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MOBILITY MEASURES FOR THE RESPONSIBILITY CUT

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ABSTRACT. We explore measures of relative social mobility in terms of equality of opportunity. This study aligns with the principles of responsibility–sensitive egalitarianism, distinguishing between responsibility and non-responsibility factors. We introduce an additive decomposability property when these factors are distinguishable and independent. We subsequently provide axiomatic characterizations of mobility indices that evaluate doubly stochastic transition matrices In addition to the conditions for the index values to be the same, we employ the postulate that equalization of life chances is desirable. Moreover, we demonstrate incompatibility in the invariance properties of dimensional changes; that is, the values of the indices cannot be constant according to the number of social ranking categories. Thus, two corresponding compromised measures are proposed.

Keywords: Mobility, Responsibility, Opportunity, Equality

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1. INTRODUCTION

In the current opportunity egalitarian paradigm, it is important to distinguish the sources of inequality; that is, to identify whether it is the outcome for which individuals should be held responsible. This prominent theory of distributive justice stems from seminal works of Dworkin (1981a,b) which suppose that "we are responsible for the consequence of the choices we make out of those convictions or preferences or personality" (Dworkin, 2000, p. 7). Equality of opportunity is achieved when inequality that

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does not stem from individual responsibility is compensated for, but inequality because of individual responsibility is kept untouched.

Meanwhile, social mobility is often used as a proxy measure of equality of opportunity (e.g., Chetty et al., 2017). The importance of focusing on process over outcomes has been argued for some time such as in Stiglitz (1999) who states "(u)nequal outcomes that serve a social function, are arrived at fairly, or are a consequence of individual exercise of responsibility are more acceptable than those that are not" (p. 46). We can indeed take note of individual responsibility by focusing on processes rather than outcomes; nevertheless, the process, or social mobility, also includes both factors that individuals should and should not be responsible for. Mobility concepts themselves follow a "more movement, more mobility" principle according to Cowell and Flachaire (2019); however, "more movement" does not necessarily mean "more equality of opportunity."

In this study, we investigate to construct measures of relative social mobility focusing on the distinction between responsibility and non-responsibility factors because "if social mobility is understood in terms of equality of opportunity, one should rely on a notion of social welfare that embodies basic principles of responsibility–sensitive egalitarianism" (Fleurbaey, 2008, p. 231). In addition to the sources of inequality, the sources of mobility should be distinguished; thus, we propose an additive decomposability property of stochastically independent factors for indices that evaluate doubly stochastic transition matrices. Furthermore, we postulate that equalization of life chances is desirable, and introduce several conditions under which the index values should be the same. We then provide axiomatic characterizations of the mobility measures for the responsibility cut, including an impossibility result between axioms.

The remainder of this paper is organized as follows. We present definitions and preparatory results in Section 2. We provide the axioms and demonstrate the axiomatic characterizations in Section 3. We conclude in Section 4.

2. DEFINITIONS

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the set of positive integers and the set of real numbers, respectively. The set of all positive (resp. nonzero) real numbers is $\mathbb{R}_{++} = \{x > 0 | x \in \mathbb{R}\}$ (resp. $\mathbb{R}^* = \{x \gtrless 0 | x \in \mathbb{R}\}$). We describe the mobility using a stochastic matrix, following Atkinson (1983). Suppose that there are two periods and n classes of income or some other social status for $n \in \mathbb{N}$. For $k = 1, ..., n$, let m_1^k be the relative number of observations in class k in period 1. The marginal discrete distribution in period 1 is indicated by the vector $\mathbf{m}_1 = \left[m_1^1, m_1^2, ..., m_1^n\right]$, and correspondingly in period 2. Thus, the mobility pattern can be represented by an $n \times n$ *transition matrix* $\mathbf{A}$, where $\mathbf{m}_2 = \mathbf{m}_1 \mathbf{A}$.

For $i, j = 1, ..., n$, let $a_{i,j}$ be i-th row and j-th column element of $\mathbf{A}$. We focus on changes in relative positions, or pure exchange mobility, so that $\mathbf{A}$ is *doubly stochastic*; that is, $\sum_{i=1}^n a_{i,j} = \sum_{j=1}^n a_{i,j} = 1$. The typical element $a_{i,j}$ is the relative frequency of observations with income or status class i in period 1 and class j in period 2. The set of $n \times n$ doubly stochastic matrices is denoted as $\mathscr{A}$.

For $i = 1, ..., n$, let λ_i denote *eigenvalue* of $\mathbf{A}$.[1] The bar notation $|\cdot|$ is used to denote absolute values. The transpose of a vector is denoted by superscript $\mathbf{T}$. The set of eigenvalues of a square matrix $\mathbf{A}$ is denoted as $\sigma(\mathbf{A})$.

The Kronecker product is indicated by $\otimes$; for example, for 2×2 matrices,

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}, \tag{1}$$

the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \otimes \mathbf{B} \equiv \begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}. \tag{2}$$

[1] There are n eigenvalues including algebraic multiplicity.

A *block matrix* is a matrix defined using smaller matrices called blocks. For example,

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \tag{3}$$

where $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, and $A_{2,2}$ are themselves matrices, is a block matrix. A matrix of the form

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & \bigstar & \cdots & \bigstar \\ & A_{2,2} & & \vdots \\ & & \ddots & \bigstar \\ \mathbf{0} & & & A_{k,k} \end{bmatrix}, \tag{4}$$

where $i = 1, ..., k$ and all blocks below the block diagonals are zero is a *block upper triangular*. A block upper triangular matrix in which all the diagonal blocks are 1×1 or 2×2 is said to be *upper quasitriangular*.

A square matrix $\mathbf{P}$ is a *permutation matrix* if exactly one element in each row and column is equal to 1 and all other elements are 0. Matrix $\mathbf{A}$ is *permutation equivalent* to $\mathbf{B}$ if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}$. Moreover, $\mathbf{A}$ is called *reducible* if $\mathbf{P}^{\mathbf{T}}\mathbf{A}\mathbf{P}$ is an upper quasitriangular matrix; otherwise, $\mathbf{A}$ is called *irreducible*.

