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Identification of social effects through variations in network structures

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Abstract

Bramoullé et al. (2009) provided identification conditions for linear social interaction models through network structures. Despite the importance of their results, the authors omitted detailed mathematical discussions. Moreover, they consider cases where many identical networks are observed simultaneously within the same dataset. In reality, multiple networks with different structures, such as classrooms or villages, are repeatedly observed within the same dataset. The purpose of this paper is to fill in the mathematical gaps in their arguments and to establish identification conditions for networks with different structures. In addition, we find the smallest network size as a necessary condition for identifying social effects. We also discuss the identification conditions of network models with a fixed network effect.

Keywords: identification; network model; social interactions; network size.
Journal of Economic Literature Classification: C31, D85
2020 Mathematics Subject Classification: 62P20, 91B72

1 Introduction

The social interaction model describes the case where the individual outcome is affected by the outcomes of other individuals. Manski (1993) showed that social effects are not identified in a linear social interaction model, where each individual is affected by all other individuals in the same group[1]. A parameter is said to be identified if it is uniquely recovered from observed data. If social effects are not identified, two distinct parameters are consistent with a model, making it impossible to estimate the model. Several approaches are proposed to address this identification problem. Brock and Durlauf (2001, 2007) exploited non-linearities emerging from discrete choice models to identify social effects. Lee (2007) showed that social effects are identified through variations in group sizes in a linear social interaction model.

Bramoullé et al. (2009) provided crucial identification results through network structures. First, they showed that the social effects are identified in linear models. The linear models

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[1]This model has many empirical applications including criminal activity (Glaeser et al., 1996), school achievement (Sacerdote, 2001), and smoking behavior (Soetevent and Kooreman, 2007).

are considered in most papers on social interactions since they are naturally related to the standard simultaneous linear model (Moffitt, 2001). Second, they analyzed the identification problem under a more realistic interaction pattern. Most papers on social interactions assumed a group interaction such that individuals are affected by all others in their group (Manski, 1993; Moffitt, 2001; Lee, 2007). This type of interaction pattern is not likely to represent most forms of relationships between individuals. In contrast, Bramoullé et al. (2009) considered a more general interaction pattern through network structures. Due to these contributions, their identification results are extended in many directions[2]: the network model with multivariate choices (Cohen-Cole et al., 2018); the endogenous network model (Goldsmith-Pinkham and Imbens, 2013; Hsieh and Lee, 2016; Johnsson and Moon, 2021; Auerbach, 2022; Jochmans, 2023); and the panel data network model (Comola and Prina, 2021).

The purpose of this paper is to fill in the gaps in the mathematical details of Bramoullé et al. (2009) and to provide identification conditions for networks with different structures. While the identification conditions provided by Bramoullé et al. (2009) are essential in econometrics, the paper skips details of mathematical discussions. We rigorously verify the validity of their claims by filling in the omitted discussions. Bramoullé et al. (2009) considered the case where many identical network structures are observed simultaneously within the same dataset. In reality, multiple networks with different structures, such as classrooms or villages, are observed simultaneously within the same dataset. Although they applied their identification results to a single large network constructed by stacking heterogeneous networks, they implicitly assumed that a single large network is repeatedly observed, which is not consistent with network observations. In this paper, we provide identification conditions for each different network without stacking different network structures. While our identification conditions are mathematically equivalent to those in Bramoullé et al. (2009), our results are consistent with network observations. We present examples of network structures satisfying identification conditions. We also provide an identification condition of a network model with a network fixed effect when we observe different networks simultaneously within the same dataset.

We reveal the smallest network size as a necessary condition for identifying social effects. For example, consider a network model with a network fixed effect. If the identification condition is satisfied under identical network observations, the network must contain at least four individuals. We verify through direct calculation that networks with two or three individuals do not satisfy the identification condition. Similarly, if the identification condition is satisfied under heterogeneous network structures, the network size is at least three. If the identification conditions are satisfied for network models with or without a network fixed effect, the smallest network size is smaller than that for identical network structures.

The remainder of this paper is organized as follows. In Section 2, we provide the identification condition of the network model when we observe different structures of the networks simultaneously within the same dataset. Identification is obtained from the variations of network structures, which create exogenous variations of the reduced-form parameters. Section 3 addresses the correlated effects in the form of the network fixed effect. Section 4 concludes the paper.

[2]The network model has a broad range of empirical applications including network of coauthorships among economists (Goyal et al., 2006), network of check-out workers (Mas and Moretti, 2009), and network of farmers for adopting new technologies (Conley and Udry, 2010).

2 Identification of the basic network model

Suppose that we observe L independent networks. Each network $\ell = 1, \dots, L$ has n_ℓ individuals. A peer group $P_{\ell i} \subset \{1, \dots, n_\ell\}$ of individual i in network ℓ is a set of $n_{\ell i}$ individuals who affect individual i. Individual i is excluded from his or her own peer group: that is, $i \notin P_{\ell i}$. An individual is said to be isolated if his or her peer group is empty. The adjacency matrix is an $n_\ell \times n_\ell$ matrix G_ℓ whose elements are given by

$$(G_\ell)_{ij} = \begin{cases} \frac{1}{n_{\ell i}} & j \in P_{\ell i} \\ 0 & j \notin P_{\ell i}, \end{cases}$$

where $P_{\ell i} \neq \emptyset$. We set $(G_\ell)_{ij} = 0$ for isolated individual i. The adjacency matrix of the network illustrated in Figure 1 is given below.

$$G_m = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The peer group of individual 1 is $P_{m1} = \{2, 3, 4\}$, while the peer groups of others are $P_{m2} = P_{m3} = P_{m4} = \{1\}$. In the following part of the paper, we assume that the network structure is exogenously given.

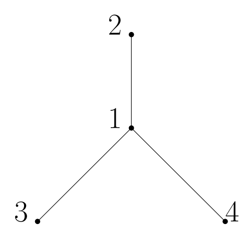

Figure 1: Network m

The individual-level network model is given by

$$y_{\ell i} = \alpha + \beta \left(\sum_{j \in P_{\ell i}} (G_\ell)_{ij} y_{\ell j} \right) + x_{\ell i}^\top \gamma + \left(\sum_{j \in P_{\ell i}} (G_\ell)_{ij} x_{\ell j} \right)^\top \delta + \varepsilon_{\ell i},\ \ell = 1, \dots, L,\ i = 1, \dots, n_\ell,$$

where $y_{\ell i} \in \mathbb{R}$ is an outcome variable for individual i in network ℓ, $x_{\ell i} = (x_{\ell i1}, \dots, x_{\ell iK})^\top \in \mathbb{R}^K$ is a covariate vector for individual i in network ℓ, and $\varepsilon_{\ell i}$ is an error term satisfying $\mathbb{E}[\varepsilon_{\ell i} | x_{\ell 1}, \dots, x_{\ell n_\ell}] = 0$. Let $(\alpha, \beta, \gamma, \delta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^K \times \mathbb{R}^K$ be parameters such that $|\beta| < 1$. The network-level model is given by

$$y_\ell = \alpha \iota + \beta G_\ell y_\ell + X_\ell \gamma + G_\ell X_\ell \delta + \varepsilon_\ell,\ \mathbb{E}[\varepsilon_\ell | X_\ell] = 0, \tag{2.1}$$

where $y_\ell = (y_{\ell 1}, \dots, y_{\ell n_\ell})^\top$ is an outcome vector, $X_\ell = (x_{\ell 1}^\top, \dots, x_{\ell n_\ell}^\top)^\top$ is an $n_\ell \times K$ covariate matrix, $\iota = (1, \dots, 1)^\top$ is a vector of 1s, and $\varepsilon_\ell = (\varepsilon_{\ell 1}, \dots, \varepsilon_{\ell n_\ell})^\top$ is a vector of error terms.

Assumption 1. The support of the distribution of X_ℓ is not contained in any proper linear subspaces of $\mathbb{R}^{n_\ell}$.

