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Stability and strategy-proofness for matching with interval constraints

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We consider a matching problem with interval constraints under the hierarchical region structure. We proposes new stability, interval respecting stability, for matching problems with interval constraints, which defines ceiling respecting stability (Kamada and Kojima, 2018) using a blocking coalition instead of a pair, following floor respecting stability (Akin, 2021). Interval respecting stability coincides with floor respecting stability in problems with floor constraints and implies ceiling respecting stability in problems with ceiling constraints. In addition, interval respecting stability generally implies Pareto efficiency, unlike ceiling respecting stability.

We also propose a generalized flexible deferred acceptance algorithm for a problem with interval constraints, which is a flexible deferred acceptance algorithm (i.e., cumulative offer process) that allocates quotas between regions, reserving additional numbers for doctors' future offers needed to fill the floor constraints even if there are no offers now. Under acceptability, we show that further combining the above algorithm with the serial dictatorship yields an algorithm that satisfies interval respecting stability. We also show that the combinded algorithm is strategy-proof for doctors.

Keywords: Matching; Interval constraints; Stability; Strategy-proofness; Cumulative offer process

JEL classification: C78; D47; D61; D63

1 Introduction

In this paper, we show the existence of stable matching in the matching problem with interval constraints and the algorithm that produces it.

Recently, the matching theory is applied to many real-world problems, and increasingly research is being conducted toward this end. The simplest matching models, while tractable, maybe too naive to be applied to real-world problems. For example, a school may have a preference not only for individual students but also for diversity. [1] In this case, a school may prefer if one student enrolls with another student, but not if she enrolls without him. We cannot deal with this situation in the most elementary matching problem. Alternatively, it is also essential to consider matching in cases with no standard stable matching. For example, in such cases, it is known that the larger the market, the less dissatisfied the participants are. Moreover, we often face problems where there are ceiling constraints or floor constraints on the number of matches an agent or object can make. We do not doubt that such matching models with constraints are indispensable for solving real-world problems like residency matching or laboratory assignment problems. [2]

While matching models with ceiling constraints and matching models with floor constraints have been studied respectively, there are not many models that exist together, but they are vital. In residency matching, doctor applications may be concentrated in certain regions. In many cases, they have solved the problem by imposing ceiling constraints on hospitals or their regions. In addition, Japanese medical schools have "regional quotas," or quotas that force applicants to train and work in a particular region at the time of admission. Students admitted under this quota may be forced to pay a large penalty if they do not find employment in that region. However, many students currently break the terms and conditions and take residency and employment in other areas, criticizing such restrictions as improperly harsh. In such cases, it may be possible to distribute the doctor centrally to the regions by considering regional floor constraints rather than ceiling constraints as in the past. Furthermore, for example, when assigning new employees to a department, it may be desirable to set a minimum and a maximum number of employees for a specific department in a specific branch and then match them. Thus, there appear to be many matching problems for which it is more natural to consider both ceiling and floor constraints, i.e., interval constraints.

In the matching model with constraints, it is crucial to determine how much to weaken stability. This is because stability in the usual sense may not work well under constraints, and "too weak" stability is not practical. Ehlers. et al. (2014) showed that there is no stable matching in the usual sense in matching with floor constraints. On the other hand, Akin (2021) shows that floor respecting stable matching exists if reassignment to another hospital is not allowed for agents outside a blocking coalition. Kamada and Kojima (2017) discuss to what extent stability can be weakened in matching with ceiling constraints to the extent that stable matching exists. Kamada and Kojima (2018) shows that if a blocking pair exceeds the regional ceiling in a newly created matching, they allow it to exist as a non-legitimate blocking pair and that in matching with ceiling constraints, there is always a ceiling respecting stable matching. We show that it is not always appropriate to use the above stabilities under interval constraints. Floor respecting stability is too strong, and ceiling respecting stability is inefficient in matching interval constraints. We show that there is always a stable matching under interval constraints by allowing some movement

[1] Diversity problems in schools are usually more complicated than the model in this paper: see, e.g., Kurata et al. (2017), Ayğun and Turhan (2020), Correa et al. (2019), Ayğun and Bó (2021).

[2] In very recent years, some have tried to apply matching with constraints to healthcare rationing as well; see, e. g., Aziz and Brandl (2020), Pathak et al. (2021).

inside the blocking coalition. On the other hand, if we consider even stronger stability, we show that such stable matching may not exist under interval constraints.

We introduce interval constraints flexible differed acceptance algorithm. The algorithm is a generalization of the ceiling constraints flexible differed acceptance algorithm introduced by Kamada and Kojima (2018) for the matching problem with ceiling constraints. When the accumulated applications by doctors currently fall below the ceiling constraints for a region, a quota is allocated to each region, leaving a margin for future applications to that region. We show that the algorithm, defined as a combination of interval constraints flexible differed acceptance algorithm and serial dictatorship, satisfies interval respecting stability and strategy-proofness. We also show that we need to combine the generalized flexible differed acceptance algorithm with the serial dictatorship. In other words, simply adopting the interval constraints flexible differed acceptance algorithm instead of the ceiling constraints flexible differed acceptance algorithm achieves feasible matching but does not necessarily satisfy interval respecting stability. On the other hand, directly combining the ceiling constraints flexible differed acceptance algorithm with serial dictatorship, without generalizing the flexible differed acceptance algorithm, may not result in a feasible matching, unlike with floor constraints.

In the remainder of this section, we describe related research. In section 2, we introduce the basic setting. In section 3, we discuss several types of stability and introduce the IRS. In section 4, we propose the GFDA and GFDA+SD algorithms and explain that it satisfies our stability and strategy-proofness. In section 5, we discuss further issues. All proofs are in the Appendix.

1.1 Related Literature

Matching problems with constraints can be divided into problems with ceiling constraints and floor constraints. The matching with ceiling constraints problem has been studied primarily in the context of the residency matching problem. The model is characterized by the presence of ceiling constraints for each hospital in a given region. Such a problem originated by Kamada and Kojima (2015), which has discussed various notions of regional structure and stability and the mechanisms that produce stable matching. Kamada and Kojima (2017) introduces two types of stability when regions have ceiling constraints: strong stability and weak stability. Both stabilities allow for some blocking pairs and propose matching that does not override the ceiling constraints and does not cause anyone's deviations. They proved that while strong stable matching may not exist, weak stable matching always exists. [3] Kamada and Kojima (2018) proves that a necessary and sufficient condition for the existence of weak stable matching is that the structure of the region is a hierarchy. Intuitively, the structure of a region is a hierarchy, meaning that only some of the regions or hospitals belonging to one region do not belong to another region. We also adopt the framework in their work but propose stronger stability than the one they introduce. Kamada and Kojima (2018) also proves that the constrained matching problem can be associated with the contracted matching problem (Hatfield and Milgrom, 2005). Note that the choice function in contracted matching has been variously characterized, e.g., Hatfield and Milgrom (2015), Hirata and Kasuya (2017), and Hatfield, Kominers, and Westkamp (2021), but independently of their approach. We define the choice according to Kamada and Kojima (2018).

[3]Aziz, Baychkov, and Biró (2020) also showed that checking whether strong stable matching exists or not is NP-hard.

Ehler (2010) and Ehlers et al. (2014) split the notion of stability into non-wastefulness and fairness and show that there may be no matching that satisfies both of them in the matching with floor constraints problem. [4] Biró et al. (2010) and Huang (2010) also proved that it is NP-hard to determine whether stable matching exists with floor constraints. Thus, the usefulness of analyzing the model and a matching model with ceiling constraints, where stability in the usual sense may exist, is weakened. This is why there have been few studies of matching models with interval constraints.

Many studies have examined whether stable matching or some of its properties are present in more restricted situations. Fragiadakis et al. (2016) developed a mechanism that would require a tradeoff between the two concepts proposed by Ehlers et al. (2014). Yokoi (2020) proves that the existence of justified envy-free matching is NP-hard but solvable under certain conditions. Tomoeda (2018) showed that there is always a stable matching by restricting hospital preferences. Akin (2021) shows that by devising a definition of blocking coalition, there exists stable matching and a strategy-proof mechanism that produces it in the matching with floor constraints problem. They require that doctors not in the blocking coalition be matched with the same partner in the new matching as in the previous matching or be unemployed. Such blocking is consistent with that in the model with ceiling constraints. We, therefore, build on Akin (2021) and integrate the matching with ceiling constraints problem with the model with floor constraints to analyze reasonable stability.

2 Model

2.1 Preliminaries

Let there be a finite set of doctors D and hospitals H. Doctors are numbered from 1 to $|D|$. Thus, $D = \{1, 2, \dots, |D|\}$. Each doctor $d \in D$ has a strict preference $\succ_d$ over hospitals and remaining unassigned, denoted by $\emptyset$. For any $h, h' \in H \cup \{\emptyset\}$, we write $h \succeq_d h'$ if and only if $h \succ_d h'$ or $h = h'$. Let $\boldsymbol{P}_d$ is the set of all preferences. Doctor d is said to be acceptable to h if $d \succ_h \emptyset$. For any $\succ_d \in \boldsymbol{P}_d$, let $Ac_{\succ_d}$ be the set of all acceptable hospitals: i.e., $Ac_{\succ_d} = \{h \in H | h \succ_d \emptyset$. .To simplify the notation, we write the list of all acceptable partners representing the preference. For example,

$$\succ_d : h, h'.$$

In particular, we say that a preference $\succ_d \in \boldsymbol{P}_d$ is **acceptable** iff $Ac_{\succ_d} = H$.

Each hospital h is endowed with a physical capacity $q_h > 0$ and has a strict preference $\succ_h$ over the subsets of doctors. We assume that the hospital preferences are acceptable and **responsive** (Roth, 1985): formally, for every $D' \subseteq D$ with $|D'| < q_h$ and $d, d' \in D \setminus D'$, we have $D' \cup \{d\} \succ_h D' \cup \{d'\} \succ_h D'$ iff $\{d\} \succ_h \{d'\}$.

A subset $\mu \subset D \times H$ is a **matching** iff (i) every $d \in D$ has at most one $h \in H$ such that $(d, h) \in \mu$ and (ii) for every $h \in H$, doctors such that $(d, h) \in \mu$ are at most q_h. Here, $(d, h) \in \mu$ means that doctor d is assigned to hospital h. Let $\mathcal{M}$ denote the set of all matchings. For each $\mu \in \mathcal{M}$, let μ_D and μ_H respectively denote the doctors who are assigned to some hospital, and the hospitals that some doctor are assigned to in the matching: i.e., $\mu_D = \{d \in D | \exists h \in H, (d, h) \in \mu\}$

[4]Fairness is also called justified envy-freeness (Abdulkadiroğlu and Sönmez, 2003).

and $\mu_H = \{h \in H | \exists d \in D, (d,h) \in \mu\}$. In addition, for each $(d,h) \in D \times H$, let μ_d and μ_h respectively denote the hospital and doctors whose (one of) partners are d and h: i.e., $\mu_d = h$ if $(d,h) \in \mu$ for some $h \in H$, otherwise, $\mu_d = \emptyset$; and $\mu_h = \{d \in D | (d,h) \in \mu\}$.

Since hospital preferences are acceptable, there is no individual rationality constraint for hospitals. We say a matching $\mu \in \mathcal{M}$ is **individual rational** if and only if for each $d \in D$, $\mu_d \succ_d \emptyset$.

2.2 Regions

There is a set of regions $R \subseteq 2^H$. We say that R is **hierarchy** if $r, r' \in R$ implies $r \subseteq r'$ or $r' \subseteq r$ or $r \cap r' = \emptyset$. We also assume that $\{h\} \in R$ for each $h \in H$ and $H \in R$. The set of regions $s(r) \subset R$ is the **subregion** of r if $\bigcup_{r' \in s(r)} r' = r$ and for any $r_1, r_2 \in s(r)$, $r_1 \cap r_2 = \emptyset$. Let $R_{\ni h} = \{r \in R | h \in r\}$ for each $h \in H$. For any $\mu \in \mathcal{M}$, let $\mu_r = \{d \in D | \mu_d \in r\}$.

Each region $r \in R$ is confronted with **floor constraints** $\underline{\kappa}_r$ and **ceiling constraints** $\bar{\kappa}_r$, where the number of doctors in the region is required to be in the interval. We say a matching $\mu \in \mathcal{M}$ is **feasible** if $|\mu_r| \in [\underline{\kappa}_r, \bar{\kappa}_r]$ for each $r \in R$. To avoid complexity, we assume that for any $h \in H$, $\bar{\kappa}_{\{h\}}$ is equal to or smaller than q_h.

To consider the condition under which a feasible matching exists, we take the following approach. Given constraints $\underline{\kappa}, \bar{\kappa} \in \mathbb{N}_+^R$, we inductively define $(\underline{\kappa}_r^*, \bar{\kappa}_r^*)$ for each region $r \in R$ as follows:

- Set $\underline{\kappa}_{\{h\}}^* = \underline{\kappa}_{\{h\}}$. For any nonsingleton $r \in R$ such that each of $\underline{\kappa}_{s(r)}^*$ has been defined, set $\underline{\kappa}_r^* = \max\{\underline{\kappa}_r, \sum_{s(r)} \underline{\kappa}_{r'}^*\}$.
- Set $\bar{\kappa}_{\{h\}}^* = \bar{\kappa}_{\{h\}}$. For any non-singleton $r \in R$ such that each of $\bar{\kappa}_{s(r)}^*$ has been defined, set $\bar{\kappa}_r^* = \min\{\bar{\kappa}_r, \sum_{s(r)} \bar{\kappa}_{r'}^*\}$.

For any $r \in R$, $\underline{\kappa}_r^*$ and $\bar{\kappa}_r^*$ respectively describe the floor and ceiling numbers of doctors for the region that are necessary to satisfy the floor and ceiling constraints of all regions included in r. We get the following result.

Proposition 1. If there is a feasible matching, then (i) $\underline{\kappa}_H^* \leq |D|$ and (ii) $\underline{\kappa}_r^* \leq \bar{\kappa}_r^*$ for each $r \in R$.

Proposition 1 gives a necessary condition for the existence of feasible matching. Hereafter, we assume that $(\underline{\kappa}, \bar{\kappa})$ satisfies this condition.

Assumption 1. (i) $\underline{\kappa}_H^* \leq |D|$ and (ii) $\underline{\kappa}_r^* \leq \bar{\kappa}_r^*$ for each $r \in R$.

It is shown as a corollary of the feasibility of our algorithm (Proposition 6) that this condition is also the sufficient condition.

