WINPEC Working Paper Series No. E2413
January 2025

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Abstract

The problem of efficient allocation of the grand coalition worth in transferable-utility games boils down to specifying how the surplus is distributed among individuals, in the situation where the individual share is well-defined. We show that the Individual Monotonicity axiom for Equal Surplus, together with Efficiency and Equal Treatment, implies Egalitarian Surplus Sharing, while the same axiom for Equal Ratio implies Proportional Division. The results thus illustrate the common structure in deriving two principles of surplus distribution, egalitarian and proportional, from the Individual Monotonicity axioms. We further show that relaxation of Equal Treatment leads to Weighted Surplus Sharing and Shifted Proportional Division, highlighting the common structure in which Individual Monotonicity characterizes the allocations that can incorporate social objectives of a redistributive nature.

Keywords: TU-games, monotonicity, egalitarian surplus sharing, proportional division, redistribution.

JEL Classification: C71 , D63

1 Introduction

Monotonicity plays a crucial role in characterizing values in cooperative games. One of the primary examples is that the strong monotonicity, together with the axioms of efficiency and symmetry, characterizes the Shapley value (Young, 1985).[1] A variety of characterizations using monotonicity conditions have also been obtained in the literature (Casajus and Yokote, 2019). Weak monotonicity, for instance, leads to the egalitarian Shapley value (Joosten, 1996; van den Brink et al, 2013; Casajus and Huettner, 2014b), while grand coalition monotonicity implies the equal division value (Casajus and Huettner, 2014b). Based on linear algebraic arguments, Yokote and Funaki (2017) provides a unified approach, in which various combinations of monotonicity conditions imply linear combinations of the corresponding values. In particular, they show that surplus+individual monotonicity, together with efficiency and symmetry, characterizes the Center of Imputation Set (CIS), introduced by Driessen and Funaki (1991).

In addition to the above result, several characterizations of the CIS value have been explored in the literature. Two kinds of characterizations are provided in van den Brink and Funaki (2009), one with consistency and standardness, and another with efficiency, symmetry, linearity and weak individual rationality. The characterization given by Casajus and Huettner (2014a) is based on the axiom of coalition surplus monotonicity, which requires that if all zero-normalized worths increase in coalitions that include a player, then the excess amount the player receives over the individual worth also increases.

Our characterization is simple and constructive. Suppose that an exogenous function specifies what is considered as a legitimate *share* of each individual in the society. The *Individual Monotonicity for Equal Surplus* (IMES) axiom requires that, if an individual's share increases, what she receives also increases, given that the surplus,

[1] In the original work by Shapley (1953), the Shapley value is characterized by the axioms of efficiency, symmetry, linearity and the null player property. The linearity axiom plays an essential role in the proof.

defined as the remainder of the grand coalition worth net of the sum of the individual shares, remains the same. Combined with the Efficiency and Equal Treatment axioms, we obtain a characterization of the Egalitarian Surplus Sharing (ESS) value. In particular, we obtain a characterization of the CIS value if the individual share is defined as the stand-alone coalition worth. Similarly, we obtain a characterization of the egalitarian non-separable contribution (ENSC) value, if the individual share is defined as the separable contribution, i.e., the increase in worth when she joins the rest of the society and forms the grand coalition (Driessen and Funaki, 1991).

A remarkable feature of our characterization results is that none of them relies on the linearity axiom. This direction of research is in line with the recent work by Nakada (2024) which uses decision-theoretic tools to provide an explanation how linearity is derived from monotonicity.

Furthermore, we show that the same proof technique can be applied to the characterization of the Proportional Division by the axiom of *Individual Monotonicity for Equal Ratio* (IMER). The proof is analogous. Instead of considering the remainder after subtracting the total individual shares from the grand coalition worth, we consider the ratio after dividing the latter by the former. We then show that the surplus should be divided proportionally. This line of characterization of the Proportional Division value is new and different from those developed in recent papers (Zou et al, 2021, 2022; van den Brink et al, 2023). A novelty of our results lies in the common structure in which the principles of egalitarian and proportional surplus sharing are derived by the Individual Monotonicity axioms.

We then provide further characterizations by dropping the Equal Treatment axiom and requiring Weak Homogeneity instead. By applying the same technique again, we show that IMES characterizes the Weighted Surplus Sharing (WSS), which includes Egalitarian Surplus Sharing as a special equal-weight case. Since the WSS is written as a class of allocations obtained by a zero-sum redistribution based on the ESS, our characterization shows that dropping equal treatment corresponds to asymmetric treatment of individuals, which allows us to incorporate social objectives of an asymmetric nature, such as minority protection, support for the disabled, consideration of seniority, and so on.

A novel finding is that the Proportional Division is also extended in an analogous way. By dropping Equal Treatment, IMER characterizes the Shifted Proportional Division, a class of allocations obtained as a zero-sum redistribution from the Proportional Division. Given that the redistribution terms are written as proportional to the Equal Surplus and Equal Ratio respectively, our results again highlight the central role of the Individual Monotonicity axioms and the common structure of the characterization.

The rest of the paper is organized as follows. The characterization results of ESS, including CIS and ENSC as special cases, are presented in Section 2. Our axiom is extended to the Individual Monotonicity for Equal Ratio, and a characterization of Proportional Division is obtained in Section 3. We relax the Equal Treatment axiom and characterize Weighted Surplus Sharing and Shifted Proportional Division in Section 4. Characterization in subdomains is considered in Section 5. Section 6 concludes.

2 Individual Monotonicity for Equal Surplus

2.1 Preliminary

Let $N = \{1, 2, \cdots, n\}$ be the set of the players. Let $\mathcal{V}^N = \left\{v : 2^N \to \mathbb{R} | v(\emptyset) = 0\right\}$ denote the set of all cooperative transferable utility games (TU-games) on N. For $S \subseteq N$, $v(S)$ is called the *worth* of coalition S. Let $\varphi : \mathcal{V}^N \to \mathbb{R}^n$ where $\varphi_i(v)$ is the *value* assigned to player $i \in N$. We say that player i and $j (\neq i)$ are *symmetric* in v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. We say that a game v is symmetric if all players are symmetric in v.

Axiom 1 (Efficiency)**.** *For any* $v \in \mathcal{V}^N$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Axiom 2 (Equal Treatment). *For any $v \in \mathcal{V}^N$, if i and j are symmetric in v, $\varphi_i(v) = \varphi_j(v)$.*

Throughout the paper, the Efficiency axiom is required. The Equal Treatment axiom is required except in Section 4, where we examine its relaxation.

2.2 A characterization of f-ESS

Suppose that there exists a function $f : \mathcal{V}^N \to \mathbb{R}^n$ which represents the *individual share*. The function f provides a general structure in what is considered as the legitimate share of each individual given the game v. It may represent the social consensus according to which how much individual share should reflect the factors such as individual contribution to the coalitions of diverse sizes. For example, f_i could be simply the stand-alone coalition worth $v(\{i\})$, or $\sum_{j \neq i} v(\{i,j\})$ where $v(\{i,j\})$ is obtained from i's total pairwise contributions, e.g., revenue from matches in a sport league (Bergantinos and Moreno-Ternero, 2020).

The surplus is defined as the remainder after subtracting the sum of individual shares from the grand coalition worth. Since efficiency is not required on f, the surplus can be positive or negative.