Finally, we recall the following propositions.

Proposition 1 (Marcus and Minc, 1964, p. 133, 5.13.3)**.** *Every eigenvalue λ_i of a doubly stochastic matrix satisfies $|\lambda_i| \leq 1$.*[2]

Proposition 2 (Matsui, 2020, Lemma 1)**.** *Each of the following statements (i)–(iii) holds.*

(i) *If a doubly stochastic matrix $\mathbf{A}$ is diagonal, or identical, that is,*

$$a_{i,j} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}, \tag{5}$$

then every eigenvalue λ_i of $\mathbf{A}$ is 1.

[2]See also, for example, Gantmacher (1959, p. 100).

(ii) *If a doubly stochastic matrix* $\mathbf{A}$ *is antidiagonal, that is,*

$$a_{i,j} = \begin{cases} 0 & (i+j \neq n+1) \\ 1 & (i+j = n+1) \end{cases}, \tag{6}$$

then the absolute value of every eigenvalue λ_i *of* $\mathbf{A}$ *is* 1.

(iii) *If each element of a doubly stochastic matrix* $\mathbf{A}$ *is the same, that is,* $a_{i,j} = 1/n$ *for all* $i, j = 1, \ldots, n$*, then only one eigenvalue of* $\mathbf{A}$ *is* 1*, and the other eigenvalues are all* 0*.*

3. Factor–decomposable mobility indices

3.1. **Axioms.** We introduce the axioms for the mobility index, represented by the function $\phi : \mathscr{A} \to \mathbb{R}$. The first axiom is required for the responsibility cut. If the two transition matrices are mutually independent, then the sum of the index values of each matrix is equal to their Kronecker product.

Axiom 1 (Decomposability of independent factors)**.** For transition matrices $\mathbf{A}$ and $\mathbf{B}$, each of which are generated from two independent factors,

$$\phi(\mathbf{A} \otimes \mathbf{B}) = \phi(\mathbf{A}) + \phi(\mathbf{B}). \tag{7}$$

Example 1. For example, assume that two independent factors generate the transition matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}$$

The Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{bmatrix}. \tag{9}$$

Suppose that we observe (9) as a mobility matrix and that it is composed of **A** as a non-responsibility factor and **B** as a responsibility factor. Since **A** and **B** are independent,[3] we obtain the Kronecker product of them (not a normal matrix product). It seems that some of the "0" elements of (9) are undesirable because they may indicate no opportunity for individuals to move to another category, but all of them are because of individual responsibility. Thus, we require the index value of (9) to be additively decomposed into two factors, so that we can consider reducing opportunity inequality only because of the non-responsibility factor based on the index values.

The second axiom requires that the value of the index is invariant to a change of basis.

Axiom 2 (Permutation equivalence)**.** If the transition matrices **A** and **B** are permutation equivalent; that is, for a permutation matrix **P**,

$$\mathbf{A} = \mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}, \tag{10}$$

then

$$\phi\left(\mathbf{A}\right) = \phi\left(\mathbf{B}\right). \tag{11}$$

Example 2. $\phi\left(\mathbf{A}\right) = \phi\left(\mathbf{B}\right)$ holds for the transition matrices **A** and **B**:

$$\mathbf{A} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}, \tag{12}$$

[3]We can also say that **A** and **B** is a pair of independent Markov chains.

where

$$
\mathbf{P^T B P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = \mathbf{A}. \tag{13}
$$

Suppose that the first, second, and third rows and columns represent the first, second, and third income classes, respectively. On the one hand, the transition matrix **A** describes the situation in which $2/3$ of the first class will be the same and $1/3$ the second class in the next period; **B** describes the situation in which $1/3$ of the first class will be the same and $2/3$ the second class in the next period. In this respect, **B** may be considered better because it is more mobile. On the other hand, **A** describes the situation in which $1/3$ of the third class is persistent and $2/3$ will be the second class in the next period; **B** describes the situation in which $2/3$ of third class is persistent and $1/3$ will be the second class in the next period. This time, **A** is more mobile. That is, there is a trade-off, a kind of symmetric situation, between **A** and **B** because they are only "permuted." The index is real-valued and should be of complete order. We therefore postulate that they have the same value in such a situation.

The third axiom requires the index values to be the same in the two extreme situations: perfect immobility and perfect mobility.

Axiom 3 (Symmetry)**.** The index values of $n \times n$ matrices that are diagonal and antidiagonal are the same.

Example 3. For the transition matrices,

$$\mathbf{A}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \tag{14}$$

$\phi(\mathbf{A})=\phi(\mathbf{B})$.

A perfect mobility situation (i.e., **B**) is one in which the children of the wealthiest parents are the poorest and those of the poorest parents are the wealthiest. Aside from the fact that this society is unstable, it is the same as perfect immobility in that the status of children is determined by the status of their parents. Therefore, we consider at least these two situations, **A** and **B** to be equal (or equally bad). We part way with the mobility ordering on the principle that "more movement, more mobility" (Cowell and Flachaire, 2018, 2019), and create a normative measure for mobility *evaluation* in terms of opportunity equality.

Note that the axioms *Permutation equivalence* and *Symmetry* are independent. If **A** and **B** satisfy *Permutation equivalence*, then they are similar matrices; that is, they share the same eigenvalues, trace, determinant, etc. However, diagonal and antidiagonal matrices do not have such properties; for example, it is easy to confirm that **A** and **B** in (14) have different traces: $\mathrm{tr}(\mathbf{A})=3$ and $\mathrm{tr}(\mathbf{B})=1$. Thus, *symmetric* matrices are not *permutation equivalent* in general.

The fourth axiom requires the index to be a continuous function of the elements of the transition matrix.

