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AnAxiomaticAnalysisofIntervalShapleyValues

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Abstract

Interval games are an extension of cooperative coalitional games in which players are assumed to face payoff uncertainty as represented by a closed interval. In this study, we examine two interval-game versions of Shapley values (i.e., the interval Shapley value and the interval Shapley-like value), and characterize them using an axiomatic approach. For the interval Shapley value, we show that the existing axiomatization can be generalized to a wider subclass of interval games called size monotonic games. For the interval Shapley-like value, we show that a standard axiomatization using Young's strong monotonicity holds on the whole class of interval games.

Keywords: cooperative interval games; interval uncertainty; Shapley value; axiomatization

1 Introduction

This paper examines interval cooperative games in which players face payoff uncertainty. Characteristic functions thus assign a closed interval rather than a real number that would be assigned in traditional coalitional games. Interval games were introduced by Branzei et al. [7] and various solution concepts have subsequently been proposed and examined by Alparslan Gök et al. [5], Liang and Li [11], Meng et al. [12], and Shino et al. [16], among others.[1]

In this paper, we revisit existing solution concepts for interval games and investigate their properties using an axiomatic approach. More particularly, we focus on the interval Shapley value (ISV) developed by Alparslan Gök et al. [4] and the interval Shapley-like value (ISLV) by Han et al. [9], both of which are based on the Shapley value [14] for classical coalitional games. For the ISV, we show that the domain of interval games covered by existing axiomatizations can be widened substantially. For the ISLV, we show that a standard axiomatization using the strong monotonicity by Young [17] holds for all interval games.

The remainder of the paper is as organized as follows. Section 2 briefly reviews the models and solution concepts. The main results are presented in Section 3. Section 4 concludes.

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[1] For more details on the literature, including applications of interval games, see Alparslan Gök [1], Branzei et al. [6], and Ishihara and Shino [10].

2 Models and Solution Concepts

2.1 Coalitional games and interval games

An n-person coalitional game or a transferable utility game is a pair (N,v), where $N = \{1,2,\dots,n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that associates a real number $v(S) \in \mathbb{R}$ with each set $S \subset N$, with the condition that $v(\emptyset) = 0$. For a coalition S, let $|S|$ be the number of players in S. A number $v(S)$ is called the worth of S. We refer to S and N as a coalition and grand coalition, respectively. Let CG be the set of all coalitional games with player set N.

Similar to an n-person coalitional game (N,v), an n-person interval game is defined as a pair (N,w), where N is a set of players and w is a characteristic function of type $2^N \to I(\mathbb{R})$ with $w(\emptyset) = [0,0]$, where $I(\mathbb{R})$ is the set of all closed and bounded intervals in $\mathbb{R}$. Therefore, an interval game differs from a coalitional form game in that w assigns a closed interval to each coalition (instead of a real number). Interval $w(S)$ is called the worth set of S and the minimum and the maximum of $w(S)$ are denoted by $\underline{w}(S)$ and $\overline{w}(S)$, respectively, that is, $w(S) = [\underline{w}(S), \overline{w}(S)]$. An interval game (N,w) considers a situation in which the players face "interval uncertainty," in that they know that a coalition S could have $\underline{w}(S)$ as the minimal reward and $\overline{w}(S)$ as the maximal reward, but they do not know which of these will be realized. Let IG be the set of all interval games with player set N. For simplicity, we denote n-person interval games (N,w) by w.