We consider the following axiom of Individual Monotonicity for Equal Surplus. The axiom requires the value to be monotonic in the individual share, if the surplus is equal.

Axiom 3 (f-Individual Monotonicity for Equal Surplus: f-IMES). *For any $v, w \in \mathcal{V}^N$, if*

$$v(N) - \sum_{k \in N} f_k(v) = w(N) - \sum_{k \in N} f_k(w)$$

and $f_i(v) \geq f_i(w)$, then $\varphi_i(v) \geq \varphi_i(w)$.

We also introduce the following axiom, Individualistic property for Equal Surplus (IES), which requires the value to depend solely on the individual share under equal surplus.

Axiom 4 (f-Individualistic property for Equal Surplus: f-IES)**.** *For any $v, w \in \mathcal{V}^N$, if $v(N) - \sum_{k \in N} f_k(v) = w(N) - \sum_{k \in N} f_k(w)$ and $f_i(v) = f_i(w)$, then $\varphi_i(v) = \varphi_i(w)$.*

It is straightforward to see that (f-IMES) implies (f-IES), by applying the inequality in the definition of (f-IMES) to both directions.

Our first result is that the f-IMES axiom, together with Efficiency and Equal Treatment, characterizes the Egalitarian Surplus Sharing, defined as follows:

Definition 1 (Egalitarian Surplus Sharing: ESS)**.** *We say that φ is the Egalitarian Surplus Sharing value with respect to f, if*

$$\varphi_i(v) = f_i(v) + \frac{1}{n}\left(v(N) - \sum_{k \in N} f_k(v)\right), \forall i \in N. \qquad (f\text{-ESS})$$

Before stating the theorem, we provide the following assumptions on f.

Assumption 1. *f is symmetric, that is, for any permutation π on N and for any $i \in N$, $f_i(v) = f_{\pi(i)}(\pi v)$ where the game πv is defined by $v(S) = \pi v(\pi S)$ for all $S \subseteq N$.*

Assumption 2. *For any $v \in \mathcal{V}^N$, $\exists c \in \mathbb{R}$ such that for any $x \in \mathbb{R}^n$ whose i-th component x_i is either c or $f_i(v)$ for all i, there exists $w \in \mathcal{V}^N$ which satisfies $f(w) = x$. Moreover, if $x_i = c$ for all i, then there exists a symmetric w.*

Assumption 3. *For any $v \in \mathcal{V}^N$, $f(v)$ does not depend on $v(N)$.*

Assumption 1 stipulates that the definition of individual share does not depend on the label of the players. Assumption 2 guarantees a minimal variety for games. The condition is fairly mild: it is satisfied if f is linear and symmetric.[2] Assumption 3 is imposed in order to avoid double-counting the grand coalition worth when the surplus is defined and the Efficiency axiom is considered. Function f can be non-linear. For example, $f_i(v) = v(i)^\alpha$ with $\alpha > 0$ satisfies these assumptions.

The following theorem provides a characterization of the f-ESS value.

Theorem 1. *Suppose that f satisfies Assumptions 1, 2 and 3. A value φ satisfies (Efficiency), (Equal Treatment) and (f-IMES), if and only if it is the f-ESS value.*

[2] A formal proof of this claim, extending Theorem 1 of Weber (1988), is in the Appendix.

Proof. By Assumption 1, the "if" part is obvious. We show the "only if" part.

By Assumption 2, for any v, there exist a constant $c \in \mathbb{R}$ and a sequence of games $w^0, w^1, \cdots, w^{n-1}$ such that, for $k = 0, 1, \cdots, n-1$,

$$f_i\left(w^k\right) = \begin{cases} f_i\left(v\right) & \text{if } i \leq k \\ c & \text{if } i > k \end{cases}. \tag{1}$$

For each k, let v^k be the game constructed from w^k by replacing the grand coalition worth as follows:

$$v^k(S) := \begin{cases} w^k(S) & \text{if } S \subsetneq N, \\ v\left(N\right) - \sum_{i \in N} f_i(v) + \sum_{i \in N} f_i(w^k) & \text{if } S = N. \end{cases} \tag{2}$$

By Assumption 3, we have

$$f_i(v^k) = f_i(w^k) \ \forall i, k. \tag{3}$$

For $k = n$, let $v^n = w^n = v$ and (1), (2) and (3) are satisfied. For $k = 0$, Assumption 2 guarantees that w^0 is symmetric, and thus v^0 is also symmetric, by construction.

By (2) and (3), the sequence $(v_k)_{k=0}^n$ is constructed so that the surplus remains constant for all k: $v^k\left(N\right) - \sum_{i \in N} f_i\left(v^k\right) = v^k\left(N\right) - \sum_{i \in N} f_i\left(w^k\right) = v\left(N\right) - \sum_{i \in N} f_i\left(v\right).$ Notice that the last part is independent of k. Hence, for $k = 1, 2, \cdots, n$, we have: $v^k\left(N\right) - \sum_{i \in N} f_i\left(v^k\right) = v^{k-1}\left(N\right) - \sum_{i \in N} f_i\left(v^{k-1}\right)$. Also by (1) and (3), we have $f_i\left(v^k\right) = f_i\left(v^{k-1}\right)$ for all $i \neq k$. Hence, we can apply (f-IES) to v^k and v^{k-1}, and obtain

$$\varphi_i\left(v^k\right) = \varphi_i\left(v^{k-1}\right) \ \text{for all } i \neq k. \tag{4}$$

Then, for $i \in N$,

$$\begin{cases} \varphi_i\left(v^n\right) = \varphi_i\left(v^{n-1}\right) = \cdots = \varphi_i\left(v^i\right), \\ \varphi_i\left(v^{i-1}\right) = \varphi_i\left(v^{i-2}\right) = \cdots = \varphi_i\left(v^0\right). \end{cases} \tag{5}$$

Now, consider any $k \in N$. By (Efficiency),

$$v^k(N) = \sum_{i \in N} \varphi_i\left(v^k\right) = \varphi_k\left(v^k\right) + \sum_{i \neq k} \varphi_i\left(v^k\right),$$
$$v^{k-1}(N) = \sum_{i \in N} \varphi_i\left(v^{k-1}\right) = \varphi_k\left(v^{k-1}\right) + \sum_{i \neq k} \varphi_i\left(v^{k-1}\right).$$

By (4) and (5),

$$v^k(N) - v^{k-1}(N) = \varphi_k\left(v^k\right) - \varphi_k\left(v^{k-1}\right) = \varphi_k\left(v^n\right) - \varphi_k\left(v^0\right). \tag{6}$$

By (2), $v^k(N) - v^{k-1}(N) = f_k(v) - c$. Since $v^n = v$,

$$\varphi_k(v) = f_k(v) - c + \varphi_k\left(v^0\right). \tag{7}$$

Now, consider v^0. Since v^0 is symmetric, by (Equal Treatment) and (Efficiency), $\varphi(v^0)$ should be the equal division: $\varphi_i\left(v^0\right) = v^0(N)/n$. Then, by (2),

$$\varphi_i\left(v^0\right) = \frac{1}{n}\left(v(N) + cn - \sum_{k \in N} f_k(v)\right), \forall i.$$

Combined with (7), we obtain:

$$\varphi_k(v) = f_k(v) + \frac{1}{n}\left(v(N) - \sum_{i \in N} f_i(v)\right).$$

□

Theorem 1 can be applied to the characterization of several values studied in the literature. First, suppose that the legitimate individual share is defined as the stand-alone coalition worth. By letting $f_i(v) = v(\{i\})$ for all i, above assumptions are satisfied.[3] We thus obtain a characterization of the Center of Imputation Set (CIS) value (Driessen and Funaki, 1991).