Axiom 4 (Continuity)**.** $\phi(\mathbf{A})$ is a continuous function of $a_{i,j}\in\mathbf{A}$.

The fifth axiom requires that the value of the index be maximized when individuals have equal probabilities of moving to other social status categories.

Axiom 5 (Equalization of life chances)**.** The index value is maximized when all the elements of the $n\times n$ transition matrix **A** are the same; that is, $a_{i,j}=1/n$.

The term *Equalization of life chances* is introduced by Van de gaer et al. (2001). This is often considered a requirement for the equality of opportunity for transition matrices (e.g., Shorrocks, 1978; Kanbur and Stiglitz, 2016; Matsui, 2020). However, our purpose is to incorporate responsibility-sensitive egalitarianism into mobility evaluation to measure (in)equality of opportunity. Hence, this is just one of the possible desirable properties for mobility indices.

The sixth and seventh axioms require the index to be constant, regardless of the dimensions of the transition matrices.

Axiom 6 (Maximum invariance to dimensions)**.** The maximum value of the index of $n \times n$ matrix is constant for any $n \in \mathbb{N}$.

Axiom 7 (Minimum invariance to dimensions)**.** The minimum value of the index of $n \times n$ matrix is constant for any $n \in \mathbb{N}$.

These invariance properties to dimensions require the values of mobility indices to be identical if the actual situation in a society is the same, regardless of how categories are created at least in the extreme cases.

3.2. **Characterization results.** We derive the mobility indices, each of which is a function $\phi : \mathscr{A} \to \mathbb{R}$.

Theorem 1. *If the mobility index satisfies Decomposability of independent factors, Permutation equivalence, Symmetry, and Continuity, then it is represented by the function* $\phi : \mathscr{A} \to \mathbb{R}$ *such that*

$$K \log \left(\sum_{i=1}^{n} |\lambda_i|^{\alpha} \right), \tag{15}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, *and* $K, \alpha \in \mathbb{R}$.

Lemma 1. *If the index satisfies Permutation equivalence, then it is represented by a function* $\psi : \sigma(\cdot) \to \mathbb{R}$ *such that*

$$\phi(\mathbf{A}) = \psi(\sigma(\mathbf{A})). \tag{16}$$

Proof of Lemma 1. By *Permutation equivalence*, for

$$\mathbf{A} = \mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}, \tag{17}$$

we have

$$\phi(\mathbf{A}) = \phi(\mathbf{B}). \tag{18}$$

For a doubly stochastic matrix **B**, the following real Shur form exists:[4]

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{U}\mathbf{S}, \tag{19}$$

where **S** is a nonsingular matrix, and **U** is an upper quasitriangular matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_3 \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}. \tag{20}$$

Since

$$\det[\mathbf{U} - \lambda_i\mathbf{I}] = \det[\mathbf{U}_1 - \lambda_i\mathbf{I}] \cdot \det[\mathbf{U}_2 - \lambda_i\mathbf{I}], \tag{21}$$

the eigenvalues of **U** are given by the following:[5]

$$\sigma(\mathbf{U}) = \sigma(\mathbf{U}_1) \cup \sigma(\mathbf{U}_2). \tag{22}$$

We consider cases in which (i) **B** is reducible and (ii) **B** is irreducible.

(i) When **B** is reducible, $\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P} = \mathbf{U}$, or

$$\mathbf{B} = \mathbf{P}\mathbf{U}\mathbf{P}^{\mathbf{T}}. \tag{23}$$

Comparing (19) and (23), we have $\mathbf{S}^{-1} = \mathbf{P}$ and $\mathbf{S} = \mathbf{P}^{\mathbf{T}}$. Since permutation matrices **P** and $\mathbf{P}^{\mathbf{T}}$ are orthogonal, by substituting (23) into (17),

$$\mathbf{A} = \mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P} = \mathbf{P}\mathbf{P}^{\mathbf{T}}\mathbf{U}\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{U}, \tag{24}$$

[4]See, for example, Horn and Johnson (2012, p. 103, Theorem 2.3.4).
[5]See Marcus and Minc (1964, p. 23, 2.15.1) and Silvester (2000).

and by (18) and (24), the following equation holds:

$$\phi(\mathbf{A}) = \phi(\mathbf{U}) = \phi(\mathbf{B}). \tag{25}$$

Now, for the Shur form (19), $\mathbf{B} = \mathbf{S}^{-1}\mathbf{U}\mathbf{S}$, the "diagonal blocks [of $\mathbf{U}$] are completely determined by the eigenvalues [of $\mathbf{B}$]" (Horn and Johnson, 2012, p. 103). Moreover, by Lemma 3 of Perfect and Mirsky (1965),[6] if a doubly stochastic matrix $\mathbf{B}$ is reducible, then $\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}\,(=\mathbf{U})$ is a direct sum of doubly stochastic matrices; that is,

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_3 \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}. \tag{26}$$

Thus, the matrix $\mathbf{U}$, which is a direct sum of $\mathbf{U}_1$ and $\mathbf{U}_2$, is determined by the eigenvalues of $\mathbf{B}$. This implies that there exists a function ψ such that

$$\phi(\mathbf{U}) = \psi(\sigma(\mathbf{B})). \tag{27}$$

Also, by (22), the set of eigenvalues of $\mathbf{U}$ is the same as those of $\mathbf{B}$:

$$\sigma(\mathbf{U}) = \sigma(\mathbf{B}); \tag{28}$$

thus, we have

$$\phi(\mathbf{U}) = \psi(\sigma(\mathbf{U})) = \psi(\sigma(\mathbf{B})). \tag{29}$$

By (25),

$$\phi(\mathbf{A}) = \phi(\mathbf{U}) = \psi(\sigma(\mathbf{U})) = \psi(\sigma(\mathbf{B})) = \phi(\mathbf{B}). \tag{30}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are permutation equivalent, $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$.[7] Hence,

$$\phi(\mathbf{A}) = \psi(\sigma(\mathbf{A})) = \psi(\sigma(\mathbf{B})) = \phi(\mathbf{B}); \tag{31}$$

[6] See also Marcus and Minc (1964, p. 123, 5.3.1) for reducible matrix.