We provide some interval calculus notations. For a positive number a and a closed interval $I = [\underline{I}, \overline{I}]$, we define $aI = [a\underline{I}, a\overline{I}]$. Let $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ be two closed intervals. First, when $(\underline{I}+\overline{I})/2 = (\underline{J}+\overline{J})/2$, which means that the medians of the two intervals are identical, we denote this by $I \sim J$. Second, if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$, we denote it by $I \geq J$. Third, if $(\underline{I} + \overline{I})/2 \geq \underline{J} + \overline{J}/2$, then we denote it by $I \gtrsim J$. The sum of I and J, denoted by $I + J$, is given as $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$. For subtraction between intervals, on the other hand, there are different definitions. First, following Alparslan Gök et al. [3], the partial subtraction operator denoted by "$-$" is defined as $I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. Note that the partial subtraction operator is only defined for an ordered interval pair, i.e., $(I,J) \in I(\mathbb{R}) \times I(\mathbb{R})$ satisfying $\overline{J} - \underline{J} \leq \overline{I} - \underline{I}$. Alternatively, Moore's [13] subtraction operator which we denote by "$\ominus$" is given by: $I \ominus J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$. In contrast to the partial subtraction operator, Moore's operator can be defined for any interval pairs $(I,J) \in I(\mathbb{R}) \times I(\mathbb{R})$.

Players i and j are symmetric if $w(S \cup \{i\}) = w(S \cup \{j\})$ for every $S \subset N \setminus \{i,j\}$. i is a dummy player if $w(S \cup \{i\}) = w(S) + w(\{i\})$ for every $S \in 2^{N \setminus \{i\}}$. For different interval games $w', w'' \in IG$, the sum of the interval games $w' + w'' \in IG$ is also an interval game itself, defined by $(w' + w'')(S) = w'(S) + w''(S)$ for every $S \in 2^N$. $w \in IG$ is called size monotonic if $\overline{w}(S) - \underline{w}(S) \leq \overline{w}(T) - \underline{w}(T)$ for every $S,T \in 2^N$ with $S \subset T$. Let $SMIG$ be the set of all size monotonic interval games. For $S \in 2^N \setminus \{\emptyset\}$ and $I_S \in I(\mathbb{R})$, the unanimous interval game $I_S u_S$ is defined as:

$$I_S u_S(T) = \begin{cases} I_S & \text{if} \quad T \supset S \\ [0,0] & \text{otherwise.} \end{cases}$$

Let KIG be the set of all interval games that can be expressed as a sum of unanimous interval games. The following remark and example indicate that KIG covers only a small range of interval games.

Remark 2.1 *It holds that $KIG \subset SMIG$.*[2]

[2]Alparslan Gök. [2] and Alparslan Gök et al.[4] noted this property without proof. Our proof is available on request.

Example 2.1 *For an arbitrary three-person coalitional game $v \in CG$ and for positive real numbers ϵ and δ, we define the three-person interval game $w_{v,\epsilon,\delta}$ as follows: $w(\emptyset) = [0,0]$, $w(\{1\}) = [v(\{1\})-\epsilon, v(\{1\})+\epsilon]$, $w(\{2\}) = [v(\{2\})-\epsilon, v(\{2\})+\epsilon]$, $w(\{3\}) = [v(\{3\})-\epsilon, v(\{3\})+\epsilon]$, $w(\{1,2\}) = [v(\{1,2\})-\delta, v(\{1,2\})+\delta]$, $w(\{1,3\}) = [v(\{1,3\})-\delta, v(\{1,3\})+\delta]$, $w(\{2,3\}) = [v(\{2,3\})-\delta, v(\{2,3\})+\delta]$, $w(\{1,2,3\}) = [v(\{1,2,3\})-3\epsilon, v(\{1,2,3\})+3\epsilon]$. $w_{v,\epsilon,\delta}$ corresponds to the situation in which the degree of uncertainty depends only on the number of coalitions, and the uncertainty regarding the worth of the grand coalition is three times larger than that of the singleton coalition. Note that $w_{v,\epsilon,\delta} \in SMIG$ when $\epsilon \leq \delta \leq 3\epsilon$, but $w_{v,\epsilon,\delta} \in KIG$ only when $\delta = 2\epsilon$.*

2.2 Solution concepts

Let a subset of IG be K. A (single-valued) interval solution on K is a function f that associates a single n-dimensional interval vector $f(w) \in I(\mathbb{R})^n$ with each game $w \in K$. This study focuses on two existing interval solutions, i.e., the ISV and ISLV, and investigates their axiomatic characterization. Whereas the ISLV is an interval solution on IG, the ISV is an interval solution on SMIG, i.e., a subclass of IG, because it is defined by using the partial subtraction operator.