In order to derive the reduced-form equation of the model (2.1), we need to show the existence of $(I-\beta G_\ell)^{-1}$.

Proposition 2.1. $\|G_\ell\|^* \leq 1$ holds, where $\|\cdot\|^*$ is the operator norm.

Proof. By the definition of the operator norm,

$$\|G_\ell^\top\|^* = \left[\max_{v\in\mathbb{R}^{n_\ell}} \sqrt{(G_\ell^\top v)^\top G_\ell^\top v} \quad \text{s.t.} \quad \sqrt{v^\top v}=1\right]. \tag{2.2}$$

The Lagrangian function of (2.2) is

$$\mathcal{L} = v^\top G_\ell G_\ell^\top v + \lambda(1-v^\top v),$$

where λ is the Lagrange multiplier. The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial v} = 2G_\ell G_\ell^\top v - 2\lambda v = 0,$$

which implies

$$\sum_{i=1}^{n_\ell}\sum_{j=1}^{n_\ell}(\tilde{G}_\ell)_{ij} v_i v_j = \lambda,$$

where $\tilde{G}_\ell \coloneqq G_\ell G_\ell^\top$. Note that λ is an eigenvalue of $\tilde{G}_\ell$. Since each element of G_ℓ is less than or equal to 1 and each row sum of G_ℓ is 1 or 0,

$$|(\tilde{G}_\ell)_{ij}| = \left|\sum_{k=1}^{n_\ell}(G_\ell)_{ik}(G_\ell)_{jk}\right| \leq \sum_{k=1}^{n_\ell}|(G_\ell)_{ik}||(G_\ell)_{jk}| \leq \sum_{k=1}^{n_\ell}|(G_\ell)_{ik}| \leq 1.$$

By the Cauchy-Schwarz inequality,

$$|\lambda| = \left|\sum_{i=1}^{n_\ell}\sum_{j=1}^{n_\ell}(\tilde{G}_\ell)_{ij} v_i v_j\right| \leq \sum_{i=1}^{n_\ell}\sum_{j=1}^{n_\ell}|(\tilde{G}_\ell)_{ij}||v_i||v_j| \leq \left(\sum_{i=1}^{n_\ell}|v_i|\right)^2 \leq \sum_{i=1}^{n_\ell} v_i^2 = 1.$$

Since $\|G_\ell^\top\|^* = \|G_\ell\|^*$ and $\|G_\ell^\top\|^* = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of $\tilde{G}_\ell$,

$$\|G_\ell\|^* = \sqrt{\lambda_{\max}} \leq 1$$

is shown. ■

If $\|\beta G_\ell\|^* < 1$, $I-\beta G_\ell$ is invertible. To see this, assume that $I-\beta G_\ell$ is not invertible under $\|\beta G_\ell\|^* < 1$. Then,

$$\exists u \in \mathbb{R}^{n_\ell}\setminus\{0\} \quad \text{s.t.} \quad (I-\beta G_\ell)u = 0.$$

Without loss of generality, assume $\|u\| = 1$. Then, $u = \beta G_\ell u$ implies $\|\beta G_\ell\|^* \geq \|\beta G_\ell u\| = \|u\| = 1$, which leads to a contradiction.

Since $|\beta| < 1$ is assumed, $(I-\beta G_\ell)^{-1}$ exists. The reduced-form equation of the model (2.1) is given by

$$y_\ell = \alpha(I-\beta G_\ell)^{-1}\iota + (I-\beta G_\ell)^{-1}(X_\ell\gamma + G_\ell X_\ell \delta) + (I-\beta G_\ell)^{-1}\varepsilon_\ell. \tag{2.3}$$

Theorem 2.1 (Bramoullé et al., 2009)**.** Suppose that for some k_0, $\gamma_{k_0}\beta + \delta_{k_0} \neq 0$.

(1) If the matrices I, G_ℓ, and G_ℓ^2 are linearly independent, the structural parameters $(\alpha, \beta, \gamma, \delta)$ are identified.

(2) If the matrices I, G_ℓ, and G_ℓ^2 are linearly dependent, and no individual is isolated, the structural parameters $(\alpha, \beta, \gamma, \delta)$ are not identified.

Proof. (1) The expectation of the reduced-form equation (2.3) conditioned on X_ℓ is

$$\mathbb{E}[y_\ell|X_\ell] = \alpha(I - \beta G_\ell)^{-1}\iota + (I - \beta G_\ell)^{-1}(X_\ell\gamma + G_\ell X_\ell \delta).$$

Assume that for $\theta = (\alpha, \beta, \gamma, \delta)$ and $\theta' = (\alpha', \beta', \gamma', \delta')$,

$$\alpha(I - \beta G_\ell)^{-1}\iota + (I - \beta G_\ell)^{-1}(X_\ell\gamma + G_\ell X_\ell \delta) = \alpha'(I - \beta' G_\ell)^{-1}\iota + (I - \beta' G_\ell)^{-1}(X_\ell\gamma' + G_\ell X_\ell \delta') \tag{2.4}$$

holds with probability one. By Assumption 1, (2.4) implies

$$\begin{cases} \alpha(I - \beta G_\ell)^{-1}\iota = \alpha'(I - \beta' G_\ell)^{-1}\iota \\ (I - \beta G_\ell)^{-1}(X_\ell\gamma + G_\ell X_\ell \delta) = (I - \beta' G_\ell)^{-1}(X_\ell\gamma' + G_\ell X_\ell \delta'). \end{cases} \tag{2.5}$$

Let $x_\ell^k := (x_{\ell 1k}, \ldots, x_{\ell n_\ell k})^\top$ be a covariate vector of characteristics k in network ℓ. Then, the second equation of (2.5) implies

$$(I - \beta G_\ell)^{-1}(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta' G_\ell)^{-1}(\gamma_k' I + \delta_k' G_\ell)x_\ell^k \tag{2.6}$$

for all $k = 1, \ldots, K$, which yields

$$(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta G_\ell)(I - \beta' G_\ell)^{-1}(\gamma_k' I + \delta_k' G_\ell)x_\ell^k. \tag{2.7}$$

By Proposition A.1 in Appendix, we can write (2.7) as

$$(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta' G_\ell)^{-1}(I - \beta G_\ell)(\gamma_k' I + \delta_k' G_\ell)x_\ell^k$$

and multiplying $(I - \beta' G_\ell)$ on both sides, we get

$$(I - \beta' G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta G_\ell)(\gamma_k' I + \delta_k' G_\ell)x_\ell^k. \tag{2.8}$$

Arranging (2.8) results in

$$[(\gamma_k - \gamma_k')I + (\delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta')G_\ell + (\delta_k'\beta - \delta_k\beta')G_\ell^2]x_\ell^k = \mathbf{0}. \tag{2.9}$$

If I, G_ℓ, and G_ℓ^2 are linearly independent, (2.9) implies

$$\begin{cases} \gamma_k = \gamma_k' \\ \delta_k + \gamma_k'\beta = \delta_k' + \gamma_k\beta' \\ \delta_k'\beta = \delta_k\beta'. \end{cases}$$

If $\delta_k'\beta \neq 0$, $\delta_k'\beta = \delta_k\beta'$ implies

$$\exists \lambda \neq 0 \text{ s.t. } \beta' = \lambda\beta \text{ and } \delta_k' = \lambda\delta_k.$$

Since $\gamma_k = \gamma'_k$ and $\delta_k + \gamma'_k\beta = \delta'_k + \gamma_k\beta'$, we get

$$\delta_k + \gamma_k\beta = \lambda(\delta_k + \gamma_k\beta).$$

Since $\delta_{k_0} + \gamma_{k_0}\beta \neq 0$ for some k_0, $\lambda = 1$ holds for k_0. Thus, $\beta' = \beta$ and $\delta'_{k_0} = \delta_{k_0}$ are shown. Moreover, $\alpha(I - \beta G_\ell)^{-1}\iota = \alpha'(I - \beta' G_\ell)^{-1}\iota$ implies

$$\alpha(I - \beta G_\ell)^{-1}\iota = \alpha'(I - \beta G_\ell)^{-1}\iota,$$

which results in $\alpha' = \alpha$. Therefore, in the case of $\delta'_k\beta \neq 0$, $(\alpha, \beta, \gamma_{k_0}, \delta_{k_0}) = (\alpha', \beta', \gamma'_{k_0}, \delta'_{k_0})$ holds.