[7] See, for example, Horn and Johnson (2012, p. 58, Corollary 1.3.4 (a)).

that is, when **B** is reducible, we must have a function ψ, as the index, such that $\psi : \sigma(\mathbf{A}) \to \mathbb{R}$.

(ii) When **A** is irreducible, $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$ also holds because **A** and **B** are permutation equivalent. Hence, the following equation also holds:

$$\phi(\mathbf{A}) = \psi(\sigma(\mathbf{A})) = \psi(\sigma(\mathbf{B})) = \phi(\mathbf{B}). \tag{32}$$

In general, **A** can be reducible or irreducible. Therefore, if ϕ satisfies *Permutation equivalence*, then there exists a function $\psi : \sigma(\cdot) \to \mathbb{R}$ such that

$$\phi(\mathbf{A}) = \psi(\sigma(\mathbf{A})). \tag{33}$$

□

Lemma 2. *If a function $\psi : \sigma(\cdot) \to \mathbb{R}$ satisfies Decomposability of independent factors and Continuity, then it is represented by*

$$\psi(\lambda_1, ..., \lambda_n) = K \log\left(\left|\sum_{i=1}^{n} \lambda_i^{\alpha}\right|\right), \tag{34}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $K \in \mathbb{R}$, and $\sum_{i=1}^{n} \lambda_i^{\alpha} \in \mathbb{R}^*$.

Proof of Lemma 2. By *Decomposability of independent factors*,

$$\psi(\sigma(\mathbf{A} \otimes \mathbf{B})) = \psi(\sigma(\mathbf{A})) + \psi(\sigma(\mathbf{B})). \tag{35}$$

First, we consider $\mathbb{R}^n$ as the product of the lower product spaces:

$$\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q, \tag{36}$$

where $p \in \mathbb{N}$, $q \in \mathbb{N}$, and $p + q = n$. Every $\mathbf{x} \in \mathbb{R}^n$ can be represented as $\mathbf{x} = [\mathbf{x}_p, \mathbf{x}_q]$, with $\mathbf{x}_p \in \mathbb{R}^p$, $\mathbf{x}_q \in \mathbb{R}^q$, and if $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} = [\mathbf{y}_p, \mathbf{y}_q]$, $\mathbf{y}_p \in \mathbb{R}^p$, $\mathbf{y}_p \in \mathbb{R}^p$, then

$$\mathbf{x} + \mathbf{y} = [\mathbf{x}_p, \mathbf{x}_q] + [\mathbf{y}_p, \mathbf{y}_q] = [\mathbf{x}_p + \mathbf{y}_p, \mathbf{x}_q + \mathbf{y}_q]. \tag{37}$$

By Theorem 5.5.1 of Kuczma (2009, p. 138–139), if $\xi : \mathbb{R}^n \to \mathbb{R}$ is an additive function and $\mathbb{R}^n$ has decomposition (36), then there exist additive functions $\xi_p : \mathbb{R}^p \to \mathbb{R}$ and $\xi_q : \mathbb{R}^q \to \mathbb{R}$ such that[8]

$$\xi(\mathbf{x}) = \xi\left(\mathbf{x}_p, \mathbf{x}_q\right) = \xi_p(\mathbf{x}_p) + \xi_q\left(\mathbf{x}_q\right). \tag{38}$$

Put $\zeta_p(\mathbf{x}_p) = \log \xi_p(\mathbf{x}_p)$ and $\zeta_q(\mathbf{x}_p) = \log \xi_q(\mathbf{x}_p)$. Then, *Continuity* yields, by Theorem 5.5.2 of Kuczma (2009, p. 139),

$$\begin{aligned}\zeta_p(\mathbf{x}_p) + \zeta_q\left(\mathbf{x}_q\right) &= \log \xi_p(\mathbf{x}_p) + \log \xi_q\left(\mathbf{x}_q\right) \\ &= \log\left(\xi_p(\mathbf{x}_p) \times \xi_q\left(\mathbf{x}_q\right)\right).\end{aligned} \tag{39}$$

Since ξ_p and ξ_q are additive functions,

$$\begin{aligned}\log\left(\xi_p(\mathbf{x}_p) \times \xi_q\left(\mathbf{x}_q\right)\right) &= \log \xi\left(\mathbf{x}_p \otimes \mathbf{x}_q\right) \\ &= \zeta\left(\mathbf{x}_p \otimes \mathbf{x}_q\right),\end{aligned} \tag{40}$$

where $\zeta : \mathbb{R}^{p\times q} \to \mathbb{R}$, which is a composition of logarithmic and additive functions.[9] Summarizing the above, we have the following claim.

Claim 1. *If a function $\xi : \mathbb{R}^{p+q} \to \mathbb{R}$ is additive and continuous, then $\zeta : \mathbb{R}^{p\times q} \to \mathbb{R}$ is a composite of logarithmic and additive functions:*

$$\zeta\left(\mathbf{x}_p \otimes \mathbf{x}_q\right) = \log\left(x_1y_1 + x_1y_2 + x_2y_1 + ... + x_py_q\right), \tag{41}$$

where $\mathbf{x}_p = [x_1, x_2, ..., x_p]$ and $\mathbf{y}_q = [y_1, y_2, ..., y_q]$.