Furthermore, each nonsingleton region r is assumed to have a (preorder) preference $\tilde{\succsim}_r$ on $\mathbb{N}_+^{s(r)}$, where each $w_{s(r)} \in \mathbb{N}_+^{s(r)}$ is considered a distribution of numbers of doctors among the subregions. We assume that regional preference $\tilde{\succ}_r$ satisfies the following two properties. The first assumption is **distributional acceptance**. That is, $w' \lneq w$ implies $w \tilde{\succ}_r w'$ for any $w, w' \in \mathbb{N}_+^{s(r)}$. The second assumption is **distributional substitutability with floor constraint**

$\underline{\kappa}_r^*$. That is, $\tilde{\succsim}_r$ has a quasi-choice function $\tilde{\mathrm{Ch}}_r : \mathcal{V}_r \to \mathbb{N}_+^{s(r)}$, where

$$\mathcal{V}_r \stackrel{\text{def}}{=} \left\{ (\underline{v}, v, c) \in \mathbb{N}_+^{s(r)} \times \mathbb{N}_+^{s(r)} \times \mathbb{N}_+ \middle| \{w \in \mathbb{N}_+^{s(r)} | \underline{v} \leq w \leq v \ \& \sum_{s(r)} w_{r'} \leq c\} \neq \emptyset \right\},$$

such that

(A) $\tilde{\mathrm{Ch}}_r((\underline{v}, v), c)$ is one of the best distributions among all distributions in $\{w \in [\underline{v}, v] | \sum_{s(r)} w_{r'} \leq c\}$ with respect to $\tilde{\succsim}_r$:

$$\forall (\underline{v}, v, c) \in \mathcal{B} \times \mathbb{N}_+, \quad \tilde{\mathrm{Ch}}_r(\underline{v}, v, c) \in [\underline{v}, v] \quad \& \quad \sum_{s(r)} (\tilde{\mathrm{Ch}}_r(\underline{v}, v, c))_{r'} \leq c$$
$$\& \ \forall w \in [\underline{v}, v] \text{ s.t. } \sum_{s(r)} w_{r'} \leq c, \quad \tilde{\mathrm{Ch}}_r(\underline{v}, v, c) \tilde{\succsim}_r w.$$

(B) Especially, if $\underline{v} = \underline{\kappa}_r^*$, the chosen numbers of doctors are substitutive among the subregions:

$$v \geq v' \& c \geq c' \Longrightarrow \tilde{\mathrm{Ch}}_r(\underline{\kappa}_r^*, v, c) \geq \min\{\tilde{\mathrm{Ch}}_r(\underline{\kappa}_r^*, v', c'), v\}.$$

This assumption requires that $\tilde{\succsim}_r$ reveals a type of substitutability specified in (B) under a choice problem where the range is an $|s(r)|$-dimensional interval with the cap of the total numbers. Intuitively, (B) means that when the lower bound of the interval is $\underline{\kappa}_{s(r)}^*$, (i) if the cap is relaxed, the number of doctors does not decrease in any subregion, and (ii) if the upper bound of the interval is relaxed and the original number of doctors is smaller than the upper bound in a subregion, the number of doctors in the subregion does not increase. Here, the fact that the lower bound of the interval is $\underline{\kappa}_{s(r)}^*$ means that region r chooses the distribution in the range where all regions included in r can satisfy their floor constraints. If $\underline{\kappa}_{s(r)}^* = \mathbf{0}$, then this assumption is equivalent to that introduced in Kamada and Kojima (2018).

Note that this condition is similar to the standard substitutability by Roth and Sotomayor (1990) and Hatfield Milgrom (2005) but differs in that (i) it does not distinguish between different doctors and (ii) it can handle multi-unit supply. However, Kamada and Kojima (2018) shows that if the quasi-choice function satisfies their substitutability, then it also satisfies the law of aggregate demand and substitutability introduced by Hatfield and Milgrom (2005).

A matching problem (with interval constraints) is given as a list of $(D, H, \succ_D, \succ_H, q_H, \underline{\kappa}, \bar{\kappa}, \succsim_R)$. In particular, a problem with ceiling constraints is a problem such that $\underline{\kappa} = \mathbf{0}$; a problem with floor constraints is a problem such that $\bar{\kappa}_r = +\infty$ for any non-singleton $r \in R$ and $\bar{\kappa}_{\{h\}} = q_h$ for any $h \in H$; a problem with disjoint regions is a problem such that there exists a partition P of R such that for any $r \in R \setminus P$, $(\underline{\kappa}_r, \bar{\kappa}_r)$ is $(0, +\infty)$ if $|r| \geq 2$, and $(0, q_r)$ if $r = \{h\}$ for some $h \in H$.

3 Stability

This section discusses the stability in a problem with interval constraints.

First, we introduce two types of stability: floor and ceiling respecting stability, which was proposed to analyze problems with floor and ceiling constraints, respectively (Kamada and Kojima, 2017; Akin, 2021). Under both problems, stable matching may not exist if we require blocking not to exist at all. Instead, both stability types consider making as many blockings as possibly valid to the extent that the existence of a corresponding stable matching is guaranteed. We then propose stability in problems with interval constraints, interval respecting stability, following the same principles, based on the two types of stability. Finally, we discuss impossibility of stronger stability.

3.1 Floor respecting stability

When considering blocking in a problem with floor constraints, unlike the case of ceiling constraints, we have to consider whether the doctor's original region meets the floor constraint after the blocking. Although the original hospital is not part of the blocking pair, it is questionable whether blocking that does not result in feasible matching is valid in society. Thus, we need to consider what moves outside the blocking pair for a feasible matching that is reasonable. Floor respecting stability considers that if the pair can be considered as a part of a larger coalition and the coalition can form a feasible matching as a blocking coalition, then the blocking is valid. In other words, floor respecting stability requires the non-existence of blocking by the general form of coalition, including non-pair-type coalition.

Definition 1. A nonempty coalition $B = B_D \times B_H \subseteq D \times H$ **blocks** μ with μ' if

(i) μ' is feasible,

(ii) For every $i \in B_D \cup B_H$, $\mu'_i \succ_i \mu_i$,

(iii) For every $d \notin D \setminus B_D$, $\mu'_d \in \{\mu_d, \varnothing\}$.

(iv) For every $h \notin H \setminus B_H$, $\mu'_h = \mu_h \setminus B_D$.[5]

Especially, a coalition B is a blocking pair iff B is a singleton. For notational simplicity, a blocking pair is denoted by (d, h) instead of $(\{d\}, \{h\})$.

In μ' formed by blocking, individuals other than B either have the same partner as that of μ or lose a member of B as a partner; only members of B can have new partners, but they must strictly improve.

[5]There are two differences between our and Akin (2021)'s definitions. First, they refer to the combination of Babove and the doctors and hospitals that did not move as a coalition. Since their stability does not depend on what a coalition refers to, this change is simply a matter of terminology.

Secondly, in their original definition, (iv) is $\mu'_h \subseteq \mu_h \setminus B_D$. Thus, a hospital not in the coalition can fire some currently matched doctors regardless of whether it improves its welfare. Although Akin (2021) considers problems with floor constraints, when there are also ceiling constraints, some matchings can be unreasonably blocked since we can make a blocking satisfying ceiling constraints by forcing irrelevant hospitals to fire some of their doctors. In this sense, to reasonably generalize the concept to problems other than problems with floor constraints, we add the latter part to (iv).

Note that under problems with floor constraints, a matching has a blocking in the original definition if and only if it has a blocking in our definition, since if a matching with such dismissal is feasible, then the matching is also feasible without the dismissal.

The definition of floor respecting stability is as follows:

Definition 2. We say a matching μ is **floor respecting stable** if and only if μ is feasible, individually rational, and there is no blocking coalition.

Akin (2021) shows that in a matching problem with floor constraints, there exists a floor respecting stable matching. Ehlers (2014) shows that if we further consider blocking to also be valid, including individuals with new but not improving partners, there is no corresponding stable matching.

3.2 Ceiling respecting stability

It is known that in problems with ceiling constraints, even considering only blocking pairs to be valid may lead to the non-existence of stable matching (Kamada and Kojima, 2015). For each $(d, h) \in D \times H$, let μ^{dh} be a matching such that $\mu^{dh}_{d'} = \mu_{d'}$ for $d' \neq d$ and $\mu^{dh}_{d} = h$. That is, μ^{dh} is a matching where only doctor d moves to hospital h from μ. To ensure the existence of stable matching, ceiling respecting stability restricts valid blocking pairs in the following way: for any blocking pair (d, h), first, if μ_h contains a doctor d' who is less preferable for h to d, then blocking with μ' where h dismiss d' and employs d is valid *(justified envyness)*; secondly, if $m_d = \emptyset$, blocking with m^{dh} is valid *(strongly wastefullness)* ;[6] otherwise, blocking by μ' is vaild only if the smallest region r containing both the original hospital μ_d and h is (a) included in some region reaching the ceiling and (b) strictly prefers $\mu'_{s(r)}$ to $\mu_{s(r)}$ *(legitimacy)*. The intuition for legitimacy is the following. Note here that the fact that r is the smallest such region is equivalent to the fact that r is the (unique) largest region whose subregion distribution differs between μ and μ^{dh}. In other words, r is the highest region affected by the change from μ to μ^{dh}. Thus, this condition requires that the highest region affected by the blocking should prefer the blocking matching to the original matching.

The definition of ceiling respecting stability is as follows.

Definition 3. We say a matching μ is **ceiling respecting stable** if and only if μ is feasible, individual rational, and if $(d, h) \in D \times H$ blocks μ with μ', (i) $d' \succ_h d$ for every $d' \in \mu_h$, and (ii) $\mu = \mu^{dh}$ and there exists $r \in R$ satisfying $\mu_d, h \in r$ and $(|\mu_{r'}|)_{s(r)} \neq (|\mu^{dh}_{r'}|)_{s(r)}$, and it follows that $|\mu_{\bar{r}}| = \bar{\kappa}_r$ for some $\bar{r} \in R$ with $\bar{r} \supseteq r$, and $(|\mu_{r'}|)_{s(r)} \tilde{\succsim}_r (|\mu^{dh}_{r'}|)_{s(r)}$.

Here, (i) corresponds to μ being justified envy-free. (ii) corresponds to μ being neither strongly wasteful nor legitimate.[7] Kamada and Kojima (2018) show that the ceiling respecting stable matching always exists in a matching problem with ceiling constraints.

3.3 Interval respecting stability

Floor respecting stability requires that all blocking, including non-pair-type blockings, is valid and that no valid blocking exists. On the other hand, ceiling respecting stability requires that only a further part of blocking pairs is valid and that no valid blocking exists. Thus, the set of

[6] Note that for m^{dh} to be a blocking matching, it must be feasible (condition (i)). Therefore, $m_d = \emptyset$ implies the addition of d does not violate any constraint.

[7] (ii) means non strongly wastfulness since if $\mu_d = \emptyset$, then there is no region satisfying $m_d, h \in r$.

valid blockings for floor respecting stability includes that for ceiling respecting stability. The following result holds.

Remark 1. Floor respecting stability implies ceiling respecting stability.

However, each has its deficiencies in problems with interval constraints. First, as mentioned earlier, even in problems with ceiling constraints, if all pair-type blockings are valid, there may be no stable matching. On the other hand, Akin (2021) shows that in the problem with floor constraints, considering only blocking pairs is too weak to satisfy Pareto efficiency among doctors and hospitals.

A matching μ is **Pareto efficient** if there is no other feasible matching μ' such that $\mu'_i \succeq_i \mu_i$ for all $i \in D \cup H$ and $\mu'_i \succ_i \mu_i$ for some $i \in D \cup H$.

Fact 1. (i) If the problem is not a matching with floor constraints, floor respecting stable matching may not exist. (ii) If the problem is not a matching with ceiling constraints, ceiling respecting stable matching may not be Pareto efficient.

We now explore stability that considers as many blockings as possible to be valid, to the extent that the existence of stable matching can be ensured, based on the above two types of stability in the corresponding constrained problems. We have two issues to discuss.

First, should we consider a non-pair-type blocking coalition outside of problems with floor constraints? The following example shows that even in problems with only ceiling constraints, unlike in problems with no constraint, there is a matching that is not blocked by any blocking pair but is blocked by some non-pair type coalition.

Example 1. Let $D = \{1, 2\}$ and $H = \{h_1, h'_1, h_2, h'_2\}$. The physical cap q_h is $+\infty$ for every $h \in H$. Their preferences are defined as

$$\succ_1\colon h_1, h'_1, h_2, h'_2$$
$$\succ_2\colon h_2, h'_2, h_1, h'_1$$

$$\succ_{h_1}\colon 1, 2$$
$$\succ_{h'_1}\colon 2, 1$$
$$\succ_{h_2}\colon 2, 1$$
$$\succ_{h'_2}\colon 1, 2.$$

The region structure R is $\{\{h_1\}, \{h'_1\}, \{h_2\}, \{h'_2\}, r_1, r_2, H\}$. Hospitals h_1 and h'_1 is located in r_1 and h_2 and h'_2 in r_2. Formally, $\{h_1\}, \{h'_1\} \subset r_1$ and $\{h_2\}, \{h'_2\} \subset r_2$. There is no ceiling constraint: $\bar{\kappa}_r = +\infty$ for every $r \in R$. Floor constraints are given as $\underline{\kappa}_{r_1} = \underline{\kappa}_{r_2} = 1$.

The regional preference is

$$\tilde{\succ}_{r_1} : (1, 0), (0, 1)$$
$$\tilde{\succ}_{r_2} : (1, 0), (0, 1).$$

Consider the following matching:

$$\mu = \{(1, h_2'), (2, h_1')\}.$$

There is no pair blocking μ; although $\{1\} \succ_{h_1} \emptyset$ and $h_1 \succ_1 h_2'$, any matching that they can form is not feasible, and the same is true for 2 and h_2. However, coalition $\{1, 2\} \times \{h_1, h_2\}$ can block μ with $\mu' = \{(1, h_1), (2, h_2)\}$.

Note that in this example, all regions also prefer μ' to μ. Therefore, it is also reasonable to regard this blocking as valid from the perspective of regional preferences. Thus, we will assume blockings including non-pair-type coalition for interval constraints, as long as stable matching exists.[8]

Next, we must consider what non-pair type blockings are to be considered valid since considering all blocking to be valid leads to the non-existence of stable matching. The classification of blockings in ceiling respecting stability is for blocking pairs. In non-pair-type blocking, they can be complex. According to our aim to make as many blockings as possibly valid, for the first two categories, we consider in the following way: first, for at least one hospital h in B_H and a doctor in $\mu'_h \cap B_D$, blocking with μ' is valid if μ_h contains a doctor d' who is less preferable for h to d *(justified envy)*; secondly, if $\mu'_D \supsetneq \mu_D$, then blocking with μ' is valid *(strongly wasteful)*. Note that if B is a blocking pair, these are equivalent to those in ceiling respecting stability.

Finally, we consider legitimacy. Unlike the case of a blocking pair, there can be multiple hospitals in B_H. Therefore, there is not necessarily one highest affected region in the sense that the sub-region's distribution has changed and the distribution of any superior region has not changed. Thus, there may be conflicting preferences among the highest regions. Again, we will make as much blocking as possible valid for this. That is, if at least one of the highest regions in the above sense prefers blocking, the blocking is valid; in other words, only when none of the highest regions prefers blocking, the blocking is not legitimate.

We define interval respecting stability in the following way.[9]

Definition 4. A matching μ is **interval respecting stable** if and only if μ is feasible, individual rational, and if $B \subseteq D \times H$ blocks μ with μ' (i) $d' \succ_h d$ for every $h \in B_H$, $d \in \mu'_h \cap B_D$, and $d' \in \mu_h$, and (ii) for each $h \in B_H$, there exist $r^* \in R_{\ni h}$ satisfying $(|\mu_{r'}|)_{s(r^*)} \neq (|\mu'_{r'}|)_{s(r^*)}$, and it follows that $|\mu_{\bar{r}}| = \bar{\kappa}_r$ for some $\bar{r} \in R$ with $\bar{r} \supseteq r^*$, and $(|\mu_{r'}|)_{s(r)} \tilde{\succsim}_r (|\mu'_{r'}|)_{s(r)}$ for any $r \in R$ with $r^* \subseteq r \subseteq \bar{r}$.