For $w \in SMIG$, the ISV, denoted by $\Psi(w) = (\Psi_1(w), ..., \Psi_n(w))$, is defined as:

$$\text{For } i \in N,\ \ \Psi_i(w) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \{w(S \cup \{i\}) - w(S)\}.$$

For $w \in IG$, the ISLV, denoted by $\Phi(w) = (\Phi_1(w), ..., \Phi_n(w))$, is defined as:

$$\text{For } i \in N,\ \ \Phi_i(w) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \{w(S \cup \{i\}) \ominus w(S)\}.$$

It is worth noting the following regarding the relationship between the ISV and the ISLV:

Lemma 2.1 *For any $w \in SMIG$ and $i \in N$, $\Psi_i(w) \subset \Phi_i(w)$.*[3]

3 Main Results

This section reviews existing axiomatizations of the ISV and ISLV and investigates their properties further using a new axiomatic approach. First, following the analysis of strong monotonicity in coalitional games by Peleg and Sudhölter [15], we define the following:

$$\begin{aligned}
\mathcal{D}(w) &= \{S \subset N \mid \text{there exists } T \subset S \text{ with } w(T) \neq [0,0]\} \\
\mathcal{D}^m(w) &= \{S \in \mathcal{D}(w) \mid \nexists T \in \mathcal{D}(w) \text{ with } T \subsetneq S\} \\
S_0(w) &= \bigcap \{S \mid S \in \mathcal{D}^m(w)\}
\end{aligned}$$

$\mathcal{D}^m(w)$ is the set of minimal coalitions in $\mathcal{D}(w)$. Note that at least one player in $S_0(w)$ is not included in T. Then, $w(T) = [0,0]$.

[3]The proof is available on request.

3.1 Results for the interval Shapley value

For the ISV, we consider the following axioms.

- Axiom 1: Efficiency **[EF]**

$$\left(\sum_{i\in N} f_i(w) = w(N)\right)\left(\forall w \in IG\right)$$

- Axiom 2: Symmetry **[SYM]**

$$\left(f_i(w) = f_j(w)\right)\left(\text{if } w(S\cup\{i\}) = w(S\cup\{j\})\ \forall S \in 2^{N\setminus\{i,\,j\}}\right)$$

- Axiom 3: Dummy Player Property **[DP]**

$$\left(f_i(w) = w(\{i\})\right)\left(\text{if } w(S\cup\{i\}) = w(S) + w(\{i\})\ \forall S \in 2^{N\setminus\{i\}}\right)$$

- Axiom 4: Additivity **[AD]**

$$\left(f_i(w'+w'') = f_i(w') + f_i(w'')\right)\left(\forall w', w'' \in IG\right)\left(\forall i \in N\right)$$

- Axiom 5: Strong Monotonicity w.r.t. the Partial Operator **[SM-P]**

$$\left(f_i(w) \geq f_i(w')\right)\left(\text{if } w(S\cup\{i\}) - w(S) \geq w'(S\cup\{i\}) - w'(S)\ \forall S \in 2^{N\setminus\{i\}}\right)$$

The existing axiomatizations for the ISV have been implemented only for KIG games. Namely, Alparslan Gök et al. [4] showed that, within KIG games, the ISV is the unique solution that satisfies EF, SYM, DP, and AD. Similarly, Alparslan Gök [2] showed that, within KIG games, the ISV is the unique solution that satisfies EF, SYM, and SM-P.

For the ISV, our main results are Theorem 3.1 and Theorem 3.2 below. Note that because the ISV is defined on SMIG, each theorem shows that its associated axiomatization is implemented on the largest possible domain of the interval games.

