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Abstract

The Nash bargaining solution (Nash, 1950, 1953) is the most used game theory tool for analyzing bargaining problems. Its validity is examined from an equilibrium analysis using a non-cooperative game such as Nash's demand game (NDG). Since the NDG has multiple equilibria, we need an equilibrium selection. In this note, we apply the Harsanyi and Selten (1988)'s risk-dominance criterion to the NDG. We show that in a wide class of utility functions, the risk-dominant equilibrium of the NDG coincides with the Nash bargaining solution.

Keywords: Bargaining theory, Nash bargaining solution, Nash demand game, Risk dominance, Risk aversion

JEL Classification: C78, D1

1. Introduction

The Nash bargaining solution (Nash, 1950) is the most used game theory tool for analyzing bargaining problems. Its application is enormous such as the analysis of wage negotiations (Grout, 1984; Svejnar, 1986), the formation of trade unions (Abrego et al., 2001), and international agreements on global warming (Yu et al., 2017).

The validity of the Nash bargaining solution is examined from an equilibrium analysis using a non-cooperative game such as the demand game introduced by Nash (1953). Since there are many equilibria in Nash's demand game (NDG), Nash uses the smoothing approach to narrow down the equilibria and shows the consistency with the Nash bargaining solution.

Harsanyi and Selten (1988) established a theory of equilibrium selection in a non-cooperative game. Among their ideas, the risk-dominance criterion has been shown to help explain experimental data. Harsanyi and Selten (1988) found that in the unanimous bargaining game, which is a simplified version of the NDG, the risk dominance leads to an outcome of the Nash bargaining solution. However, the relationship between the risk-dominant equilibrium of the NDG and the Nash bargaining solution has not been clarified.

This note aims to bridge the gap between the two outstanding achievements in game theory. We reveal the relationship between the risk-dominant equilibrium of the NDG and the Nash bargaining solution. We show that the risk-dominant equilibrium in NDG is consistent with the Nash bargaining solution for a comprehensive class of utility functions. This result is a generalization of Anbarci and Feltovich (2013), who showed consistency in a particular class of utility functions.

The paper is organized as follows. Next section explains our setup of the bargaining problem. Section 3 gives our main results.

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2. Simple bargaining problem

We consider a simple bargaining problem for a fixed amount of divisible goods like money, assets, and real estate. The pie size is M that can be split between two bargainers 1 and 2. Let each share be x_1, x_2 with $x_1 + x_2 = M$. If negotiations break down, each gets $v_1 \geqq 0, v_2 \geqq 0$ of the divisible goods. We assume there is room to bargain for the division of M, that is, $v_1 + v_2 < M$. Let $u_i, i = 1, 2$ be a utility function of bargainer i, which satisfies the continuity and $u' > 0$. Let $d = (u_1(v_1), u_2(v_2))$ be the pair of utilities when the bargaining breaks down. Then, the set of possible pair of utilities through bargaining agreement is

$$Z = \{(z_1, z_2) \ : \ \exists(x_1, x_2) \text{ with } x_1 + x_2 = M \text{ such that } (z_1, z_2) \leqq d\}.$$

The bargaining problem is which point in Z the bargainers agree on as the bargaining outcome. Nash bargaining solution is widely accepted as its several normative properties (Nash, 1950, 1953; Rubinstein et al., 1992) and positive interpretations (Nash, 1953; Binmore et al., 1986; Young, 1993; Rubinstein et al., 1992). The Nash bargaining solution chooses the utility pair that maximizes the Nash product that is defined as the product of utility differences from disagreement to agreement among the two bargainers:

$$NP(x_1, x_2) = (u_1(x_1) - u_1(v_1)) \times (u_2(x_2) - u_2(v_2)).$$

Then, the Nash bargaining solution is the solution to the following maximization problem.[1]

$$\max \ NP(x_1, x_2)$$
$$s.t. \ \ x_1 + x_2 = M, \ \ x_1 \geqq v_1, \quad x_2 \geqq v_2.$$

We assume that there exists a unique maximizer of this problem.

3. Risk dominance and the Nash demand game

In the Nash demand game (NDG), two players simultaneously report their demand x_i for $i = 1, 2$, $0 \leqq x_i \leqq M$, and if it matches to the constraint (i.e., $x_1 + x_2 \leqq M$), they obtain the demanded amount. In contrast, if the pair of their demand is not feasible ($x_1 + x_2 > M$), they obtain their disagreement outcome v_i for $i = 1, 2$.

