}\left[\frac{\dot{v}(x_\ell)^2}{v_0(x_\ell)^2}\right].$$

For $\dot{\tau}_1 = (\dot{\alpha}_1, \dot{\theta}_1, \dot{v}_1, \dot{\varphi}_1),\ \dot{\tau}_2 = (\dot{\alpha}_2, \dot{\theta}_2, \dot{v}_2, \dot{\varphi}_2) \in \overline{T(\Lambda, \lambda_0)}$, the Fisher information metric on the tangent space $\overline{T(\Lambda, \lambda_0)}$ is given by

$$\begin{aligned}
&\langle \dot{\tau}_1, \dot{\tau}_2\rangle_{\mathrm{F}} \\
&= 4\mathbb{E}\left[\frac{\{\dot{\varphi}_1(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}_1\iota + X_\ell\dot{\theta}_1)\}\{\dot{\varphi}_2(\varepsilon_\ell) - \nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}_2\iota + X_\ell\dot{\theta}_2)\}}{\varphi_0(\varepsilon_\ell)^2}\right] \\
&\quad + 4\mathbb{E}\left[\frac{\dot{v}_1(x_\ell)\dot{v}_2(x_\ell)}{v_0(x_\ell)^2}\right].
\end{aligned}$$

We show that the Riesz representer $\dot{\tau}^* = (\dot{\alpha}^*, \dot{\theta}^*, \dot{v}^*, \dot{\varphi}^*)^\top$ of the gradient of $\rho(\lambda_0)$ is given by

$$\begin{bmatrix}\dot{\alpha}^* \\ \dot{\theta}^* \\ \dot{v}^* \\ \dot{\varphi}^*\end{bmatrix} = \begin{bmatrix} -\frac{1}{n}\iota^\top(I-\beta_0 G)\mathbb{E}[X_\ell]\dot{\theta}^* \\ \left(4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}\mathrm{Var}(X_\ell^\top\nabla\varphi_0(\varepsilon_\ell))\right]\right)^{-1} c \\ 0 \\ 0\end{bmatrix}. \tag{3.4}$$

That is,

$$c^\top\dot{\theta} = \langle \dot{\tau}^*, \dot{\tau}\rangle_{\mathrm{F}}. \tag{3.5}$$

Since we have $\dot{\alpha}^*\iota = -(I-\beta_0 G)\mathbb{E}[X_\ell]\dot{\theta}^*$, plugging (3.4) into the right hand side of (3.5) yields

$$\begin{aligned}
\langle \dot{\tau}^*, \dot{\tau}\rangle_{\mathrm{F}} &= -4\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell)}{\varphi_0(\varepsilon_\ell)^2}\nabla\varphi_0(\varepsilon_\ell)^\top(X_\ell - \mathbb{E}[X_\ell])\dot{\theta}^*\right] \\
&\quad + 4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}(\dot{\theta}^*)^\top(\nabla\varphi_0(\varepsilon_\ell)^\top(X_\ell - \mathbb{E}[X_\ell]))^\top\nabla\varphi_0(\varepsilon_\ell)^\top(I-\beta_0 G)^{-1}\dot{\alpha}\iota\right] \\
&\quad + 4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}(\dot{\theta}^*)^\top(\nabla\varphi_0(\varepsilon_\ell)^\top(X_\ell - \mathbb{E}[X_\ell]))^\top\nabla\varphi_0(\varepsilon_\ell)^\top X_\ell\right]\dot{\theta} \\
&= -4\mathbb{E}\left[\frac{\dot{\varphi}(\varepsilon_\ell)}{\varphi_0(\varepsilon_\ell)^2}\nabla\varphi_0(\varepsilon_\ell)^\top\underbrace{\mathbb{E}[X_\ell - \mathbb{E}[X_\ell]|\varepsilon_\ell]}_{=0}\dot{\theta}^*\right] \\
&\quad + 4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}(\dot{\theta}^*)^\top(\nabla\varphi_0(\varepsilon_\ell)^\top\underbrace{\mathbb{E}[X_\ell - \mathbb{E}[X_\ell]|\varepsilon_\ell]}_{=0})^\top\nabla\varphi_0(\varepsilon_\ell)^\top(I-\beta_0 G)^{-1}\dot{\alpha}\iota\right] \\
&\quad + 4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}(\dot{\theta}^*)^\top\mathrm{Var}(X_\ell^\top\nabla\varphi_0(\varepsilon_\ell))\right]\dot{\theta} \\
&= (\dot{\theta}^*)^\top\left(4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}\mathrm{Var}(X_\ell^\top\nabla\varphi_0(\varepsilon_\ell))\right]\right)\dot{\theta} \\
&= c^\top\dot{\theta},
\end{aligned}$$