Theorem 3.1 *For any $w \in SMIG$, the ISV is the unique solution that satisfies EF, SYM, DP, and AD.*

Theorem 3.2 *For any $w \in SMIG$, the ISV is the unique solution that satisfies EF, SYM, and SM-P.*

Proof of Theorem 3.1. Alparslan Gök et al.[4] showed that the ISV satisfies EF, SYM, DP, and AD on SMIG. Therefore, it suffices to show its uniqueness. The following Lemmas 3.1 to 3.5 are necessary to show the uniqueness, and all proofs are from Shino et al. [16].

Lemma 3.1 *For a coalition R, we define a coalitional form game $v_R \in G$ as:*

$$v_R(S) = \begin{cases} 1 & \text{if} \quad R \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $w \in IG$, there uniquely exists $2(2^n - 1)$ real numbers $(\underline{c_R},\ \overline{c_R} : R \subset N)$ that satisfy

$$v_{\underline{w}} = \sum_{R\subset N} \underline{c_R} v_R, \quad v_{\overline{w}} = \sum_{R\subset N} \overline{c_R} v_R \quad \text{where}$$

$$\underline{c_R} = \sum_{T \subset R} (-1)^{|R|-|T|} v_{\underline{w}}(T), \quad \overline{c_R} = \sum_{T \subset R} (-1)^{|R|-|T|} v_{\overline{w}}(T).$$

Lemma 3.2 *For any coalition $R \subset N$,* $w + \sum_{R:\underline{c_R} > \overline{c_R}} [-\underline{c_R},\ -\overline{c_R}] v_R = \sum_{R:\underline{c_R} \leq \overline{c_R}} [\underline{c_R},\ \overline{c_R}] v_R.$

Lemma 3.3 *Suppose that a solution for SMIG f satisfies EF, SYM, and DP. Then, for the interval game $[\underline{c},\ \overline{c}]v_R$ $(Note : \underline{c} \leq \overline{c})$,*

$$f_i([\underline{c},\ \overline{c}]v_R) = \begin{cases} [\underline{c},\ \overline{c}]/|R| & if \quad i \in R \\ [0,\ 0] & otherwise. \end{cases}$$

Proof. See Alparslan Gök et al.[4].

Lemma 3.4 *Let ϕ be the Shapley value for coalitional games. Then, it holds that $\phi_i(v_{\underline{w}}) = \sum_{R \ni i} (\underline{c_R}/|R|)$ and $\phi_i(v_{\overline{w}}) = \sum_{R \ni i} (\overline{c_R}/|R|)$.*

Lemma 3.5 *Let ϕ be the Shapley value for coalitional games. Then, the ISV for $w \in SMIG$ is $\Psi_i(w) = [\phi_i(v_{\underline{w}}),\ \phi_i(v_{\overline{w}})]$ and $\phi_i(v_{\underline{w}}) \leq \phi_i(v_{\overline{w}})$.*

Proof of Theorem 3.1. (cont.) Suppose that solution f satisfies EF, SYM, DP, and AD. Then, it suffices to show that $f = \psi$. For an interval game $w \in SMIG$, from Lemma 3.1, Lemma 3.2, and AD, it follows that $f_i(w) + \sum_{R:\underline{c_R} > \overline{c_R}} f_i([-\underline{c_R},\ -\overline{c_R}]v_R) = \sum_{R:\underline{c_R} \leq \overline{c_R}} f_i([\underline{c_R},\ \overline{c_R}]v_R)$. From Lemma 3.3, it also holds that $f_i(w) + \sum_{R \ni i:\underline{c_R} > \overline{c_R}} ([-\underline{c_R},\ -\overline{c_R}]/|R|) = \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} ([\underline{c_R},\ \overline{c_R}]/|R|)$. Now, from Lemma 3.4,

$$\begin{aligned} \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{\overline{c_R}}{|R|} - \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{-\overline{c_R}}{|R|} &= \sum_{R \ni i} \frac{\overline{c_R}}{|R|} = \phi_i(v_{\overline{w}}) \\ \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{\underline{c_R}}{|R|} - \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{-\underline{c_R}}{|R|} &= \sum_{R \ni i} \frac{\underline{c_R}}{|R|} = \phi_i(v_{\overline{w}}). \end{aligned}$$