As in ceiling respecting stability, (i) corresponds to μ being justified envy-free. (ii) means that for any hospital h in B_H, (a) there is a region $\bar{r}$ that has reached its ceiling that includes it, and (b) either the highest affected region or the region $\bar{r}$ in itself does not prefer blocking (illegitimate). (ii) also means that μ is not strongly wasteful.[10]

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[8] Note that the algorithm proposed by Kamada and Kojima (2018) consequently generates a matching satisfying interval respective stability, which assumes blocking including the non-pair coalition, in problems with ceiling constraints. This is shown by the fact that our algorithm, which generates an interval respective stable matching, coincides with theirs in problems with ceiling constraints. In other words, our result shows that the matchings generated by their algorithm satisfies stability stronger than what they consider.

[9] We discuss variants of the definition in Section 5.1.

[10] The proof is as follows. By (ii), there is $R' \subset R$ that covers B_H, is mutually disjoint, and satisfies $\mu_r = \bar{\kappa}_r$ for every

Interval respecting stability considers blocking coalitions, but unlike floor respecting stable, it considers only those coalitions that satisfy additional conditions to be valid. Therefore, floor respecting stable is as strong as or stronger than interval respecting stable. In particular, the additional condition is always satisfied in the matching problem with floor constraints, so floor respecting stable and interval respecting stable are equivalent. Next, if the coalition in interval respecting stable is a pair, then (i) and (ii) in interval respecting stable and ceiling respecting stable are equivalent. Thus, interval respecting stable is as strong as or stronger than ceiling respecting stable.

Now, we have the following remark.

Remark 2. Floor respecting stable implies interval respecting stable. Interval respecting stable implies ceiling respecting stable. Especially in matching problem with floor constraints, floor respecting stable is equivalent to interval respecting stable.

Floor respecting stable is the strongest requirement, but as shown in Fact 1 (i), it cannot be guaranteed to exist even for matching problems with ceiling constraints. On the other hand, ceiling respecting stable is a weak requirement; this condition does not imply Pareto efficiency (Fact 1 (ii)).

In contrast to floor respecting stability, interval respecting stability always satisfies Pareto efficiency.

Proposition 2. Interval respecting stability implies Pareto efficiency.

3.4 Strong interval respecting stability

Both illegitimacy of ceiling and interval respecting stability require $h\ inB_H$ to be included in the region reaching the ceiling. However, there is a difference between them as to what this means. In the case of ceiling respecting stability, legitimacy implies that d in B_D moves within the region reaching its ceiling. On the other hand, legitimacy does not imply this in the case of interval respecting stability. The blocking can be illegitimate even if the same number of doctors move in and out between different regions reaching their ceilings, rather than within one region that has reached its ceiling. For stronger stability, we might consider regarding the blocking as legitimate unless movement is inside a region that has reached its ceiling, even if the coalition is not a pair. We call it strong interval respecting stability.

Definition 5. A matching μ is **strongly interval respecting stable** if μ is feasible, individually rational, and if $B \in D \times H$ blocks μ with μ', then (i) $d' \succ_h d$ for every $h \in B_H$, $d' \in \mu_h$ and $d \in \mu'_h \cap B$ and (ii) there exists $\bar{r} \in R$ such that $|\mu_{\bar{r}}| = \bar{\kappa}_r$, and for every $h \in B_H$, there exists $r_h \in R_{\ni h} \cap 2^{\bar{r}}$ such that $(|\mu_{r'}|)_{s(r_h)} \neq (|\mu'_{r'}|)_{s(r_h)}$ and $(|\mu_{r'}|)_{s(r)} \tilde{\succsim}_r (|\mu'_{r'}|)_{s(r)}$ for every $r \in R$ with $r_h \subseteq r \subseteq \bar{r}$.

$r \in R'$. By definition of blocking, $\mu'_h \leq \mu_h$ for any $h \in H \setminus B_H$. Thus,

$$\cup_{h\in\cup R'}|\mu'_h| = \cup_H|\mu'_h| - \cup_{h\notin\cup R'}|\mu'_h| \geq \cup_H|\mu'_h| - \cup_{h\notin\cup R'}|\mu_h| > \cup_H|\mu_h| - \cup_{h\notin\cup R'}|\mu_h| = \cup_{h\in\cup R'}|\mu_h|.$$

Since R' is mutually disjoint, $\mu'_r > \mu_r = \bar{\kappa}_r$ for some $r \in R'$, which contradicts feasibility of μ'.

This stability requires all hospitals in B_H to be included in the same region reaching its ceiling: that is, $\bar{r}$ in the definition of interval respecting stability must be the same for every $h \in B_H$. Thus, while this stability regards some blocking as invalid, it is stronger than interval respecting stability.

Remark 3. Floor respecting stable implies strong interval respecting stable. strong interval respecting stable implies interval respecting stable.

The following example illustrates that strongly interval respecting stable matching does not always exist.

Example 2. Suppose that there are four doctors $D = \{d_1, d_2, d_3, d_4\}$ and three hospitals $H = \{h_1, h_2, h_3, h_4\}$. The preference profile is

$$\begin{aligned}
\succ_{d_1} &: h_2, h_3, h_4, h_1 \\
\succ_{d_2} &: h_1, h_3, h_4, h_2 \\
\succ_{d_3} &: h_2, h_1, h_4, h_3 \\
\succ_{d_4} &: h_1, h_3, h_2, h_4
\end{aligned}$$

$$\begin{aligned}
\succ_{h_1} &: d_1, d_2, d_3, d_4 \\
\succ_{h_2} &: d_2, d_1, d_3, d_4 \\
\succ_{h_3} &: d_3, d_4, d_1, d_2 \\
\succ_{h_3} &: d_4, d_3, d_1, d_2.
\end{aligned}$$

The regional structure is $R = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_4\}, r, r', H\}$ where $r = \{h_1, h_2\}$ and $r' = \{h_3, h_4\}$. There is no floor constraint: $\underline{\kappa}_{r''} = 0$ for every $r'' \in R$. Only r and r' have ceiling constraints given by $\bar{\kappa}_r = \bar{\kappa}_{r'} = 1$; thus, $\bar{\kappa}_{r''} = +\infty$ for $r'' \notin \{r, r'\}$. We assume that regional preferences are indifferent. That is, each region cares only about the total number of slots filled in its own subregion.

Note that when considering either stable matching, each hospital never matches with a doctor other than the most preferred one. For example, suppose d_2 is matched with h_1. In this case, d_1 is matched with h_3 or h_4 or is unemployed. If he is matched with h_3, the pair (d_3, h_3) blocks this match because d_3, the most preferred doctor for h_3, is unemployed. Similarly, if he is matched with hospital h_4, the pair (d_4, h_4) blocks this matching. Also, when d_1 is unemployed, the pair (d_1, h_1) blocks this matching. These blockings are not valid for considering either stability. Therefore, we consider only the following feasible matchings:

$$\mu_1 = \{(d_1, h_1), (d_3, h_3)\}, \quad \mu_2 = \{(d_2, h_2), (d_3, h_3)\},$$
$$\mu_3 = \{(d_1, h_1), (d_4, h_4)\}, \quad \mu_4 = \{(d_2, h_2), (d_4, h_4)\}$$

These matchings are interval respecting stable because each region indifferently prefers each allocation if the total number of doctors in its subregions is the same.

On the other hand, none of these are strongly interval respecting stable. This is because the strongly interval respecting stable also considers blocking valid across regions. For example, the matching μ_1 is blocked by the matching

$$\mu_1' = \{(d_1, h_3), (d_3, h_1)\}.$$

Similarly, the other matchings are not strongly interval respecting stable. Thus, strongly interval respecting stable matching may not exist.

Fact 2. In the matching problem with ceiling constraint, there may not exist a strongly interval respecting stable matching.

This means that the non-existence of stable matching arises not only from making all moves within a single region that has reached its ceiling valid but also from making moves between regions that have reached their ceilings valid. This is consistent with the notion of interval respecting stability.

In Section 4, we will show constructively that interval constraints respecting stable matching always exist and that a mechanism satisfies strategy-proofness.

4 Generalized FDA + serial dictatorship (GFDA+SD) algorithm

In this section, we define the generalized FDA + serial dictatorship (GFDA+SD) algorithm and show that it satisfies stability and strategy proofness. The definition of this algorithm consists of two parts. First, we formulate the generalized choice function Ch for the FDA in interval constraint problems. However, the FDA obtained by this choice function (GFDA) does not produce a feasible matching. Hence, we next consider a combination of this and the serial dictatorship process (GFDA+SD).

4.1 Reserved quota allocations

First, we introduce a concept for matching related to the process of making the lower constraint satisfied by adding doctors to that matching, which has an essential role in our algorithm.

For any matching $\mu \in \mathcal{M}$, inductively define **reserved quota** $\kappa_r \in$ for any $r \in R$ in the following way:

- Set $\kappa_{\{h\}} = \max\{\underline{\kappa}_r, |\mu_h|\}$. For any $r \in R$ such that each of $\kappa_{s(r)}$ has been defined, set $\kappa_r = \max\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}\}$.

We can consider κ_r the minimum numbers of doctors belonging to each region when doctors are added to μ to satisfy the floor constraint in the future. κ_r is different from $\max\{\underline{\kappa}_r^*, |\mu_r|\}$, which is the number of doctors ensuring that all regions included in r can satisfy the floor constraints by reallocation of doctors in r.

Example 3. There are three regions $r_A, r_B, r_C \in R$, and r_A and r_B are located in r_C. The floor constraints for each region are given as $X = Y$. Assume that r_A and r_B belong to an arbitrarily large number of hospitals and that many doctors apply to those hospitals.

Suppose that there are 110 doctor applicants in region r_A and 30 in region r_B. In this case, $\underline{\kappa}_{r_C}^* = 100$, which simply tells us that at least 100 doctors are needed to satisfy the floor constraints for regions below region r_C. Also, $\max\{\underline{\kappa}^*_{r_C}, |\mu_{r_C}|\} = 140$, which requests a quota to be reserved for r_C when doctor movement from r_A to r_B is allowed for. On the other hand, since we have $\kappa_{r_A} = 110$ and $\kappa_{r_B} = 50$, we obtain $\kappa_{r_C} = 160$. In other words, κ_{r_C} requests that we need to reserve an additional 20 slots, i.e., 160 slots overall, to meet all the floor constraints in the region below it without reducing the number of doctors who are currently matched with hospitals in r_A.

Thus, κ_r is the minimum number of doctors to satisfy the floor constraints by adding new doctors without reducing the current numbers of doctors for all regions included in r.

For any $r \in R$, define δ_r as $\delta_r = \kappa_r - \sum_{s(r)} \kappa_{r'}$ if $|r| \geq 2$, $\delta_{\{h\}} = \kappa_{\{h\}} - |\mu_h|$ if $r = \{h\}$ for some $h \in H$. Note that δ_r is non-negative and $\kappa_r = \underline{\kappa}_r$ if $\delta_r > 0$. δ_r is the number of doctors that need to be added to region r to satisfy the floor constraint. In other words, it is **the marginal shortfall** of doctors for region r.

For any $r' \in s(r)$, $\kappa_{r'}$ is the sum of doctors who are currently assigned to r' and the shortfalls of all regions included in r'. Thus, δ_r is the *marginal* shortfall of doctors that takes place between r and $s(r)$ (or $\{h\}$ and h). Let $\sum_R \delta_r$ be **the total shortfall** of doctors for μ. We can obtain the following equation:

$$\forall \mu \in \mathcal{M}, \quad \kappa_H - |\mu| = \left(\delta_H + \sum_{s(H)} \kappa_{r'}\right) - |\mu| = \delta_H + \sum_{s(H)} (\kappa_{r'} - |\mu_{r'}|) = \cdots = \sum_R \delta_r.$$

Thus, $\kappa_H - |\mu|$ is an alternative expression of the total shortfall for μ.

Note that $\kappa_H - |\mu| = 0$ is equivalent to that μ satisfies the floor constraints.

Proposition 3. For any $\mu \in \mathcal{M}$, $\kappa_H - |\mu| = 0$ iff $|\mu_r| \geq \underline{\kappa}_r$ for every $r \in R$. Thus, μ is feasible iff $\kappa_H - |\mu| = 0$ and $\kappa_r \leq \bar{\kappa}_r$ for every $r \in R$.

Thus, how to add doctors to the doctor shortage regions is essential when a matching does not satisfy the floor constraints. The following two sets will play an important role in this analysis: for any matching $\mu \in \mathcal{M}$, define $R^-(\mu) \subseteq R$ and $H^-(\mu) \subseteq H$ as follows.

- $r \in R^-(\mu)$ iff $\delta_r > 0$ or $[r \in R^-(\mu)$ and $\kappa_r < \bar{\kappa}_r]$.
- $h \in H^-(\mu)$ iff $\{h\} \in R^-(\mu)$.

$R^-(\mu)$ (or $H^-(\mu)$) is a region (or hospital) that is connected to a superior doctor shortage region without any region reaching ceiling constraints. In other words, $H^-(\mu)$ is the set of hospitals that can add doctors to a doctor shortage region without violating the ceiling constraint of any region.

Lemma 1. For any $\mu \in \mathcal{M}$, if $\kappa_H - |\mu| > 0$, then $H^-(\mu)$ is non empty.

Lemma 2. For any $\mu \in \mathcal{M}$, $d \notin \mu_D$, and $h \in H^-(\mu)$, when we consider $\mu' = \mu \cup \{(d, h)\}$, it follows that $\kappa'_r = \kappa_r$ for any $r \in R$ such that $r = H$ or $\kappa_r = \bar{\kappa}_r$.

Lemma 1 implies that (when Assumption 1 is satisfied,) any matching with a shortage of doctors has such a hospital that can add a doctor.

Lemma 2 implies two facts. First, $\kappa'_H = \kappa_H$, which means that if we add a doctor to such a hospital, then the reserved quota of H does not change while the total number of doctors increases (since $\mu'_D = \mu_D \cup \{d\}$). That is, such an addition must decrease the total shortfall $\kappa_H - |\mu|$ by one. Secondly, $\kappa'_r = \kappa_r$ when $\kappa_r = \bar{\kappa}_r$, which means that such an addition must not increase the reserved quota of any region reaching the ceiling constraint. Since $\kappa'_r \geq |\mu'_r|$, this ensures that such an addition does not create a new violation of ceiling constraints.

Since the second fact also holds for any $h \in H$ and we assume $\bar{\kappa}_r \leq q_r$, μ' satisfies the hospital's physical caps. Thus, Lemma 2 also ensures that $\mu' \in \mathcal{M}$.

4.2 Construction of choice function Ch in the interval constraint problem.

In this section, we consider how to generalize FDA by Kamada and Kojima (2018),CFDA, for the problem with ceiling constraints. The idea of CFDA is as follows. First, instead of the original problem, consider a matching problem with contracts between one fictitious hospital and D, where H is the set of contracts. Then, we use the *cumulative offer process* (COP) to generate a matching. Finally, we interpret the matching in the original problem, where H is the set of hospitals rather than contracts.

Key in this idea is how to construct the choice function of a single fictitious hospital, which is needed for COP. Based on the preferences of each region and hospital, we have to cleverly construct the choice function to derive the good behavior of the COP and the stability of the resulting matches. We propose how to construct the function generalized for the interval constraint problem.