Definition 2 (CIS)**.** *The Center of the Imputation Set is:*

$$CIS_i(v) = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{k\in N} v(\{k\})\right), \ \forall i \in N.$$

The f-IMES axiom becomes as follows:

Axiom 5 (Individual Monotonicity for Equal Surplus: IMES)**.** *For any $v, w \in \mathcal{V}^N$, if*

$$v(N) - \sum_{k\in N} v(\{k\}) = w(N) - \sum_{k\in N} w(\{k\})$$

and $v(\{i\}) \geq w(\{i\})$, then $\varphi_i(v) \geq \varphi_i(w)$.

Since Assumptions 1, 2 and 3 are satisfied for $f_i(v) = v(\{i\})$, we obtain:

Corollary 2. *A value satisfies (Efficiency), (Equal Treatment) and (IMES), if and only if it is the CIS value.*

Notice that this characterization does not hinge on the linearity axiom, which is often used in the literature.

The dual concept of the CIS is the ENSC, the Egalitarian Non-Separable Contribution (Driessen and Funaki, 1991). In the ENSC, the non-separable contribution, defined as the remaining part of the grand coalition value net of the total separable contributions of all players, is distributed equally among all players.

Definition 3 (SC, NSC)**.** *The separable contribution of player i in game v is $SC_i(v) = v(N) - v(N\backslash\{i\})$. The non-separable contribution of game v is $NSC(v) = v(N) - \sum_{i\in N} SC_i(v)$.*

[3] Moreover, Assumption 2 is satisfied with an arbitrary $c \in \mathbb{R}$, since w is obtained from v by simply replacing $v(\{i\})$ with c.

Definition 4 (ENSC)**.** *The Egalitarian Non-Separable Contribution is defined by:*

$$ENSC_i(v) = SC_i(v) + \frac{1}{n} NSC(v), \ \forall i \in N.$$

A characterization of ENSC, analogous to Corollary 2, is provided by the Individual Monotonicity axiom defined over v^*, the dual of v.

Definition 5 (Dual)**.** *The dual v^* of game v is defined by $v^*(S) = v(N) - v(N \backslash S)$, for any $S \subseteq N$.*

In particular, $v^*(\{i\}) = SC_i(v), \forall i$. The surplus to be shared among the players is also defined by the dual. The IMES axiom for v^* becomes as follows:

Axiom 6 (IMES*)**.** *For any $v, w \in \mathcal{V}^N$, if*

$$v(N) - \sum_{k \in N} v^*(\{k\}) = w(N) - \sum_{k \in N} w^*(\{k\})$$

and $v^*(\{i\}) \geq w^*(\{i\})$, then $\varphi_i(v) \geq \varphi_i(w)$.

By definition, $v^*(\{i\})$ coincides with SC_i and the surplus coincides with NSC. It is straightforward to verify that this axiom is equivalent to f-IMES by letting $f_i(v) = -v(N \backslash \{i\})$.[4] We thus obtain the following characterization as another corollary of Theorem 1.

Corollary 3. *A value φ satisfies (Efficiency), (Equal Treatment) and (IMES*), if and only if it is the ENSC value.*

Simplicity of our constructive proof allows us to apply the same technique to various situations. We call our proof technique as the IM (Individual Monotonicity) method and provide its outline in the Appendix.

[4]It is equivalent to set $f(v) = v^*(\{i\})$, but the above definition guarantees that Assumption 3 is satisfied so that the proof of Theorem 1 is applied directly.

2.3 Characterization of the f-ESS family

In the previous subsection, we have first fixed a specific individual share represented by function f, and then provided a characterization of the f-ESS value. Instead, we provide here a characterization of the *f-ESS family*, the set of values that can be obtained as a result of the egalitarian surplus sharing from *some* individual share f.

First, if no restriction is imposed on f, the answer becomes trivial. Any efficient value φ can be written as an f-ESS by regarding φ itself as f. On the other hand, any f-ESS value is efficient by definition. Therefore, the set of values which can be written as an f-ESS by any f coincides with the set of all efficient values.

Second, if f is restricted to be linear, the answer is straightforward: a value φ is f-ESS for some linear f, if and only if φ is efficient and linear. This follows from linearity of (f-ESS) in Definition 1.

Now, suppose that f is linear and symmetric. Then, the set of f-ESS values turns out to include known values such as CIS, ENSC and the Equal Division. To provide a full description of the result, define a sequence of values $\left(\psi^k\right)_{k=1}^n$ as follows:

Definition 6. *For each k, define $\psi^k : \mathcal{V}^N \to \mathbb{R}^n$ by:*

$$\psi_i^k(v) = \left(1 - \frac{k}{n}\right)\left(\sum_{S:|S|=k, S\ni i} v(S)\right) - \frac{k}{n}\left(\sum_{S:|S|=k, S\not\ni i} v(S)\right) + \frac{1}{n} v(N). \tag{8}$$

In particular, note that ψ^1 coincides with the CIS, ψ^{n-1} coincides with the ENSC, and ψ^n coincides with the Equal Division: $ED_i(v) = v(N)/n \ \forall i, \forall v$.[5]

Proposition 4. *There exists a linear and symmetric function $f : \mathcal{V}^N \to \mathbb{R}^n$ such that a value φ is f-ESS, if and only if φ is written as an affine combination of $\left(\psi^k(v)\right)_{k=1}^n$,*

[5]Note that ψ^k satisfies the projection axiom if and only if $k = 1$ or $k = n - 1$: $\varphi_i(v) = v(i)\ \forall i$, for any additive game v.

that is, there exist coefficients $\left(\lambda^k\right)_{k=1}^n$ *such that* $\sum_{k=1}^n \lambda^k = 1$ *and*

$$\varphi\left(v\right) = \sum_{k=1}^{n} \lambda^k \psi^k\left(v\right). \tag{9}$$

Proof is in the Appendix. It is known that the set of values described by (9) coincides with the set of all linear, symmetric and efficient values (Ruiz et al (1998), Lemma 9).

As can be seen from the proposition, the process of deriving an efficient value φ by ESS from an arbitrary individual share f can be viewed as an efficient extension operator using the principle of egalitarian surplus sharing. Further discussion on the characterization of extension operators is beyond the scope of the current paper and readers are invited to refer to Funaki et al (2024).

3 Individual Monotonicity for Equal Ratio

3.1 A characterization of Proportional Division

The IMES axiom can be extended to the one which requires monotonicity with respect to the equal *ratio*, rather than the equal *surplus*. We then obtain a characterization of the Proportional Division value. We limit our attention to the following class

$$\mathcal{V}_+^N := \left\{ v \in \mathcal{V}^N \,\middle|\, \sum_{k\in N} v\left(\{k\}\right) > 0 \right\},$$

and let $\mathcal{F}_+ := \left\{ \mathcal{V}_+^N \to \mathbb{R}^n \middle| \sum_{k\in N} f_k(v) > 0, \forall v \in \mathcal{V}_+^N \right\}$, in order to avoid division by zero.