Table 1: Pairs of parameters satisfying $\delta_{k_0}\beta = \delta_{k_0}\beta' = 0$

Pairs of parameters	$\gamma_{k_0}\beta + \delta_{k_0} \neq 0$	$\gamma'_{k_0}\beta' + \delta'_{k_0} \neq 0$
$(\delta'_{k_0}, \delta_{k_0}) = (0, 0)$	possible	possible
$(\delta'_{k_0}, \beta') = (0, 0)$	possible	impossible
$(\beta, \delta_{k_0}) = (0, 0)$	impossible	possible
$(\beta, \beta') = (0, 0)$	possible	possible

If $\delta'_k\beta = 0$, $\delta'_k\beta = \delta_k\beta' = 0$ is satisfied. Since $\gamma_{k_0}\beta + \delta_{k_0} \neq 0$ and $\gamma'_{k_0}\beta' + \delta'_{k_0} \neq 0$ for some k_0,

$$\beta = \delta_{k_0} = 0 \text{ or } \beta' = \delta'_{k_0} = 0$$

cannot be satisfied. Thus, $\beta = \beta' = 0$ or $\delta_{k_0} = \delta'_{k_0} = 0$ holds (Table 1). If $\beta = \beta' = 0$ holds, $\delta_{k_0} + \gamma'_{k_0}\beta = \delta'_{k_0} + \gamma_{k_0}\beta'$ implies $\delta_{k_0} = \delta'_{k_0}$. If $\delta_{k_0} = \delta'_{k_0} = 0$ holds, $\gamma_{k_0} = \gamma'_{k_0}$ and $\delta_{k_0} + \gamma'_{k_0}\beta = \delta'_{k_0} + \gamma_{k_0}\beta'$ lead to $\beta = \beta'$. For both cases, $\beta' = \beta$ results in $\alpha' = \alpha$.

For any other k, by multiplying $(I - \beta G_\ell)$ on both sides of (2.6), we get

$$(\gamma_k I + \delta_k G_\ell)x_\ell^k = (\gamma'_k I + \delta'_k G_\ell)x_\ell^k,$$

which implies

$$[(\gamma_k - \gamma'_k)I + (\delta_k - \delta'_k)G_\ell]x_\ell^k = \mathbf{0}.$$

Since I and G_ℓ are linearly independent, $(\gamma_k, \delta_k) = (\gamma'_k, \delta'_k)$ is obtained.

(2) By the same argument as in the proof of (1) in Theorem 2.1, we obtain

$$\begin{cases} \alpha(I - \beta G_\ell)^{-1}\iota = \alpha'(I - \beta' G_\ell)^{-1}\iota \\ [(\gamma_k - \gamma'_k)I + (\delta_k - \delta'_k + \gamma'_k\beta - \gamma_k\beta')G_\ell + (\delta'_k\beta - \delta_k\beta')G_\ell^2]x_\ell^k = \mathbf{0}, \end{cases} \tag{2.10}$$

where $x_\ell^k := (x_{\ell 1k}, \ldots, x_{\ell n_\ell k})^\top$ is a covariate vector of characteristics k in network ℓ. Since no individual is isolated, $G_\ell\iota = \iota$ is satisfied, which implies

$$\alpha(I - \beta G_\ell)^{-1}\iota = \alpha \sum_{t=0}^{\infty} \beta^t G_\ell^t \iota = \alpha \sum_{t=0}^{\infty} \beta^t \iota = \frac{\alpha}{1 - \beta}\iota.$$

The first equation of (2.10) becomes

$$\frac{\alpha}{1 - \beta} = \frac{\alpha'}{1 - \beta'}. \tag{2.11}$$

By substituting $G_\ell^2 = \lambda I + \lambda' G_\ell,\ \lambda, \lambda' \in \mathbb{R}$ into the second equation of (2.10), we get

$$[(\gamma_k - \gamma_k' + \lambda(\delta_k'\beta - \delta_k\beta'))I + (\delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta' + \lambda'(\delta_k'\beta - \delta_k\beta'))G_\ell]x_\ell^k = \mathbf{0}.$$

The linear independence of I and G_ℓ implies

$$\gamma_k - \gamma_k' + \lambda(\delta_k'\beta - \delta_k\beta') = 0 \tag{2.12}$$

and

$$\delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta' + \lambda'(\delta_k'\beta - \delta_k\beta') = 0. \tag{2.13}$$

Only the three equations (2.11), (2.12), and (2.13) need to be satisfied for identifying four parameters $(\alpha, \beta, \gamma_k, \delta_k)$, which implies that the structural parameters are not identified. ■

If the identification condition in Theorem 2.1 is satisfied, the network size is at least three (Example 2.1). For networks with two individuals, the matrices I, G_ℓ, and G_ℓ^2 are always linearly dependent. There are four possible adjacency matrices for such networks:

$$G_a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ G_b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ G_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ G_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the case of G_a, the square of the matrix is

$$G_a^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

which implies that I, G_a, and G_a^2 are linearly dependent. In the case of G_b, we have

$$G_b^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0I + 0G_b,$$

which implies that I, G_b, and G_b^2 are linearly dependent. Similarly, for G_c and G_d, we can verify that the matrices I, G_ℓ, and G_ℓ^2 are linearly dependent.

Corollary 2.1. If the identification condition in Theorem 2.1 is satisfied, a network must contain at least three individuals.

Example 2.1. The adjacency matrix G_1 of network 1 illustrated in Figure 2 is given by

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

G_1^2 is calculated as

$$G_1^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Apparently, I, G_1, and G_1^2 are linearly independent. By Theorem 2.1, the structural parameters are identified.

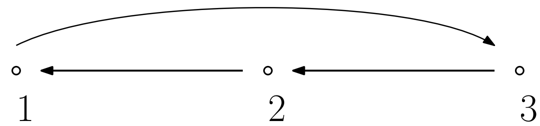

Figure 2: Network 1

The identification condition of Theorem 2.1 is satisfied if there exists an *intransitive triad* in the network. An intransitive triad is a set of three individuals i, j, k such that i is affected by j, j is affected by k, but i is not affected by k (Figure 3). Then, $(G_\ell)_{ik} = 0$ and $(G_\ell^2)_{ik} = (G_\ell)_{ij}(G_\ell)_{jk} > 0$. If we assume $G_\ell^2 = \lambda I + \lambda' G_\ell$ for $\lambda, \lambda' \in \mathbb{R}$,

$$(G_\ell^2)_{ik} = \lambda(I)_{ik} + \lambda'(G_\ell)_{ik} = 0,$$

which contradicts with $(G_\ell^2)_{ik} > 0$. Therefore, the existence of an intransitive triad implies linear independence of I, G_ℓ, and G_ℓ^2. In Example 2.1, network 1 in Figure 2 contains an intransitive triad: that is, 1 is affected by 2, 2 is affected by 3, but 1 is not affected by 3. Thus, we can identify the structural parameters.