Let the function that we are about to construct be denoted by Ch. Ch is a function assigning a subset of X to any set $X \subseteq D \times H$, where X is considered a set of accumulated contracts offered by doctors to the fictitious hospital and $Ch(X)$ is considered the set of provisionally accepted contracts.

Ch is constructed through three procedures. (1) a procedure whereby, based on the number of contracts applied for each hospital, each region calculates the contract quantity adjusted by the constraints from the singleton regions towards the top region; (2) a procedure whereby, based on subregions' adjusted contract quantities, the upper region r determines the quota allocation from the top region towards the singleton regions; and (3) a procedure whereby, based on its quota, each hospital chooses doctors from contracts addressed to itself.

The three are formulated below:

Adjusted contract quantity vector v: For any $X \subseteq D \times H$, inductively define the Adjusted contract quantity vector $\in \mathbb{N}_+^R$ towards the upper regions as follows.

- For each $h \in H$,

$$v_{\{h\}} = \begin{cases} \underline{\kappa}_{\{h\}} & \text{if } |X_h| < \underline{\kappa}_{\{h\}}, \\ |X_h| & \text{if } |X_h| \in [\underline{\kappa}_{\{h\}}, \bar{\kappa}_{\{h\}}], \\ \bar{\kappa}_{\{h\}} & \text{if } |X_h| > \bar{\kappa}_{\{h\}} \end{cases}$$

- For each $r \in R$ such that $\nu_{s(r)}$ has been defined,

$$\nu_r = \begin{cases} \underline{\kappa}_r & \text{if } \sum_{s(r)} \nu_{r'} < \underline{\kappa}_r, \\ \sum_{s(r)} \nu_{r'} & \text{if } \sum_{s(r)} \nu_{r'} \in [\underline{\kappa}_r, \bar{\kappa}_r], \\ \bar{\kappa}_r & \text{if } \sum_{s(r)} \nu_{r'} > \bar{\kappa}_r \end{cases}$$

Reserved quota allocation function $\bar{Ch}$ For ν obtained above, inductively define the quota $\bar{Ch}_r(X) \in \mathbb{N}_+$ of each region $r \in R$ as follows:

- $\bar{Ch}_H(X) = \nu_H$.
- For $r \in R$ such that $\bar{Ch}_r$ has been defined,

$$\bar{Ch}_{s(r)}(X) = \tilde{Ch}_r(\underline{\kappa}^*_{s(r)}, \nu_{s(r)}; \bar{Ch}_r(X)).$$

Choice function Ch For each $h \in H$, using $\bar{Ch}_h(X)$, which is the last number obtained above, as the cap, each hospital h chooses desirable contracts from X_h as $Ch_h(X)$:

$$\begin{aligned} \forall h \in H, \quad & |Ch_h(X)| = \min\{X_h, \bar{Ch}_{\{h\}}(X)\} \\ & \quad \& \quad \forall d, d' \in X_h, \ [\{d\} \succ_h \{d'\} \,\&\, d' \in Ch_h(X)] \implies d \in Ch_h(X). \end{aligned}$$

Note that from the definition, we have $\underline{\kappa}^*_r \leq \nu_r$ for every $r \in R$. Thus, we can show inductively from $r = H$ that $\sum_{s(r)} \underline{\kappa}^*_{r'} \leq Ch_r(X)$. Thus, $(\underline{\kappa}^*_{s(r)}, \nu_{s(r)}; \bar{Ch}_r(X)) \in \mathcal{V}$ in the definition of $\bar{Ch}_r$.

In addition, by distributional acceptance, condition (A) of $\tilde{Ch}$, and hospitals' acceptance, it follows that $\bar{Ch}_r(X) = \sum_{s(r)} \bar{Ch}_{r'}(X)$ unless (i) $|r| \geq 2$ and $\bar{Ch}_r(X) = \underline{\kappa}_r > \sum_{s(r)} \nu_{s(r)} = \sum_{s(r)} \bar{Ch}_{r'}(X)$ or (ii) $r = \{h\}$ for some $h \in H$ and $\bar{Ch}_r(X) = \underline{\kappa}_{\{h\}} > |X_h| = |Ch_r(X)|$. Therefore, $\bar{Ch}(X)$ is the reserved quota allocation of $Ch(X)$.

Remark 4. For each $X \subseteq D \times H$ and $r \in R$, (i) if $|r| \geq 2$, then $\bar{Ch}_r(X) = \min\{\underline{\kappa}_r, \sum_{s(r)} \bar{Ch}_{r'}(X)\}$, and (ii) if $r = \{h\}$ for some $h \in H$, $\bar{Ch}_r(X) = \min\{\underline{\kappa}_r, |Ch_{\{h\}(X)}|\}$. In other words, when $Ch(X) = \mu$, $\bar{Ch}(X) = \kappa$.

Furthermore, note that $\bar{Ch}(X) \leq \bar{\kappa}$ since $\nu \leq \bar{\kappa}$.

Remark 5. For each $X \subseteq D \times H$, $\bar{Ch}(X) \leq \bar{\kappa}$.

From these remarks, we can consider Ch as a procedure that first determines the reserved quota allocation κ satisfying the ceiling constraint based on X, and then lets the hospital choose doctors following $(\kappa_{\{h\}})_H$.

Thus, $Ch(X)$ satisfies the ceiling constraints but not necessarily the floor constraints. To satisfying the floor constraints, from Proposition 3, its total shortfall $\bar{Ch}_H(X) - |Ch(X)|$ must be zero.

There are two differences between our choice function and that of Kamada and Kojima (2018). First, $\nu_r = \underline{\kappa}_r$ if $\sum_{s(r)} \nu_{r'} < \underline{\kappa}_r$. That is, if the adjusted contract quantities of subregions fall below the floor constraint, the region requests to the upper regions by *accumulating* the minimum number of additional doctors required. Second, $\bar{Ch}_{s(r)} \geq \underline{\kappa}^*_{s(r)}$. This means that each region r is forced not to choose an allocation smaller than $\underline{\kappa}^*_r$.

In other words, a feature of our procedure is that it requires regions to choose the subregion quota allocation in advance, considering the quota to accept additional doctors to meet the floor constraint in the future (even if there are currently no applications from the corresponding doctors). If each region does not have a floor constraint, then our procedure is equivalent to those in Kamada and Kojima (2018).

The following properties ensure that our process, defined in the next section, behaves well. Property (i) is called *substitutability* in the literature. (ii) implies two properties. First, $|Ch_h(X \cup \{x\})| - |Ch_h(X)|$ is not negative, which is called *the law of aggregate demand* in the literature. Second, since $\bar{Ch}_H(X) - |Ch_h(X)|$ means the total shortfall, this means that adding a contract x must decrease the total shortfall by 1 or 0.

Proposition 4. For any $X \subset D \times H$ and $\{x\} \notin X$,
(I) $Ch(X \cup \{x\}) \subseteq Ch(X) \cup \{x\}$,
(II) $(\bar{Ch}_H(X \cup \{x\}) - \bar{Ch}_H(X), |Ch(X \cup \{x\})| - |Ch(X)|)$ is $(1,1)$, $(0,1)$, or $(0,0)$.

4.3 The GFDA+SD algorithm

Kamada and Kojima (2018) consider a constrained matching problem as a matching problem with contracts between one hospital with the choice function including no floor constraints and a set D of doctors, where H is the set of contractual contents, and consider the COP in that problem; we call this algorithm using Ch defined above *the generalized FDA algorithm.* [11]

The generalized FDA (GFDA) algorithm Let $X^0 = \mu^0 = D^0 = \emptyset$ and move to Step 1.

Step d: Let $D^d = D^{d-1} \cup \{d\}$, $X^{d,0} = X^{d-1}$, and $\mu^{d,0} = \mu^{d-1}$. Move to Sub-step $(d.1)$

Sub-step $(d.n)$: Choose any $d' \in D^d \setminus \mu_D^{d,n-1}$ such that $Ac_{\succ_{d'}} \setminus X_{d'}^{d,n-1} \neq \emptyset$, and let $X^{d,n} = X^{d,n-1} \cup \{(d', h)\}$ where h satisfies that $h \notin X_{d'}^{d,n-1}$ and $h \succeq_d h'$ for every $h' \notin X_{d'}^{d,n-1}$. Let $\mu^{d,n} = Ch(X^{d,n})$ and move to Sub-step $(d, n+1)$.

If there is no such d', then let $X^d = X^{d,n-1}$ and $\mu^d = \mu^{d,n-1}$ and terminates this step: if $d < |D|$, then move to Step $d+1$; if $d = |D|$, then let $\mu = \mu^d$ and terminates the algorithm.

The COP with the choice function satisfying substitutability and the law of aggregate demand generates a matching (Hirata and Kasuya, 2014). Thus, the GFDA algorithm generates a matching. In addition, the matching is individually rational since for each Sub-step (d, n), h is the best for $\succ_{d'}$ among $H \setminus X_{d'}^{d,n-1}$ and $Ac_{\succ_{d'}} \setminus X_{d'}^{d,n-1} \neq \emptyset$.

[11] When Ch satisfies substitutability and the law of aggregate demand, the order of the doctors' offers does not affect what matching the COP finally generates. [12] Therefore, instead of starting from an empty matching and cumulative offer set, we regard the COP up to $\{1, ..., d-1\}$ as "the intermediate stage of the COP in $\{1, ..., d\}$, where offers of doctor d are processed after all offers of doctors previous to d have been completed" and start from the last doctor d's offer.

For interval constraint problems, even if doctors' preferences are acceptable, GFDA does not always lead to feasible matching (see Section 5.3). Following Akin (2021), we consider an algorithm that combines Serial Dictatorship. The intuition of our algorithm is as follows. Note that we can regard each Step $d \in D$ in the GFDA algorithm as the process generating the COP matching when $\{1, \ldots, d\}$ is the set of all doctors (see also footnote 11). Increase d one by one (i.e. add one doctor at a time) and observe the series of total shortfalls, $\{\sum_R \delta_r^1, \sum_R \delta_r^2, \ldots, \sum_R \delta_r^{|D|}\}$. Since additional doctors are not always placed to make up for shortfalls in the doctor shortage regions, it does not necessarily follow that it will be zero (i.e., feasible) when we reach $d = |D|$. Our algorithm initially increases d by 1 and observes the total shortfall one by one, starting from the state with no doctor. However, if the remaining number of doctors, $|D| - d$, is in line with the total shortfall, to ensure feasibility, in subsequent stages, we force the added doctors to be matched in the doctor shortage hospital (the "+SD" process).

The formal definition of the GFDA+SD algorithm is as follows.

The generalized FDA+SD algorithm Let $X^0 = \mu^0 = D^0 = \emptyset$ and move to Step 1 of the GFDA phase.

GFDA phase

Step d**:** Execute Step d of GFDA algorithm. When the step terminates, if $d < |D|$ and $\sum_R \delta_r^{d,n-1} < |D| - d$, then move to Step $d+1$ of GFDA phase; if $d < |D|$ and $\sum_R \delta_r^{d,n-1} \geq |D| - d$, then move to Step $d+1$ of Serial-dictatorship phase; if $d = |D|$, then let $\mu = \mu^d$ and terminates the algorithm.

Serial-dictatorship (SD) phase

Step d: If $H^-(\mu^{d-1}) \cap Ac_{>_d} \neq \emptyset$, then let $\mu^d = \mu^{d-1} \cup \{(d, h)\}$ where h satisfies that $h \geq_d h'$ for every $h' \in H^-(\mu^{d-1})$, otherwise, let $\mu^d = \mu^{d-1}$. This step is terminated: if $d < |D|$, then move to Step $d+1$ of Serial-dictatorship phase; if $d = |D|$, then let $\mu = \mu^d$ and terminates the algorithm.

Let d^* denote the final step of the GFDA phase. In a problem with ceiling constraints, the total shortfall is zero, so $d^* = |D|$, and the matching produced by this algorithm is equal to the FDA by Kamada and Kojima (2018).

The following proposition explains the change when the step proceeds by 1 in the GFDA (or GFDA-phase of the GFDA-SD) algorithm, corresponding to the results to those of Proposition 4.

Proposition 5. For any $d \in D$ in the GFDA algorithm or any $d \leq d^*$ of GFDA phase in GFDA+SD algorithm,
(i) $\mu_D^d \subseteq \mu_D^{d-1} \cup \{d\}$, and
(ii) $(\kappa_H^d - \kappa_H^{d-1}, |\mu^d| - |\mu^{d-1}|)$ is $(1, 1)$, $(0, 1)$, or $(0, 0)$.

Proposition 5 (i) implies that doctors who are unmatched in Step $d-1$ cannot match any hospital in Step d.[13] Proposition 5 (ii) implies the total shortfall decreases by 0 or 1 when step proceeds by 1. Thus, if the algorithm transitions to SD phase, it follows that $\sum_R \delta_r^{d^*} = |D| - d^*$.

From Lemmas 1 and 2, we know that this implies that the algorithm must proceed to Step $|D|$, and for each Step d of SD phase, (a) $H^-(\mu^{d-1})$ is not empty, (b) the total shortfall decreases by 0 or 1, and (c) $\mu^d \in \mathcal{M}$. In particular, (c) inductively implies that the GFDA+SD algorithm generates a matching, since μ^{d^*} generated by the GFDA phase is a matching. Further, it is individually rational since μ^{d^*} is individually rational and $h \in Ac_{\succ_d}$ for (d, h) added in each step of the SD phase.

When doctors' preferences are acceptable ($Ac_{\succ_d} = H$), (a) implies that doctor d matches some hospital for any Step $d > d^*$, and thus the total shortfall must decrease by 1. Thus, the total shortfall becomes 0 at Step $|D|$. In this way, this algorithm generates a feasible matching.

Proposition 6. When $\succ_D$ is acceptable, $\sum_R \delta_r^{d^*,n-1} = |D| - d^*$ for d^* in the GFDA+SD algorithm and the generated matching is feasible.

The algorithm satisfies a stronger requirement. The generated matching satisfies our stability.

Theorem 1. When $\succ_D$ is acceptable, the matching generated by the GFDA+SD algorithm is interval respecting stable.

4.4 Strategy-proofness

Unlike feasibility and stability, the GFDA+SD (and GFDA) algorithms are strategy-proof for doctors even without acceptance. Let a **matching function** f be a function from $\boldsymbol{P}_D$ to $\mathcal{M}$.

Definition 6 (Strategy-proof)**.** A matching function f is strategy-proof (for doctors) iff for every $\succ_D \in \boldsymbol{P}_D$, $d \in D$, and $\succ'_d \in \boldsymbol{P}_d$, $f(\succ_D) \succeq_d f(\succ'_d, \succ_{D\setminus\{d\}})$.

Hatfield and Milgrom (2005) showed that a function assigning a matching generated by the COP when the choice functions satisfy substitutability and the law of aggregate demand to each preference profile is strategy-proof. Therefore, from Proposition 4, GFDA is strategy-proof: let *GFDA* be the matching function such that for any $\succ_D \in GFDA(\succ_D)$ is the matching generated by the GFDA algorithm when the preference profile of doctors is $\succ_D$.

Remark 6. Matching function *GFDA* is strategy-proof.

