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Optimal Schooling for Economic Growth

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Abstract: It goes without saying that education matters to promote economic growth. To examine the importance of education or schooling in economic growth, the Uzawa-Lucas model is the most popular in economics. It regards the accumulation of human capital through schooling (i.e., going to school) as the engine of economic growth. The current paper uses a generalized version of the Uzawa-Lucas model and studies the relationship between schooling-related parameters and economic growth. It is concluded that the growth rate of a macroeconomy becomes higher if workers become more patient, population grows faster, the rate of human capital depreciation becomes smaller, or the potentially maximum growth rate of human capital becomes bigger. These results may be expected intuitively. But the effect of the schooling-time elasticity of the growth rate of human capital (i.e., the exponent of the learning function) is not clear. It is shown that it depends on some conditions on schooling time.

Key words: Education, Schooling, Economic Growth, Generalized Uzawa-Lucas Model

JEL classification: E13, O41, O43

1. Introduction

Since Adam Smith (1776) emphasized the positive effect of education on the productivity of workers, education has been regarded as one of the sources of economic growth. Many macroeconomists supported Smith's idea by macro data, and Barro and Lee (2015) reconfirmed empirically that education or schooling mattered to promote economic growth in both advanced and developing countries.

When it comes to theoretical considerations which this paper is concerned about, it is Uzawa (1965) who constructed a model for the first time to examine the importance of education in economic growth. Then, after a long break, Lucas (1988) modified Uzawa's model to explain why growth rates across countries differed so much. He regarded the accumulation of human capital through schooling (i.e., going to school) as the fundamental factor of economic growth. A crucial part of the Uzawa model is a labor efficiency function which makes it possible that there exists a

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persistent growth in a two-sector model of production and education. In the Lucas model too, an indispensable part is a learning function which is the counterpart of Uzawa's labor efficiency function. The Uzawa model with a labor efficiency function and the Lucas model with a learning function have much in common both economically and mathematically. Hence the name Uzawa-Lucas model. The Uzawa-Lucas model has been extensively used to analyze the relationship between education and economic growth.

Unlike in the Uzawa model, however, there is an external effect in the production sector of the Lucas model. The growth rate is depressed due to the effect. Such a situation corresponds, Lucas (1988) argues, to actual economies. As for examples with an external effect in the production sector as in Lucas (1988), Mulligan and Sala-i-Martin (1993) calculate the steady state and simulate transitional paths toward it. Xie (1994) shows that when the external effect is relatively strong, there exists a continuum of equilibrium paths starting from the same initial condition against Lucas's conjecture. Gómez (2003) derives a fiscal policy which leads to the first-best optimum equilibrium. Hiraguchi (2009) obtains a closed-form solution by applying the method of Boucekkine and Ruiz-Tamarit (2008). Finally, as for other examples, Chamley (1993) and Kuwahara (2017) prove the existence of multiple steady states in the Uzawa-Lucas model with an external effect in the education sector.

As for examples without an external effect, on the other hand, Caballé and Santos (1993) show that every positive initial condition converges to some steady state. Faig (1995) introduces government consumption as well as private consumption and analyzes the response to technology and government spending shocks. Ortigueira (1998) examines the implications of tax policies. Boucekkine and Ruiz-Tamarit (2008) and Chilarescu (2011) pursue rigorously transitional dynamics toward a unique steady state by virtue of closed-form solutions. Canton (2002) (in discrete time) and Tsuboi (2018) (in usual continuous time) analyze the stochastic Uzawa-Lucas model with uncertainty in the education sector. And among them is Lucas (1990) too who proposes the best structure of income taxation using the CES production function and the utility function with leisure.

This paper focuses on the model by Lucas (1990) and Caballé and Santos (1993) mentioned above since they are a generalization of the original Uzawa-Lucas model and there remains much to be considered for a better understanding of the role of schooling in economic growth. The remainder of the paper is organized as follows. Section 2 introduces the model in a general form. Sections 3 and 4 specifies it to obtain clear results on the relationship between various parameters in the model and economic growth. Section 5 is a conclusion.

2. The Generalized Uzawa-Lucas Model

In the model by Lucas (1990) and Caballé and Santos (1993), there are N workers at time t whose number grows at a constatnt rate n:[1]

$$\dot{N} = nN,\ N(0) > 0.$$

Each worker is endowed with one unit of time and makes a decision between working for income and going to school for accumullation of human capital. Let u be the time used for working. Then, $1-u$ is the time used for schooling. The evolution of human capital (or skill level) h per worker is governed by the learning function

$$\frac{\dot{h}}{h} = G(1-u) - \theta,\ G' > 0,\ G'' \leq 0,\ \theta \geq 0,\ h(0) > 0, \tag{1}$$

where θ is the rate of human capital depreciation.