which implies (3.5).

Therefore, the efficiency bound of $\rho(\lambda_0)$ is given by

$$\begin{aligned}
\text{l.b.}(\rho(\lambda_0)) &= \|(\dot{\alpha}^*, \dot{\theta}^*, \dot{\upsilon}^*, \dot{\varphi}^*)\|_{\mathrm{F}}^2 \\
&= 4\mathbb{E}\left[\frac{\{\nabla\varphi_0(\varepsilon_\ell)^\top((I-\beta_0 G)^{-1}\dot{\alpha}^*\iota + X_\ell\dot{\theta}^*)\}^2}{\varphi_0(\varepsilon_\ell)^2}\right] \\
&= 4\mathbb{E}\left[\frac{\{\nabla\varphi_0(\varepsilon_\ell)^\top(-\mathbb{E}[X_\ell]\dot{\theta}^* + X_\ell\dot{\theta}^*)\}^2}{\varphi_0(\varepsilon_\ell)^2}\right] \\
&= (\dot{\theta}^*)^\top\left(4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}\mathrm{Var}(X_\ell^\top\nabla\varphi_0(\varepsilon_\ell))\right]\right)\dot{\theta}^* \\
&= c^\top\left(4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}\mathrm{Var}(X_\ell^\top\nabla\varphi_0(\varepsilon_\ell))\right]\right)^{-1} c.
\end{aligned}$$

Since $c \in \mathbb{R}^3$ is arbitrary, the efficiency bound for θ_0, denoted by l.b.(θ_0), is given by

$$\text{l.b.}(\theta_0) = \left(4\mathbb{E}\left[\frac{1}{\varphi_0(\varepsilon_\ell)^2}\mathrm{Var}(X_\ell^\top\nabla\varphi_0(\varepsilon_\ell))\right]\right)^{-1} = \Sigma^{-1},$$

which completes the proof. ■

4 A Sufficient Condition for the Existence of the Efficiency Bound

In this section, we show that the efficiency bound Σ^{-1} exists under the identification condition provided by Bramoullé et al. (2009). Their identification condition is stated in the following proposition.

Proposition 4.1 (Bramoullé et al., 2009)**.** Suppose that $\gamma\beta + \delta \neq 0$ holds. If I, G, and G^2 are linearly independent, the structural parameters $\theta = (\alpha, \beta, \gamma, \delta)$ are identified.

In the following proposition, we show that the identification condition ensures that Σ is non-singular.

Proposition 4.2. If $\gamma_0\beta_0 + \delta_0 \neq 0$ holds and I, G, and G^2 are linearly independent, the matrix Σ is invertible.