A unanimous bargaining game (UBG) considered in Harsanyi and Selten (1988) is very close to the NDG but different in that it requires the perfect match of the two demands. That is, they obtain x_1 and x_2 only when $x_1 + x_2 = M$ in the UBG.

These two games have many Nash equilibria. Among them, the efficient one is characterized as follows: (x_1, x_2) with $x_1 + x_2 = M$ and $x_i \geqq v_i$ for $i = 1, 2$.

Since there still exists an enormous number of efficient Nash equilibria, the efficiency is still insufficient to select one equilibrium. In this paper, we focus on the problem of choosing one equilibrium among the efficient equilibria.

Harsanyi (1982) consider the player's thought process that generates the prior of the opponent's choice. The following is the two-player version of such a process.[2]

[1] In this paper, an outcome (x_1^*, x_2^*) that maximizes $NP(x_1, x_2)$ is called the Nash bargaining solution because we focus on the outcome of the bargaining.

[2] n-player version of the process is almost the same as the one we explain here. A critical assumption on the n-player version is that in i's belief, j believes that all other players choose x_{-j} in probability z, and y_{-j} in probability $1 - z$. Thus, j believes in the perfect coordination of the other players. In addition, it is assumed that after the thought process, all $i \neq j$ reach the same prior distribution about j's choice. In other words, all players

- Take arbitrary two efficient equilibrium $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Consider a situation called a $x : y$ uncertainty, where both players are wondering between equilibria x and y, and face uncertainty about the opponent's choice
- Player $i \in \{1, 2\}$ thinks that
  - player $j \neq i$ believes that player i chooses x_i in probability q, and chooses y_i in $1 - q$
  - given q, player j chooses the best response to the mixed strategy. Here j's choice is not restricted to $\{x_j, y_j\}$ and can choose any strategy from j's strategy set. If there are multiple best responses, j chooses the centroid best reply where j chooses every pure-strategy best response in equal probability
  - this q follows the uniform distribution on $[0, 1]$
- From such belief, player i calculates a prior probability distribution about the choice of player j

Player i best replies to a prior distribution of the opponent's behavior, where i's choice is not restricted to $\{x_i, y_j\}$. The best reply is denoted by s_i^*, and we tell that player i has a risk preference to s_i^* for the $x : y$ uncertainty.

Both players consider the above way and determine their risk preference to the situation of the $x : y$ uncertainty. When every player has a unique risk preference, the results are one of the following three situations:

- (R1) if $(s_1^*, s_2^*) = x$, x risk-dominates y,
- (R2) if $(s_1^*, s_2^*) = y$, y risk-dominates x,
- (R3) otherwise, we need a tracing procedure of Harsanyi (1975) that starts from an initial prior (s_1^*, s_2^*) and determines which equilibrium is achieved as the result of players' continuous thought process.

Although the above method is complicated, it is well known that there is a convenient way to detect the risk dominance in a two-by-two game (see Table 1). Suppose that (s_1, s_2) and (t_1, t_2) are strict Nash equilibria (i.e., $a_{ss} > a_{ts}$, $b_{ss} > b_{st}$ and $a_{tt} > a_{st}$, $b_{tt} > b_{ts}$ hold true).

		P2	
		s_2	t_2
P1	s_1	a_{ss}, b_{ss}	a_{st}, b_{st}
	t_1	a_{ts}, b_{ts}	a_{tt}, b_{tt}

Table 1: Two-by-two game

In a two-by-two game, an equilibrium (s_1, s_2) risk-dominates another equilibrium (t_1, t_2) if and only if

$$(a_{ss} - a_{ts})(b_{ss} - b_{st}) > (a_{tt} - a_{st})(b_{tt} - b_{ts}). \tag{1}$$

Therefore, an equilibrium with a higher product of the deviation loss from the equilibrium risk-dominates the one with the lower product of the deviation loss.