The production of output Y is described by a neoclassical production function

$$Y = F(K, huN),$$

where K is physical capital as a whole and huN represents effective labor as a whole. Define per capita output as $y = Y/N$ and capital per unit of effective labor as $x = K/hN$. Then, the aggregate production function can be written per capita as follows:

$$y = huf\left(\frac{x}{u}\right),\ f\left(\frac{x}{u}\right) = F\left(\frac{x}{u}, 1\right). \tag{2}$$

Let s be the gross rate of saving. Then, the physical capital accumulation equation can be written per unit of effective labor as

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[1] The following formulation is à la Uzawa (1965).

$$\dot{x} = suf\left(\frac{x}{u}\right) - [n + \pi + G(1-u) - \theta]x,\ \ \pi \geq 0,\ \ x(0) > 0, \tag{3}$$

where π is the rate of physical capital depreciation and $x(0) = K(0)/h(0)N(0)$.

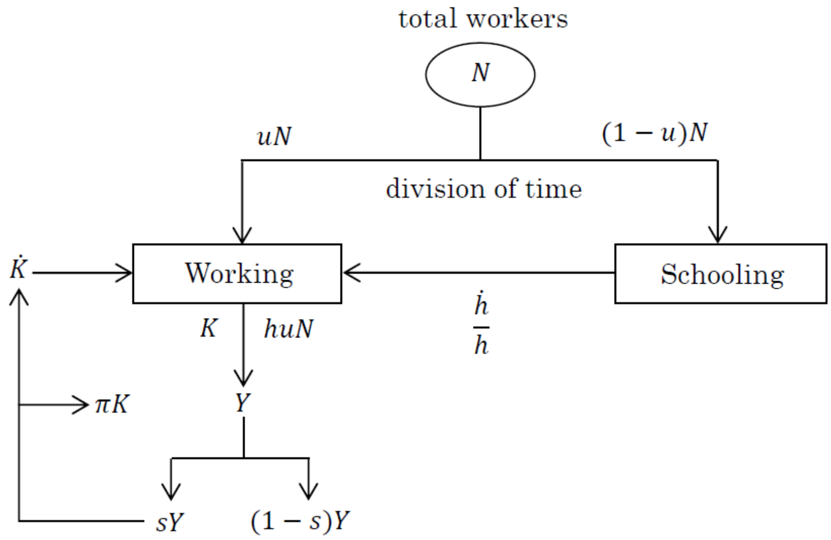

Figure 1. Division of Time between Working and Schooling

The instantaneous utility of each worker is described by the CRRA utility function

$$\frac{[(1-s)y]^{1-\sigma}}{1-\sigma},\ \ \sigma > 0,$$

where $(1-s)y$ is per capita consumption and the inverse of σ represents the intertemporal elasticity of substitution between consumptions. The purpose of workers is to maximize the sum of utilities discounted by the rate of time preference ρ

$$U = \int_0^\infty \frac{[(1-s)y]^{1-\sigma}}{1-\sigma} e^{-(\rho-n)t} dt$$

subject to equations (1) and (3) as well as the constraint

$$(1-\sigma)G(1) < \rho - n + (1-\sigma)\theta < (1-\sigma)G(0) + G'(0). \tag{4}$$

Inequality (4) is necessary for u to lie between 0 and 1. Figure 1 outlines the model under consideration.[2]

It is shown by Lucas (1990) and Caballé and Santos (1993) that in this economy there exist steady states, designated by a superscript $*$, which satisfy the following conditions:

$$\rho - n = (1-\sigma)[G(1-u^*) - \theta] + u^* G'(1-u^*), \tag{5}$$

$$f'\left(\frac{x^*}{u^*}\right) - \pi = \rho + \sigma[G(1-u^*) - \theta], \tag{6}$$

$$s^* f\left(\frac{x^*}{u^*}\right) = \frac{x^*}{u^*}[n + \pi + G(1-u^*) - \theta]. \tag{7}$$

Equation (5) represents the arbitrage condition for wages. Simply speaking, if the left-hand side exceeds the right-hand side, it is advantageous for a worker to work more for current wages by inceasing working time u. On the other hand, if the right-hand side exceeds the left-hand side, it is advantageous for a worker to go to school more for future wages by increasing schooling time $1-u$. Equation (5) must hold in steady state. Equation (6) comes from the Euler equation which represents the arbitrage condition between current and future consumption. Equation (7) is obtained by setting $\dot{x} = 0$ in equation (3).

It is easy to see that a steady-state value u^* of working time is derived from equation (5) alone. At the same time an optimal schooling time $1-u^*$ is obtained. Next, a steady-state value x^* of capital per unit of effective labor is derived from equation (5) given the value of u^*. Remember here that the number N of workers is exogenous at each momnet of time. Then, equation (6) determines a steady-state value of the *ratio* of physical capital to human capital $K^*/(h^*N)$, not their levels. Finally, a steady-state value s^* of the gross saving rate is calculated from equation (7) given the values of u^* and x^*. Thus, a steady state (or a balanced growth path) in this model is characterized by a unique set of u^*, $K^*/(h^*N)$, and s^*.[3]

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[2] $(1-u)N$ in Figure 1 represents the total amount of workers' time devoted to schooling, while in Uzawa's (1965) model it is interpreted as the total number of workers employed in the education sector, teachers in a word. For details see Sasakura (2022).