Therefore, from Lemma3.5, we can subtract the interval $\sum_{R \ni i:\underline{c_R} > \overline{c_R}} ([\underline{-c_R},\ \overline{-c_R}]/|R|)$ from the interval $\sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} ([\underline{c_R},\ \overline{c_R}]/|R|)$, and it follows that:

$$f_i(w) = \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{[\underline{c_R},\ \overline{c_R}]}{|R|} - \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{[-\underline{c_R},\ -\overline{c_R}]}{|R|} = [\phi_i(v_{\underline{w}}),\ \phi_i(v_{\overline{w}})].$$

Therefore, from Lemma 3.5, $f_i(w) = \Psi_i(w)$. □

Next, we prove Theorem 3.2.

Proof of Theorem 3.2. For SMIG, Alparslan Gök et al.[4] showed that the ISV satisfies EF and SYM, and Alparslan Gök.[2] showed that it satisfies SM-P. Therefore, it suffices to show its uniqueness, i.e., a solution f for $w \in SMIG$ satisfying EF, SYM, and SM-P must be identical to Ψ.

Following Peleg and Sudhölter.[15], we use mathematical induction regarding $|\mathcal{D}(w)|$. If $|\mathcal{D}(w)| = 0$, then $f_i(w) = [0,0]$ for every $i \in N$ by EF and SYM. Because $\Psi_i(w) = [0,0]$ for every i, $f(w) = \Psi(w)$. Now, assume that $f(w) = \Psi(w)$ for any $w \in SMIG$ satisfying $|\mathcal{D}(w)| \leq k$ and consider any $w \in SMIG$ satisfying $|\mathcal{D}(w)| = k + 1$. For $S \in \mathcal{D}^m(w)$, we define $w_S \in IG$ as $w_S(T) = w(S \cap T)$ for all $T \subset N$ and let $w' \in IG$ be defined by $w' = w - w_S$. Because $S \in \mathcal{D}^m(w)$, the following holds:

$$w_S(T) = \begin{cases} w(S) & \text{if} \quad T \supset S \\ [0,0] & \text{otherwise} \end{cases} \qquad w'(T) = \begin{cases} w(T) - w(S) & \text{if} \quad T \supset S \\ w(T) & \text{otherwise} \end{cases}$$

Note that if $T \supset S$, then $w(T) - w(S)$ is an interval because $w \in SMIG$.

$w'(T \cup \{i\}) - w'(T) = w(T \cup \{i\}) - w(T)$ holds for all $T \subset N$ because for every $i \in N \backslash S$ the following is true:

$$w'(T \cup \{i\}) = \begin{cases} w(T \cup \{i\}) - w(S) & \text{if} \quad T \supset S \\ w(T \cup \{i\}) & \text{otherwise.} \end{cases}$$

First, because f satisfies SM-P, (i) $f_i(w') = f_i(w)$ for all $i \in N \backslash S$. Second, because $|\mathcal{D}(w')| \leq k$, and from the assumptions, (ii) $f_i(w') = \Psi_i(w')$ for all $i \in N \backslash S$. Finally, because Ψ satisfies SM-P, (iii) $\Psi_i(w') = \Psi_i(w)$ for all $i \in N \backslash S$. From (i)–(iii), it holds that $f_i(w) = \Psi_i(w)$ for every $i \in N \backslash S$. As this holds for every $S \in \mathcal{D}^m(w)$, we have:

$$f_i(w) = \Psi_i(w) \quad \forall i \in N \backslash S_0(w). \tag{1}$$

As $w(T) = [0, 0]$ for every T satisfying $S_0(w) \backslash T \neq \emptyset$, $w(S \cup \{i\}) = w(S \cup \{j\}) = [0, 0]$ for every $i, j \in S_0(w)$ and all $S \in 2^{N \backslash \{i, j\}}$. Furthermore, $f_i(w) = f_j(w)$ and $\Psi_i(w) = \Psi_j(w)$ hold because f and Ψ satisfy SYM. Therefore, from EF and (1), we have:

$$f_i(w) = \Psi_i(w) \quad \forall i \in S_0(w). \tag{2}$$

(1) and (2) imply that $f(w) = \Psi(w)$. □

3.2 Results for the interval Shapley-like value

For the ISLV, in addition to SYM and AD, the following axioms are considered.