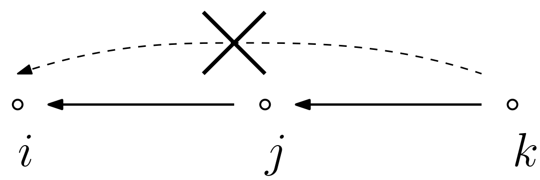

Figure 3: An intransitive triad

In reality, we often observe multiple networks with different structures, such as classrooms or villages, simultaneously within the same dataset. Bramoullé et al. (2009) applied the identification condition in Theorem 2.1 to a single large network $G \coloneqq \mathrm{diag}\{G_1, \ldots, G_L\}$ by stacking different network structures $G_1, \ldots, G_L$. However, this approach implicitly assumes that a single large network G is repeatedly observed, which is not consistent with network observations. In Theorem 2.2 below, we provide an identification condition for each heterogeneous network without stacking different network structures. While our identification condition is mathematically equivalent to the result in Bramoullé et al. (2009), our result is consistent with network observations.

Theorem 2.2. Suppose that $\gamma_{k_0}\beta + \delta_{k_0} \neq 0$ for some k_0. For all networks ℓ, the matrices I, G_ℓ, and G_ℓ^2 are linearly dependent, and

$$G_\ell^2 = \lambda_\ell I + \mu_\ell G_\ell,\ \ \ell = 1, \ldots, L,$$

where $\lambda_\ell, \mu_\ell \in \mathbb{R}$. If there exist two networks r and s such that $(\lambda_r, \mu_r) \neq (\lambda_s, \mu_s)$, the structural parameter $\theta = (\alpha, \beta, \gamma, \delta)$ is identified.

Proof. By the same argument as in the proof of Theorem 2.1, we obtain

$$[(\gamma_k - \gamma_k')I + (\delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta')G_\ell + (\delta_k'\beta - \delta_k\beta')G_\ell^2]x_\ell^k = \mathbf{0}, \tag{2.14}$$

where $x_\ell^k := (x_{\ell 1k}, \dots, x_{\ell n_\ell k})^\top$. Since we assumed $G_\ell^2 = \lambda_\ell I + \mu_\ell G_\ell$, we can write (2.14) as

$$[(\gamma_k - \gamma_k' + \lambda_\ell(\delta_k'\beta - \delta_k\beta'))I + (\delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta' + \mu_\ell(\delta_k'\beta - \delta_k\beta'))G_\ell]x_\ell^k = \mathbf{0}.$$

By linear independence of I and G_ℓ, we obtain

$$\begin{cases} \gamma_k - \gamma_k' + \lambda_\ell(\delta_k'\beta - \delta_k\beta') = 0 \\ \delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta' + \mu_\ell(\delta_k'\beta - \delta_k\beta') = 0. \end{cases} \tag{2.15}$$

If $\lambda_r \neq \lambda_s$, the first equation of (2.15) implies $\delta_k'\beta = \delta_k\beta'$. If $\mu_r \neq \mu_s$, the second equation of (2.15) implies $\delta_k'\beta = \delta_k\beta'$. In any cases, by substituting $\delta_k'\beta = \delta_k\beta'$ into (2.15), we get

$$\begin{cases} \gamma_k = \gamma_k' \\ \delta_k + \gamma_k'\beta = \delta_k' + \gamma_k\beta'. \end{cases}$$

By the same argument as in the proof of Theorem 2.1, we obtain $\theta = \theta'$. ■

In Theorem 2.2, the identification arises because the variations of the network structures create exogenous variations in the reduced-form parameters. By multiplying G_ℓ on both sides of the reduced-form equation (2.3),

$$G_\ell y_\ell = \alpha G_\ell (I - \beta G_\ell)^{-1}\iota + (I - \beta G_\ell)^{-1} G_\ell X_\ell \gamma + (I - \beta G_\ell)^{-1} G_\ell^2 X_\ell \delta + (I - \beta G_\ell)^{-1} G_\ell \varepsilon_\ell.$$

For each covariate x_ℓ^k, the reduced-form equation is given by

$$G_\ell y_\ell = \alpha G_\ell (I - \beta G_\ell)^{-1}\iota + (I - \beta G_\ell)^{-1} G_\ell x_\ell^k \gamma_k + (I - \beta G_\ell)^{-1} G_\ell^2 x_\ell^k \delta_k + (I - \beta G_\ell)^{-1} G_\ell \varepsilon_\ell.$$

Substituting $G_\ell^2 = \lambda_\ell I + \mu_\ell G_\ell$ into the above equation gives

$$G_\ell y_\ell = \alpha G_\ell (I - \beta G_\ell)^{-1}\iota + \delta_k \lambda_\ell (I - \beta G_\ell)^{-1} x_\ell^k + (\gamma_k + \delta_k \mu_\ell)(I - \beta G_\ell)^{-1} G_\ell x_\ell^k + (I - \beta G_\ell)^{-1} G_\ell \varepsilon_\ell. \tag{2.16}$$

Furthermore, by substituting (2.16) into the structural model (2.1), we obtain the reduced-form equation given by

$$\begin{aligned} y_\ell = {} & [\alpha I + \alpha\beta G_\ell (I - \beta G_\ell)^{-1}]\iota + [\gamma_k I + \beta \delta_k \lambda_\ell (I - \beta G_\ell)^{-1}] x_\ell^k \\ & + [\delta_k I + \beta(\gamma_k + \delta_k \mu_\ell)(I - \beta G_\ell)^{-1}] G_\ell x_\ell^k + (\beta (I - \beta G_\ell)^{-1} G_\ell \varepsilon_\ell + \varepsilon_\ell). \end{aligned}$$

The reduced-form parameters for network ℓ

$$\left(\alpha I + \alpha\beta G_\ell (I - \beta G_\ell)^{-1}, \gamma_k I + \beta \delta_k \lambda_\ell (I - \beta G_\ell)^{-1}, \delta_k I + \beta(\gamma_k + \delta_k \mu_\ell)(I - \beta G_\ell)^{-1}\right)$$

are uniquely recovered from the data. If we observe two networks r and s such that $(\lambda_r, \mu_r) \neq (\lambda_s, \mu_s)$, we can introduce exogenous variations in the reduced-form parameters, which results in the identification of the structural parameters.

Corollary 2.2. If the identification condition in Theorem 2.2 is satisfied, a network must contain at least two individuals.

Example 2.2. Suppose that we simultaneously observe two networks with different structures, as illustrated in Figure 4, within the same dataset. The adjacency matrices G_2 and G_3 for each network are given by

$$G_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

G_2^2 and G_3^2 are calculated as

$$G_2^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_3^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $G_2^2 = I + 0G_2$ holds while $G_3^2 = 0I + 0G_3$ is satisfied, the structural parameters are identified.

Figure 4: Network 2 (left) and network 3 (right)

3 Identification of the network model with the network fixed effect

To account for the correlation between unobserved network attributes and the covariate, we introduce network-specific unobservables in the model (2.1). In the schooling example, wealthy parents tend to live in areas with good schools (Black, 1999). As a result, students from high-income families are more likely to attend schools with teachers who possess better unobserved teaching skills. To control for the network-level unobservables, we include a network fixed effect, which may be correlated with the covariates.