Proof. We show that if

$$a^\top \mathrm{Var}(X_\ell^\top \nabla\varphi_0(\varepsilon_\ell))a = 0 \tag{4.1}$$

holds for $a = (a_1, a_2, a_3)^\top \in \mathbb{R}^3$, $a = (0,0,0)^\top$ is satisfied. By (4.1),

$$\begin{aligned}
&a^\top \mathrm{Var}(X_\ell^\top \nabla\varphi_0(\varepsilon_\ell))a \\
&= \mathrm{Var}((X_\ell a)^\top \nabla\varphi_0(\varepsilon_\ell)) \\
&= \mathrm{Var}\Big(\Big\{a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}(\alpha_0\iota + \gamma_0 x_\ell + \delta_0 G x_\ell) \\
&\qquad + a_2(I-\beta_0 G)^{-1}x_\ell + a_3(I-\beta_0 G)^{-1}Gx_\ell\Big\}^\top \nabla\varphi_0(\varepsilon_\ell)\Big) \\
&= \mathrm{Var}\Big(\Big\{a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}\alpha_0\iota + [a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}(\gamma_0 I + \delta_0 G) \\
&\qquad + a_2(I-\beta_0 G)^{-1} + a_3(I-\beta_0 G)^{-1}G]x_\ell\Big\}^\top \nabla\varphi_0(\varepsilon_\ell)\Big) \\
&= 0.
\end{aligned} \tag{4.2}$$

For the variance in (4.2) to be zero, we need

$$\begin{aligned}
a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}\alpha_0\iota + [a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}(\gamma_0 I + \delta_0 G) \\
+ a_2(I-\beta_0 G)^{-1} + a_3(I-\beta_0 G)^{-1}G]x_\ell = 0.
\end{aligned}$$

Since ι and x_ℓ are linearly independent by Assumption 1 (1), we obtain

$$\begin{cases}
a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}\alpha_0 = O \\
a_1(I-\beta_0 G)^{-1}G(I-\beta_0 G)^{-1}(\gamma_0 I + \delta_0 G) + a_2(I-\beta_0 G)^{-1} + a_3(I-\beta_0 G)^{-1}G = O.
\end{cases} \tag{4.3}$$

By multiplying $I - \beta_0 G$ on both sides of the second equation in (4.3), we have

$$a_1 G(I-\beta_0 G)^{-1}(\gamma_0 I + \delta_0 G) + a_2 I + a_3 G = O. \tag{4.4}$$

By Proposition A.3 in the Appendix, $G(I-\beta_0 G)^{-1} = (I-\beta_0 G)^{-1}G$ holds. We can rewrite (4.4) as

$$a_1(I-\beta_0 G)^{-1}G(\gamma_0 I + \delta_0 G) + a_2 I + a_3 G = O. \tag{4.5}$$

By multiplying $(I - \beta_0 G)$ on both sides of (4.5), we get

$$a_2 I + (a_1\gamma_0 - a_2\beta_0 + a_3)G + (a_1\delta_0 - a_3\beta_0)G^2 = O.$$

Since I, G, and G^2 are linearly independent, we have

$$\begin{cases}
a_2 = 0 \\
a_1\gamma_0 - a_2\beta_0 + a_3 = 0 \\
a_1\delta_0 - a_3\beta_0 = 0.
\end{cases} \tag{4.6}$$

By plugging $a_2 = 0$ into the second equation of (4.6), we get

$$a_1\gamma_0 + a_3 = 0. \tag{4.7}$$

Summing (4.7)$\times\beta_0$ and the third equation of (4.6) yields

$$a_1(\gamma_0\beta_0 + \delta_0) = 0.$$

Since $\gamma_0\beta_0 + \delta_0 \neq 0$, we have $a_1 = 0$. By plugging $a_1 = 0$ into (4.7), we obtain $a_3 = 0$.

Therefore, if $a \neq (0,0,0)^\top$ holds,

$$a^\top \mathrm{Var}(X_\ell^\top \nabla\varphi_0(\varepsilon_\ell))a \neq 0$$

is satisfied, which implies that $\mathrm{Var}(X_\ell^\top \nabla\varphi_0(\varepsilon_\ell))$ is positive definite. Therefore, Σ is positive definite, which implies the non-singularity of Σ. ■

5 Conclusion

In this paper, we derived the efficiency bound for the social interaction model with network structures under incomplete information. Bramoullé et al. (2009) considered a linear social interaction model under complete information. However, their model is not appropriate for the case where the outcomes of others are not completely observed or not precisely predictable by the individuals. In our paper, we considered the linear social interaction model with network structures under incomplete information and derived the efficiency bound. The efficiency bound for the model considered in this paper had not been derived before. Moreover, we provided a sufficient condition for the existence of the efficiency bound. We showed that the identification condition provided by Bramoullé et al. (2009) is sufficient for the existence of the efficiency bound.