The IMER axiom and the PD value with respect to f are defined as follows:

Axiom 7 (f-Individual Monotonicity for Equal Ratio: f-IMER)**.** *Fix $f \in \mathcal{F}_+$. For any $v, w \in \mathcal{V}_+^N$, if*

$$\frac{v(N)}{\sum_{k \in N} f_k(v)} = \frac{w(N)}{\sum_{k \in N} f_k(w)}$$

and $f_i(v) \geq f_i(w)$, then $\varphi_i(v) \geq \varphi_i(w)$.

Definition 7 (f-PD)**.** *The Proportional Division value with respect to f is:*

$$\varphi_i(v) = \frac{f_i(v)}{\sum_{k \in N} f_k(v)} v(N). \tag{f-PD}$$

Theorem 5. *Suppose that $f \in \mathcal{F}_+$ satisfies Assumptions 1, 2 and 3.*[6] *A value φ satisfies (Efficiency), (Equal Treatment) and (f-IMER), if and only if it is the f-PD value.*

Proof. Proof is analogous to that of Theorem 1. For any $v \in \mathcal{V}_+^N$, define a sequence of games as in (1). The only difference is that we now define:

$$v^k(N) = \frac{\sum_{i \leq k} f_i(v) + (n-k)c}{\sum_{i \in N} f_i(v)} v(N) \tag{10}$$

for $k = 0, 1, \cdots, n$, instead of (2). Then, we obtain equal ratio instead of equal surplus between v^k and v^{k-1}, for $k = 1, \cdots, n$. We can therefore apply (f-IMER) and obtain (4), (5) and (6).

As in the proof of Theorem 1, apply (Efficiency) and (Equal Treatment) to v^0 and we obtain $\varphi_i\left(v^0\right) = v^0(N)/n$ for all $i \in N$. Letting $k = 0$ in (10), we have:

$$\varphi_i\left(v^0\right) = \frac{cv(N)}{\sum_{j \in N} f_j(v)}.$$

Also by (10),

$$v^k(N) - v^{k-1}(N) = \frac{f_k(v) - c}{\sum_{i \in N} f_i(v)} v(N).$$

[6] It is sufficient to require them for $v \in \mathcal{V}_+^N$.

Therefore, by (6), we obtain:

$$\varphi_k(v) = \frac{cv(N)}{\sum_{i\in N} f_i(v)} + \frac{f_k(v) - c}{\sum_{i\in N} f_i(v)} v(N) = \frac{f_k(v)}{\sum_{i\in N} f_i(v)} v(N),$$

which is equal to (f-PD). □

In particular, by letting $f_i(v) = v(\{i\})\ \forall i$, we obtain a characterization of the Proportional Division (PD) value.

Definition 8 (PD)**.** *The Proportional Division value is defined as:*

$$\varphi_i(v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N), \ \forall i \in N. \tag{PD}$$

The f-IMER axiom becomes:

Axiom 8 (Individual Monotonicity for Equal Ratio: IMER)**.** *For any $v, w \in \mathcal{V}_+^N$, if*

$$\frac{v(N)}{\sum_{k\in N} v(\{k\})} = \frac{w(N)}{\sum_{k\in N} w(\{k\})}$$

and $v(\{i\}) \geq w(\{i\})$, then $\varphi_i(v) \geq \varphi_i(w)$.

Corollary 6. *A value φ satisfies (Efficiency), (Equal Treatment) and (IMER), if and only if it is the PD value.*

4 Characterization without Equal Treatment

We have required Equal Treatment in all the characterization results obtained above. By dropping it, the value can incorporate social objectives of an asymmetric nature. In this section, we consider characterization of the values *without* the Equal Treatment axiom.

4.1 Weighted Surplus Sharing

We first drop the Equal Treatment axiom used in the characterization of the Egalitarian Surplus Sharing, and consider the Weak Homogeneity axiom instead. Let u^N be the standard N-unanimity game, that is, $u^N(S) = 1$ if $S = N$, and $u^N(S) = 0$ otherwise.

Axiom 9 (Weak Homogeneity: WH)**.** *For any* $\lambda \in \mathbb{R}$, $\varphi\left(\lambda u^N\right) = \lambda\varphi\left(u^N\right)$.

Recall that the surplus is shared equally among all individuals in the ESS. Instead, we consider Weighted Surplus Sharing (WSS), in which the surplus is shared in proportion to a constant weight vector which sums up to one (Kongo, 2019; Yang et al, 2019). More precisely, WSS is defined as follows:

Definition 9 (Weighted Surplus Sharing: WSS)**.** *A value* φ *is a Weighted Surplus Sharing value, if there exists a constant vector* $a = (a_i)_{i\in N}$ *satisfying* $\sum_{i\in N} a_i = 1$, *such that*

$$\varphi_i(v) = v(\{i\}) + a_i\left(v(N) - \sum_{k\in N} v(\{k\})\right), \ \forall i \in N, \forall v \in \mathcal{V}^N. \tag{WSS}$$

The ESS value is a special case of WSS in which $a_i = 1/n$ for all i. Also, notice that we do not impose the restriction of $a_i \geq 0, \forall i \in N$, although the weight is often assumed to be non-negative in the literature (Kongo, 2019; Yang et al, 2019). The reason is that we do not exclude general forms of surplus sharing, in which certain individuals are taxed in order to achieve a desirable redistribution. The following theorem provides a characterization of WSS.

Theorem 7. *The value* φ *satisfies (Efficiency), (IMES) and (WH) if and only if it is a WSS value.*

Proof. The proof is analogous to that of Theorem 1. Set $f_i(v) = v(\{i\})$ for all i, and $c = 0$. The only difference is the last part, which specifies the value of game v^0. Now, define a game $w := v^0(N)\,u^N$ where u^N is the standard N-unanimity game. Then,

$w(N) = v^0(N)$ and $w(\{i\}) = v(\{i\}) = 0$ for all $i \in N$. Hence, by (IMES), we obtain $\varphi_k\left(v^0\right) = \varphi_k(w), \forall k \in N$.

By (WH), we have $\varphi_k(v^0) = v^0(N)a_k, \forall k \in N$, where $a_k := \varphi_k(u^N)$. Then, by (7),

$$\begin{aligned}\varphi_i(v) &= v(\{i\}) + v^0(N)\,a_i \\ &= v(\{i\}) + a_i\left(v(N) - \sum_{k\in N} v(\{k\})\right).\end{aligned}$$

This is exactly (WSS) with $a_i = \varphi_i\left(u^N\right)$. □

In Theorem 7, we have extended the set of characterized allocations from ESS to WSS by weakening Equal Treatment to Weak Homogeneity. In order to obtain more precise mathematical boundary of the characterization between the two values, consider Weak Symmetry (van den Brink, 2007):

Axiom 10 (Weak Symmetry)**.** *For every $v \in \mathcal{V}^N$, if $v(S \cup \{i\}) = v(S \cup \{j\})$, $\forall S \subseteq N\backslash\{i, j\}$, $\forall i, j \in N$ with $i \neq j$, then there exists a constant $c \in \mathbb{R}$ such that $\varphi_i(v) = c$ for all $i \in N$.*

By definition, Equal Treatment implies Weak Symmetry. Since the Equal Treatment axiom is applied only to the unanimity game in the proof of Theorem 1 with $f_i(v) = v(\{i\})$, we obtain another characterization of ESS, by weakening Equal Treatment to Weak Symmetry. In turn, under the assumption of Efficiency, Weak Symmetry implies Weak Homogeneity. Therefore, Theorem 7 indicates that a characterization boundary between ESS and WSS lies between the requirement of Weak Symmetry and that of Weak Homogeneity.