[3] If the economy is on an optimal path and the initial value of x, $x(0)(= K(0)/h(0)N(0))$, is equal to x^*, the *levels* of physical and human capital at time t can be written respectively as $K(0)e^{[n+G(1-u^*)-\theta]t}$ and $h(0)e^{[G(1-u^*)-\theta]t}$. For a steady-state value of U, U^*, and the effects of initial conditions of per capita capital stocks $h(0)$ and $k(0)(=$

3. Specifications and Some Results (1)

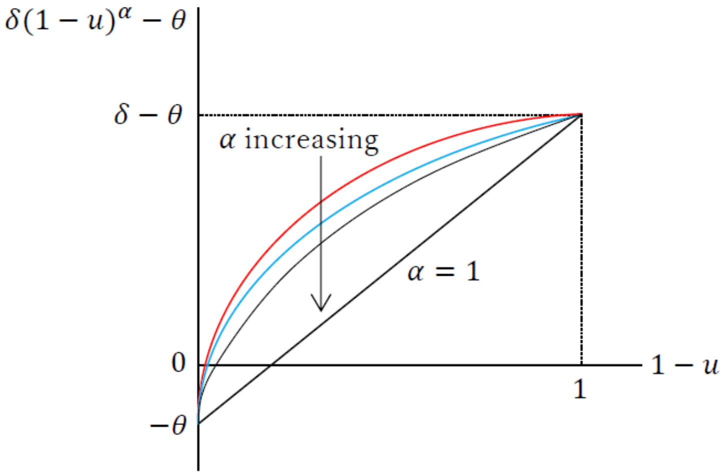

Figure 2. Learning Function

As is apparent, the steady-state growth rate in this economy is the sum of the (endogenous) steady-state growth rate of human capital and the (exogenous) growth rate of population (i.e., the number of workers). In what follows, for simplicity, let the growth rate refer to the steady-state growth rate in the economy, and let us examine the features of the growth rate by specifying the learning function (1) as

$$\frac{\dot{h}}{h} = \delta(1-u)^{\alpha} - \theta, \ \ \delta > 0, \ \ 0 < \alpha \leq 1, \tag{8}$$

where $G(1-u) = \delta(1-u)^{\alpha}$, δ is the potentially maximum growth rate of human capital,[4] and α is the schooling-time elasticity of the growth rate of human capital.[5] Figure 2 describes the relationship between schooling time $1-u$ and the corresponding growth rate $\dot{h}/h$ of human capital under a reasonable assumption that $\delta - \theta > 0$.

$K(0)/N(0)$) on it, see Appendix.

[4] Correctly speaking, the potentially maximum growth rate of human capital should be $\delta - \theta$ for $u = 0$, but for convenience it refers to δ alone in what follows.

[5] Lucas (1990) deals with the case of $\theta = 0$. Caballé and Santos (1993) take into account the case of $\theta > 0$ too, but they just give an example of the learning function with $\alpha = 1$ as in Lucas (1988).

As said in the previous section, an optimal schooling time is calculated by equation (4) alone. Then, write $g(u) = (1-\sigma)[G(1-u) - \theta] + uG'(1-u)$, which corresponds to the right-hand side of equation (5). In the case of $G(1-u) = \delta(1-u)^{\alpha}$ for $0 < u < 1$,

$$g(u) = \delta(1-u)^{\alpha-1}[\alpha u + (1-\sigma)(1-u)] - (1-\sigma)\theta, \tag{9}$$
$$g'(u) = \delta\alpha(1-u)^{\alpha-2}[\sigma(1-u) + (1-\alpha)u] > 0,$$
$$g''(u) = \delta\alpha(1-\alpha)(1-u)^{\alpha-3}[1 + \sigma(1-u) + (1-\alpha)u] > 0,$$

$g(0) = (1-\sigma)(\delta-\theta)$, $g(1) = \infty$, $g'(0) = \delta\alpha\sigma > 0$, and $g'(1) = \infty$. And constraint (4) becomes

$$(1-\sigma)(\delta-\theta) < \rho - n < \infty. \tag{10}$$

Figure 3 shows how a unique steady-state value of workig time (as well as schooling time) can be found on the basis of the above information. The graph of $g(u)$ is an upward sloping curve starting from the intercept $g(0) = (1-\sigma)(\delta-\theta)$ and tending to infinity from the left of the vertical line $u = 1$. The value of u^* is that of the horizontal axis at the interception of the graph of $g(u)$ and the horizontal line $\rho - n$. Thus, inequality $(1-\sigma)(\delta-\theta) < \rho - n$ in (10) gurantees that $u^* > 0$, which implies that workers are engaged in producing goods without fail in steady state. On the other hand, inequality $\rho - n < \infty$ in (10) gurantees that $u^* < 1$, which implies that workers goes to school without fail in steady state.