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[8]For example, for $\mathbf{x}_p = [p_1, p_2, p_3]$, $\mathbf{x}_q = [q_1, q_2]$,

$$\xi(p_1, p_2, p_3, q_1, q_2) = \xi_p(p_1, p_2, p_3) + \xi_q(q_1, q_2).$$

[9]For example, for $\mathbf{x}_p = [p_1, p_2, p_3]$, $\mathbf{x}_q = [q_1, q_2]$,

$$\begin{aligned}\zeta_p(p_1, p_2, p_3) + \zeta_q(q_1, q_2) &= \log(p_1 + p_2 + p_3) + \log(q_1 + q_2) \\ &= \log((p_1 + p_2 + p_3) \times (q_1 + q_2)) \\ &= \log(p_1q_1 + p_1q_2 + p_2q_1 + p_2q_2 + p_3q_1 + p_3q_2) \\ &= \zeta(p_1q_1, p_1q_2, p_2q_1, p_2q_2, p_3q_1, p_3q_2).\end{aligned}$$

Now, since eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ are $\lambda_i \mu_j$,[10] we have the following functional equation:[11]

$$F_{nm}(\lambda_1\mu_1, \lambda_1\mu_2, \lambda_2\mu_1, \lambda_2\mu_2, ..., \lambda_n\mu_m) = F_n(\lambda_1, ..., \lambda_n) + F_m(\mu_1, ..., \mu_m), \tag{42}$$

where $F_{nm} : \mathbb{R}^{n \times m} \to \mathbb{R}$, $F_n : \mathbb{R}^n \to \mathbb{R}$, and $F_m : \mathbb{R}^m \to \mathbb{R}$.

Moreover, for $\mathbf{t}_{nm} = [t_1, ..., t_{nm}]$, put $G(\mathbf{t}_{nm}) = F\left(e^{\mathbf{t}_{nm}}\right)$. By (42), for arbitrary $\mathbf{u}_n = [u_1, ..., u_n]$ and $\mathbf{v}_m = [v_1, ..., v_m]$,

$$\begin{aligned} G_n(\mathbf{u}_n) + G_m(\mathbf{v}_m) &= F_n(e^{\mathbf{u}_n}) + F_m(e^{\mathbf{v}_m}) \\ &= F_{nm}(e^{\mathbf{u}_n} e^{\mathbf{v}_m}) \\ &= F_{nm}\left(e^{\mathbf{u}_n + \mathbf{v}_m}\right) \\ &= G_{nm}(\mathbf{u}_n + \mathbf{v}_m). \end{aligned} \tag{43}$$

That is, G_{nm} is additive. Here, G_{nm} corresponds to ξ and F_{nm} corresponds to ζ in the previous argument. Hence, by Claim 1, F_{nm} is a composite of logarithmic and additive functions. We obtain, for $\sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j > 0$, $\sum_{i=1}^n \lambda_i > 0$, $\sum_{j=1}^m \mu_j > 0$, and $K, \alpha \in \mathbb{R}$,[12]

$$K \log\left(\sum_{i=1}^n \sum_{j=1}^m \lambda_i^\alpha \mu_j^\alpha\right) = K \log\left(\sum_{i=1}^n \lambda_i^\alpha\right) + K \log\left(\sum_{j=1}^m \mu_j^\alpha\right). \tag{44}$$

For $\sum_{i=1}^n \lambda_i = 1$,

$$F_n(1) = \log(1) = 0. \tag{45}$$

[10]See, for example, Mac Duffee (1933, p. 84, Corollary 43.81), Marcus and Minc (1964, p. 24, 2.15.11), and Horn and Johnson (1991, p. 245, Theorem 4.2.12).

[11]The functional equation (42) is not recognized as one of the Cauchy equations, “because of the operation of multiplication occurring in the argument” (Kuczma, 2009, p. 343). See also Aczél (1966, p. 37). Lemma 2 can be regarded as an extension of Theorem 5.5.1, 5.5.2, 13.1.2, and 13.1.5 of Kuczma (2009, pp. 139–140, 344, 348).

[12]We need two arbitrary constants because it involves two operations.

Furthermore, for $\lambda_i = -1/n$ for all $i = 1,...,n$, by (42),

$$F_{nn}(1/nn, 1/nn, ..., 1/nn) = F_n(-1/n, -1/n, ..., -1/n) + F_n(-1/n, -1/n, ..., -1/n). \tag{46}$$

Since

$$F_{nn}(1/nn, 1/nn, ..., 1/nn) = \log(1/nn + 1/nn + \cdots + 1/nn) = \log(1) = 0, \tag{47}$$

we obtain

$$F_n(-1/n, -1/n, ..., -1/n) = 0. \tag{48}$$

Thus, for $\sum_{i=1}^n \lambda_i < 0$ and $\mu_j = -1/m$ for all $j = 1,...,m$,

$$\begin{aligned} KF_n\left(\left|\sum_{i=1}^n \lambda_i^\alpha\right|\right) &= KF_{nm}\left(-\sum_{i=1}^n \lambda_i^\alpha \mu_j^\alpha\right) \\ &= KF_n\left(\sum_{i=1}^n \lambda_i^\alpha\right) + F_m(-1/m, -1/m, ..., -1/m) \\ &= KF_n\left(\sum_{i=1}^n \lambda_i^\alpha\right). \end{aligned} \tag{49}$$

Therefore, for $\sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j \in \mathbb{R}^*$, $\sum_{i=1}^n \lambda_i \in \mathbb{R}^*$, and $\sum_{j=1}^m \mu_j \in \mathbb{R}^*$,

$$K\log\left(\left|\sum_{i=1}^n \sum_{j=1}^m \lambda_i^\alpha \mu_j^\alpha\right|\right) = K\log\left(\left|\sum_{i=1}^n \lambda_i^\alpha\right|\right) = K\log\left(\left|\sum_{j=1}^m \mu_j^\alpha\right|\right), \tag{50}$$

which completes the proof. □

Proof of Theorem 1. From Lemma 1 and 2, for $\sum_{i=1}^n \lambda_i \in \mathbb{R}^*$ and $\sum_{j=1}^m \mu_j \in \mathbb{R}^*$, there exists an additive function $F : \mathbb{R}^{n\times m} \to \mathbb{R}$ such that

$$\begin{aligned} K\log(|\lambda_1^\alpha \mu_1^\alpha + \lambda_1^\alpha \mu_2^\alpha + \lambda_2^\alpha \mu_1^\alpha + \cdots + \lambda_n^\alpha \mu_m^\alpha|) =& K\log(|\lambda_1^\alpha + \cdots + \lambda_n^\alpha|) \\ &+ K\log(|\mu_1^\alpha + \cdots \mu_m^\alpha|). \end{aligned} \tag{51}$$