Fortunately, in the UBG, this simple method to detect risk dominance is functional. Consider the $x : y$ restricted normal form game in which their strategy is either x_i or y_i for $i = 1, 2$ (see Tables 2 and 3). Then, the risk-dominance relation of the UBG is well captured by the risk-dominance relation of the $x : y$ restricted normal form game (we omit the proof of this statement, but a similar proof will appear in the proof of Theorem 2). Thus, an equilibrium x risk-dominates y in the UBG if and only if

$$(u_1(x_1) - u_1(v_1))(u_2(x_2) - u_2(v_2)) > (u_1(y_1) - u_1(v_1))(u_2(y_2) - u_2(v_2)).$$

other than j have a common belief about j's behavior.

This simplification is because, in the UBG, any unmatched strategy pair leads to a bargaining breakdown.

Here, in the restricted normal form game, the product of the deviation loss of the two players from x is $NP(x_1, x_2)$, and that from y is $NP(y_1, y_2)$. Thus, we can say that an equilibrium x risk-dominates y in the UBG if and only if

$$NP(x_1, x_2) > NP(y_1, y_2).$$

Therefore, the equilibrium with a higher Nash product risk-dominates another equilibrium.

P1 \ P2	x_2	y_2
x_1	x_1, x_2	v_1, v_2
y_1	v_1, v_2	y_1, y_2

Table 2: Outcomes table of UBG

P1 \ P2	x_2	y_2
x_1	$u_1(x_1), u_2(x_2)$	$u_1(v_1), u_2(v_2)$
y_1	$u_1(v_1), u_2(v_2)$	$u_1(y_1), u_2(y_2)$

Table 3: Payoff table of UBG

Since the risk-dominance relation is a pairwise concept, we can define the maximal point based on this pairwise relation. Unfortunately, since the pairwise risk-dominance relation may violate the transitivity, we cannot choose one equilibrium from this criterion in general. However, for a particular class of games, we may find that one equilibrium risk-dominates any other equilibrium. If this is true, such an equilibrium should be selected from risk-dominance criterion and called a risk-dominant equilibrium.

Harsanyi and Selten (1988) have shown that in the UBG, there exists a risk-dominant equilibrium that risk-dominates any other efficient equilibrium. In addition, that equilibrium is the one that maximizes the Nash product.

Theorem 1 (Harsanyi and Selten (1988)). Assume there exists a unique maximizer of the Nash product. In the UBG, an equilibrium (x_1, x_2) is a risk-dominant equilibrium if and only if it is the maximizer of the Nash product.

Proof. Since the risk-dominance relation is connected to the values of the Nash product, the equilibrium with the highest Nash product risk-dominates any other equilibria. □

In the NDG, the situation is more complex compared with the UBG because the outcome of the strategy pair is one of the following: an efficient agreement, the breakdown of the negotiation, and an inefficient agreement. Thus, even for the restricted game by two efficient equilibria x and y, the off-diagonal results are asymmetric due to the feasibility of the pairs (see Tables 4 and 5). Nonetheless, we will show that in the class of the NDG, a risk-dominant equilibrium exists. Furthermore, the risk-dominant equilibrium is the one that maximizes the Nash product.

P1 \ P2	x_2	y_2
x_1	x_1, x_2	v_1, v_2
y_1	y_1, x_2	y_1, y_2

Table 4: Outcome table of NDG ($x_1 > y_1$)

P1 \ P2	x_2	y_2
x_1	$u_1(x_1), u_2(x_2)$	$u_1(v_1), u_2(v_2)$
y_1	$u_1(y_1), u_2(x_2)$	$u_1(y_1), u_2(y_2)$

Table 5: Payoff table of NDG ($x_1 > y_1$)

Theorem 2. Assume there exists a unique maximizer of the Nash product. In the NDG, an equilibrium (x_1, x_2) is a risk-dominant equilibrium if and only if it is the maximizer of the Nash product.

Proof. We first show that the simplification result still holds even for the Nash demand game. In other words, the following claim holds. Take any efficient equilibria x and y.

Claim 1: x risk-dominates y in the NDG if and only if x risk-dominates y in the $x : y$ restricted normal form game of the NDG.

Without loss of generality, we assume $x_1 > y_1$, which implies $x_2 < y_2$ by the assumption of the efficiency of x and y.

Let us consider the player 2's best reply to the situation where player 2 chooses x_1 in probability q and y_1 in $1-q$ with $0 < q < 1$.