Appendix

Proposition A.1. The differentiation of $A(t)^{-1}$ of a matrix $A(t)$ with respect to t is given by

$$\frac{d}{dt}A(t)^{-1} = -A(t)^{-1}\left(\frac{d}{dt}A(t)\right)A(t)^{-1}.$$

Proof. By differentiating both sides of $A(t)A(t)^{-1} = I$ with respect to t, we get

$$\left(\frac{d}{dt}A(t)\right)A(t)^{-1} + A(t)\left(\frac{d}{dt}A(t)^{-1}\right) = O.$$

Thus, we obtain

$$\frac{d}{dt}A(t)^{-1} = -A(t)^{-1}\left(\frac{d}{dt}A(t)\right)A(t)^{-1},$$

which completes the proof. ■

Proposition A.2. For an adjacency matrix G, $\|G\|^* \leq 1$ holds, where $\|\cdot\|^*$ is an operator norm.

Proof. By the definition of the operator norm,

$$\|G^\top\|^* = \left[\max_{u\in\mathbb{R}^n} \sqrt{(G^\top u)^\top G^\top u} \quad \text{s.t.} \quad \sqrt{u^\top u} = 1\right]. \tag{A.1}$$

The Lagrangian function of (A.1) is

$$\mathcal{L} = u^\top GG^\top u + \lambda(1 - u^\top u),$$

where λ is the Lagrange multiplier. The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial u} = 2GG^\top u - 2\lambda u = 0,$$

which implies

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{G})_{ij}u_i u_j = \lambda,$$

where $\tilde{G} \coloneqq GG^\top$. Note that λ is an eigenvalue of $\tilde{G}$. Since each element of G is less than or equal to 1 and each row sum of G is 1 or 0,

$$|(\tilde{G})_{ij}| = \left|\sum_{k=1}^{n}(G)_{ik}(G)_{jk}\right| \leq \sum_{k=1}^{n}|(G)_{ik}||(G)_{jk}| \leq \sum_{k=1}^{n}|(G)_{ik}| \leq 1.$$

By the Cauchy-Schwarz inequality,

$$|\lambda| = \left|\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{G})_{ij}u_i u_j\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{n}|(\tilde{G})_{ij}||u_i||u_j| \leq \left(\sum_{i=1}^{n}|u_i|\right)^2 \leq \sum_{i=1}^{n}u_i^2 = 1.$$

Since $\|G^\top\|^* = \|G\|^*$ and $\|G^\top\|^* = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of $\tilde{G}$,

$$\|G\|^* = \sqrt{\lambda_{\max}} \leq 1$$

is shown. ■

Proposition A.3. For an adjacency matrix G,

$$G(I - \beta_0 G)^{-1} = (I - \beta_0 G)^{-1}G \tag{A.2}$$

holds.

Proof. In the case of $\beta_0 = 0$, (A.2) is trivially satisfied. In the case of $\beta_0 \neq 0$,

$$\begin{aligned}
(I - \beta_0 G)^{-1}G &= -\frac{1}{\beta_0}(I - \beta_0 G)^{-1}(-\beta_0 G) \\
&= -\frac{1}{\beta_0}(I - \beta_0 G)^{-1}((I - \beta_0 G) - I) \\
&= -\frac{1}{\beta_0}(I - (I - \beta_0 G)^{-1}) \\
&= -\frac{1}{\beta_0}((I - \beta_0 G) - I)(I - \beta_0 G)^{-1} \\
&= G(I - \beta_0 G)^{-1}
\end{aligned}$$

holds, implying (A.2). ■

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