The individual-level network model is given by

$$y_{\ell i} = \alpha_\ell + \beta \left(\sum_{j \in P_{\ell i}} (G_\ell)_{ij} y_{\ell j} \right) + x_{\ell i}^\top \gamma + \left(\sum_{j \in P_{\ell i}} (G_\ell)_{ij} x_{\ell j} \right)^\top \delta + \varepsilon_{\ell i}, \ \ell = 1, \ldots, L, \ i = 1, \ldots, n_\ell, \tag{3.1}$$

where $y_{\ell i} \in \mathbb{R}$ is an outcome variable for individual i in network ℓ, $x_{\ell i} = (x_{\ell i1}, \ldots, x_{\ell iK})^\top \in \mathbb{R}^K$ is a covariate vector for individual i in network ℓ, α_ℓ is a network fixed effect, and $\varepsilon_{\ell i}$ is an error term satisfying $\mathbb{E}[\varepsilon_{\ell i} | x_{\ell 1}, \ldots, x_{\ell n_\ell}, \alpha_\ell] = 0$. Let $(\beta, \gamma, \delta) \in \mathbb{R} \times \mathbb{R}^K \times \mathbb{R}^K$ be parameters such that $|\beta| < 1$. In this section, we assume that no individual is isolated: that is, $P_{\ell i} \neq \emptyset$ for all networks ℓ. The network-level model is given by

$$y_\ell = \alpha_\ell \iota + \beta G_\ell y_\ell + X_\ell \gamma + G_\ell X_\ell \delta + \varepsilon_\ell, \ \mathbb{E}[\varepsilon_\ell | X_\ell, \alpha_\ell] = 0,$$

where $y_\ell = (y_{\ell 1}, \ldots, y_{\ell n_\ell})^\top$ is an outcome vector, $X_\ell = (x_{\ell 1}^\top, \ldots, x_{\ell n_\ell}^\top)^\top$ is an $n_\ell \times K$ covariate matrix, $\iota = (1, \ldots, 1)^\top$ is a vector of 1s, and $\varepsilon_\ell = (\varepsilon_{\ell 1}, \ldots, \varepsilon_{\ell n_\ell})^\top$ is a vector of error terms.

We eliminate the network fixed effect by taking local differences. We average the individual-level model (3.1) over all peers of individual i, and subtract it from i's model. This approach is local since it does not fully exploit the fact that the fixed effect is not only the same for all individual i's peers but also for all individuals in the network. In the matrix form, the local difference is expressed by

$$(I - G_\ell)y_\ell = \alpha_\ell(I - G_\ell)\iota + (I - G_\ell)\beta G_\ell y_\ell + (I - G_\ell)X_\ell\gamma + (I - G_\ell)G_\ell X_\ell\delta + (I - G_\ell)\varepsilon_\ell. \quad (3.2)$$

Since there is no isolated individual, the sum of the row vector of G_ℓ is 1, and $G_\ell\iota = \iota$ holds. Thus, we can write (3.2) as

$$(I - G_\ell)y_\ell = (I - G_\ell)\beta G_\ell y_\ell + (I - G_\ell)X_\ell\gamma + (I - G_\ell)G_\ell X_\ell\delta + (I - G_\ell)\varepsilon_\ell.$$

By Proposition A.1 in Appendix, the reduced-form equation is given by

$$(I - G_\ell)y_\ell = (I - \beta G_\ell)^{-1}(I - G_\ell)(X_\ell\gamma + G_\ell X_\ell\delta) + (I - \beta G_\ell)^{-1}(I - G_\ell)\varepsilon_\ell. \quad (3.3)$$

Theorem 3.1 (Bramoullé et al., 2009)**.** Suppose that for some k_0, $\gamma_{k_0}\beta + \delta_{k_0} \neq 0$. The structural parameter $\theta = (\beta, \gamma, \delta)$ is identified if and only if the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly independent.

Proof. The expectation of the reduced-form equation (3.3) conditioned on X_ℓ and α_ℓ is

$$(I - G_\ell)\mathbb{E}[y_\ell | X_\ell, \alpha_\ell] = (I - \beta G_\ell)^{-1}(I - G_\ell)(X_\ell\gamma + G_\ell X_\ell\delta).$$

Assume that for $\theta = (\beta, \gamma, \delta)$ and $\theta' = (\beta', \gamma', \delta')$,

$$(I - \beta G_\ell)^{-1}(I - G_\ell)(X_\ell\gamma + G_\ell X_\ell\delta) = (I - \beta' G_\ell)^{-1}(I - G_\ell)(X_\ell\gamma' + G_\ell X_\ell\delta') \quad (3.4)$$

holds with probability one. Let $x_\ell^k \coloneqq (x_{\ell 1k}, \ldots, x_{\ell n_\ell k})^\top$ be a covariate vector of characteristics k in network ℓ. Then, (3.4) implies

$$(I - \beta G_\ell)^{-1}(I - G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta' G_\ell)^{-1}(I - G_\ell)(\gamma'_k I + \delta'_k G_\ell)x_\ell^k$$

for all $k = 1, \ldots, K$, which yields

$$(I - G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta G_\ell)(I - \beta' G_\ell)^{-1}(I - G_\ell)(\gamma'_k I + \delta'_k G_\ell)x_\ell^k. \quad (3.5)$$

By Proposition A.1, we can write (3.5) as

$$(I - G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta G_\ell)(I - \beta' G_\ell)^{-1}(\gamma'_k I + \delta'_k G_\ell)(I - G_\ell)x_\ell^k,$$

which implies

$$(I - \beta' G_\ell)(I - G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta G_\ell)(I - G_\ell)(\gamma'_k I + \delta'_k G_\ell)x_\ell^k. \quad (3.6)$$

Arranging (3.6) results in

$$\Big[(\gamma_k - \gamma'_k)I + \{\delta_k - \delta'_k - (\gamma_k - \gamma'_k) + \gamma'_k\beta - \gamma_k\beta'\}G_\ell - \{\delta'_k - \delta_k + \beta'(\delta_k - \gamma_k) - \beta(\delta'_k - \gamma'_k)\}G_\ell^2 + (\beta'\delta_k - \beta\delta'_k)G_\ell^3\Big]x_\ell^k = \mathbf{0}. \quad (3.7)$$

If I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly independent, (3.7) implies

$$\begin{cases} \gamma_k = \gamma'_k \\ \delta_k + \gamma'_k\beta = \delta'_k + \gamma_k\beta' \\ \delta'_k\beta = \delta_k\beta'. \end{cases}$$

By the same argument as in the proof of Theorem 2.1, we obtain $\beta = \beta'$ and $\delta_{k_0} = \delta'_{k_0}$.

For any other k, by multiplying $(I - \beta G_\ell)$ on both sides of (3.4), we get

$$(I - G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - G_\ell)(\gamma'_k I + \delta'_k G_\ell)x_\ell^k,$$

which implies

$$[(\gamma_k - \gamma'_k)I + (\delta_k - \delta'_k + \gamma'_k\beta - \gamma_k\beta')G_\ell + (\delta'_k\beta - \delta_k\beta')G_\ell^2]x_\ell^k = \mathbf{0}.$$

Since I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly independent, we obtain $(\gamma_k, \delta_k) = (\gamma'_k, \delta'_k)$.

Suppose that I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly dependent. If we assume $G_\ell^2 = \lambda I + \mu G_\ell$ for $\lambda, \mu \in \mathbb{R}$, we can write G_ℓ^3 as

$$G_\ell^3 = \lambda G_\ell + \mu G_\ell^2 = \lambda G_\ell + \mu(\lambda I + \mu G_\ell) = \lambda\mu I + (\lambda + \mu^2)G_\ell.$$

By substituting $G_\ell^2 = \lambda I + \mu G_\ell$ and $G_\ell^3 = \lambda\mu I + (\lambda + \mu^2)G_\ell$ into (3.7), we get

$$\Big[\{\gamma_k - \gamma'_k - \lambda(\delta'_k - \delta_k + \beta'(\delta_k - \gamma_k) - \beta(\delta'_k - \gamma'_k)) + (\beta'\delta_k - \beta\delta'_k)\lambda\mu\}I \\ + \{\delta_k - \delta'_k - (\gamma_k - \gamma'_k) + \gamma'_k\beta - \gamma_k\beta' - \mu(\delta'_k - \delta_k + \beta'(\delta_k - \gamma_k) - \beta(\delta'_k - \gamma'_k)) \\ + (\beta'\delta_k - \beta\delta'_k)(\lambda + \mu^2)\}G_\ell\Big]x_\ell^k = \mathbf{0}.$$

By linear independence of I and G_ℓ, we get

$$\begin{cases} \gamma_k - \gamma'_k - \lambda(\delta'_k - \delta_k + \beta'(\delta_k - \gamma_k) - \beta(\delta'_k - \gamma'_k)) + (\beta'\delta_k - \beta\delta'_k)\lambda\mu = 0 \\ \delta_k - \delta'_k - (\gamma_k - \gamma'_k) + \gamma'_k\beta - \gamma_k\beta' - \mu(\delta'_k - \delta_k + \beta'(\delta_k - \gamma_k) - \beta(\delta'_k - \gamma'_k)) + (\beta'\delta_k - \beta\delta'_k)(\lambda + \mu^2) = 0. \end{cases}$$

Only the two equations are needed for identifying three parameters $(\beta, \gamma_k, \delta_k)$, which implies that the structural parameters are not identified.