It is shown that 2's best reply is either x_2 or y_2 because for any $z_2 < x_2$, z_2 assures the bargaining agreement, implying that z_2 gives a lower expected payoff than x_2; for any $x_2 < z_2 < y_2$, z_2 leads to the bargaining agreement in probability $1-q$ which is the same as choice y_2 but the payoff at the agreement is less than choice y_2, implying that z_2 gives a lower expected payoff than y_2; for any $z_2 > y_2$, z_2 always leads to the bargaining breakdown, implying that z_2 gives a lower expected payoff than y_2.

Since 2's best reply is either x_2 or y_2, x_2 is the strict best reply if

$$u_2(x_2) > qu_2(v_2) + (1-q)u_2(y_2) \iff q > \frac{u_2(y_2) - u_2(x_2)}{(u_2(y_2) - u_2(x_2)) + (u_2(x_2) - u_2(v_2))} := p.$$

Next, we consider 1's belief about the prior probability distribution of 2's choice. Since q follows a uniform distribution on $[0, 1]$, player 1 believes that player 2 chooses x_2 in probability $1-p$ and y_2 in probability p.

We consider the best reply of player 1 against this prior. Then, it is easily checked that the best reply of 1 is either x_1 or y_1.

Moreover, x_1 is the strict best reply if

$$(1-p)u_1(x_1) + pu_1(v_1) > u_1(y_1) \iff p < \frac{u_1(x_1) - u_1(y_1)}{(u_1(x_1) - u_1(y_1)) + (u_1(y_1) - u_1(v_1))}.$$

From some calculations, this is equal to[3]

$$\Big(u_1(x_1) - u_1(y_1)\Big)\Big(u_2(x_2) - u_2(v_2)\Big) > \Big(u_1(y_1) - u_1(v_1)\Big)\Big(u_2(y_2) - u_2(x_2)\Big). \qquad (2)$$

Thus, player 1 has risk-preference for x if the product of the deviation losses from x is larger than that from y (see Table 5).

Similarly, we can show that player 2 also has risk-preference for x if condition (2) is satisfied. Therefore, x risk-dominates y if (2) is satisfied. Thus, the proof of the claim ends.

[3]To obtain this result, we set

$$A = u_1(x_1) - u_1(y_1), B = u_1(y_1) - u_1(v_1), C = u_2(y_2) - u_2(x_2), D = u_2(x_2) - u_2(v_2).$$

Then, we have

$$p < \frac{u_1(x_1) - u_1(y_1)}{(u_1(x_1) - u_1(y_1)) + (u_1(y_1) - u_1(v_1))} \iff \frac{C}{C+D} < \frac{A}{A+B} \iff BC < AD.$$

Next, we show that the risk-dominance in the $x : y$ restricted game of the NDG is captured by comparing their Nash products. The following claim holds true.

Claim 2: In the $x : y$ restricted game of the NDG, an equilibrium x risk-dominates another equilibrium y if and only if $NP(x_1, x_2) > NP(y_1, y_2)$.

Without loss of generality, we consider the case that $x_1 > y_1$ and $x_2 < y_1$. Then, we have Table 5.

By Claim 1, in the $x : y$ restricted game of the NDG, an equilibrium x risk-dominates another equilibrium y if and only if

$$\Big(u_1(x_1) - u_1(y_1)\Big)\Big(u_2(x_2) - u_2(v_2)\Big) - \Big(u_1(y_1) - u_1(v_1)\Big)\Big(u_2(y_2) - u_2(x_2)\Big) > 0.$$

We now show that this inequality is equivalent to the inequality in their Nash products. Let $g(z_1, z_2) = u_1(z_1)u_2(z_2)$. Then, the left-hand side is reduced to

$$\Big(g(x_1, x_2) - g(x_1, v_2) - g(y_1, x_2) + g(y_1, v_2)\Big) - \Big(g(y_1, y_2) - g(y_1, x_2) - g(v_1, y_2) + g(v_1, x_2)\Big)$$

$$= \Big(g(x_1, x_2) - g(x_1, v_2) + g(y_1, v_2)\Big) - \Big(g(y_1, y_2) - g(v_1, y_2) + g(v_1, x_2)\Big).$$

Rearranging this, we have

$$= \Big(g(x_1, x_2) - g(x_1, v_2) - g(v_1, x_2)\Big) - \Big(g(y_1, y_2) - g(v_1, y_2) - g(y_1, v_2)\Big).$$