4.2 Shifted Proportional Division

An analogous extension can be applied to characterization of the Proportional Division value. Instead of Equal Treatment, we require Weak Grand Coalition Homogeneity.[7] For $\lambda \in \mathbb{R}^+$, let $\tilde{u}^\lambda$ be the game such that:

$$\tilde{u}^\lambda (S) = \begin{cases} 1 & \text{if } S \neq N, \emptyset \\ \lambda & \text{if } S = N \end{cases}.$$

Axiom 11. *(Weak Grand Coalition Homogeneity: WGCH) For $\lambda \in \mathbb{R}^+$, $\varphi\left(\tilde{u}^\lambda\right) = \lambda\varphi\left(\tilde{u}^1\right)$.*

Definition 10 (Shifted Proportional Division: SPD)**.** *We say that φ is a Shifted Proportional Division value, if $\exists\, (b_i)_{i\in N}$ such that $\sum_i b_i = 0$ and*

$$\varphi_i (v) = \frac{v(\{i\}) + b_i}{\sum_{k\in N} v(\{k\})} v(N), \ \forall i \in N. \tag{SPD}$$

Theorem 8. *A value satisfies (Efficiency), (IMER) and (WGCH), if and only if it is a SPD value.*

Proof. The proof is again analogous to that of Theorem 5, with $f_i(v) = v(\{i\})$. Since such an f satisfies Assumption 2 with an arbitrary c, let $c = 1$. In particular, (10) holds with $c = 1$.

Now, consider $\tilde{u}^1$ and let $a_i := \varphi_i\left(\tilde{u}^1\right)$, $\forall i \in N$. By Efficiency, $\sum_{i\in N} a_i = 1$. Consider the game $\tilde{u}^\lambda$ with $\lambda = v^0(N)$. Then, by WGCH, $\varphi_i\left(\tilde{u}^\lambda\right) = \lambda\varphi_i\left(\tilde{u}^1\right) = \lambda a_i = a_i v^0(N), \forall i \in N$. Since $v^0(N) = \tilde{u}^\lambda(N)$ and $v^0(\{i\}) = \tilde{u}^\lambda(\{i\}) = 1, \forall i \in N$, we can

[7] This axiom is weaker than Grand Coalition Homogeneity, which requires homogeneity of the value when only the grand coalition worth is multiplied by a constant.

apply IMER to v^0 and $\tilde{u}^\lambda$ and obtain $\varphi_i\left(v^0\right) = \varphi_i\left(\tilde{u}^\lambda\right), \forall i \in N$. Hence,

$$\varphi_i\left(v^0\right) = \varphi_i\left(\tilde{u}^\lambda\right) = a_i v^0(N) = \frac{a_i n}{\sum_{j\in N} v(\{j\})} v(N), \forall i \in N. \tag{11}$$

By (10),

$$\varphi_k(v) = v^k(N) - v^{k-1}(N) + \varphi_k\left(v^0\right) = \frac{v(k) - 1 + a_k n}{\sum_{j\in N} v(\{j\})} v(N).$$

By letting $b_k = a_k n - 1$, we have $\sum_{k\in N} b_k = n \sum_{k\in N} a_k - n = 0$, and we obtain the result. □

Under the Efficiency assumption, if a value satisfies Equal Treatment, then it also satisfies WGCH. Our results therefore indicate the extent to which the Equal Treatment axiom can be relaxed so that the set of characterized values extends from PD to SPD.

4.3 Interpretation of the weakening of ET

We have seen above that the set of characterized values is expanded from the ESS to the WSS when the Equal Treatment axiom is relaxed. It is worth emphasizing that the WSS can be written as a shifted allocation based on the ESS. To see that, let $b_i := a_i - 1/n$, $\forall i \in N$ in (WSS). Then, we have:

$$\begin{aligned} WSS_i(v) &= v(\{i\}) + \left(\frac{1}{n} + b_i\right)\left(v(N) - \sum_{k\in N} v(\{k\})\right) \\ &= ESS_i(v) + b_i\left(v(N) - \sum_{k\in N} v(\{k\})\right). \end{aligned}$$

Similarly, the Shifted Proportional Division can be written as:

$$\begin{aligned} SPD_i(v) &= \frac{v(\{i\}) + b_i}{\sum_{k\in N} v(\{k\})} v(N). \\ &= PD_i(v) + b_i \frac{v(N)}{\sum_{k\in N} v(\{k\})}. \end{aligned}$$

In both cases, the vector of coefficients $b = (b_i)_{i\in N}$ satisfies $\sum_{i\in N} b_i = 0$, and $b = 0$ is the special case in which the Equal Treatment axiom is satisfied. Therefore, relaxing the Equal Treatment axiom corresponds to an adjustment by a zero-sum transfer proportional to the vector b, which is fixed and applied to all games v.

Our results thus imply that the extended sets of allocations can incorporate social objectives of an asymmetric nature, such as redistribution, minority protection, support for the disabled, consideration of seniority, and so on. The coefficient vector b is fixed exogeneously in each society, but the same b is applied to all games v. As seen from the expressions above, the resulting allocation is written as a redistribution based on the Egalitarian Surplus Sharing or the Proportional Division, which represents the egalitarian or proportional principle, respectively. What is common in both cases is the structure in which relaxation of the Equal Treatment axiom leads to the redistribution term in the above expressions. Notice that the term multiplied by b_i corresponds to the equal surplus and the equal ratio, specified in the Individual Monotonicity axiom, respectively. Our characterization results thus highlight the common structure in the characterization of ESS and PD, and the central role played by the Individual Monotonicity axioms.

5 Characterization in subdomains

In this section, we consider the characterization in subdomains, which would highlight a broader applicability of the IM method. We show that the same f-ESS characterization is obtained in the subdomains of convex games (Shapley, 1971), superadditive games, exact games (Schmeidler, 1972), balanced games (Shapley, 1967), and totally balanced games, or equivalently, market games (Shapley and Shubik, 1969), for $f_i(v) = v(\{i\})$.

Theorem 9. *Let $\bar{\mathcal{V}}^N$ be a subdomain of $\mathcal{V}^N$, either the set of convex games, superadditive games, exact games, balanced games, or totally balanced games. A value φ satisfies (Efficiency), (ET) and (IMES) for all $v \in \bar{\mathcal{V}}^N$, if and only if it is the CIS value for all $v \in \bar{\mathcal{V}}^N$.*

To prove the theorem, we use the following lemma (proof is in the Appendix).

Lemma 1. *Suppose that $v(S) = \sum_{i \in S} v(\{i\}), \forall S \subsetneq N$.[8] Then, v is convex, superadditive, exact, balanced and totally balanced, if and only if $v(N) \geq \sum_{i \in N} v(\{i\})$.*

Proof of Theorem 9. The if part is trivial. We show the only if part. The proof is again by the IM method. Suppose $v \in \bar{\mathcal{V}}^N$. We construct a sequence of games $\left(v^k\right)_{k=0}^n$ in $\bar{\mathcal{V}}^N$. For the stand-alone coalitions, define $v^k(\{i\})$ as in (1) with $f_i(v) = v(\{i\})$ and $c = 0$. Hence, $\sum_{i \in N} v^k(\{i\}) = \sum_{i \leq k} v(\{i\})$. Define the grand coalition worth as in (2): $v^k(N) = v(N) - \sum_{i > k} v(\{i\})$. For the coalitions of size 2 to $n-1$, define $v^k(S) := \sum_{i \in S} v^k(\{i\})$. Since the coalition worth of these sizes does not matter, the proof of Theorem 1 goes through.