Assume that I, G_ℓ, and G_ℓ^2 are linearly independent, and $G_\ell^3 = \lambda I + \mu G_\ell + \nu G_\ell^2$, where $\lambda, \mu, \nu \in \mathbb{R}$. By substituting $G_\ell^3 = \lambda I + \mu G_\ell + \nu G_\ell^2$ into (3.7), we get

$$\Big[(\gamma_k - \gamma'_k + \lambda(\beta'\delta_k - \beta\delta'_k))I + (\delta_k - \delta'_k - (\gamma_k - \gamma'_k) + \gamma'_k\beta - \gamma_k\beta' + \mu(\beta'\delta_k - \beta\delta'_k))G_\ell \\ + (-(\delta'_k - \delta_k) - \beta'(\delta_k - \gamma_k) + \beta(\delta'_k - \gamma'_k) + \nu(\beta'\delta_k - \beta\delta'_k))G_\ell^2\Big]x_\ell^k = \mathbf{0}.$$

The linear independence of I, G_ℓ, and G_ℓ^2 implies

$$\begin{cases} \gamma_k - \gamma'_k + \lambda(\beta'\delta_k - \beta\delta'_k) = 0 \\ \delta_k - \delta'_k - (\gamma_k - \gamma'_k) + \gamma'_k\beta - \gamma_k\beta' + \mu(\beta'\delta_k - \beta\delta'_k) = 0 \\ -(\delta'_k - \delta_k) - \beta'(\delta_k - \gamma_k) + \beta(\delta'_k - \gamma'_k) + \nu(\beta'\delta_k - \beta\delta'_k) = 0. \end{cases} \tag{3.8}$$

Since no individual is isolated, $G_\ell \iota = \iota$ holds. By multiplying both sides of $G_\ell^3 = \lambda I + \mu G_\ell + \nu G_\ell^2$ by ι, we get

$$\lambda + \mu + \nu = 1.$$

By summing the first and second equations in (3.8), we get

$$-(\delta_k' - \delta_k) - \beta'(\delta_k - \gamma_k) + \beta(\delta_k' - \gamma_k') + (1 - \lambda - \mu)(\beta'\delta_k - \beta\delta_k') = 0,$$

which implies the third equation in (3.8). Only the two equations are necessary for identifying three parameters $(\beta, \gamma_k, \delta_k)$, which implies that the structural parameters are not identified. ■

If the identification condition in Theorem 3.1 is satisfied, the network size is at least four (Example 3.1). For networks with two individuals, the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are always linearly dependent. There are four possible adjacency matrices for such networks:

$$G_a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ G_b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ G_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ G_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the case of G_a, the square and cube of the matrix are

$$G_a^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ G_a^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = G_a,$$

respectively, which implies that I, G_a, G_a^2, and G_a^3 are linearly dependent. In the case of G_b, we have

$$G_b^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ G_b^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0I + 0G_b + 0G_b^2,$$

which implies that I, G_b, G_b^2, and G_b^3 are linearly dependent. Similarly, for G_c and G_d, we can verify that the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly dependent. For a network with three individuals, the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are always linearly dependent as shown in Appendix A.2.

Corollary 3.1. If the identification condition in Theorem 3.1 is satisfied, a network must contain at least four individuals.

Example 3.1. The adjacency matrix of network 4 illustrated in Figure 5 is given by

$$G_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

G_4^2 and G_4^3 are calculated as

$$G_4^2 = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_4^3 = \begin{bmatrix} 0 & 3/4 & 0 & 1/4 \\ 3/8 & 0 & 5/8 & 0 \\ 0 & 5/8 & 0 & 3/8 \\ 0 & 1/4 & 0 & 3/4 \end{bmatrix}.$$

One can check the linear independence of I, G_4, G_4^2, and G_4^3. By Theorem 3.1, the structural parameters are identified.

Figure 5: Network 4

Intransitive triads have a natural counterpart. Define the distance between individuals i and j in the network as the number of links connecting i and j in the shortest chain of students $i_1, \dots, i_\ell$ such that i is affected by i_1, i_1 is affected by i_2, $\dots$, and i_ℓ is affected by j. For example, an intransitive triad is a network with distance 2. Define the diameter of the network as the maximum distance between any two individuals in the same network. The identification condition in Theorem 3.1 is satisfied if the diameter of the network is greater than or equal to 3. Suppose that the diameter of network ℓ is greater than or equal to 3. Then, we can find two individuals i and j separated by a distance 3 in the network, where i_1 and i_2 are individuals between them (Figure 6). In this case, $(G_\ell)_{ij} = 0$, $(G_\ell^2)_{ij} = 0$, and $(G_\ell^3)_{ij} = (G_\ell)_{ii_1}(G_\ell)_{i_1 i_2}(G_\ell)_{i_2 j} > 0$, implying $G_\ell^3 \neq \mu I + \mu' G_\ell + \mu'' G_\ell^2$, where $\mu, \mu', \mu'' \in \mathbb{R}$. Therefore, the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly independent.

Figure 6: A network with distance 3

Theorem 3.2. Suppose that for some k_0, $\gamma_{k_0}\beta + \delta_{k_0} \neq 0$. For all networks ℓ, the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly dependent, and

$$G_\ell^3 = a_\ell I + b_\ell G_\ell + c_\ell G_\ell^2,\ \ \ell = 1, \dots, L,$$

where $a_\ell, b_\ell, c_\ell \in \mathbb{R}$, and I, G_ℓ, and G_ℓ^2 are linearly independent. If there exist two networks r and s such that $(a_r, b_r, c_r) \neq (a_s, b_s, c_s)$, the structural parameter $\theta = (\beta, \gamma, \delta)$ is identified.

Proof. By the same argument as in the proof of Theorem 3.1, we get

$$(I - \beta G_\ell)^{-1}(I - G_\ell)(\gamma_k I + \delta_k G_\ell)x_\ell^k = (I - \beta' G_\ell)^{-1}(I - G_\ell)(\gamma_k' I + \delta_k' G_\ell)x_\ell^k \tag{3.9}$$

for all $k = 1, \dots, K$, where $x_\ell^k := (x_{\ell 1 k}, \dots, x_{\ell n_\ell k})^\top$. Arranging (3.9) results in

$$\Big[(\gamma_k - \gamma_k')I + \{\delta_k - \delta_k' - (\gamma_k - \gamma_k') + \gamma_k'\beta - \gamma_k\beta'\}G_\ell \\ - \{\delta_k' - \delta_k + \beta'(\delta_k - \gamma_k) - \beta(\delta_k' - \gamma_k')\}G_\ell^2 + (\beta'\delta_k - \beta\delta_k')G_\ell^3\Big]x_\ell^k = \mathbf{0}. \tag{3.10}$$

Since we assumed $G_\ell^3 = a_\ell I + b_\ell G_\ell + c_\ell G_\ell^2$, we can write (3.10) as

$$\Big[(\gamma_k - \gamma_k' + a_\ell(\beta'\delta_k - \beta\delta_k'))I + \{\delta_k - \delta_k' - (\gamma_k - \gamma_k') + \gamma_k'\beta - \gamma_k\beta' + b_\ell(\beta'\delta_k - \beta\delta_k')\}G_\ell \\ + \{-(\delta_k' - \delta_k) - \beta'(\delta_k - \gamma_k) + \beta(\delta_k' - \gamma_k') + c_\ell(\beta'\delta_k - \beta\delta_k')\}G_\ell^2\Big]x_\ell^k = \mathbf{0}.$$