Add $g(v_1, v_2)$ in the left parenthesis and subtract it in the right parenthesis, we have

$$= \Big(g(x_1, x_2) - g(x_1, v_2) - g(v_1, x_2) + g(v_1, v_2)\Big) - \Big(g(y_1, y_2) - g(v_1, y_2) - g(y_1, v_2) + g(v_1, v_2)\Big).$$

This is equal to

$$\Big(u_1(x_1) - u_1(v_1)\Big)\Big(u_2(x_2) - u_2(v_2)\Big) - \Big(u_1(y_1) - u_1(v_1)\Big)\Big(u_2(y_2) - u_2(v_2)\Big)$$

$$= NP(x_1, x_2) - NP(y_1, y_2).$$

Hence, the risk-dominance relation of the restricted game is equivalent to the relation of Nash products. Thus, the proof of the claim ends.

From Claims 1 and 2, an x risk-dominates y in the NDG if and only if $NP(x_1, x_2) > NP(y_1, y_2)$. Thus, an equilibrium with the highest Nash product is the risk-dominant equilibrium of the NDG. □

This theorem states that the coincidence of the Nash bargaining solution and the risk-dominant equilibrium of the Nash demand game hold for a vast class of utility functions. We do not need the continuity, differentiability, and increasing-ness of utility functions.

This opens up the validity and the applicability of the Nash bargaining solution to several domains, such as a reference-dependent utility (Tversky and Kahneman (1979); its application to the bargaining problem is Kamijo and Yokote (2022)), and other-regarding preferences (Fehr and

Schmidt (1999) and Bolton and Ockenfels (2000); their applications to the bargaining problem are found in Anbarci and Feltovich (2013) and Birkeland and Tungodden (2014)).

It is well known that one of the sufficient conditions for the existence and uniqueness of the Nash product maximizer is u_i is concave for both $i = 1, 2$. In fact, for the case of concave CARA or CRRA utility functions,[4] Anbarci and Feltovich (2013) have shown the coincidence of the Nash bargaining solution and the risk-dominant equilibrium of the Nash demand game. Thus, our result means that the finding of Anbarci and Feltovich (2013) holds for any concave utility function.

Moreover, this coincidence holds even when the utility function is not concave. Kamijo and Yokote (2022) show that if the utility function satisfies the log-concavity, there exists a unique maximizer of the Nash product. Since some convex utility function is log-concave, the coincidence holds even for risk-loving players. For instance, power-type utility functions $u(x) = x^\alpha$ satisfies log-concavity for any $\alpha > 0$.[5] This validates the use of the Nash bargaining solution for the bargaining in a loss domain (i.e., the bankruptcy problem by O'Neill (1982), Aumann and Maschler (1985), Thomson (2003)).

Finally, the uniqueness itself is not essential in this statement. If there exist multiple maximizers of the Nash products, they are equivalent in the sense of risk dominance, and at the same time, each of them risk-dominates any other equilibrium other than them.

Let E denote the set of all efficient Nash equilibria of the Nash demand game. We say that the set of Nash equilibria E^R is the set of risk-dominant equilibria if and only if (i) any two equilibria in E^R are equivalent in the sense of the risk dominance (the equality holds for the inequality (1) in a restricted normal form game), and (ii) any equilibrium $x \in E^R$ risk-dominates any other equilibrium $y \notin E^R$.

The following is the corollary of Theorem 2.

Corollary 1. In the Nash demand game, E^R is the set of risk-dominant equilibra if and only if E^R is the set of every equilibrium that maximizes Nash product.

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[4] A utility function u satisfies a constant absolute risk aversion (CARA) iff for any $x \geqq 0$, $u(x) = 1 - \exp(-rx)$ for $r > 0$. A utility function u satisfies a constant relative risk aversion (CARA) iff for any $x \geqq 0$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ if $\gamma \neq 1$ and $u(x) = \log(x)$ if $\gamma = 1$.

[5] Since

$$\log u(x) = \log x^\alpha = \alpha \log x,$$

we have

$$(\log u(x))' = \alpha \frac{1}{x}, \quad \text{and} \quad (\log u(x))'' = -\alpha \frac{1}{x^2} < 0.$$

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