It remains to verify that v^k is constructed in the subdomain $\bar{\mathcal{V}}^N$ for each k. It suffices to show that v^k is convex, since convexity implies superadditivity, exactness, balancedness and total balancedness. By Lemma 1, it remains to verify $v^k(N) \geq \sum_{i \in N} v^k(\{i\})$. By (2), $v^k(N) - \sum_{i \in N} v^k(\{i\}) = v(N) - \sum_{i \in N} v(\{i\})$. Since v is convex, v^k is also convex. □

Characterization in subdomains also applies to the Proportional Division value.

Theorem 10. *Let $\bar{\mathcal{V}}_+^N$ be a subdomain of $\mathcal{V}_+^N$, either the set of convex games, superadditive games, exact games, balanced games, or totally balanced games. A value φ*

[8] We say that "v is additive except for the grand coalition" for such a game.

satisfies (Efficiency), (ET) and (IMER) for all $v \in \bar{\mathcal{V}}_{+}^{N}$, if and only if it is the PD value for all $v \in \bar{\mathcal{V}}_{+}^{N}$.

Proof. Suppose that v is convex. For each $k = 0, \cdots n$, pick any $c > 0$ and define v^k by

$$v^k(\{i\}) = \begin{cases} v(\{i\}) & \text{if } i \leq k \\ c & \text{if } i > k \end{cases},$$

$$v^k(S) = \sum_{i \in S} v^k(\{i\}) \text{ if } S \subsetneq N, \tag{12}$$

$$v^k(N) = \frac{\sum_{i \in N} v^k(\{i\})}{\sum_{i \in N} v(\{i\})} v(N). \tag{13}$$

In this way, definition of the stand-alone and the grand-coalition worths of v^k is identical to that in Theorem 5. Since the proof does not depend on the coalition worth of size 2 to $n-1$, it goes through as in Theorem 5.

By (13), $\sum_{i \in N} v^k(\{i\}) \leq v^k(N)$ is equivalent to $\sum_{i \in N} v(\{i\}) \leq v(N)$. By (12), we can apply Lemma 1. Since v is convex, v^k is also convex for each k. □

6 Concluding remarks

In this paper, we provide a characterization of the Egalitarian Surplus Sharing value using the axioms of Individual Monotonicity for Equal Surplus, Efficiency and Equal Treatment. Our characterization demonstrates that the three axioms lead to the *egalitarian allocation principle*, according to which each individual receives the sum of the two terms, the individual share and the egalitarian share of the surplus.

When the Individual Monotonicity axiom is required for Equal Ratio, again combined with the Efficiency and Equal Treatment, we obtain the *proportional principle*, according to which each individual receives the payoff proportional to the individual share. The main structure is the same: what each individual receives is the sum of the two terms, the individual share itself and the portion of the surplus distributed proportionally to the individual share. Our characterization thus highlights the essential role of the Individual Monotonicity axioms played in the characterization of two allocational principles.

We then relax the Equal Treatment axiom and show that the set of characterized allocations is extended to the Weighted Surplus Sharing and the Shifted Proportional Division, respectively. These allocations can be written as the consequence of a zero-sum redistribution based on Egalitarian Surplus Sharing and the Proportional Division respectively. Our characterizations therefore explicitly demonstrate how the relaxation of Equal Treatment corresponds to the redistribution term in the resulting allocation. It turns out that redistribution is proportional to the equal surplus and equal ratio specified in the Individual Monotonicity axioms. Consequently, our results show that integrating social objectives of an asymmetric nature boils down to how to redistribute the equal surplus and equal ratio, respectively.

Our results are applicable to the discussion on the efficient allocation where there is a social agreement concerning the individual share which does not necessarily satisfy efficiency. A natural application is the bankruptcy problem. As our characterization relies on the monotonicity axiom based on the individual shares and the grand coalition worth, the structure of our problem fits well to the characterization of the allocations in the bankruptcy problem. Another example is the Banzhaf index. It reflects individual's influence on the social outcome, and does not satisfy efficiency in general. In the commonly used normalization, the surplus is distributed proportionally to the individual share. While characterizations of the normalized Banzhaf value are available in the literature (van den Brink and van der Laan, 1998), our characterization provides a common ground for the analysis of proportional and egalitarian surplus sharing. Although the direct comparison of two types of normalization based on the common feature of Individual Monotonicity axioms is intriguing, further analysis is beyond the scope of the current paper and we leave it for future research.

Acknowledgements. We are grateful to Hervé Moulin, Satoshi Nakada and Yuki Tamura for fruitful discussions and helpful comments.

Declarations

- Funding: The research leading to these results received funding from Investissements d'Avenir, ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047, JSPS Core-to-Core Program (JPJSCCA20200001), and JSPS KAKENHI (17H02503, 18KK004, 22H00829).
- Competing interests: None.
- Ethics approval and consent to participate: Not applicable.
- Consent for publication: Not applicable.
- Data availability: Not applicable.
- Materials availability: Not applicable.
- Code availability: Not applicable.
- Author contribution: The authors contributed equally to this work.

Appendix A Appendix

A.1 An outline of the IM method

Our proof technique, which we call the IM (Individual Monotonicity) method, is simple and easy to apply in various situations. We provide an outline to clarify the proof structure, hoping that it enhances transparency and applicability of the method. Here we use the example of the simplest case, a characterization of the CIS value.

- Suppose that φ satisfies (Efficiency), (ET) and (IMES).
- For any $v \in \mathcal{V}^N$, we construct a sequence of games $v^0, v^1, \cdots, v^n$. Set $v^n = v$.
- Let $k \in N$. Given v^k, construct v^{k-1} by (i) replacing $v(\{k\})$ by 0, (ii) subtracting $v(\{k\})$ from $v(N)$, and (iii) keeping the rest of the game unchanged. In this way,

the surplus remains constant, and thus (IMES) can be applied to all $i \neq k$. By (IMES), $\varphi_i\left(v^k\right) = \varphi_i\left(v^{k-1}\right), \forall i \neq k$.

- Since the grand coalition worth decreases by $v(\{k\})$, Efficiency implies $\varphi_k\left(v^k\right) = \varphi_k\left(v^{k-1}\right) + v(\{k\})$.
- Additionally, set $v^0(S) = 0$ for all $S \subsetneq N$ such that $2 \leq |S| < n$. Then, v^0 is symmetric, and thus (ET) implies $\varphi_i\left(v^0\right) = v^0(N)/n = \left(v(N) - \sum_{k \in N} v(\{k\})\right)/n$.
- Therefore, for each $i \in N$, $\varphi_i(v^n) = \varphi_i\left(v^i\right) = v(\{i\}) + \varphi_i\left(v^{i-1}\right) = v(\{i\}) + \varphi_i\left(v^0\right) = v(\{i\}) + \left(v(N) - \sum_{k \in N} v(\{k\})\right)/n$. The surplus is equally shared.

For the PD characterization, $v(\{k\})$ is replaced by 1, and $v(N)$ is adjusted proportionally to $\sum_{i \in N} v^k(\{i\})$, at each step. The rest of the proof is the same. For a general function f, replace f_i instead of $v(\{i\})$. For the characterizations without equal treatment, $\varphi_i\left(v^0\right)$ is not the equal division anymore, and the rest is the same.