By *Symmetry*, the index values of the diagonal and antidiagonal matrices must be the same. From Proposition 2 (i) (ii), the absolute value of every eigenvalue of them is 1, but the sign of an eigenvalue of antidiagonal matrices can be negative. To ensure that the values of indices are constant regardless of the sign of each eigenvalue, we have

$$K\log\left(|\lambda_1|^{\alpha}+\cdots+|\lambda_n|^{\alpha}\right), \tag{52}$$

for any $K,\alpha\in\mathbb{R}$. □

Theorem 2. *If the mobility index is represented by (15), then it satisfies (i) Decomposability of independent factors, (ii) Permutation equivalence, (iii) Symmetry, and (iv) Continuity.*

Proof of Theorem 2. We show that the index (15) satisfies each axiom.

(i) Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices where $\lambda_i\in\sigma(\mathbf{A})$ and $\mu_j\in\sigma(\mathbf{B})$. The eigenvalues of the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is $\lambda_i\mu_j$. Thus, $K\log\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\lambda_i\mu_j\right|^{\alpha}\right)=K\log\left(\sum_{i=1}^{n}|\lambda_i|^{\alpha}\right)+K\log\left(\sum_{j=1}^{m}\left|\mu_j\right|^{\alpha}\right)$, and we have $\phi(\mathbf{A}\otimes\mathbf{B})=\phi(\mathbf{A})+(\mathbf{B})$.

(ii) Since $\mathbf{A}$ is permutation equivalent to $\mathbf{B}$, we have $\sigma(\mathbf{A})=\sigma\left(\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}\right)$.[13] Thus, we obtain $\phi(\mathbf{A})=\phi\left(\mathbf{P}^{\mathbf{T}}\mathbf{B}\mathbf{P}\right)$.

(iii) From Proposition 2 (i) (ii), for any diagonal and antidiagonal matrix, the absolute values of the eigenvalues are 1. Hence, the index values of diagonal and antidiagonal matrices with the same dimensions are the same.

(iv) By "the facts that the (complex) roots of a polynomial depend continuously on the coefficients of the polynomial and that the eigenvalues of a matrix depend continuously on the entries of the matrix" (Uherka and Sergott, 1977), λ_i is continuous on $a_{i,j}\in\mathbf{A}$.[14] The continuity of ϕ follows from the continuity of logarithmic functions. □

[13]See, for example, Horn and Johnson (2012, p. 58, Corollary 1.3.4 (a)).

[14]See, for example, Franklin (1968, p. 191–192, Theorem 1) for the proof using Rouché's theorem, and see also Horn and Johnson (2012, p. 122, Theorem 2.4.9.2) for the proof using Shur's unitary triangularization theorem. Uherka and Sergott (1977) provide an elementary proof.

Theorem 3. *For the mobility index represented by (15), (i) if $K \leq 0$, then it satisfies Equalization of life chances and Maximum invariance to dimensions, and (ii) if $K \geq 0$, then it satisfies Minimum invariance to dimensions.*

Proof of Theorem 3. We show that the index (15) satisfies each statement.

(i) From Proposition 2 (iii), if all elements of the transition matrix are the same, only one eigenvalue is 1 and the other eigenvalues are 0, so that the index value is $K \log 1 = 0$. In other cases, since $K \leq 0$, the index value is negative; thus, it satisfies *Equalization of life chances*. Moreover, the maximum value is always 0 for any dimensions; hence, it satisfies *Maximum invariance to dimensions*.

(ii) Since $K \geq 0$, the index value is always nonnegative, and the minimum value is 0; thus, it satisfies *Minimum invariance to dimensions*. □

Theorem 4. *If the mobility index satisfies Decomposability of independent factors, Permutation equivalence, Symmetry, Continuity, and Maximum (resp. Minimum) invariance to dimensions, then it is represented by a function $\phi : \mathscr{A} \to \mathbb{R}$ such that*

$$K \log \left(\sum_{i=1}^{n} |\lambda_i|^{\alpha} \right), \tag{53}$$

where $\lambda_i \in \sigma(\mathbf{A})$, and $\mathbf{A} \in \mathscr{A}$, $\alpha \in \mathbb{R}$, and $K \leq 0$ (resp. $K \geq 0$).

Proof of Theorem 4. According to Theorem 1, the functional form (53) is derived from the first four axioms. From Proposition 1, $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$ and one of which is always 1. Thus, $\log \left(\sum_{i=1}^{n} |\lambda_i|^{\alpha} \right)$ is minimized at 0 when only one eigenvalue is 1 and the other eigenvalues are 0, and, from Proposition 2 (iii), that is the case when all the elements of the transition matrix are the same. This index value 0 does not depend on n. Thus, by *Maximum* (resp. *Minimum*) *invariance to dimensions*, we have $K \leq 0$ (resp. $K \geq 0$). □

From the results of this section, we have the following factor–decomposable mobility index that satisfies Axioms 1–6.

Definition 1 (index I)**.**

$$K \log \left(\sum_{i=1}^{n} |\lambda_i|^{\alpha} \right), \tag{54}$$

where $K, \alpha \in \mathbb{R}$. We refer to it as "index I $(-)$" if $K \leq 0$ and "index I $(+)$" if $K \geq 0$.

Example 4. We consider the 2×2 transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \tag{55}$$

where p is real-valued and $0 \leq p \leq 1$. The values of the mobility index $\phi(\mathbf{A})$ obtained using index I $(-)$ when $K = 1$ and the base is e are plotted in Figure 1.