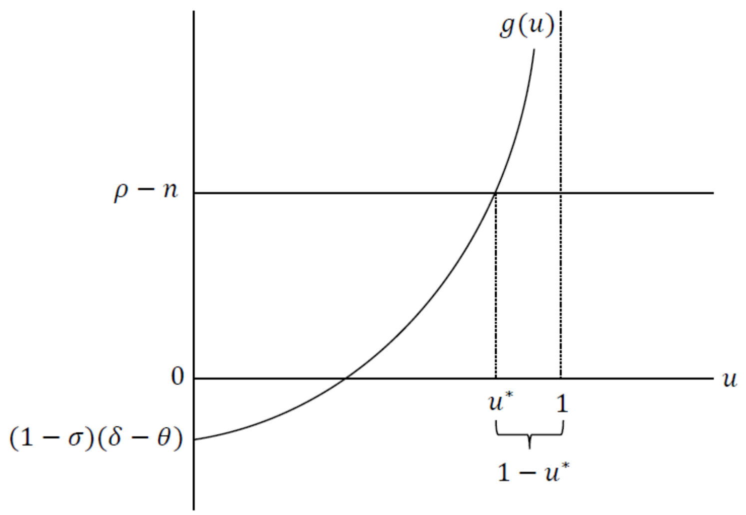

Figure 3. Optimal Schooling Time $1 - u^*$

It is seen from Figure 3 that a decrease in $\rho - n$ leads to a downward shift of the horizontal line $\rho - n$ which in turn causes an increase in optimal schooling time $1 - u^*$. Since the growth rate is an increasing function of schooling time as in Figure 2, the following results are established.

Result 1. The more patient workers are (i.e., the smaller ρ is), the higher the growth rate is.
Result 2. The faster population grows (i.e., the larger n is), the higher the growth rate is.

An increase in the rate θ of human capital depreciation shifts the graph of $g(u)$ down for $1 - \sigma > 0$ or up for $1 - \sigma < 0$. The graph does not move for $1 - \sigma = 0$. As is known from equation (8), the steady-state growth rate of human capital is the difference between $\delta(1 - u^*)^\alpha$ and θ. Since in the case of $1 - \sigma \geq 0$ schooling time $1 - u^*$ decreases or remains unchanged in response to an increase in θ, the growth rate becomes lower. In the case of $1 - \sigma < 0$, however, the two terms in equation (8) increase. So the effect on the growth rate of human capital cannot be seen at once. But simple calculations show that it is negative in the case of $1 - \sigma < 0$ too.[6] Hence the following result.

Result 3. The smaller the rate of human capital depreciation is (i.e., the smaller θ is), the higher the growth rate is.

How about the effect of the potentially maximum growth rate δ of human capital on the growth rate? First notice that $g'(u)$ is always an increasing function of δ. Next notice that when δ rises the intercept of $g(u)$, $(1 - \sigma)(\delta - \theta)$, goes up for $1 - \sigma > 0$ and stays at the origin for $1 - \sigma = 0$. Then, it is easy to know that a rise in δ leads to an increase in schooling time which in turn brings about a higher growth rate of human capital for $1 - \sigma \geq 0$. If $1 - \sigma < 0$, the intercept goes down in response to a rise in δ. This makes the effect ambiguous. In fact, by total differentiation of equation $g(u) = \rho - n$ the following results on the relationship between δ and

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[6] Differentiating equation $g(u) = \rho - n$ totally gives $\frac{du^*}{d\theta} = \frac{1-\sigma}{\delta\alpha(1-u^*)^{\alpha-2}[\sigma(1-u^*)+(1-\alpha)u^*]} < 0$ for $1 - \sigma < 0$. Using it, it is shown that $d\left(\frac{\dot{h}^*}{h^*}\right)/d\theta = -\delta\alpha(1 - u^*)^{\alpha-1}\frac{du^*}{d\theta} - 1 = -\frac{(1-u^*)+(1-\alpha)u^*}{\sigma(1-u^*)+(1-\alpha)u^*} < 0$.

schooling time $1-u^*$ can be established.[7]

Result 4. If $1-\sigma<0$,

(1) the greater the potentially maximum growth rate of human capital is (i.e., the greater δ is), the longer the schooling time is for $u^*>\frac{-(1-\sigma)}{\alpha-(1-\sigma)}$ or $1-u^*<\frac{\alpha}{\alpha-(1-\sigma)}$,

(2) the potentially maximum growth rate of human capital does not affect the schooling time for $u^*=\frac{-(1-\sigma)}{\alpha-(1-\sigma)}$ or $1-u^*=\frac{\alpha}{\alpha-(1-\sigma)}$,

(3) the greater the potentially maximum growth rate of human capital is (i.e., the greater δ is), the shorter the schooling time is for $u^*<\frac{-(1-\sigma)}{\alpha-(1-\sigma)}$ or $1-u^*>\frac{\alpha}{\alpha-(1-\sigma)}$.

As is seen from equation (8), the effect of δ on the growth rate of human capital appears in the product of two terms δ and $(1-u^*)^\alpha$. Thus, in the case of (1) and (2) above, the effect of δ on the growth rate of human capital is the same as in the case of $1-\sigma\geq 0$. In the case of (3), however, the two terms change in the opposite directions. So the effect cannot be seen at once. But some calculations reveal that the effect is positive in the case of (3) too.[8] Finally the considerations so far can be summarized as follows.

Result 5. The greater the potentially maximum growth rate of human capital is (i.e., the greater δ is), the higher the growth rate is.