A.2 Proofs

Proof of Proposition 4. We start with the "only if" part. Suppose there exists a linear and symmetric function $f : \mathcal{V}^N \to \mathbb{R}^n$. Then, there exist constants $\left(\alpha^k\right)_{k=1}^n \in \mathbb{R}^n$ and $\left(\beta^k\right)_{k=1}^n \in \mathbb{R}^n$ such that[9]

$$f_i(v) = \sum_{S \ni i} \alpha^{|S|} v(S) + \sum_{S \not\ni i} \beta^{|S|} v(S).$$

Then, we have:

$$\begin{aligned} \sum_{i \in N} f_i(v) &= \sum_{i \in N} \left(\sum_{S \ni i} \alpha^{|S|} v(S) + \sum_{S \not\ni i} \beta^{|S|} v(S) \right) \\ &= \sum_{S \subseteq N} \left\{ |S| \, \alpha^{|S|} + (n - |S|) \, \beta^{|S|} \right\} v(S). \end{aligned}$$

[9] This claim is implied from the same extension of Weber (1988) mentioned above.

Since φ is the f-ESS value,

$$\begin{aligned}
\varphi_i(v) &= f_i(v) + \frac{1}{n}\left(v(N) - \sum_{j\in N} f_j(v)\right) \\
&= \left(\sum_{S\ni i} \alpha^{|S|} v(S) + \sum_{S\not\ni i} \beta^{|S|} v(S)\right) + \frac{1}{n}\left(v(N) - \sum_{j\in N} f_j(v)\right) \\
&= \begin{pmatrix} \sum_{S\ni i}\left\{\alpha^{|S|} - \frac{|S|\alpha^{|S|}+(n-|S|)\beta^{|S|}}{n}\right\} v(S) \\ +\sum_{S\not\ni i}\left\{\beta^{|S|} - \frac{|S|\alpha^{|S|}+(n-|S|)\beta^{|S|}}{n}\right\} v(S) \end{pmatrix} + \frac{1}{n} v(N) \\
&= \begin{pmatrix} \sum_{S\ni i}\left\{\frac{n-|S|}{n}\left(\alpha^{|S|} - \beta^{|S|}\right)\right\} v(S) \\ +\sum_{S\not\ni i}\left\{\frac{|S|}{n}\left(\beta^{|S|} - \alpha^{|S|}\right)\right\} v(S) \end{pmatrix} + \frac{1}{n} v(N).
\end{aligned}$$

Let $\gamma^k := \alpha^k - \beta^k$ for $k = 1, \cdots, n-1$. We thus obtain:

$$\varphi_i(v) = \sum_{S\ni i}\left(1 - \frac{|S|}{n}\right)\gamma^{|S|} v(S) - \sum_{S\not\ni i}\frac{|S|}{n}\gamma^{|S|} v(S) + \frac{1}{n} v(N). \tag{A1}$$

Now, let $\bar{\varphi} := \varphi - ED$ and $\bar{\psi}^k := \psi^k - ED$. Then, (A1) implies that $\bar{\varphi}$ is written as a linear combination of $\left(\bar{\psi}^k(v)\right)_{k=1}^{n-1}$ as follows: $\bar{\varphi}(v) = \sum_{k=1}^{n-1} \gamma^k \bar{\psi}^k(v)$. Therefore, φ is written as an affine combination of $\left(\psi^k(v)\right)_{k=1}^{n-1}$ and ED.

Now we show the "if" part. Suppose that φ is written as an affine combination of $\left(\psi^k(v)\right)_{k=1}^{n-1}$ and ED as in (9). Let $f_i(v) = \sum_{S\ni i} \lambda^{|S|} v(S)$. Then, f is linear and symmetric. We show that the induced f-ESS value coincides with φ.

First, notice that when $(f_j)_{j\in N}$ are summed up, each $S \subseteq N$ is counted exactly $|S|$ times. Hence, we have

$$\begin{aligned}
\sum_{j\in N} f_j(v) &= \sum_{S\subseteq N} |S|\lambda^{|S|} v(S) \\
&= \sum_{S\ni i} |S|\lambda^{|S|} v(S) + \sum_{S\not\ni i} |S|\lambda^{|S|} v(S).
\end{aligned}$$

The f-ESS value is then:

$$
\begin{aligned}
& f_i(v) + \frac{1}{n}\left\{ v(N) - \sum_{j \in N} f_j(v) \right\} \\
= & \sum_{S \ni i} \lambda^{|S|} v(S) + \frac{1}{n}\left\{ v(N) - \left(\sum_{S \ni i} |S| \lambda^{|S|} v(S) + \sum_{S \not\ni i} |S| \lambda^{|S|} v(S) \right) \right\} \\
= & \sum_{S \ni i} \left(1 - \frac{|S|}{n} \right) \lambda^{|S|} v(S) - \sum_{S \not\ni i} \frac{|S|}{n} \lambda^{|S|} v(S) + \frac{1}{n} v(N) . \qquad \text{(A2)}
\end{aligned}
$$

By (8), (A2) is equal to:

$$
\begin{aligned}
& \sum_{k=1}^{n-1} \lambda^k \psi_i^k(v) + \left(1 - \sum_{k=1}^{n-1} \lambda^k \right) \frac{1}{n} v(N) \\
= & \sum_{k=1}^{n-1} \lambda^k \psi_i^k(v) + \lambda^n \frac{1}{n} v(N) ,
\end{aligned}
$$

which is equal to $\varphi_i(v)$ in (9). □

Proof of Lemma 1. The only if part is immediate: $v(N) < \sum_{i \in N} v(\{i\})$ implies that the core is empty. Hence, v is neither convex, nor superadditive, nor exact, nor balanced, nor totally balanced.

For the if part, it suffices to show that v is convex, since any convex game is also superadditive, exact, balanced and totally balanced. Recall additivity: $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subsetneq N$. We thus have $v(S) + v(T) = v(S \cap T) + \sum_{i \in S \cup T} v(\{i\})$ for $S, T \subsetneq N$. If $S \cup T \subsetneq N$, $\sum_{i \in S \cup T} v(\{i\}) = v(S \cup T)$. If $S \cup T = N$, the assumption implies $v(S) + v(T) \leq v(S \cap T) + v(N)$. □

A.3 Linear and symmetric f

The following is a formal statement and proof which specify the linear form of the value when f is linear and symmetric.

Proposition 11. *Suspect f is linear and symmetric, then there exist constants $\left(\alpha^k\right)_{k=1}^{n} \in \mathbb{R}^n$ and $\left(\beta^k\right)_{k=1}^{n-1} \in \mathbb{R}^{n-1}$ such that*

$$f_i(v) = \sum_{S \ni i} \alpha^{|S|} v(S) + \sum_{S \not\ni i} \beta^{|S|} v(S). \tag{A3}$$

Proof of Proposition 11. By Theorem 1 of Weber (1988), linearity of f implies that there exists a set of constants $\left(\alpha_i^S\right)_{i \in N, S \subseteq N}$ such that:

$$f_i(v) = \sum_{S \subseteq N} \alpha_i^S v(S), \forall i \in N, \forall v \in \mathcal{V}^N. \tag{A4}$$

We first show the following lemma.