By the linear independence of I, G_ℓ, and G_ℓ^2, we obtain

$$\begin{cases} \gamma_k - \gamma_k' + a_\ell(\beta'\delta_k - \beta\delta_k') = 0 \\ \delta_k - \delta_k' - (\gamma_k - \gamma_k') + \gamma_k'\beta - \gamma_k\beta' + b_\ell(\beta'\delta_k - \beta\delta_k') = 0 \\ -(\delta_k' - \delta_k) - \beta'(\delta_k - \gamma_k) + \beta(\delta_k' - \gamma_k') + c_\ell(\beta'\delta_k - \beta\delta_k') = 0. \end{cases} \tag{3.11}$$

If $a_r \neq a_s$, the first equation of (3.11) implies $\beta'\delta_k = \beta\delta_k'$. If $b_r \neq b_s$, the second equation of (3.11) implies $\beta'\delta_k = \beta\delta_k'$. If $c_r \neq c_s$, the third equation of (3.11) implies $\beta'\delta_k = \beta\delta_k'$. In any case, by substituting $\beta'\delta_k = \beta\delta_k'$ into (3.11), we get

$$\begin{cases} \gamma_k = \gamma_k' \\ \delta_k + \gamma_k'\beta = \delta_k' + \gamma_k\beta'. \end{cases}$$

Using the same argument as in the proof of Theorem 2.1, we obtain $\beta = \beta'$ and $\delta_{k_0} = \delta_{k_0}'$.

For any other k, by multiplying $(I - \beta G_\ell)$ on both sides of (3.9), we get

$$[(\gamma_k - \gamma_k')I + (\delta_k - \delta_k' + \gamma_k'\beta - \gamma_k\beta')G_\ell + (\delta_k'\beta - \delta_k\beta')G_\ell^2]x_\ell^k = \mathbf{0}.$$

The linear independence of I, G_ℓ, and G_ℓ^2 implies $(\gamma_k, \delta_k) = (\gamma_k', \delta_k')$. ■

If the identification condition in Theorem 3.2 is satisfied, the network size is at least three (Example 3.2). Since Theorem 3.2 requires the linear independence of I, G_ℓ, and G_ℓ^2, the structural parameters are not identified for a network with two individuals as shown in the previous section.

Corollary 3.2. If the identification condition in Theorem 3.2 is satisfied, a network must contain at least three individuals.

Example 3.2. Suppose that we simultaneously observe two networks with different structures, as illustrated in Figure 7, within the same dataset.

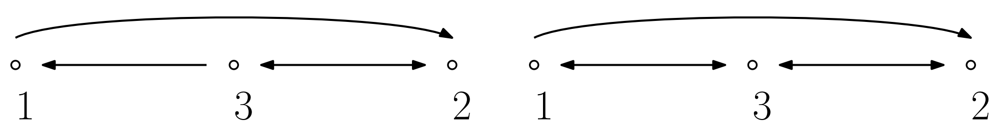

Figure 7: Network 5 (left) and network 6 (right)

Each network has a diameter of 2 and contains an intransitive triad: that is, 1 is affected by 3, 3 is affected by 2, but 1 is not affected by 2. The adjacency matrices G_5 and G_6 for each network are given by

$$G_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_6 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

G_5^2 and G_6^2 are calculated as

$$G_5^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_6^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 0 & 3/4 \end{bmatrix}.$$

Then, G_5^3 and G_6^3 are calculated as

$$G_5^3 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad G_6^3 = \begin{bmatrix} 1/4 & 0 & 3/4 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/8 & 1/4 \end{bmatrix}.$$

The matrices I, G_ℓ, and G_ℓ^2 are linearly independent for $\ell = 5, 6$. Besides, while $G_5^3 = \frac{1}{2}I + \frac{1}{2}G_5$ holds for network 5, $G_6^3 = \frac{1}{4}I + \frac{3}{4}G_6$ holds for network 6. By Theorem 3.2, the structural parameters are identified.

4 Conclusion

In this paper, we study the identification of social effects through network structures. The identification conditions provided by Bramoullé et al. (2009) are foundational in econometrics because linear models under a general interaction pattern are considered. Despite the significance of their contributions, there are mathematical gaps in their arguments. We bridge these gaps by providing the omitted mathematical discussions. Moreover, Bramoullé et al. (2009) considered the case where we observe identical network structures simultaneously within the same dataset. In reality, however, we often observe many networks with different structures, such as classrooms or villages, simultaneously within the same dataset. Bramoullé et al. (2009) applied their identification conditions to a single large network by stacking different networks. Under this framework, a single large network is repeatedly observed, which is not consistent with network observations. We provided identification conditions for network observations with various structures. While our identification conditions are mathematically equivalent to those in Bramoullé et al. (2009), our results are consistent with network observations. We illustrate examples of networks satisfying the identification conditions. We also discuss the identification of network models with a network fixed effect when we observe many networks with different structures.

For network models, with or without a network fixed effect, we characterize the smallest network size as a necessary condition for identifying social effects. For example, in a network model with a fixed effect, if the identification condition is satisfied under identical network observations, the network must contain at least four individuals. This is verified by considering all networks with two or three individuals and showing that they do not satisfy the identification condition. If the identification conditions are satisfied, the smallest network size is smaller when observing multiple different networks than when observing many identical network structures.

This paper is subject to several limitations. First, we assume that the network is exogenous, which is restrictive in many empirical applications. In the case of an endogenous network, the error term remains in the conditional expectation of the reduced-form equation, complicating the identification of structural parameters. One possible approach to address this issue is to explicitly model the formation of network links, as discussed in Johnsson and Moon (2021). Second, we confirm the smallest network size necessary for identification by directly calculating all possible adjacency matrices. For instance, in the case of a network with three individuals, we verify the linear dependence of I, G_ℓ, G_ℓ^2, and G_ℓ^3 by examining all 64 possible cases, as shown in Appendix A.2. Developing a more systematic method to verify linear dependence based on network structures remains an interesting direction for future research.

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A Appendix

A.1 Commutativity of $(I - \beta G_\ell)$ and $(I - \beta' G_\ell)^{-1}$

Proposition A.1. For an adjacency matrix G_ℓ,

$$(I - \beta G_\ell)(I - \beta' G_\ell)^{-1} = (I - \beta' G_\ell)^{-1}(I - \beta G_\ell)$$

holds.

Proof. First of all, we show that

$$(I - \beta' G_\ell)^{-1} G_\ell = G_\ell (I - \beta' G_\ell)^{-1} \tag{A.1}$$

holds. In the case of $\beta' = 0$, (A.1) is trivially satisfied. In the case of $\beta' \neq 0$,

$$\begin{aligned}(I - \beta' G_\ell)^{-1} G_\ell &= -\frac{1}{\beta'}(I - \beta' G_\ell)^{-1}(-\beta' G_\ell) \\ &= -\frac{1}{\beta'}(I - \beta' G_\ell)^{-1}((I - \beta' G_\ell) - I) \\ &= -\frac{1}{\beta'}(I - (I - \beta' G_\ell)^{-1}) \\ &= -\frac{1}{\beta'}((I - \beta' G_\ell) - I)(I - \beta' G_\ell)^{-1} \\ &= G_\ell (I - \beta' G_\ell)^{-1}\end{aligned}$$

holds, implying (A.1).