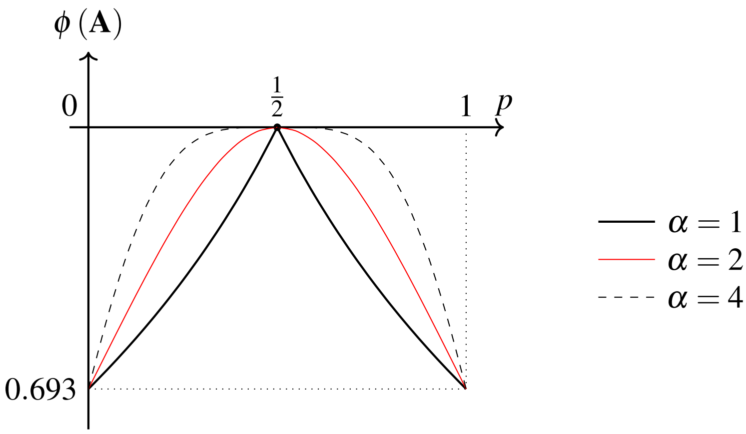

FIGURE 1. The values of the mobility index I $(-)$ in the two-dimensional case.

We can observe that the index value is a maximum when each element of the transition matrix is the same (i.e., $\forall i, j = 1, 2$, $a_{i,j} = 1/2$). Moreover, it is always nonpositive; that is, the smaller value of this index represents the more "inequality" of opportunity. Furthermore, the sensitivity of the index to deviations from the maximum varies according to the parameter α.

3.3. **Incompatibility.** We state a difficulty in our requirements as a corollary of Theorem 4.

Corollary 1. *Maximum invariance to dimensions and Minimum invariance to dimensions are incompatible for the index satisfying Decomposability of independent factors, Permutation equivalence, Symmetry, Continuity, and* $K \neq 0$*.*

Proof of Corollary 1. According to Theorem 4, if the mobility index satisfies *Decomposability of independent factors*, *Permutation equivalence*, *Symmetry*, *Continuity*, and *Maximum* (resp. *Minimum*) *invariance to dimensions*, then we have the index (53) with $K \leq 0$ (resp. $K \geq 0$). Since $K \neq 0$, the index can either satisfy *Maximum invariance to dimensions* when $K < 0$ or *Minimum invariance to dimensions* when $K > 0$. □

Given the incompatibility between *Maximum* and *Minimum invariance to dimensions*, we provide an index as a counterpart to index I $(-)$.

Theorem 5. *If, and only if, the mobility index satisfies (i) Decomposability of independent factors, (ii) Permutation equivalence, (iii) Symmetry, (iv) Continuity, (v) Equalization of life chances, and (vi) Minimum invariance to dimensions, then it is represented by a function* $\phi : \mathscr{A} \to \mathbb{R}$ *such that*

$$K \log \left(\sum_{i=1}^{n} |\lambda_i|^{\alpha} / n \right), \tag{56}$$

where $\lambda_i \in \sigma(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, $\alpha \in \mathbb{R}$, *and* $K \leq 0$.

Proof of Theorem 5 (Sufficiency). Equalization of life chances requires the index value is maximum when all elements of the transition matrix are the same. From Proposition 2 (iii), That is the case when only one eigenvalue is 1 and the other eigenvalues are 0.

From Proposition 2 (i) (ii), if the transition matrix is diagonal or antidiagonal, all absolute values of the eigenvalues are 1. From Proposition 1, $|\lambda_i| \leq 1$ for any doubly stochastic matrix, the index values of these cases are extreme values. *Minimum invariance to dimensions* requires such cases to be minimum and constant with respect to n.

From Theorem 1, for $i = 1,...,n$ and $j = 1,...,m$, we have

$$K\log\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\lambda_i\mu_j\right|^{\alpha}\right) = K\log\left(\sum_{i=1}^{n}\left|\lambda_i\right|^{\alpha}\right) + K\log\left(\sum_{j=1}^{m}\left|\mu_j\right|^{\alpha}\right). \quad (57)$$

Now we consider the case in which $\lambda_i = \mu_j = 1$ for every i, j, so that

$$K\log(nm) = K\log(n) + K\log(m). \quad (58)$$

The only condition under which $K\log(nm) = K\log(n) = K\log(m)$ is held is $n = m = 1$. Since equation 58 itself generally holds, we consider the following operation.

Subtracting (58) from (57) each side, we have

$$K\log\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\lambda_i\mu_j\right|^{\alpha}/nm\right) = K\log\left(\sum_{i=1}^{n}\left|\lambda_i\right|^{\alpha}/n\right) + K\log\left(\sum_{j=1}^{m}\left|\mu_j\right|^{\alpha}/m\right). \quad (59)$$

Thus, we obtain

$$K\log\left(\sum_{i=1}^{n}\left|\lambda_i\right|^{\alpha}/n\right) \quad (60)$$

as an index, the value of which is $K\log n/n = 0$ when $\lambda_i = 1$ for every i.

We recall that *Equalization of life chances* requires the index value to be maximum when all the elements of the transition matrix are the same, and that is the case when the only one eigenvalue is 1 and the other eigenvalues are 0. Then, we have $\log(1/n) < 0$. Therefore, we obtain $K \leq 0$. □

Proof of Theorem 6 (Necessity). We show that the index (56) satisfies each axiom.

(i) It follows from (59).

(ii) The same argument as Theorem 2 (ii) applies.

(iii) The same argument as Theorem 2 (iii) applies.

(iv) From Theorem 2 (iv), index I is a continuous function of the elements of the transition matrix. Since n is a constant, the index (56) is also continuous.

(v) From Proposition (2) (iii), when all the elements of the transition matrix are the same, then only one eigenvalue is 1 and the other eigenvalues are 0, we have

$K\log\left(\sum_{i=1}^{n}|\lambda_i|^{\alpha}/n\right) = K\log(1/n)$. In other cases, $\sum_{i=1}^{n}|\lambda_i|^{\alpha} > 1$; therefore, we have maximum value under the given condition: $K \leq 0$.