In order to calculate the gross rate s^* of saving in equation (7) explicitly, specify the production function as $Y=K^\beta(huN)^{1-\beta}$ or $f\left(\frac{x}{u}\right)=\left(\frac{x}{u}\right)^\beta$, $0<\beta<1$. Tables 1 and 2 illustrate numerical examples of Result 5, in which $\sigma=2$, $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $\rho-n=0.05$, and $\rho+\pi=0.11$. Table 1 corresponds to Result 4 (1), where δ and the schooling time move in the same direction. On the other hand, Table 2 corresponds

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[7] Differentiating equation $g(u)=\rho-n$ totally gives $\frac{du^*}{d\delta}=-\frac{(1-u^*)\{(1-\sigma)+[\alpha-(1-\sigma)]u^*\}}{\delta\alpha[\sigma(1-u^*)+(1-\alpha)u^*]}\lesseqgtr 0$ for $u^*\gtreqless\frac{-(1-\sigma)}{\alpha-(1-\sigma)}$ or $1-u^*\lesseqgtr\frac{\alpha}{\alpha-(1-\sigma)}$.

[8] Using the result of the previous footnote, it is shown that $d\left(\frac{\dot{h}^*}{h^*}\right)/d\delta=(1-u^*)^{\alpha-1}-\delta\alpha(1-u^*)^{\alpha-1}\frac{du^*}{d\delta}=\frac{(1-u^*)^\alpha}{\sigma(1-u^*)+(1-\alpha)u^*}>0$ for $u^*<\frac{-(1-\sigma)}{\alpha-(1-\sigma)}$ or $1-u^*>\frac{\alpha}{\alpha-(1-\sigma)}$.

to Result 4 (3), where δ and the schooling time move in the opposite directions.[9] But in either table δ and the growth rate move in the same direction as Result 5 shows.[10, 11]

δ	θ	α	u^*	$\delta(1-u^*)^\alpha - \theta$	s^*
0.06	0.01	0.5	0.85	0.01	0.18
0.09	0.01	0.5	0.80	0.03	0.18
0.15	0.01	0.5	0.76	0.06	0.17

Table 1. Numerical Examples for $u^* > \frac{-(1-\sigma)}{\alpha-(1-\sigma)} = \frac{2}{3}$ or $1-u^* < \frac{\alpha}{\alpha-(1-\sigma)} = \frac{1}{3}$

Source: Author's calculations.

δ	θ	α	u^*	$\delta(1-u^*)^\alpha - \theta$	s^*
0.06	0.05	0.5	0.60	− 0.01	0.19
0.09	0.05	0.5	0.63	0.00	0.18
0.15	0.05	0.5	0.64	0.04	0.18

Table 2. Numerical Examples for $u^* < \frac{-(1-\sigma)}{\alpha-(1-\sigma)} = \frac{2}{3}$ or $1-u^* > \frac{\alpha}{\alpha-(1-\sigma)} = \frac{1}{3}$

Source: Author's calculations.

[9] As Dinerstein et al. (2022) explain, estimates of the rate of human capital depreciation vary considerably among the models used. They estimate the skill depreciation rate of teachers in Greece as 0.043.

[10] Numerical examples of Result 3 are also obtained by comparing Tables 1 and 2. For example, the growth rate increases from 0.04 to 0.06 if the rate θ of human capital depreciation falls from 0.05 to 0.01 for $\delta = 0.15$.

[11] This paper focuses on steady states in the model, as is usual in the literrature. But Caballé and Santos (1993) show that there exist three kinds of transitional dynamics, the normal, paradoxical, and exogenous growth cases. According to them, the numerical examples in this paper belongs to the normal case because $\sigma = 2 > \frac{1}{3} = \beta$. In such a case, for example, a sudden increase in the steady-state physical capital moves the economy toward a new steady state with higher amount of both physical and human capital than a previous one. As in Footnote 3 let an initial condition of a previous steady state be $x(0)(= K(0)/h(0)N(0))$ and that of a new steady state be $x'(0)(= K'(0)/h'(0)N(0))$. Then, the normal case implies that $K(0) < K'(0)$, $h(0) < h'(0)$, and $x(0) = x'(0)$.

4. Specifications and Some Results (2)

This section examines the effect of the schooling-time elasticity α of the growth rate of human capital on the growth rate (of the economy as a whole). To do so, calculate the derivative of $g(u)$ with respect to α. Then,

$$\frac{dg(u)}{d\alpha} = \delta(1-u)^{\alpha-1}\{[(1-\sigma)-(1-\sigma-\alpha)u]\log(1-u)+u\}. \quad (11)$$

In order to know the sign of the above derivative, write

$$a(u) = [(1-\sigma)-(1-\sigma-\alpha)u]\log(1-u)+u,$$

which is part of the right-hand side of equation (11), and differentiate it once and twice with respect to u. Then,

$$a'(u) = -(1-\sigma-\alpha)\log(1-u) - \frac{\alpha}{1-u} + \alpha + \sigma,$$

$$a''(u) = \frac{1-\sigma-\alpha}{1-u} - \frac{\alpha}{(1-u)^2}.$$ [12]

It is easy to check that $a'(0) = \sigma$. And further calculations give the following two kinds of facts about the slope of the graph of $a'(u)$ (i.e., the sign of $a''(u)$):

Case 1. When $0 < \alpha < \frac{1}{2}$ and $0 < \sigma < 1-2\alpha$,

$$a''(u)\begin{cases} > 0 & \text{for } 0 < u < \bar{u} \\ = 0 & \text{for } u = \bar{u} \\ < 0 & \text{for } \bar{u} < u < 1 \end{cases}$$

where $\bar{u} = \frac{1-\sigma-2\alpha}{1-\sigma-\alpha}$.