Lemma 2. *Suppose that f is linear and symmetric. Then, there exists a set of constants $\left(\hat{\alpha}^S, \hat{\beta}^S\right)_{S \subseteq N}$ such that:*

$$f_i(v) = \sum_{S \ni i} \hat{\alpha}^S v(S) + \sum_{S \not\ni i} \hat{\beta}^S v(S), \forall i \in N, \forall v \in \mathcal{V}^N. \tag{A5}$$

Proof of Lemma 2. Take an arbitrary $S \subseteq N$. Consider a game v such that $v(S) \neq 0$ and $v(T) = 0$ for all $T \neq S$, $T \subseteq N$. Then, by (A4), $f_i(v) = \alpha_i^S v(S)$ and $f_j(v) = \alpha_j^S v(S)$. Now, suppose either $i, j \in S$ or $i, j \notin S$. Then, we have $\pi v = v$ for the permutation π which only exchanges i and j. Hence, symmetry implies $f_i(v) = f_j(v)$, which implies $\alpha_i^S = \alpha_j^S$. Now, for each $S \subseteq N$, replace α_i^S by $\hat{\alpha}^S$ for any i such that $i \in S$, and by $\hat{\beta}^S$ for any i such that $i \notin S$. From (A4), we obtain (A5). □

In order to proceed with the proof of Proposition 11, fix a permutation π on N. Take any $i, j \in N$ such that $\pi(i) = j$. Then, by symmetry,

$$f_i(v) = f_{\pi(i)}(\pi v) = f_j(\pi v), \forall v. \tag{A6}$$

Using Lemma 2,

$$f_j(\pi v) = \sum_{S \ni j} \hat{\alpha}^S \pi v(S) + \sum_{S \not\ni j} \hat{\beta}^S \pi v(S). \tag{A7}$$

Since $\pi v(\pi S) = v(S)$ by definition, (A7) is equal to:

$$\sum_{S \ni j} \hat{\alpha}^S v\left(\pi^{-1}(S)\right) + \sum_{S \not\ni j} \hat{\beta}^S v\left(\pi^{-1}(S)\right). \tag{A8}$$

For each S such that $S \ni j$, let $S' = \pi^{-1}(S \setminus \{j\}) \cup \{i\}$. Since $\pi(i) = j$, this induces a bijection from $\{S | S \ni j\}$ to $\{S' | S' \ni i\}$. Moreover, $\pi(S') = S$. Therefore, the first term of (A8) becomes $\sum_{S \ni j} \hat{\alpha}^S v\left(\pi^{-1}(S)\right) = \sum_{S' \ni i} \hat{\alpha}^{\pi(S')} v(S')$. Similarly, by setting $S'' = \pi^{-1}(S)$, the second term of (A8) becomes $\sum_{S \not\ni j} \hat{\beta}^S v\left(\pi^{-1}(S)\right) = \sum_{S'' \not\ni i} \hat{\beta}^{\pi(S'')} v(S'')$. Therefore, by (A7), we have $f_j(\pi v) = \sum_{S' \ni i} \hat{\alpha}^{\pi(S')} v(S') + \sum_{S'' \not\ni i} \hat{\beta}^{\pi(S'')} v(S'')$. By (A5) and (A6), the following equality should hold for any v:

$$\sum_{S \ni i} \hat{\alpha}^S v(S) + \sum_{S \not\ni i} \hat{\beta}^S v(S) = \sum_{S' \ni i} \hat{\alpha}^{\pi(S')} v(S') + \sum_{S'' \not\ni i} \hat{\beta}^{\pi(S'')} v(S'').$$

This is an identity with respect to $\{v(S)\}_{S \subseteq N}$. By comparing the coefficients of $v(S)$ on both sides for any S such that $S \ni i$, we have $\hat{\alpha}^S = \hat{\alpha}^{\pi(S)}$. Since this should hold for any $i, j \in N$ such that $\pi(i) = j$ and $i \in S$, there exists a constant α^k such that $\alpha^k = \hat{\alpha}^S$ for any $S \ni i$ such that $|S| = k$. Similarly, by comparing the coefficients of $v(S)$ for any S such that $S \not\ni i$, we obtain $\hat{\beta}^S = \hat{\beta}^{\pi(S)}$. Hence, there exists a constant β^k such that $\beta^k = \hat{\beta}^S$ for any $S \not\ni i$ such that $|S| = k$. Finally, we obtain (A3). □

In order to show that Assumption 2 is satisfied for any linear and symmetric f, we use the following lemma.

Lemma 3. *Let $i \in N$ and $S \subseteq N$. Define a matrix by $A = (a_{iS})_{i \in N, 1 \leq |S| < n}$ where $a_{iS} = \alpha^{|S|}$ if $i \in S$, and $a_{iS} = \beta^{|S|}$ if $i \notin S$. If there exists $k \in \{1, 2, \cdots, n-1\}$ such that $\alpha^k \neq \beta^k$, then the matrix A has full rank.*

Proof of Lemma 3. Suppose that A has a rank less than n. Then, there exists a linear combination of n row vectors which is equal to the zero vector, that is, there exists a non-zero vector $t = (t_i)_{i=1}^n \in \mathbb{R}^n$ such that

$$\sum_i t_i a_{iS} = 0, \forall S \subsetneq N. \tag{A9}$$

Then, for any $S \subsetneq N$ such that $|S| = k$, $\alpha^k \left(\sum_{i \in S} t_i\right) + \beta^k \left(\sum_{i \notin S} t_i\right) = 0$. Take any $i, j \in N$ such that $i \neq j$, and $S' \subseteq N \backslash \{i, j\}$ such that $|S'| = k - 1$. Let $S_1 = S' \cup \{i\}$, $S_2 = S' \cup \{j\}$, and we have:

$$\alpha^k \left(\sum_{i' \in S_1} t_{i'}\right) + \beta^k \left(\sum_{i' \notin S_1} t_{i'}\right) = 0,$$
$$\alpha^k \left(\sum_{i' \in S_2} t_{i'}\right) + \beta^k \left(\sum_{i' \notin S_2} t_{i'}\right) = 0,$$

By subtracting one from the other, we obtain $\alpha^k (t_i - t_j) - \beta^k (t_i - t_j) = 0$. Since $\alpha^k \neq \beta^k$, we have $t_i = t_j$. Since the choice of i and j was arbitrary, we have $t_1 = t_2 = \cdots = t_n$. Together with (A9), we have $t_i = 0, \forall i$, which is a contradiction. □

Finally, we have the following proposition:

Proposition 12. *Suppose f is linear and symmetric. Then, Assumption 2 is satisfied.*

Proof of Proposition 12. First, suppose $\alpha^k = \beta^k$ for all $k \in \{1, 2, \cdots, n-1\}$. Then, (A3) becomes $f_i(v) = \sum_{S \subseteq N} \alpha^{|S|} v(S)$, and thus, for any fixed v, $f_i(v)$ is a constant independent of i. Then, by letting c be this constant, Assumption 2 is satisfied.

Suppose now that $\exists k \in \{1, 2, \cdots, n-1\}$ such that $\alpha^k \neq \beta^k$. Then by Lemma 3, matrix $A = (a_{iS})$ has full rank. Then, the solution of $f(w) = x$ is given by $w = A^T(AA^T)^{-1}x$ for any $x \in \mathbb{R}^n$. When x is symmetric, so is w. Again, Assumption 2 is satisfied. □

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