(vi) Since $|\lambda_i| \leq 1$, we have $K\log\left(\left(|\lambda_1|^{\alpha} + \cdots + |\lambda_n|^{\alpha}\right)/n\right) = K\log\left((1+\cdots+1)/n\right) = K\log(n/n) = K\log 1 = 0$, and this is the minimum value which is constant regardless of n. □

Now, we have the second factor–decomposable mobility index that satisfies Axioms 1–5 and 7 as follows.

Definition 2 (index II)**.**

$$K\log\left(\sum_{i=1}^{n}|\lambda_i|^{\alpha}/n\right), \tag{61}$$

where $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $K \leq 0$.

Example 5. For the transition matrix (55), the values of the mobility index $\phi(\mathbf{A})$ obtained using index II when $K = 1$ and the base is e are plotted in Figure 2. The graphical representation is almost the same as Figure 1, only shifted upward.

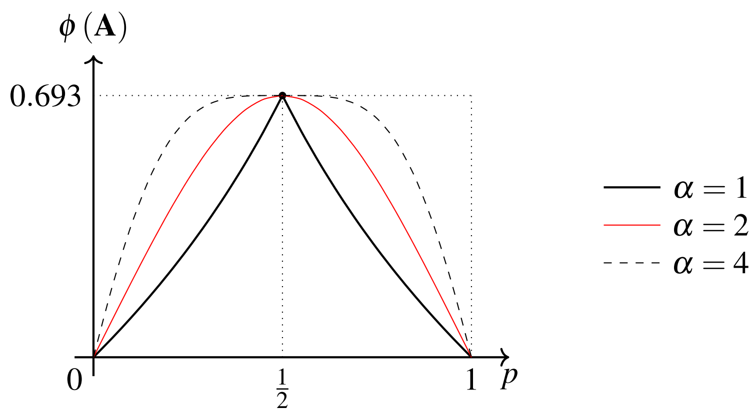

FIGURE 2. The values of the mobility index II in the two-dimensional case.

The logical relationships between the axioms and indices we provided is summarized in Figure 3.

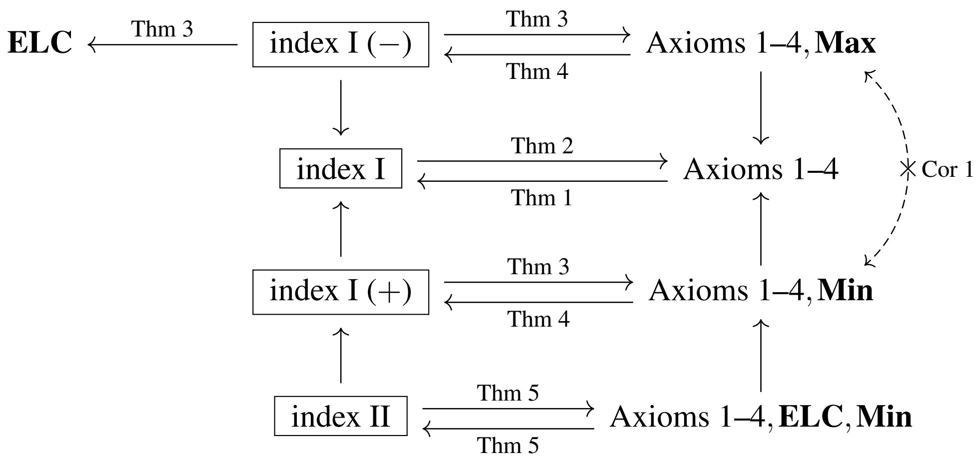

FIGURE 3. Logical implications between indices and axioms.

Axiom 1: Decomposability of independent factors
Axiom 2: Permutation equivalence
Axiom 3: Symmetry
Axiom 4: Continuity
Axiom 5: Equalization of life chances (**ELC**)
Axiom 6: Maximum invariance to dimensions (**Max**)
Axiom 7: Minimum invariance to dimensions (**Min**)

4. CONCLUDING REMARKS

We provided a fundamental framework to evaluate social mobility for the measurement of equality of opportunity, with emphasis on the additive decomposability property. Normative mobility indices for the responsibility cut were proposed and characterized axiomatically. We conclude by providing an interpretation of the incompatibility stated in Corollary 1, and by suggesting future studies to measure and evaluate mobility.

The incompatibility may be counterintuitve for indices because the index values vary according to the number of social ranking categories despite the social situation being the same. As our indices measure the dispersion of probabilities in a transition matrix, more categories provide more information on dispersion. This results in a larger index value for a larger number of categories. Our theorems and proofs show that each index is maximum or minimum invariant to dimensions only because its values are always zero at the maximum or minimum; hence, the invariance cannot be guaranteed in most cases. Moreover, if "universal" invariance is required, the following crucial disadvantage occurs. Consider reducing the number of categories of the transition matrix of the society. As the number of categories decreases, eventually any mobility pattern is described by the 1×1 transition matrix: $[1]$. Therefore, in our setting, fulfilling the conditions of *invariance to dimensions* is nothing more than deriving an index of no significance such that every index value is identical. In general, there is no comparability between the transition matrices with different number of categories; we must compare the transition matrices with the same number of categories.

The other inconvenience of our indices may stem from the axiom *Decomposability of independent factors*, which can be applied to situations in which responsibility and non-responsibility factors are independently distinguishable. By this strict requirement, although ideally demanded in political philosophy, there is difficulty in using our decomposability property for data analysis. In future studies, as sources of inequality have been estimated in recent empirical literature (e.g., Brunori et al., 2023), sources of mobility are also expected to be studied empirically. Certainly, more theoretical examinations are needed to evaluate social mobility while incorporating the notion of responsibility–sensitive egalitarianism.

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