Case 2. When $0 < \alpha < \frac{1}{2}$ and $1-2\alpha \leq \sigma$, or $\frac{1}{2} \leq \alpha \leq 1$ and $\sigma > 0$, $a''(u) < 0$.

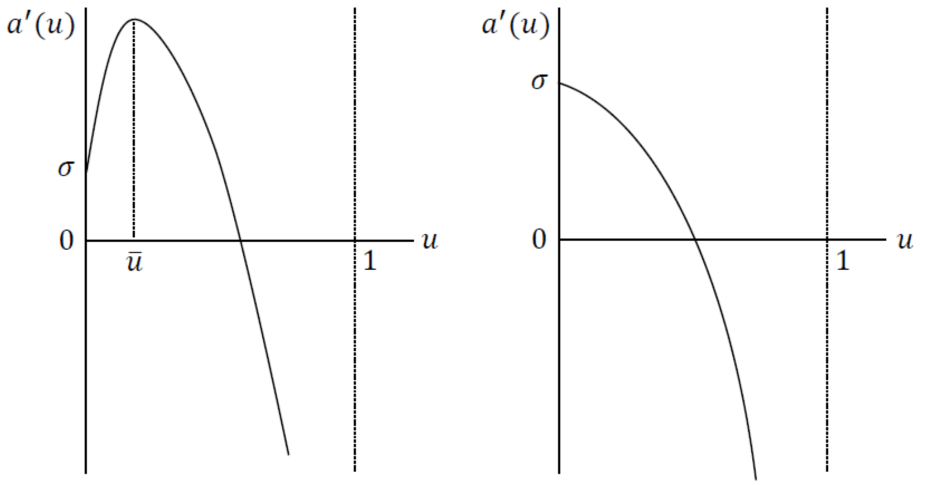
It can also be checked that $a''(1) = -\infty$ in either case. Using these considerations, the graphs of $a'(u)$ are drawn according to Cases 1 and 2 as Figure

[12] The third derivative becomes $a'''(u) = \frac{(1-\sigma-3\alpha)-(1-\sigma-\alpha)u}{(1-u)^3}$, the sign of which can be negative, zero, or positive depending on the magnitudes of σ and α.

4 (1) and (2), respectively. Using Figure 4 as well as the facts that $a(0) = 0$ and $a(1) = -\infty$, the graphs of $a(u)$ are drawn according to Cases 1 and 2 as Figure 5 (1) and (2), respectively. Finally, on the basis of Figure 5 the following result about equation (11) is obtained.

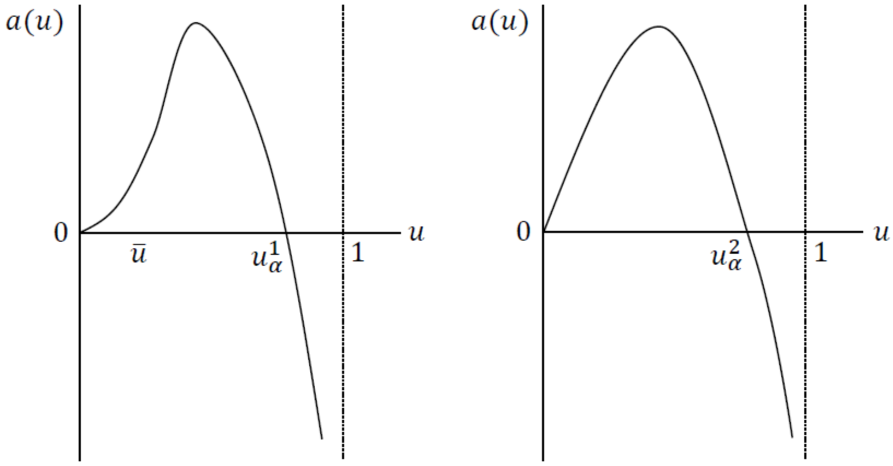
Result 6. There always exists u_α such that

$$\frac{dg(u)}{d\alpha} = \delta(1-u)^{\alpha-1}a(u) \begin{cases} > 0 & \text{for } 0 < u < u_\alpha \\ = 0 & \text{for } u = u_\alpha \\ < 0 & \text{for } u_\alpha < u < 1 \end{cases}$$



(1) $0 < \alpha < \frac{1}{2}, 0 < \sigma < 1 - 2\alpha$ (2) $0 < \alpha < \frac{1}{2}, 1 - 2\alpha \le \sigma$ or $\frac{1}{2} \le \alpha \le 1, \sigma > 0$

Figure 4. Graphs of $a'(u)$



(1) $0 < \alpha < \frac{1}{2}, 0 < \sigma < 1 - 2\alpha$ (2) $0 < \alpha < \frac{1}{2}, 1 - 2\alpha \leq \sigma$ or $\frac{1}{2} \leq \alpha \leq 1, \sigma > 0$