- Axiom 6: Indifference Efficiency **[IEFF]**

$$\left(\sum_{i \in N} f_i(w) \sim w(N)\right) \left(\forall w \in IG\right)$$

- Axiom 7: Indifference Null Player Property **[INP]**

$$\exists t \in \mathbb{R} \text{ with } t \geq 0 \text{ s.t. } \left(f_i(w) = [-t, t]\right) \left(\text{if } w(S \cup \{i\}) = w(S) \; \forall S \in 2^{N \backslash \{i\}}\right)$$

- Axiom 8: Strong Monotonicity w.r.t. Moore's operator **[SM-M]**

$$\left(f_i(w) \gtrsim f_i(w')\right) \left(\text{if } w(S \cup \{i\}) \ominus w(S) \gtrsim w'(S \cup \{i\}) \ominus w'(S) \; \forall S \in 2^{N \backslash \{i\}}\right)$$

Gallardo and Jiménez-Losada[8] showed that, in any interval game w, the ISLV is the unique solution that satisfies IEFF, SYM, INP, and AD.

Newly introduced in our study, Axiom SM-M is a natural extension of strong monotonicity using Moore's subtraction operator. The main result for the ISLV is as follows:

Theorem 3.3 *For any $w \in IG$, the ISLV is the unique solution that satisfies IEFF, SYM, and SM-M.*

We prove Theorem 3.3 by using the following Lemma 3.6 to Lemma 3.8.

Lemma 3.6 *For intervals I_1,I_2 and J, $(I_1 \ominus J) \ominus (I_2 \ominus J) \sim I_1 \ominus I_2$.*

Proof. Since $I_1 \ominus J = [\underline{I_1} - \overline{J}, \overline{I_1} - \underline{J}]$ and $I_2 \ominus J = [\underline{I_2} - \overline{J}, \overline{I_2} - \underline{J}]$, the median of $(I_1 \ominus J) \ominus (I_2 \ominus J) = [\underline{I_1} - \overline{J} - \overline{I_2} + \underline{J}, \overline{I_1} - \underline{J} - \underline{I_2} + \overline{J}]$ is $(\underline{I_1} - \overline{J} - \overline{I_2} + \underline{J} + \overline{I_1} - \underline{J} - \underline{I_2} + \overline{J})/2 = (\underline{I_1} - \overline{I_2} + \overline{I_1} - \underline{I_2})/2$. Also, the median of $I_1 \ominus I_2 = [\underline{I_1} - \overline{I_2}, \overline{I_1} - \underline{I_2}]$ is $(\underline{I_1} - \overline{I_2} + \overline{I_1} - \underline{I_2})/2$. □

Lemma 3.7 *For positive numbers a_1,a_2 and intervals I_1,I_2,J_1,J_2, if $I_1 \gtrsim J_1$ and $I_2 \gtrsim J_2$, then $a_1 I_1 + a_2 I_2 \gtrsim a_1 J_1 + a_2 J_2$.*

Proof. If $I_1 \gtrsim J_1$,$I_2 \gtrsim J_2$, then $(\underline{I_1} + \overline{I_1})/2 \geq (\underline{J_1} + \overline{J_1})/2$ and $(\underline{I_2} + \overline{I_2})/2 \geq (\underline{J_2} + \overline{J_2})/2$, implying that $(\underline{a_1 I_1 + a_2 I_2} + \overline{a_1 I_1 + a_2 I_2})/2 \geq (\underline{a_1 J_1 + a_2 J_2} + \overline{a_1 J_1 + a_2 J_2})/2$. Therefore $a_1 I_1 + a_2 I_2 \gtrsim a_1 J_1 + a_2 J_2$. □