Let *GFDA+SD* be the matching function such that for any $\succ_D$, $GFDA{+}SD(\succ_D)$ is the matching generated by the GFDA+SD algorithm when the preference profile of doctors is $\succ_D$. *GFDA+SD* also satisfies strategy-proofness.

Theorem 2. *GFDA+SD* is strategy-proof.

[13]This is also a corollary of population monotonicity of the COP. The change between two steps corresponds to the change in the COP when doctor d is added.

Even if doctors' preferences are acceptable, these algorithms are strategy-proof because they satisfy strategy-proof on a broader domain. Therefore, if doctors are acceptable, the GFDA+SD algorithm is strategy-proof and satisfies interval respecting stability.[14] In addition, our result ensures that, even if each doctor is unsure whether other doctors are acceptable, they will honestly declare their preferences and thus achieve stability at least when they are all acceptable.

5 Discussion

5.1 Variations of interval respecting stability

In this section, we consider variations of interval respecting stability by comparing it with the original definition of ceiling respecting stability.

Kamada and Kojima (2018) study a problem with ceiling constraints without assuming hierarchical region structures. They consider stability defined in the following way:

Definition 7 (Alternative representation of ceiling respecting stability)**.** We say a matching μ is **ceiling respecting stable** if and only if μ is feasible, individual rational, and if (d, h) blocks μ with μ^{dh}, then (i) $d' \succ_h d$ for every $d' \in \mu_h$ and (ii) there exists $r \in R$ satisfying $|\mu_r| = \bar{\kappa}_r$ and $R' = \{r' \in R \cap 2^r | \mu_d, h \in r'\}$ satisfies the following condition: (a) there exists $r' \in R'$ such that $(|\mu_{r''}|)_{s(r')} \tilde{\succ}_{r'} (|\mu^{dh}|_{r''})_{s(r')}$ or (b) every $r' \in R'$ satisfies $(|\mu_{r''}|)_{s(r')} \tilde{\succsim}_{r'} (|\mu^{dh}|_{r''})_{s(r')}$.

Requiring (a) or (b) means that μ^{dh} is not weakly Pareto superior to μ for all regions in R'.

Since we assume that the region structure R is hierarchical, then R' is linearly ordered with the inclusion relationship and only the smallest element r' in R' satisfies $(|\mu_{r''}|)_{s(r')} \neq (|\mu^{dh}|_{r''})_{s(r')}$; thus, for any $r^+ \in R' \setminus \{r'\}$, $(|\mu_{r''}|)_{s(r^+)} = (|\mu^{dh}|_{r''})_{s(r^+)}$. Therefore, requiring (a) or (b) are equivalent to requiring $(|\mu_{r''}|)_{s(r)} \tilde{\succsim}_{r'} (|\mu^{dh}|_{r''})_{s(r'')}$ for the smallest element r' in R', which is our representation of ceiling respecting stability.

When we define interval respecting stability, even if we assume hierarchical R, we have to consider invalidity of blocking based on preferences of a set of regions, like R' in the above definition. There, we only require (b) for the set of authoritative regions to decide the invalidity. One may consider to require (a) or (b) similarly to the above definition. However, since it enlarges invalid blocking, it gives us weaker stability. Therefore, our result does not change even if we consider it.

Next, in the above definition of ceiling respecting stability, if there exists $r \in R$ satisfying (ii), for every region outside of R', we have $(|\mu_{r''}|)_{s(r)} = (|\mu^{dh}|_{r''})_{s(r'')}$ since for any subject other than d and h, their partner does not change. However, this property does not hold in interval respecting stability even if condition (ii) in the definition holds. If the coalition is not a pair, there can be a feasible matching where doctors move from hospitals outside the regions reaching the ceiling to those regions. Therefore, for stronger stability, we can consider blocking to be valid if there is a region that is not included in the region where the ceiling has been reached, but which would prefer the blocking more.

[14] Note that the fact that GFDA and SD separately satisfy strategy proofness does not directly imply strategy-proofness of GFDA+SD, since doctors before d^* may improve by controlling the timing of the shift from GFDA to SD through false reporting.

However, this does not strengthen stability. Note that condition (ii) implies that all hospitals $h \in B$ are located in regions that reach the ceiling. Hence, the number of doctors in hospitals with no region reaching the ceiling will remain the same or decrease. This implies the following.

Proposition 7. For any interval stable matching μ, if $B \in D \cup H$ blocks μ with μ' and satisfies condition (ii), then $|\mu'_{s(r)}| \leq |\mu_{s(r)}|$ for every $r \in R$ such that $|\mu_{r^+}| < \bar{\kappa}_{r^+}$ for every $r^- \subseteq r^+$.

Therefore, if the matching is interval respecting stable, in blocking that satisfies condition (ii), the regions that are not included in the regions reaching the ceiling also do not prefer blocking.

5.2 CFDA+SD algorithm

Akin (2021) showed that DA+SD, an algorithm that combines the unconstrained DA algorithm with the serial-dictatorship, similarly to our study, satisfies floor respecting stability in the problem with floor constraints. On the other hand, Kamada and Kojima (2018) showed that CFDA, which is the FDA ignoring floor constraints, satisfies ceiling respecting stability in the problem with ceiling constraints. Thus, we are interested in the performance of an algorithm that combines CFDA and SD rather than GFDA in interval constraint problems.

Define Ch^c as follows.

Ceiling only adjusted contract quantity vector v^c: For any X, inductively define $v^c \in \mathbb{N}_+^R$ towards the upper regions as follows.

- For each $h \in H$

$$v^c_{\{h\}} = \begin{cases} |X_h| & \text{if } |X_h| \in [\underline{\kappa}_{\{h\}}, \bar{\kappa}_{\{h\}}], \\ \bar{\kappa}_{\{h\}} & \text{if } |X_h| > \bar{\kappa}_{\{h\}} \end{cases}$$

- For any $r \in R$ such that each of $v^c_{s(r)}$ has been defined,

$$v^c_r = \begin{cases} \sum_{s(r)} v^c_{r'} & \text{if } \sum_{s(r)} v^c_{r'} \in [\underline{\kappa}_r, \bar{\kappa}_r], \\ \bar{\kappa}_r & \text{if } \sum_{s(r)} v^c_{r'} > \bar{\kappa}_r \end{cases}$$

$\bar{Ch}^c$ and Ch^c: Replace v with v^c and construct $\bar{Ch}$ and Ch following the definitions. Let $Ch^c = Ch$.

Note that Ch^c is equivalent to Ch when there is no floor constraint. Define the algorithm that replaces Ch with Ch^c in the GFDA+SD algorithm as the CFDA+SD algorithm.

The ceiling constraint FDA+SD (CFDA+SD) algorithm The CFDA+SD algorithm is the algorithm obtained by replacing Ch with Ch^c in the GFDA+SD algorithm.

CFDA+SD may not give a feasible matching.

Example 4. Suppose that there are three doctors $D = \{d_1, d_2, d_3\}$ and three hospitals $H = \{h_1, h_2, h_3\}$. The preference profile is

$$\succ_d : h_2, h_1, h_3 \text{ for any } d \in D \text{ and}$$
$$\succ_h : d_1, d_2, d_3 \text{ for any } h \in H.$$

There are two regions $R = \{r_1, r_2\}$ and each hospital is located as $r(h_1) = r(h_2) = r_1$ and $r(h_3) = r_2$. Ceiling and floor constraints of each region are given by $\underline{\kappa}_{\{h_1\}} = 1$ and $\bar{\kappa}_{r_1} = 1$. We assume that regional preferences are indifferent. That is, each region cares only about the total number of slots filled in its own subregion.

Here, we run the CFDA+SD algorithm. We start that algorithm with an empty matching. First, we choose doctor d_1. Since each doctor prefers h_2 the most, she applies to that hospital. Given this contract (d_1, h_2) and the choice of each region based on the constraints, h_2 chooses the one most preferred doctor. Therefore, hospital h_2 accepts d_2. Since the total shortfall is 1 and the number of remaining doctors is 2, this step ends and we proceed to the next step of the CFDA.

Next, d_2 makes an offer to h_2. Since h_2 chooses the single most preferred doctor, it accepts d_1 and rejects d_2. So d_2 then applies to its next preferred hospital h_3. Similarly, h_3 chooses the one most preferred doctor, so it accepts d_2. Here, the total shortfall is 1, which equals the number of remaining doctors, so we proceed to SD. However, h_1 cannot match, although it is acceptable to d_3, because the ceiling for region r_1 is binding.

Thus, the CFDA+SD algorithm may not produce a feasible match because one region may not fulfill the floor constraint but has reached the ceiling for its upper region.

Proposition 8. There is a case where the CFDA+SD algorithm does not generate a feasible matching.

This example clarifies why we need Ch rather than Ch^c : Ch allocates quotas that take into account future additions of doctors to solve the shortage; Ch^c does not take this into account and therefore ignores the shortage and allocates doctors who are currently applying up to the ceiling. As a result, even if it moves to SD, it will not be able to add doctors to the shortage regions without violating the ceiling, thus failing to achieve a feasible matching.

5.3 GFDA with ceiling $|D|$

This section examines how to generate feasible matchings using only GFDA without SD.

In the GFDA algorithm, Ch takes into account the number of shortfalls and generates a matching that will be feasible if doctors are added in the future. However, Ch does not take into account the total number of doctors, which may result in a matching that is not feasible even if all currently unmatched doctors participate. If Ch takes into account the total number of doctors and rejects the above-mentioned allocations, GFDA will generate a feasible matching. Define $Ch^{|D|}$ below.

$|D|$-ceiling adjusted contract quantity vector $v^{|D|}$: For any X and $r \in R \setminus \{H\}$, let $v_r^{|D|} = v_r$. Let

$$v_H^{|D|} = \begin{cases} \underline{\kappa}_H & \text{if } \sum_{s(H)} v_{r'}^{|D|} < \underline{\kappa}_H, \\ \sum_{s(H)} v_{r'}^{|D|} & \text{if } \sum_{s(H)} v_{r'}^{|D|} \in [\underline{\kappa}_H, \min\{\bar{\kappa}_H, |D|\}], \\ \min\{\bar{\kappa}_H, |D|\} & \text{if } \sum_{s(H)} v_{r'}^{|D|} > \min\{\bar{\kappa}_H, |D|\} \end{cases}$$

$\bar{Ch}^{|D|}$ and $Ch^{|D|}$: Replace v with $v^{|D|}$ and construct $\bar{Ch}$ and Ch following the definitions. Let $Ch^{|D|} = Ch$.

In other words, $Ch^{|D|}$ is Ch with $\bar{\kappa}_H$ modified to the smaller of $|D|$ and the true $\bar{\kappa}_H$. $Ch^{|D|}$ does not choose an allocation where the reserved total quota κ_H is greater than $|D|$. The rest part is the same as for Ch. We name the algorithm that replaces Ch with $Ch^{|D|}$ in GFDA as the GFDA with ceiling $|D|$.

The generalized FDA algorithm with ceiling $|D|$ The GFDA with ceiling $|D|$ is the algorithm obtained by replacing Ch with $Ch^{|D|}$ in the GFDA+SD algorithm.

This algorithm generates a feasible matching.

Proposition 9. When $\succ_D$ is acceptable, the matching generated by GFDA with ceiling $|D|$ is feasible.

Note that ceiling $|D|$ for region H is a pseudo ceiling and not the true ceiling for H. Therefore, GFDA with ceiling $|D|$ may fail to satisfy not only stability, but also the weaker condition of ceiling respecting stability.

Example 5. Let $D = \{1, 2, 3\}$ and $H = \{h_1, h_2, h_3\}$. The physical cap q_h is $+\infty$ for every $h \in H$. Their preferences are defined as

$$\forall d \in D, \quad \succ_d : h_3, h_2, h_1$$
$$\forall h \in H, \quad \succ_h : 1, 2, 3$$

The region structure R is $\{\{h_1\}, \{h_2\}, \{h_3\}, H\}$. There is no ceiling constraint: $\bar{\kappa}_r = +\infty$ for every $r \in R$. In addition, $\underline{\kappa}_{\{h_1\}} = 1$ is the unique floor constraint; $\underline{\kappa}_r = 0$ for any $r \in R$ with $r \neq \{h_1\}$. The unique non-singleton region H's preference is represented by the following objective function $u_H : \mathbb{N}_+^{|s(H)|} \to \boldsymbol{R}$.

$$u_H(w_{\{h_1\}}, w_{\{h_2\}}, w_{\{h_3\}}) = 1.2w_{\{h_1\}} + 1.1w_{\{h_2\}} + w_{\{h_1\}}$$

That is, H slightly value the number in $\{h_1\}$ over that in $\{h_2\}$ and that in $\{h_2\}$ over that in $\{h_3\}$.

Consider the GFDA with ceiling $|D| = 3$. Then, $\mu^1 = \{(1, h_3)\}$ and $\kappa_H^1 = 2$; $\mu^2 = \{(1, h_3), (2, h_3)\}$ and $\kappa_H^2 = 3$; $\mu^{3,1} = \{(1, h_3), (2, h_3)\} = X^{3,1} \setminus \{(3, h_3)\}$ and $\kappa_H^{3,1} = 3$; $\mu^{3,2} = \{(1, h_3), (3, h_2)\}$; $\mu^{3,3} = \{(3, h_2), (2, h_2)\}$; $\mu^{3,4} = \{(2, h_2), (1, h_2)\}$; finally, $\mu = \mu^3 = \{(2, h_2), (1, h_2), (3, h_1)\}$.

However, μ is not ceiling respecting stable. Both $\{1, h_3\}$ and $\{2, h_3\}$ block μ with μ^{1h_3} and μ^{2h_3} but there is no ceiling constraint in this problem. The GFDA+SD matching, which is the unique interval respecting stable matching in this problem, is $\{(1, h_3), (2, h_3), (3, h_1)\}$.

Fact 3. There is a case where the GFDA with ceiling $|D|$ does not generate ceiling respecting stability.

If κ_H is guaranteed to be less than or equal to $|D|$ even under the original Ch, then replacing Ch does not affect the algorithm. This is also a necessary and sufficient condition for GFDA to function alone.[15]

Proposition 10. The following three conditons are equivalent:
(i) The matchings generated by the GFDA algorithm and the GFDA with ceiling $|D|$ are equal.
(ii) The matching generated by the GFDA algorithm is feasible.
(iii) The matching generated by the GFDA algorithm is interval respecting stable.

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[15]GFDA+DA and GFDA with ceiling $|D|$ are not the same algorithm even in this case; GFDA+DA shifts to the DA process when κ_H first reaches $|D|$ even if it never goes to $|D| > \kappa_H$. Therefore, the matching produced by the two may not also be the same.

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Appendix

1 Proof of Proposition 1

Note that for each $r \in R$,

$$\underline{\kappa}_r^* = \max_{P \subset R \text{ s.t. } P \text{ is a partition on } r} \sum_{r' \in P} \underline{\kappa}_r \tag{1}$$

$$\bar{\kappa}_r^* = \min_{P \subset R \text{ s.t. } P \text{ is a partition on } r} \sum_{r' \in P} \bar{\kappa}_r \tag{2}$$

Proof. Consider any feasible matching μ and any $r \in R$. Suppose that for some partition P on r, $|\mu_r| < \sum_{r' \in P} \underline{\kappa}_{r'}$. Then, there must be $r' \in P$ such that $|\mu_{r'}| < \underline{\kappa}_{r'}$, which contradicts feasibility. Thus, $|\mu_r| \geq \sum_{r' \in R'} \underline{\kappa}_{r'}$ for any partition $R' \subset R$ on r. By (1), $|\mu_r| \geq \underline{\kappa}_r^*$. In the same manner, we can show by (2) that $|\mu_r| \leq \bar{\kappa}_r^*$. Thus, $\bar{\kappa}_r^* \geq \underline{\kappa}_r^*$.