Figure 5. Graphs of $a(u)$

Now I am in a position to confirm the effect of the schooling-time elasticity α of the growth rate of human capital on the growth rate (of the economy). Figure 6 represents the response of the steady-state value of u to a rise in the elasticity α. The steady-state value of u increases in Panel (1), while it remains unchanged in Panel (2). In other words, the schooling time decreases in the former, while it does not change in the latter. On the other hand, it is apparent from Figure 2 that given u^* the growth rate of human capital decreases in response to a rise in α. Therefore, it can be said at once that in both cases the growth rate decreases when α rises. How about the case of Panel (3) in which the schooling time increases? A rise in α decreases the growth rate of human capital given u^*, whereas an increase in the schooling time moves it in the opposite direction. By a visual inspection only the effect of α on the growth rate of human capital cannot be seen. But some calculations reveal that the effect is positive after all.[13] Hence the following result.

Result 7. When an initial steady-state value of u is smaller than (greater than or equal to) u_α defined in Result 6, an increase in the schooling-time elasticity α of the growth rate of human capital leads to an increase (a decrease) in the growth rate.

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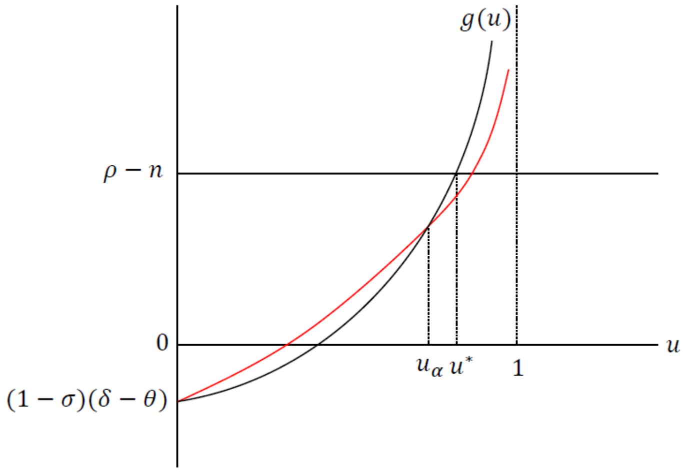
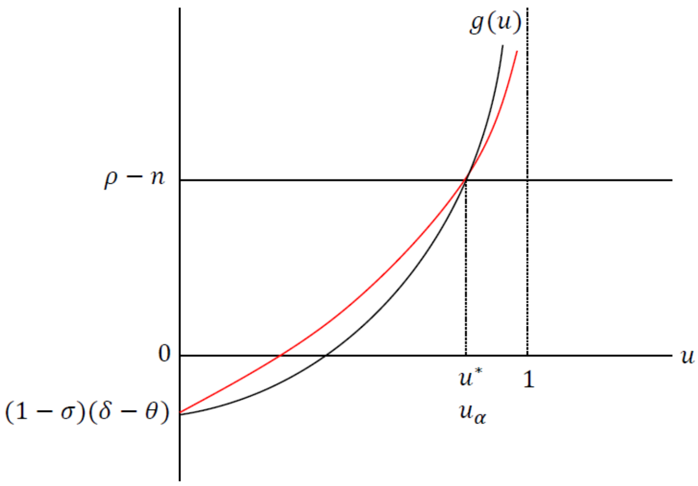
[13] Differentiating equation $g(u) = \rho - n$ totally gives $\frac{du^*}{d\alpha} = -\frac{(1-u)a(u)}{\alpha[\sigma(1-u^*)+(1-\alpha)u^*]} < 0$ for $u^* < u_\alpha$. Using it, it is shown that $d\left(\frac{\dot{h}^*}{h^*}\right)/d\alpha = \frac{du^*}{d\alpha}\delta(1-u^*)^\alpha \log(1-u^*) > 0$.

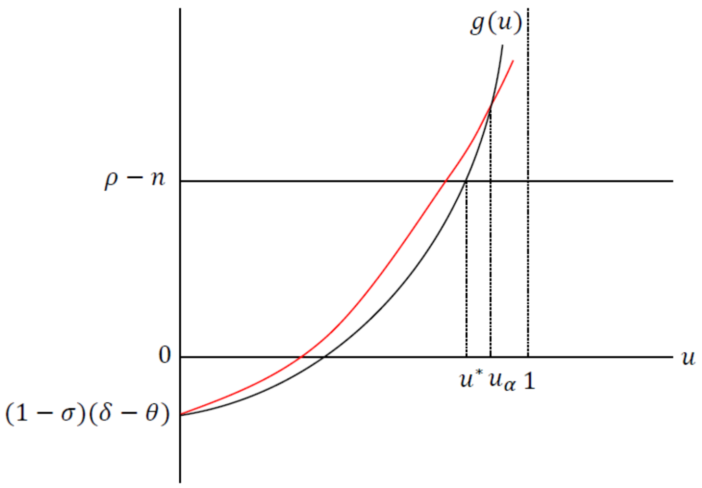
For example, when $\sigma = 2$ and $\alpha = \frac{1}{2}$ as in Tables 1 and 2, $a(u) = (-1 + \frac{3}{2}u)\log(1 - u) + u$. Thus, $u_\alpha = 0.91$ by solving $a(u) = 0$. Since $u^* < u_\alpha$ for all examples in Tables 1 and 2, Result 7 implies that an increase in α from $\frac{1}{2}$ causes each growth rate to rise *ceteris paribus*.