Lemma 3.8 *The ISLV satisfies IEFF, SYM, and SM-M.*

Proof. As Han et al. [9] showed that the ISLV satisfies IEFF and SYM, it suffices to show that the ISLV satisfies SM-M. For $S \in 2^{N \setminus \{i\}}$, if $w(S \cup \{i\}) \ominus w(S) \gtrsim w'(S \cup \{i\}) \ominus w'(S)$, then:

$$\sum_{S \in 2^{N \setminus \{i\}}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \{w(S \cup \{i\}) \ominus w(S)\} \gtrsim \sum_{S \in 2^{N \setminus \{i\}}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \{w'(S \cup \{i\}) \ominus w'(S)\}$$

is true by Lemma 3.7. Therefore, $\Phi_i(w) \gtrsim \Phi_i(w')$. □

Proof of Theorem 3.3. Suppose a solution f satisfies IEFF, SYM, and SM-M. Then, we show that $f_i(w) \sim \Phi_i(w)$ for every $i \in N$ by mathematical induction regarding $|\mathcal{D}(w)|$. If $|\mathcal{D}(w)| = 0$, then $f_i(w) = [0, 0]$ for every $i \in N$ by IEFF and SYM. As $\Phi_i(w) = [0, 0]$, $f_i(w) \sim \Phi_i(w)$ for every $i \in N$. Assume that for any $w \in IG$ satisfying $|\mathcal{D}(w)| \leq k$, $f_i(w) \sim \Phi_i(w)$ holds for every $i \in N$ and consider $w \in IG$ satisfying $|\mathcal{D}(w)| = k + 1$. For $S \in \mathcal{D}^m(w)$, we define $w_S \in IG$ as $w_S(T) = w(S \cap T)$ for all $T \subset N$. As $S \in \mathcal{D}^m(w)$,

$$w_S(T) = \begin{cases} w(S) & \text{if} \quad T \supset S \\ [0,0] & \text{otherwise} \end{cases}$$

holds. We define $w' \in IG$ as follows:

$$w'(T) = \begin{cases} w(T) \ominus w(S) & \text{if} \quad T \supsetneq S \\ [0,0] & \text{if} \quad T = S \\ w(T) & \text{otherwise.} \end{cases}$$

Note that for every $i \in N \setminus S$, the following holds:

$$w'(T \cup \{i\}) = \begin{cases} w(T \cup \{i\}) \ominus w(S) & \text{if} \quad T \supsetneq S \\ w(T \cup \{i\}) \ominus w(T) & \text{if} \quad T = S \\ w(T \cup \{i\}) & \text{otherwise.} \end{cases}$$

Therefore, from Lemma 3.6, $w'(T \cup \{i\}) \ominus w'(T) \sim w(T \cup \{i\}) \ominus w(T)$ for all $T \subset N$.

First, as f satisfies SM-M, (i) $f_i(w') \sim f_i(w)$ for all $i \in N \backslash S$. Second, as $|\mathcal{D}(w')| \leq k$ and from the assumptions, (ii) $f_i(w') \sim \Phi_i(w')$ for all $i \in N \backslash S$. Finally, because Φ satisfies SM-M from Lemma 3.8, (iii) $\Phi_i(w') \sim \Phi_i(w)$ for all $i \in N \backslash S$. From (i)–(iii), it holds that $f_i(w) \sim \Phi_i(w)$ for every $i \in N \backslash S$. Because this holds for every $S \in \mathcal{D}^m(w)$,

$$f_i(w) \sim \Phi_i(w) \ \forall i \in N \backslash S_0(w). \tag{3}$$

As $w(T) = [0, 0]$ for every T satisfying $S_0(w) \backslash T \neq \emptyset$, $w(S \cup \{i\}) = w(S \cup \{j\}) = [0, 0]$ for every $i, j \in S_0(w)$ and all $S \in 2^{N \backslash \{i, j\}}$. Furthermore, as f satisfies SYM, $f_i(w) = f_j(w)$ and Φ also satisfies SYM by Lemma 3.8, it holds that $\Phi_i(w) = \Phi_j(w)$. Therefore, from IEFF and (3),