Since we choose any $r \in R$, when $r = H$, we have $|D| \geq |\mu_H| \geq \underline{\kappa}_H^*$. □

2 Proof of Proposition 3

Proof. Suppose that $\kappa_H - |\mu| = 0$. Then, $\sum_R \delta_r = 0$. Since δ_r is non-negative, $\delta_r = 0$ for every $r \in R$. By definition, $\kappa_r = \sum_{s(r)} \kappa_{r'}$ for every $r \in R$. By construction of κ_r, $\kappa_r = |\mu_r|$ for every $r \in R$. Since $\kappa_r = \min\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}\} \geq \underline{\kappa}_r$, then $|\mu_r| \geq \underline{\kappa}_r$.

Note that for any $r' \in R$, $\kappa_{r'} \geq \sum_{s(r')} \kappa_{r''} \geq \cdots \geq \sum_{h \in r'} \kappa_{\{h\}} \geq \sum_{h \in r'} |\mu_h| = |\mu_{r'}|$. Suppose that $\kappa_H - |\mu| > 0$. Then, there exists $r \in R$ with $\delta_r > 0$. Thus, $\kappa_r - \sum_{s(r)} \kappa_{r'} > 0$ and $\kappa_r = \max\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}\} = \underline{\kappa}_r$. Since $\sum_{s(r)} \kappa_{r'} \geq \sum_{s(r)} |\mu_{r'}| = |\mu_r|$, we have $\underline{\kappa}_r = \kappa_r > \sum_{s(r)} \kappa_{r'} \geq |\mu_r|$. □

3 Proof of Proposition 2

Proof. Suppose that μ is interval respecting stable and $\mu' \in MM$ Pareto dominates μ. Let $A_D = \{d \in D | \mu_d \neq \mu'_d\}$ and $A_H = \{h \in H | \mu_h \neq \mu'_H\}$. Since each of $(\succ_D, \succ_H)$ is a strict preference, we have $\mu'_d \succ_d \mu_d$ for any $d \in A_D$ and $\mu'_h \succ_h \mu_h$ for any $h \in A_H$. Thus, $A_D \times A_H$ blocks μ with μ'.

If $A_H = \emptyset$, since $\mu' \neq \mu$ and , then $A_D \neq \emptyset$. Choose any $d \in A_D$. Since $\mu'_d \succ_d \mu_d$, then $\varnothing \succ_d \mu_d$. Thus, μ_d is not individual rational, which contradicts μ's interval respecting stability. Thus, $A_H \neq \emptyset$.

Next, suppose that there exists $h \in A_H$ such that $|\mu'_h| \leq |\mu_h|$. Choose any $D' \subseteq \mu_h \setminus \mu'_h$ with $|D'| = |\mu'_h \setminus \mu_h|$. Since μ is interval respecting stable, for any $d' \in \mu'_h \setminus \mu_h$ and any $d \in D'$, $\{d\} \succ_h \{d'\}$. Since $\succ_h$ is strict, acceptable, and responsive, then $\mu_h \succeq_h D' \cup [\mu'_h \cap \mu_h] \succ_h \mu'$, which is a contradiction. Thus, $|\mu'_h| > |\mu_h|$ for any $h \in A_H$. Since $|\mu'_h| = |\mu_h|$ for any $h \notin A_H$, then $(|\mu'_h|)_H \gtrsim (|\mu_h|)_H$.

Consider any $r \in R$ such that $(|\mu'_{r'}|)_{s(r)} \neq (|\mu_{r'}|)_{s(r)}$. Since $(|\mu'_h|)_H \gtrsim (|\mu_h|)_H$, $(|\mu'_{r'}|)_{s(r)} \gtrsim (|\mu_{r'}|)_{s(r)}$. By distribution acceptance of $\tilde{\succ}_r$, $(|\mu'_{r'}|)_{s(r)} \tilde{\succ}_r (|\mu_{r'}|)_{s(r)}$. Thus, Condition (ii) in definition of interval respecting stability does not hold, which is a contradiction. □

4 Proof of Proposition 4

In this section, we discuss the relation of $Ch(X)$ and $Ch(X \cup \{x\})$ for any X and $\{x\} \notin X$.

Let (w, v) and (w', v') be the adjusted contract quantity vectors corresponding to X and $X \cup \{x\}$ respectively. Let (κ, μ) and (κ', μ') denote $(\bar{Ch}(X), Ch(X))$ and $(\bar{Ch}(X \cup \{x\}), Ch(X \cup \{x\}))$ respectively.

Let $x_H = h$. Thus, $w_h = w'_h + 1$ and $w_{-h} = w'_{-h}$.

By definition of v and v', there exists a sequence $(n_n)_{n=1}^{n^*}$ on $R_{\ni h}$ such that $r_n \in s(r_{n-1})$ for each $n = 2, \dots, n^*$, $r_{n^*} = \{h\}$, and

$$\forall r \in R, v'_r = \begin{cases} v_r + 1 & \text{if } r \in \{r_1, \dots, r_n\} \\ v_r & \text{otherwise} \end{cases}$$

Note that r_1 can be H, otherwise, there exists r_0 such that $r_1 \in s(r_0)$ and $\sum_{s(r_0)} v_{r'} \notin [\underline{\kappa}_{r_0}, \bar{\kappa}_r - 1]$.

First, we show four relations between (κ_r, κ'_r) and $(\kappa_{s(r)}, \kappa'_{s(r)})$ for $r \in R$. (i) and (ii) are properties for any region on this sequence. (iii) and (iv) are properties for other regions.

Lemma 3. The following four properties hold:
(i) If $\kappa'_{r_n} = \kappa_{r_n}$, then (a) $[\kappa_{s(r_n)} = v_{s(r_n)}$ and $\kappa'_{s(r_n)} = v'_{s(r_n)}]$, (b) $\kappa'_{s(r_n)} = \kappa_{s(r_n)}$ or (c) $[\kappa_{r_{n+1}} = v_{r_{n+1}}$ and $\kappa'_{r_{n+1}} = v'_{r_{n+1}}]$, $\kappa'_{r^*} = \kappa^*_r - 1$ for some $r^* \in s(r_n) \setminus \{r_{n+1}\}$, and $\kappa'_{r'} = \kappa_{r'}$ for every $r' \in s(r_n) \setminus \{r_{n+1}, r^*\}$.
(ii) If $\kappa'_{r_n} = v'_{r_n}$ and $\kappa_{r_n} = v_{r_n}$, then $\kappa'_{s(r_n)} = v'_{s(r_n)}$ and $\kappa_{s(r_n)} = v_{s(r_n)}$.
(iii) For every $r \notin \cup_n \{r_n\}$ with $\kappa'_r = \kappa_r$, then $\kappa'_{s(r)} = \kappa_{s(r)}$.
(iv) For every $r \notin \cup_n \{r_n\}$ with $\kappa'_r = \kappa_r - 1$, then $\kappa'_{s(r)} = (\kappa_{r'} - 1, \kappa_{s(r_n)})$.

Proof. (i) If $\kappa_{r_n} > \sum_{s(r_n)} v_{r'}$, since $\underline{\kappa}_{r_n} \geq \sum_{s(r_n)} v_{r'} + 1 = \sum_{s(r_n)} v'_{r'}$, then we obtain (a). Suppose that $\kappa_{r_n} \leq \sum_{s(r_1)} v_{r'}$. Since $\kappa'_{r_n} = \kappa_{r_n}$, $v'_{r_{n+1}} = v_{r_{n+1}} + 1$, and $v'_{r'} = v_{r'}$ for every $r' \in s(r_n) \setminus \{r_{n+1}\}$, by Condition (i) of $\bar{C}h$, we obtain (b) or (c).

(ii) Since $\bar{\kappa}_{r_n} \geq v'_{r_n} > v_{r_n}$, then $\bar{\kappa}_{r_n} > \sum_{s(r_n)} v_{r'}$. Thus, $\bar{\kappa}_{r_n} \geq \sum_{s(r_n)} v_{r'} + 1 = \sum_{s(r_n)} v'_{r'}$. Therefore, $v_{r_n} \geq \sum_{s(r_n)} v_{r'}$ and $v'_{r_n} \geq \sum_{s(r_n)} v'_{r'}$. By assumption of (ii), $\kappa_{r_n} \geq \sum_{s(r_n)} v'_{r'}$ and $\kappa_{r_n} \geq \sum_{s(r_n)} v_{r'}$. Since $\succ_{r_n}$ is monotonic, $\kappa'_{s(r_n)} = \bar{C}h_r(v'_{s(r_n)}, \kappa'_{r_n}) = v'_{s(r_n)}$ and $\kappa_{s(r_n)} = \bar{C}h_r(v_{s(r_n)}, \kappa_{r_n}) = v_{s(r_n)}$.

(iii) Since $v'_{s(r)} = v_{s(r)}$, then $\kappa'_{s(r)} = \bar{C}h_r(v'_{s(r)}, \kappa'_r) = \bar{C}h_r(v_{s(r)}, \kappa_r) = \kappa_{s(r)}$.

(iv) Since by definition of $\bar{C}h$, $v_r \geq \kappa_r$, then $v'_r = v_r \geq \kappa_r > \kappa'_r \geq \underline{\kappa}^*_r \geq \underline{\kappa}_r$. Thus, $\sum_{s(r)} v_{r'} \geq v_r \geq \kappa_r$ and $\sum_{s(r)} v'_{r'} \geq v'_r \geq \kappa'_r$. By Condition (ii) of $\bar{C}h$, $\kappa'_{s(r)} < \kappa_{s(r)}$. Since $\succ_r$ is monotonic, we obtain this conclusion. □

Note that if $\kappa'_{r_1} \neq \kappa_{r_1}$, then $r_1 = H$ and $v'_{r_1} = v_{r_1} + 1$. Thus, $\bar{\kappa}_{r_1} \geq v'_{r_1} > v_{r_1} \geq \underline{\kappa}_{r_1}$. Since $r_1 = H$, we have $\kappa'_{r_1} = v'_{r_1}$ and $\kappa_{r_1} = v_{r_1}$]. In addition, for every r_n, if $\kappa'_{r_n} = v'_{r_n}$ and $\kappa_{r_n} = v_{r_n}$, then [$\kappa'_{r_{n+1}} = v'_{r_{n+1}}$ and $\kappa_{r_{n+1}} = v_{r_{n+1}}$] (Property (i)), and if $\kappa'_{r_n} = \kappa_{r_n}$, then [$\kappa'_{r_{n+1}} = v'_{r_{n+1}}$ and $\kappa_{r_{n+1}} = v_{r_{n+1}}$] (Properties (ii).a and c) or $\kappa'_{r_n} = \kappa_{r_n}$ (Property (ii).b). Thus, we can consider four cases.

Case 1 Property (ii) applies to every r_n. (Note that this implies [$\kappa'_{r_1} = v'_{r_1}$ and $\kappa_{r_1} = v_{r_1}$] and $r_1 = H$.)

Case 2 There exists r_n such that Property (i).b applies to every $n' < n$, Property (ii).a applies to r_n, and Property (ii) applies to every $n' > n$.

Case 3 There exists r_n such that Property (i).b applies to every $n' < n$, Property (ii).c applies to r_n, and Property (ii) applies to every $n' > n$.

Case 4 Property (ii).b applies to every r_n.

Now, we give Proof of Proposition 4 for every case.

Proof.

Claim 1. In Cases 1,2, and 4, $\kappa'_{h'} = \kappa_{h'}$ for every $h' \neq h$.

For every $h' \neq h$, there is no r_n such that $\kappa_{r''} \neq \kappa'_{r''}$ for some $r'' \in s(r'_n) \setminus \{r_n + 1\}$. Thus, by repeatedly applying Property (iii), it follows that $\kappa_{h'} = \kappa'_{h'}$.

Claim 2. In Case 3, There exists $h' \in r_n$ such that $\kappa'_{h'} = \kappa_{h'} - 1$ and $\kappa'_{h''} = \kappa_{h''}$ for every $h'' \neq h, h'$.

For $r^* \in s(r_n)$, by repeatedly applying Properties (iii) and (iv), there exists unique $h' \in r_n \setminus \{h\}$ such that $\kappa'_{h'} = \kappa_{h'} - 1$ and $\kappa'_{h''} = \kappa_{h''}$ for every $h'' \in r_n \setminus \{h, h'\}$. For every $r''' \notin r_n$, since there is no $r_{n'}$ such that $r''' \in r_{n'}$ and $\kappa'''_{r'} \neq \kappa'_{r''''}$ for some $r'''' \in s(r_{n'}) \setminus \{r_{n'+1}\}$, by repeatedly applying Property (iii), it follows that $\kappa_{h'''} = \kappa'_{h'''}$.

Claim 3. $Ch(X \cup \{x\}) \subseteq \mathrm{Ch}(X) \cup \{x\}$.

By Claims 1 and 2, for every $h' \neq h$, $\kappa_{h'} \geq \kappa'_{h'}$. By responsibility of $\succeq_{h'}$ and definition of Ch, $[Ch(X \cup \{x\})]'_h \subseteq [Ch(X)]'_h$. In Case 4, $\kappa_h = \kappa'_h$. Thus, in the same manner, $[Ch(X \cup \{x\})]_h \subseteq [Ch(X)]_h$. In Cases 1,2, and 3, $\kappa_h = w_h$. By acceptability of h, $[Ch(X)]_h \cup \{x\} = X_h \cup \{x\} \supseteq [Ch(X \cup \{x\})]_h$.

Claim 4. $(\kappa'_H - \kappa_H, |Ch(X \cup \{x\})| - |Ch(X)|)$ is $(1,1)$, $(0,1)$, or $(0,0)$.

In Case 1, $\kappa'_H - \kappa_H = 1$ since $\kappa'_H = \nu'_H = \nu_H + 1 = \kappa_H + 1$. In Cases 2,3, and 4, $\kappa'_H - \kappa_H = 0$ since $\nu'_{r'} = \nu_{r'}$ for every $r' \supseteq r_1$.

Note that by acceptability and the fact that $\kappa'_{h'}, \kappa_h \leq \bar{\kappa}_{h'} \leq q_{h'}$ for any $h' \in H$, we have $|Ch(X \cup \{x\})| = \sum_H \kappa'_{h'}$ and $|Ch(X)| = \sum_H \kappa_h$.

In Cases 1,2, and 3, since Property (i), (ii).a, or c applies to $n^* - 1$, then $\kappa'_h = w'_h = w_h + 1 = \kappa_h + 1$. In Case 4, since Property (ii).b applies to $n^* - 1$, then $\kappa'_h = \kappa_h$.