(1) $u^* > u_\alpha$



(2) $u^* = u_\alpha$



(3) $u^* < u_\alpha$

Figure 6. Response of the Steady-State Value u^* to a Rise in the Elasticity α

5. Conclusion

By specifying the learning function in the endogenous growth model of Lucas (1990) and Caballé and Santos (1993), this paper examined how the growth rate of the economy is affected by changes in various parameters in the model. It is schooling time or education that matters in the analysis. It is concluded that the growth rate becomes higher if workers become more patient, population grows faster, the rate of human capital depreciation becomes smaller, or the potentially maximum growth rate of human capital becomes bigger. These results may be intuitively expected, though they cannot always be established easily due to the existence of conflicting effects. When it comes to the effect of the schooling-time elasticity of the growth rate of human capital (i.e., the exponent of the learning function), it depends on the magnitudes of an initial steady-state value of schooling time $1-u^*$ and its critical value $1-u_\alpha$. That is, if the former exceeds (does not exceed) the latter, an increase in the schooling-time elasticity leads to an increase (a decrease) in the growth rate.

Finally two limitations should be mentioned. First, needless to speak, the results of this paper is based on a particlur form of a learning function. It is an open question whether other learning functions support them or not. Second, it is not obvious how these theoretical results are related to empirical facts. Actually it seems, according to Savvides and Stengos (2009), that there is a nonlinear relationship between human capital and economic growth. I.e., for countries with low levels of human capital (measured by mean years of schooling) the effect on economic growth is negative, while it becomes positive at middle levels. For countries with high human capital, the positive effect becomes weak. The model of this paper cannot explain all of them. Numerical examples in Tables in 1 and 2 may correspond respectively to middle-level and low-level countries to some extent.

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Appendix

Let k denote per capita capital stock $\frac{K}{N}$. Then, the steady-state utility is calculated as follows.

$$U^* = \int_0^\infty \frac{[(1-s^*)y^*]^{1-\sigma}}{1-\sigma} e^{-(\rho-n)t} dt$$

$$= \int_0^\infty \frac{\left[(1-s^*)h(0)e^{[G(1-u^*)-\theta]t} u^* f\left(\frac{x^*}{u^*}\right)\right]^{1-\sigma}}{1-\sigma} e^{-(\rho-n)t} dt$$

$$= \int_0^\infty \frac{\left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - s^* u^* f\left(\frac{x^*}{u^*}\right)\right] h(0)\right\}^{1-\sigma}}{1-\sigma} e^{-\{\rho-n-(1-\sigma)[G(1-u^*)-\theta]\}t} dt$$

$$= \frac{\left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - x^*[n+\pi+G(1-u^*)-\theta]\right] h(0)\right\}^{1-\sigma}}{1-\sigma} \int_0^\infty e^{-u^* G'(1-u^*)t} dt$$

$$= \frac{\left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - x^*[\rho+\pi+\sigma[G(1-u^*)-\theta]-u^* G'(1-u^*)]\right] h(0)\right\}^{1-\sigma}}{(1-\sigma)u^* G'(1-u^*)}$$

$$= \frac{\left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - x^*\left(f'\left(\frac{x^*}{u^*}\right) - u^* G'(1-u^*)\right)\right] h(0)\right\}^{1-\sigma}}{(1-\sigma)u^* G'(1-u^*)}$$

$$= \frac{\left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - x^* f'\left(\frac{x^*}{u^*}\right)\right] h(0) + u^* G'(1-u^*)k(0)\right\}^{1-\sigma}}{(1-\sigma)u^* G'(1-u^*)}$$

because of equations (2), (4), (5), and (6), and the assumption that $x^* = \frac{K(0)}{h(0)N(0)} = \frac{k(0)}{h(0)}$.

Thus, the effects of initial conditions of per capita capital stocks on the steady-state utility are calculated as

$$\frac{\partial U^*}{\partial h(0)} = \frac{f\left(\frac{x^*}{u^*}\right)-\frac{x^*}{u^*}f'\left(\frac{x^*}{u^*}\right)}{G'(1-u^*)}\left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - x^* f'\left(\frac{x^*}{u^*}\right)\right] h(0) + u^* G'(1-u^*)k(0)\right\}^{-\sigma},$$

$$\frac{\partial U^*}{\partial k(0)} = \left\{\left[u^* f\left(\frac{x^*}{u^*}\right) - x^* f'\left(\frac{x^*}{u^*}\right)\right] h(0) + u^* G'(1-u^*)k(0)\right\}^{-\sigma}.$$

$\frac{\partial U^*}{\partial h(0)}$ and $\frac{\partial U^*}{\partial k(0)}$ correspond respectively to v^* and q^* in Uzawa (1965) in which $\sigma = 0$.

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