Next, by (A.1), we can rewrite $(I - \beta G_\ell)(I - \beta' G_\ell)^{-1}$ as

$$\begin{aligned}(I - \beta G_\ell)(I - \beta' G_\ell)^{-1} &= (I - \beta' G_\ell)^{-1} - \beta G_\ell (I - \beta' G_\ell)^{-1} \\ &= (I - \beta' G_\ell)^{-1} - \beta (I - \beta' G_\ell)^{-1} G_\ell \\ &= (I - \beta' G_\ell)^{-1}(I - \beta G_\ell),\end{aligned}$$

which completes the proof. ■

A.2 Linear dependence of I, G_ℓ, G_ℓ^2, and G_ℓ^3 for a network with three individuals

We confirm that the matrices I, G_ℓ, G_ℓ^2, and G_ℓ^3 are linearly dependent for a network with three individuals by calculating all possible adjacency matrices. When individual 1 is not affected by others, and individual 2 is affected by individual 3,

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = G_1,$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_2^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_2 + 0G_2^2,$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_3^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad G_3^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 \end{bmatrix} = \frac{1}{2}G_3,$$

$$G_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_4^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_4^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_4 + 0G_4^2.$$

When individual 1 is not affected by others, and individual 2 is affected by individual 1,

$$G_5 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_5^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_5^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_5 + 0G_5^2,$$

$$G_6 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_6^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_6^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_6 + 0G_6^2,$$

$$G_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_7^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}, \quad G_7^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_7 + 0G_7^2,$$

$$G_8 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_8^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_8^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_8 + 0G_8^2.$$

When individual 1 is not affected by others, and individual 2 is affected by individuals 1 and 3,

$$G_9 = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_9^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_9^3 = \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/2 & 0 \end{bmatrix} = \frac{1}{2}G_9,$$

$$G_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{10}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{10}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{10} + 0G_{10}^2,$$

$$G_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{11}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \end{bmatrix}, \quad G_{11}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 1/8 & 0 & 1/8 \\ 1/8 & 1/8 & 0 \end{bmatrix} = \frac{1}{4}G_{11},$$

$$G_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{12}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{12}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{12} + 0G_{12}^2.$$

When individuals 1 and 2 are not affected by others,

$$G_{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{13}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{13}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{13} + 0G_{13}^2,$$

$$G_{14} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{14}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{14}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{14} + 0G_{14}^2,$$

$$G_{15} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{15}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{15}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{15} + 0G_{15}^2,$$

$$G_{16} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{16}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{16}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{16} + 0G_{16}^2.$$

When individuals 1 and 2 are affected by individual 3,

$$G_{17} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{17}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{17}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = G_{17},$$

$$G_{18} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{18}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{18}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = G_{18},$$

$$G_{19} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{19}^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{19}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = G_{19},$$

$$G_{20} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{20}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{20}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{20} + 0G_{20}^2.$$

When individual 1 is affected by individual 3, and individual 2 is affected by individual 1,

$$G_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{21}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{21}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$G_{22} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{22}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{22}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = G_{22},$$

$$G_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{23}^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_{23}^3 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}G_{23},$$

$$G_{24} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{24}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{24}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{24} + 0G_{24}^2.$$

When individual 1 is affected by individual 3, and individual 2 is affected by individuals 1 and 3,

$$G_{25} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{25}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_{25}^3 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}G_{25},$$

$$G_{26} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{26}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{26}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} = G_{26},$$

$$G_{27} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{27}^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 0 & 3/4 \end{bmatrix}, \quad G_{27}^3 = \begin{bmatrix} 1/4 & 0 & 3/4 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/8 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{27},$$

$$G_{28} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{28}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{28}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{28} + 0G_{28}^2.$$

When individual 1 is affected by individual 3, and individual 2 is not affected by others,

$$G_{29} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{29}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{29}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{29} + 0G_{29}^2,$$

$$G_{30} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{30}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{30}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = G_{30},$$

$$G_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{31}^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad G_{31}^3 = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/4 & 1/4 & 0 \end{bmatrix} = \frac{1}{2}G_{31},$$

$$G_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{32}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{32}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{32} + 0G_{32}^2.$$

When individual 1 is affected by individual 2, and individual 2 is affected by individual 3,

$$G_{33} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{33}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{33}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = G_{33},$$

$$G_{34} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{34}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{34}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$G_{35} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{35}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad G_{35}^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}G_{35},$$

$$G_{36} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{36}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{36}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{36} + 0G_{36}^2.$$

When individual 1 is affected by individual 2, and individual 2 is affected by individual 1,

$$G_{37} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{37}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{37}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = G_{37},$$

$$G_{38} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{38}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{38}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = G_{38},$$

$$G_{39} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{39}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{39}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} = G_{39},$$

$$G_{40} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{40}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{40}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = G_{40}.$$

When individual 1 is affected by individual 2, and individual 2 is affected by individuals 1 and 3,

$$G_{41} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{41}^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_{41}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = G_{41},$$

$$G_{42} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{42}^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{42}^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}G_{42},$$

$$G_{43} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{43}^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}, \quad G_{43}^3 = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/8 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{43},$$

$$G_{44} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{44}^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{44}^3 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}G_{44}.$$

When individual 1 is affected by individual 2, and individual 2 is not affected by others,

$$G_{45} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{45}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{45}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{45} + 0G_{45}^2,$$

$$G_{46} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{46}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{46}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{46} + 0G_{46}^2,$$

$$G_{47} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{47}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}, \quad G_{47}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{47} + 0G_{47}^2,$$

$$G_{48} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{48}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{48}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{48} + 0G_{48}^2.$$

When individual 1 is affected by individuals 2 and 3, and individual 2 is affected by individual 3,

$$G_{49} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{49}^2 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{49}^3 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = G_{49},$$

$$G_{50} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{50}^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad G_{50}^3 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}G_{50},$$

$$G_{51} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{51}^2 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix}, \quad G_{51}^3 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 0 & 1/4 & 3/4 \\ 3/8 & 3/8 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{51},$$

$$G_{52} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{52}^2 = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{52}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{52} + 0G_{52}^2.$$

When individual 1 is affected by individuals 2 and 3, and individual 2 is affected by individual 1,

$$G_{53} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{53}^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{53}^3 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}I + \frac{1}{2}G_{53},$$

$$G_{54} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{54}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad G_{54}^3 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = G_{54},$$

$$G_{55} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{55}^2 = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}, \quad G_{55}^3 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 3/4 & 1/4 & 0 \\ 3/8 & 3/8 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{55},$$

$$G_{56} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{56}^2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{56}^3 = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}G_{56}.$$

When individual 1 is affected by individuals 2 and 3, and individual 2 is affected by individuals 1 and 3,

$$G_{57} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{57}^2 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad G_{57}^3 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 0 & 3/4 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{57},$$

$$G_{58} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{58}^2 = \begin{bmatrix} 3/4 & 0 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad G_{58}^3 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/4 & 0 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{58},$$

$$G_{59} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{59}^2 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}, \quad G_{59}^3 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/8 & 1/4 \end{bmatrix} = \frac{1}{4}I + \frac{3}{4}G_{59},$$

$$G_{60} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{60}^2 = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{60}^3 = \begin{bmatrix} 0 & 1/8 & 1/8 \\ 1/8 & 0 & 1/8 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{4}G_{60}.$$

When individual 1 is affected by individuals 2 and 3, and individual 2 is not affected by others,

$$G_{61} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_{61}^2 = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{61}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{61} + 0G_{61}^2,$$

$$G_{62} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad G_{62}^2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad G_{62}^3 = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} = \frac{1}{2}G_{62},$$

$$G_{63} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad G_{63}^2 = \begin{bmatrix} 1/4 & 1/4 & 0 \\ 0 & 0 & 0 \\ 0 & 1/4 & 1/4 \end{bmatrix}, \quad G_{63}^3 = \begin{bmatrix} 0 & 1/8 & 1/8 \\ 0 & 0 & 0 \\ 1/8 & 1/8 & 0 \end{bmatrix} = \frac{1}{4}G_{63},$$

$$G_{64} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{64}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{64}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I + 0G_{64} + 0G_{64}^2.$$