Thus, in Cases 1 and 2, by Claim 1, $\sum_H \kappa'_{h'} - \sum_H \kappa_h = 1$. In Case 3, by Claim 2, $\sum_H \kappa'_{h'} - \sum_H \kappa_h = 0$. In Case 4, by Claim 1, $\sum_H \kappa'_{h'} - \sum_H \kappa_h = 0$. □

5 Proof of Proposition 7

Proof. We consider induction from singleton regions. Consider any singleton region $\{h\} \in \cup_H \{h'\}$ such that $|\mu_{r^+}| < \bar{\kappa}_{r^+}$ for every $r^+ \in R$ with $\{h\} \subseteq r^+$. If $|\mu'_{\{h\}}| > |\mu_{\{h\}}|$, then $\mu'_h \not\subset \mu_h$. Thus, $h \in B$. By condition (ii) of interval respecting stability, there exists $r^+ \in R_{\ni h}$ with $|\mu_{r^+}| = \bar{\kappa}_{r^+}$. Since $\{h\} \subseteq r^+$, this is a contradiction. Thus, $|\mu'_{\{h\}}| \leq |\mu_{\{h\}}|$ for any $\{h\} \in \cup_H \{h'\}$ such that $|\mu_{r^+}| < \bar{\kappa}_{r^+}$ for every $r^+ \in R$ with $\{h\} \subseteq r^+$.

Consider $r \in R$ such that $|\mu_{r^+}| < \bar{\kappa}_{r^+}$ for every $r^+ \in R$ with $r \subseteq r^+$. Suppose by induction that for every $r^- \in s(r)$ such that $|\mu_{r^+}| < \bar{\kappa}_{r^+}$ for every $r^+ \in R$ with $r^- \subseteq r^+$, it follows that $|\mu'_{r^-}| \leq |\mu_{r^-}|$.

Note that if $r^- \in s(r)$ has $r' \in R$ with $r^- \subseteq r'$ such that $|\mu_{r'}| = \bar{\kappa}_{r'}$, since $|\mu_{r^+}| < \bar{\kappa}_{r^+}$ for every $r^+ \in R$ with $r \subseteq r^+$, it follows that $|\mu_{r^-}| = \bar{\kappa}_{r^-}$. Since μ' is feasible, $|\mu'_{r^-}| \leq |\mu_{r^-}|$. Thus, $|\mu'_{r^-}| \leq |\mu_{r^-}|$ for every $r^- \in s(r)$, which implies that $|\mu'_r| \leq |\mu_r|$.

By induction, we obtain the desired conclusion. □

6 Properties of the GFDA and the GFDA+SD algorithms

6.1 Preliminary analysis

For any matching μ, from the definition of $H^-(\mu)$, $h \in H^-(\mu)$ has the smallest region r_1 such that $s(r_1) > 0$, and all in-between regions are in $R^-(\mu)$. If one doctor is added to h, for any in-between region r, since $\delta_r = 0$, κ_r increases by 1. κ_{r_1} remains unchanged since $\delta_{r_1} > 0$. For every other region r', it is obvious that $\kappa_{r'}$ does not change. We summarize this as a remark.

Remark 7. For any μ and $h \in H^-(\mu)$, there exists $r_1 \in R^-(\mu)$ such that $\delta_{r_1} > 0$ and for any $r \in R_{\ni h}$ such that $r \subseteq r_1$, it follows that $r \in R^-(\mu)$ and $\delta_r = 0$, and for each $d \notin \mu_H$,

$\mu' = \mu \cup \{(d,h)\}$ satisfies that

$$\forall r \in R, \kappa'_r = \begin{cases} \kappa_r + 1 & \text{if } h \in r \subsetneq r_1 \\ \kappa_r & \text{otherwise} \end{cases}$$

Note that this implies that for any $r \in R$ such that (i) $r = H$ or (ii) $r \in s(r')$ for some $r' \notin R^-(\mu)$, we have $\kappa_r = \kappa'_r$.[16]

Next, we show that under the matching obtained by Ch, a hospital in $H^-(\mu^d)$ or some region with a bounded ceiling constraint does not reject any contract.

Lemma 4. For any X, if $\mu = Ch(X)$, then for each $h \in H$ such that (i) $h \in H^-(\mu)$ or (ii) $\kappa_r < \bar{\kappa}_r$ for every $r \in R_{\ni h}$, we have $\mu_h = X_h$.

Proof. In case (i), by Remark 7, there exists $r_0 \in R_{\ni h}$ such that $\delta_{r_0} > 0$ and for each $r \in R_{\ni h}$ with $r \subseteq r_0$, $\kappa_r < \bar{\kappa}_r$. By definition of $\bar{Ch}$, $\delta_{r_0} > 0$ implies that $\kappa_{s(r_0)} = \nu_{s(r_0)}$. In case (ii), let $r_0 = H$. Since $\kappa_H < \bar{\kappa}_H$, by definition of $\bar{Ch}$, $\kappa_{s(r_0)} = \nu_{s(r_0)}$.

Define $(r_n)_{n=1}^{n^*}$ as $r_n = s(r_{n-1})$ for each $n = 1, \dots, n^*$ and $r_{n^*} = \{h\}$. Note that $\kappa_{r_n} < \bar{\kappa}_{r_n}$ for each $n = 1, \dots, n^*$, and $\kappa_{r_1} = \nu_{r_1}$. Since $\bar{\kappa}_{r_1} > \nu_{r_1}$, by definition of ν, $\sum_{s(r_1)} \nu_{r'} \leq \nu_{r_1} = \kappa_{r_1}$. By definition of $\bar{Ch}$, $\kappa_{s(r_1)} = \nu_{s(r_1)}$. Especially, $\kappa_{r_2} = \nu_{r_2}$. By repeating the above argument, we have $\kappa_{r_{n^*}} = \kappa_{\{h\}} = \nu_{\{h\}}$. Since $\nu_{\{h\}} < \bar{\kappa}_{\{h\}}$, $|\mu_h| = w_h$. □

6.2 Proof of Proposition 5

Proof. Consider any $d \leq d^*$ and let Step d terminate at Sub-step (d, n^*). By definition, Sub-step (d, n^*) lets $\mu^d = \mu^{d,n^*-1}$ and immediately finishes.

We first show that $\mu_D^{d,n} \subseteq \mu_D^{d-1} \cup \{d\}$ and $|\mu^{d,n}| - |\mu^{d-1}| = 0$ for any $n \in \{1, \dots, n^* - 2\}$. We consider induction with respect to n. By definition, $\mu_D^{d,0} = \mu^{d-1} \subsetneq \mu_D^{d-1} \cup \{d\}$ and $|\mu^{d,0}| = |\mu^{d-1}|$. We now show that for each $n \in \{1, \dots, n^* - 2\}$, if $\mu_D^{d,n-1} \subsetneq \mu_D^{d-1} \cup \{d\}$ and $|\mu^{d,n-1}| = |\mu^{d-1}|$, then $\mu_D^{d,n} \subseteq \mu_D^{d-1} \cup \{d\}$ and $|\mu^{d,n}| - |\mu^{d-1}| = 0$.

Suppose that $\mu^{d,n-1} \subseteq \mu_D^{d-1} \cup \{d\}$ and $|\mu^{d,n-1}| = |\mu^{d-1}|$. Note that for any $d' \in \{1, \dots, d-1\} \setminus \mu_D^{d-1}$, $X_{d'}^{d-1} = H$. Thus, $X^{d,n} = X^{d,n-1} \cup \{(d'', h)\}$ satisfies that $d'' \in \mu_D^{d-1} \cup \{d\}$. By Proposition 4 (i),

$$\mu_D^{d,n} = (Ch(X^{d,n}))_D \subseteq (Ch(X^{d,n-1}) \cup \{(d'', h)\})_D = \mu_D^{d,n-1} \cup (\mu_D^{d-1} \cup \{d\}) = \mu_D^{d-1} \cup \{d\},$$

By Proposition 4 (ii),

$$|\mu^{d,n}| - |\mu^{d-1}| = |\mu^{d,n}| - |\mu^{d,n-1}| = |Ch(X^{d,n})| - |Ch(X^{d,n-1})| \in \{0, 1\}$$

If $|\mu^{d,n}| - |\mu^{d-1}| = 1$, since $\mu^{d,n} \subseteq \mu^{d-1} \cup \{d\}$, then $\mu_D^{d,n} = \mu_D^{d-1} \cup \{d\}$. Since $X_{d'}^{n-1} = H$ for any $d' \in \{1, \dots, d-1\} \setminus \mu_D^{d-1}$, there is no $d' \in \{1, \dots, n\} \setminus \mu_D^d$ such that $X_{d'}^{d,n} \neq H$. Thus, $n = n^* - 1$, which is a contradiction.

[16] When (i) or (ii) holds and $h \in r \subsetneq r_1 \in R^-(\mu)$, we have $r = r_1$ and by this remark, $\kappa'_{r_1} = \kappa_{r_1}$.

By induction with respect to n, we have (a) $\mu^{d,n} \subseteq \mu_D^{d-1} \cup \{d\}$ and (b) $|\mu^{d,n}| - |\mu^{d-1}| = 0$ for every $n \in \{1, \ldots, n^* - 2\}$. (a) implies that $\mu_D^d = \mu_D^{d,n^*-2} \subseteq \mu_D^{d-1} \cup \{d\}$, which is the conclusion (i).

Next, (b) implies $|Ch(X^{d,n})| - |Ch(X^{d,n-1})| = |\mu^{d,n}| - |\mu^{d,n-1}| = 0$ for every $n \in \{1, \ldots, n^* - 2\}$. By Proposition 4 (ii), $\kappa_H^{d,n} - \kappa_H^{d,n-1} = \bar{C}h(X^{d,n}) - \bar{C}h(X^{d,n-1}) = 0$. Thus, we have $\kappa_H^{d,n^*-2} - \kappa_H^{d-1} = 0$ and $|\mu^{d,n^*-2}| - |\mu^{d-1}| = 0$.Thus,

$$\begin{aligned}(\kappa^d - \kappa^{d-1}, |\mu^d| - |\mu^{d-1}|) &= (\kappa^{d,n^*-1} - \kappa^{d,n^*-2}, |\mu^{d,n^*-1}| - |\mu^{d,n^*-2}|) \\ &= (\bar{C}h(X^{d,n^*-1}) - \bar{C}h(X^{d,n^*-2}), |Ch(X^{d,n^*-1})| - |Ch(X^{d,n^*-2})|)\end{aligned}$$

which is $(0,0)$, $(0,1)$, or $(1,1)$ by Proposition 4. □

6.3 Proof of Lemma 1

Proof. Since $\kappa_H - |\mu| > 0$, there exists $r \in R$ such that $\delta_r > 0$. Note that $\kappa_r = \underline{\kappa}_r$. Suppose that $H^-(\mu)$ is empty. Then, for any $h \in r$, there exists $r^h \in R_{\ni h}$ such that $r^h \subseteq r$ and $\kappa_{r^h} = \bar{\kappa}_{r^h}$. Since R is hierarchical, there exists $P \subseteq \{r^h\}_H$ that is a parition of r. Since $\delta_r > 0$,

$$\underline{\kappa}_r - \sum_{r' \in P} \bar{\kappa}_{r'} = \kappa_r - \sum_{r' \in P} \kappa_{r'} = \sum_{r'' \in R \text{ s.t. } \exists r' \in P, r' \subsetneq r'' \subseteq r} \delta_{r''} > 0.$$

Since $\bar{\kappa}_r^* \leq \sum_P \bar{\kappa}_{r'}$ (by (2)) and $\underline{\kappa}_r^* \geq \underline{\kappa}_r$, we have $\underline{\kappa}_r^* > \bar{\kappa}_r^*$, which contradicts Assumption 1. □

6.4 Proof of Lemma 2

Proof. By Remark 7, if $\kappa_r \neq \kappa_r'$ for some $r \in R$, then it follows that $r \in R^-(\mu)$, $\delta_r(\mu) = 0$, and $\kappa_r' - \kappa_r = 1$. □

6.5 Proof of Theorem 2

First we introduce several preliminary results and concepts.

By definition, including those in the intermediate steps, every matching generated by the GFDA or the GFDA+SD algorithm is individual rational: that is, doctor does not match hospitals worse than being unemployed.

Remark 8 (Individual rationality)**.** Any matching μ^d generated at some step in the GFDA or the GFDA+SD algorithm satisfies $\mu_{d'}^d \succeq_{d'} \emptyset$ for any $d' \in D$.

Next, we can say that for any step in the GFDA algorithm or GFDA phase of the GFDA+SD algorithm, the already existing doctors' welfares will not be improved.

Remark 9. Consider any Steps d, d' in the GFDA algorithm or $d, d' \leq d^*$ in the GFDA phase of the GFDA+SD algorithm. Suppose that $d < d'$. Then, for any $d'' \in D^d$, $\mu_{d''}^d \succeq_{d''} \mu_{d''}^{d'}$.

Proof. By definition of algorithms, for any $d'' \in D^d$, if $h \succ_{d''} \mu_{d''}^d$ for some $h \in H$, then $(d'', h) \in X^d \setminus \mu^d$. By Proposition 4 (i), $\mu^{d'} \subseteq \mu^d \cup [X^{d'} \setminus X^d]$. Thus, $(d'', h) \notin \mu^{d'}$. □

For any permutation π on D, we can consider the GFDA algorithm where the order of doctors is changed from $(1, 2, \ldots, |D|)$ to $(\pi(1), \pi(2), \ldots, \pi(|D|))$.

The reordered GFDA algorithm with order $(\pi(1), \pi(2), \ldots, \pi(|D|))$ Let $X^0 = \mu^0 = D^0 = \emptyset$ and move to Step 1 of GFDA phase.

Step d Let $D^d = D^{d-1} \cup \{\pi(d)\}$, $X^{d,0} = X^{d-1}$, and $\mu^{d,0} = \mu^{d-1}$. Move to Sub-step $(d.1)$

> **Sub-step** (d, n) Choose any $d' \in D^d \setminus \mu_D^{d,n-1}$ such that $X^{d,n-1} \neq Ac_{\geq_d}$, and let $X^{d,n} = X^{d,n-1} \cup \{(d', h)\}$ where h satisfies that $h \notin X_{d'}^{d,n-1}$ and $h \geq_d h'$ for every $h' \notin X_{d'}^{d,n-1}$. Let $\mu^{d,n} = Ch(X^{d,n})$ and move to Sub-step $(d, n+1)$.
>
> If there is no such d', then let $X^d = X^{d,n-1}$ and $\mu^d = \mu^{d,n-1}$ and terminates this step: if $d < |D|$, then move to Step $d+1$; if $d = |D|$, then let $\mu = \mu^d$ and terminates the algorithm.

For any Step d, we can regard μ^d as the COP matching when the set of doctors is $\{\pi(1), \pi(2), \ldots, \pi(d)\}$. Hirata and Kasuya (2017) show that when Ch satisfies substitutability and the law of aggregate demand, any change in the offer sequence does not affect the matching generated by COP. In addition, Hatfield and Milgrom (2005) show that when Ch satisfies these properties, the COP is strategy-proof.

Note that Ch satisfies these properties (Proposition 4). For any $D' \subseteq D$, let $GFDA^{D'}(\geq)$ be the matching generated at Step $|D'|$ in any reordered GFDA algorithm with order such that $\cup_{d=1}^{|D'|}\pi(d) = D'$ when $\geq$ is the doctors' reporting preference profile. Then, $GFDA^{D'}$ is strategy-proof for doctors in D'. Since $GFDA^{D'}(\geq) = \emptyset$ for any $d \notin D'$, it is also strategy-proof for other doctors.

Remark 10 (Strategy-proofness of *GFDA*)**.** For any $D' \subseteq D$, $GFDA^{D'}$ is strategy-proof.