$$f_i(w) \sim \Phi_i(w) \ \forall i \in S_0(w). \tag{4}$$

(3) and (4) implies $f_i(w) \sim \Phi_i(w)$ for every $i \in N$. $\square$

4 Conclusion

In this study, we investigate two interval-game versions of the Shapley value, i.e., the ISV and ISLV and characterize them with a new axiomatic approach. For the ISV, we show that the existing axiomatization can be generalized to a wider subclass of interval games called size monotonic games. For the ISLV, we show that a standard axiomatization using Young's strong monotonicity holds on the whole class of interval games. It should be noted that those results can be derived by focusing on $|D(w)|$, following Peleg and Sudhölter.[15], rather than on games expressed as a sum of unanimous interval games (KIG). Furthermore, although the existing axiomatizations employ different approaches, we axiomatize the ISV and ISLV in a unified way in the proofs. Regarding further research, Shino et al. [16] proposed a third interval-game version of the Shapley value, called Shapley mapping. It has been characterized by some axiomatizations in [16] but not yet by one that includes strong monotonicity. It would be worthwhile investigating this topic in future.

References

[1] S. Z. Alparslan Gök, Cooperative interval games: Theory and applications, Lambert Academic Publishing (2010).

[2] S. Z. Alparslan Gök, On the interval Shapley value, Optimization, 63 (2014) 747–755.

[3] S. Z. Alparslan Gök, R. Branzei, S. Tijs, Convex interval games, Journal of Applied Mathematics and Decision Sciences, 342089 (2009).

[4] S. Z. Alparslan Gök, R. Branzei, S. Tijs, The interval Shapley value: An axiomatization, Central European Journal of Operations Research, 18 (2010) 131–140.

[5] S. Z. Alparslan Gök, S. Miquel, S. Tijs, Cooperation under interval uncertainty, Mathematical Methods of Operational Research, 69 (2009) 99–109.

[6] R. Branzei, O. Branzei, S. Zeynep Alparslan Gök, S. Tijs, Cooperative interval games: A survey. Central European Journal of Operations Research, 18 (2010) 397–411.

[7] R. Branzei, D. Dimitrov, S. Tijs, Shapley-like values for interval bankruptcy games, Economic Bulletin, 3 (2003) 1–8.

[8] J.M. Gallardo, A. Jiménez-Losada, Comment on ”A new approach of cooperative interval games: The interval core and Shapley value revisited,“Operations Research Letters, 46 (2018) 434–437.

[9] W. Han, H. Sun, G. Xu, A new approach of cooperative interval games: The interval core and Shapley value revisited, Operations Research Letters, 40 (2012) 462–468.

[10] S. Ishihara, J. Shino, A solution mapping and its axiomatization in two-person interval games, Journal of the Operations Research Society of Japan, 64 (2021) 214–226.

[11] K. Liang, D. Li, A direct method of interval Banzhaf values of interval cooperative games, Journal of Systems Science and Systems Engineering, 28 (2019) 382–391.

[12] F. Meng, X. Chen, C. Tan, Cooperative fuzzy games with interval characteristic functions, International Journal of Operational Research, 16 (2016) 1–24.

[13] R. Moore, Methods and applications of interval analysis, SIAM Studies in Applied Mathematics, Philadelphia (1979).

[14] L. S. Shapley, A value for n-person games, Annals of Mathematics Studies, 28 (1953) 307–318.

[15] B. Peleg, P. Sudhölter, Introduction to the theory of cooperative games, 2nd ed, Springer (2007).

[16] J. Shino, S. Ishihara, S. Yamauchi, Shapley mapping and its axiomatization in n-person cooperative interval games, Mathematics, 10 (2022) 3963.

[17] H. P. Young, Monotonic solutions of cooperative games, International Journal of Game Theory, 14 (1985) 65–72.