Let $D_d = \{1, 2, \ldots, d\}$ for $d \in D$. Thus, for any Step d in the GFDA algorithm or $d \leq d^*$ in the GFDA phase of the GFDA+SD algorithm, the generated matching is described as $GFDA^{D_d}(\geq)$ when $\geq$ is reported.

Now, we will show that the GFDA+SD algorithm is strategy-proof.

Proof. Let μ be the matching generated by the GFDA+SD algorithm when the true preference profile $\geq$ is reported. Consider any $d \in D$ and $\geq'_d$. Let μ' and d'^* be respectively the matching and the final step in GFDA phase when $(\geq'_d, \geq_{-d})$ is reported. We will show this result by contradiction. Suppose that $\mu'_d >_d \mu_d$. Thus, $\mu'_d \neq \emptyset$. Since in the GFDA+SD algorithm, what d matches is independent of the orders of hospitals lower than μ'_d, without loss of generality, let $\emptyset >'_d h$ iff $\mu'_d >'_d h$ for any $h \in H$.

Claim 1. $d \leq \min\{d'^*, d^*\}$.

Suppose that $d > \min\{d'^*, d^*\}$. Since the preference profiles of $\{1, \ldots, d-1\}$ are equal between $\geq$ and $(\geq^*_d, \geq_{-d})$, each Step $d' \leq d-1$ is equal between the two cases. Since $d > \min\{d'^*, d^*\}$, Step d is in the SD phase for either case and, when μ^{d-1} and μ'^{d-1} are the matchings generated in Step $d-1$ under $\geq$ and $(\geq^*_d, \geq_{-d})$ respectively, we have $H^-(\mu^{d-1}) = H^-(\mu^{*d-1})$.

Thus, $\mu'_d \in H^-(\mu'^{d-1}) \cup \{\emptyset\} = H^-(\mu^{d-1}) \cup \{\emptyset\}$ Thus, $\mu_d \succeq_d \mu'_d$, which is a contradiction. Thus, $d \leq \min\{d'^*, d^*\}$.

Note that the above claim implies that $\mu'_d = \mu'^{d'^*}_d$ and $\mu_d = \mu^{d^*}_d$.

Let $d^+ \leq d^*$ be the smallest step such that $\mu^{d^+}_d = \mu^{d^*}_d (= \mu_d)$.

Claim 2. $d'^* < d^+$.

Suppose that $d'^* \geq d^+$. By strategy-proofness of *GFDA* and Remark 9, $\mu'_d = \mu'^{d'^*}_d = GFDA^{D^{d'^*}}_d(\succeq'_d, \succeq_{-d}) \preceq_d GFDA^{D^{d'^*}}_d(\succeq) \preceq_d GFDA^{D^{d^+}}_d(\succeq) = \mu_d$, which is a contradiction. Thus, $d'^* < d^+$.

By definition, $\mu^{d^+-1} \neq \mu^{d^+}$. By Remark 9, $\mu^{d^+-1}_d \succ_d \mu^{d^+}_d$. By individual rationality of *GFDA*, $\mu^{d^+-1}_d \neq \emptyset$.

Let $\succ^+$ be such that (i) for each $h, h' \in H$, $h \succeq^+_d h'$ iff $h \succeq_d h'$, and (ii) for each $h \in H$, $\emptyset \succeq^{dp}_d h$ iff $\mu^{d^+-1}_d \succeq_d h$. Let $\mu^+ = GFDA^{D^{d^+}}(\succeq^+_d, \succeq_{-d})$.

In addition, let $\mu'^+ = GFDA^{D^{d^+}}(\succeq'_d, \succeq_{-d})$.

Claim 3. $\mu^+_d = \emptyset$ and $\kappa^+_H - |\mu^+| \leq |D| - d^+$.

By strategy-proofness of *GFDA*, $\mu^+_d = GFDA^{D^{d^+}}_d(\succeq^+_d, \succeq_{-d}) \preceq_d GFDA^{D^{d^+}}(\succeq) = \mu^{d^+}_d = \mu^{d^*}_d = \mu^+_d \prec_d \mu^{d^+-1}_d$. By individual rationality of *GFDA*, $\mu^{+d}_d = \emptyset$.

Consider the GFDA+SD algorithm when $(\succeq^+_d, \succeq_{-d})$ is reported. Since $\succeq^+_d$'s preferences on hospitals at least as good as $\mu^{d^+-1}_d$ are the same as those of $\succeq_d$, each step $d' \leq d^+ - 1$ is equal between $\succeq$ and $(\succeq^+_d, \succeq_{-d})$. Thus, when μ^{+-1} is the matching generated at Step $d^+ - 1$ in the case of $(\succeq^+_d, \succeq_{-d})$, we have $\mu^{+-1} = \mu^{d^+-1}$. Thus, $\mu^{+-1}_d = \mu^{d^+-1}_d \neq \emptyset$. In addition, since $d^+ - 1 < d^+ \leq d^*$, then $\kappa^{+-1}_H - |\mu^{+-1}| < |D| - (d^+ - 1)$. By Proposition 5 (ii), $\kappa^+_H - |\mu^+| \leq |D| - d^+$.

Claim 4. $\mu'^+_d = \emptyset$ and $\kappa'^+_H - |\mu'^+| > |D| - d^+$.

By strategy-proofness of *GFDA*, $\mu'^+_d = GFDA^{D^{d^+}}(\succeq'_d, \succeq_{-d}) \preceq_d GFDA^{D^{d^+}}(\succeq) = \mu^+_d = \mu^*_d = \mu_d \prec_d \mu'_d$. By definition of μ'_d and individual rationality of *GFDA*, $\mu'^+_d = \emptyset$.

Note that $\mu'^{d'^*}_d = \mu'_d \neq \emptyset$. Thus, $d \in \mu'^{d'^*}_D \setminus \mu'^+_D$.

Further, note that $\mu'^{d'^*}$ and μ'^+ are respectively the matchings generated at Steps d'^* and d^+ in the GFDA algorithm. By Proposition 5 (ii), $\kappa'^+ \geq \kappa'^{d'^*}$. In addition, by Claim 2 and Proposition 5 (i), $\mu'^+_D \subseteq \mu'^{d'^*}_D \cup \{d'^*, \ldots, d^+\}$.Since $d \in \mu'^{d'^*}_D \setminus \mu'^+_D$, then $|\mu'^+| \leq |\mu'^{d'^*}| + (d^+ - d'^*) - 1$.

By definition of d'^*, $\kappa'^{d'^*}_H - |\mu'^{d'^*}| \geq |D| - d'^*$. Thus,

$$\begin{aligned}\kappa'^+_H - |\mu'^+| &\geq \kappa'^{d'^*}_H - (|\mu'^{d'^*}| + (d^+ - d'^*) - 1)\\ &> \kappa'^{d'^*}_H - |\mu'^{d'^*}| - (d^+ - d'^*) \geq |D| - d'^* - (d^+ - d'^*) = |D| - d^+.\end{aligned}$$

Claim 5. $\kappa^+_H - |\mu^+| = \kappa'^+_H - |\mu'^+|$, which contradicts Claims 3 and 4.

By Claims 1 and 2, $d \in D^{d^+}$. Note that $GFDA^{D^{d^+}\setminus\{d\}}(\succeq'_d, \succeq_{-d}) = GFDA^{D^{d^+}\setminus\{d\}}(\succeq^+_d, \succeq_{-d})$. Let this matching be μ^-.

Since $\mu^+ = GFDA^{D^{d^+}\setminus\{d\}}(\succeq^+_d, \succeq_{-d})$, we can regard μ^- and μ^+ respectively as the matchings generated at Step $d^+ - 1$ and d^+ in the reordered GFDA algorithm with the order replacing d with

d^+, when $(\succeq_d^+, \succeq_{-d})$ is reported. By Claim 3, the d^+-th doctor d does not match any hospital. By Proposition 5 (i), $\mu_D^- = \mu_D^+$. By Proposition 5 (ii), $\kappa_H^+ - |\mu^+| = \kappa_H^- - |\mu^-|$.

In the same manner, since $\mu'^+ = GFDA^{D^d \setminus \{d\}}(\succeq_d', \succeq_{-d})$, we can regard we can regard μ^- and μ'^+ respectively as those when $(\succeq_d', \succeq_{-d})$ is reported. Thus, by Claim 4, in the same manner, we have $\kappa_H'^+ - |\mu'^+| = \kappa_H^- - |\mu^-|$. Thus, $\kappa_H'^+ - |\mu'^+| = \kappa_H^+ - |\mu^+|$. □

6.6 Feasibility and stability of algorithms

In this section, we will give the proofs of Proposition 6, Theorem 1, and Proposition 9.

Throughout this section, we assume that *the doctors' preferences are acceptable*: i.e., $Ac_{\succ_d} = H$ for any $d \in D$.

6.6.1 Proof of Proposition 6

Proof. Since $\mu^0 = \emptyset$, by Assumption 1, $\kappa^0 - |\mu^0| = \bar{\kappa}^* \leq |D| = |D| - 0$. By Proposition 5, there exists d^* in the algorithm and it satisfies $\kappa_H^{d^*} - |\mu^{d^*}| = |D| - d^*$. In addition, Since $\mu^{d^*} = Ch(X^{d^*})$, by Remark 5, $\kappa^{d^*} \leq \bar{\kappa}$.

We consider induction. Suppose that some $d \in \{d^*, \ldots, |D|\}$ satisfies that $\kappa_H^d - |\mu^d| = |D| - d$ and $\kappa^d \leq \bar{\kappa}$. If $d = |D|$, then $\kappa_H^{|D|} - |\mu^{|d|}| = 0$. By Remark 3, $\mu = \mu^{|D|}$ is feasible.

If $d < |D|$, then $\kappa_H^d - |\mu^d| = |D| - d > 0$. By Lemma 1, $H^-(\mu^d) \neq \emptyset$. Thus, by acceptability and definition of steps in the SD phase, $\mu^{d+1} = \mu^d \cup \{(d+1, h)\}$ for some $h \in Ac_{\succ_d} = H^-(\mu^d)$. Thus, $|\mu^{d+1}| = |\mu^d| + 1$. In addition, by Lemma 2, $\kappa_H^{d+1} = \kappa_H^d$. Thus, $\kappa_H^{d+1} - |\mu^{d+1}| = \kappa_H^d - |\mu^d| - 1 = |D| - (d+1)$.

By inductive assumption, $\kappa_r^d = \bar{\kappa}_r$ or $\kappa_r^d \leq \bar{\kappa}_r - 1$ for any $r \in R$. Note that if $\kappa_r^d = \bar{\kappa}_r$, then $r \notin R^-(\mu^d)$. Thus, by Lemma 2, $\kappa^{d+1} \leq \bar{\kappa}_r$. By induction on d, we obtain the conclusion. □

6.6.2 Proof of Theorem 1

Let μ be the FDA+SD matching. Since the FDA matchin is individually rational and $h \in Ac_{\succeq_d}$ for (d, h) in each step of the SD phase, μ is individually rational. In addition, by Proposition 6, μ is feasible.

Let $A_D \cup A_H$ block μ with μ'. In this section, we will show that B and μ' satisfy (i) and (ii) in the definition of interval respecting stability. Since we choose any μ', this implies that μ is interval respecting stable.

Note that for each $d \in \{d^*, \ldots, |D|\}$, $\mu^d = \mu \cap [D^d \times H]$ since for each stage $d > d^*$, we just add one contract to the matching in the previous stage. In the same manner, let $\mu'^m = \mu' \cap [D^d \times H]$ for each $d \in \{d^*, \ldots, |D|\}$. In the proof, we will consider these two sequences $(\mu^{d^*}, \ldots, \mu^{|D|-1}, \mu)$ and $(\mu'^{d^*}, \ldots, \mu'^{|D|-1}, \mu')$ of matchings.

Note that for every $r \in R$, $\kappa_r^d \geq \kappa_r^{d'}$ if $d \geq d' \geq d^*$ because of $\mu^d \supseteq \mu^{d'}$ and definition of κ. In the same manner, for every $r \in R$, $\kappa_r'^d \geq \kappa_r'^d$ if $d \geq d' \geq d^*$.

Since μ' is feasible, $\kappa_H' - \sum_H \mu_h' = 0$. Since for every $d \in \{d^*, \ldots, |D|\}$, $\kappa_H' \geq \kappa_H'^d$ and $\sum_H \mu_h' \leq \sum_H \mu_h'^d + D - d$, we obtain the following remark.

Remark 11. For every $d^*, \ldots, |D|$, $\kappa_H'^d - \sum_H \kappa_h'^d = \sum_{r \in R} \left(\kappa_r'^d - \sum_{s(r)} \kappa_{r'}'^d\right) \leq |D| - d$. Especially, $\mu_{d'}' \neq \emptyset$ for any $d' > d$ when the equality holds.

Define $\bar{R}^{-d^*} \subseteq R$ as $r \in \bar{R}^{-d^*}$ iff $r \in R^-(\mu^{d^*})$ or there exists $r' \in R^-(\mu^{d^*})$ such that $r \in s(r')$.

In the proof, in particular, the four matchings κ^{d^*}, κ'^{d^*}, κ and κ' play an important role. The proof has two complements. Lemma 5 shows that for $r \in \bar{R}^{-d^*}$, κ_r and κ'_r are equal and that doctors after $d^* + 1$ are not in B. This is obtained from two claims. The first claim is the equality of κ^{d^*} and κ'^{d^*}. Based on this claim, the second claim inductively shows that the equality of κ^d andκ'^d for every $d \geq d^*$ and that doctors after $d^* + 1$ are not in B.

Note that interval respecting stability focuses on $r \in R$ such that $\kappa_r = \kappa'_r$ and $(\kappa_{r'})_{s(r)} \neq (\kappa'_{r'})_{s(r)}$. From Lemma 5, in the region on $R^-(\mu^{d^*})$, $(\kappa_{r'})_{s(r)} = (\kappa'_{r'})_{s(r)}$. On the other hand, direct subregion allocations can be different in regions other than $R^-(\mu^{d^*})$. In Lemma 6, in regions other than $R^-(\mu^{d^*})$, from the fact that doctors after $d^* + 1$ are not in B, we can show that $((\nu_{r'})_{s(r)} \geq (\kappa'_{r'})_{s(r)})$, i.e. the final allocations under the blocking were already in the scope of the adjusted contract quantity in stage d^* of the FDA-SD.

Finally, we show that, for the above regions, if $\kappa_r = \kappa'_r$, then $(\kappa_{r'})_{s(r)})$ is at least as good as $(\kappa'_{r'})_{s(r)})$, and every $h \in A_H$ is included in the region where the upper bound is reached. This implies that μ' satisfies the conditions in the definition of interval respecting stability.

Lemma 5. For every $r \in \bar{R}^{-d^*}$, $\kappa_r = \kappa'_r$. Especially, $\mu_h = \mu'_h$ for every $h \in H^-(\mu^{d^*})$.

Proof.

Claim 1. For every $r \in \bar{R}^{-d^*}$, $\kappa_r^{d^*} \geq \kappa_r'^{d^*}$. Especially, $\mu'_h \subseteq \mu_h$ for every $h \in H^-(\mu^{d^*})$.

We will show this claim by induction. Consider any $h \in H^-(\mu^{d^*})$. if $d \in \mu'_h \setminus \mu_h$ for some $d \leq d^*$, by definition of blocking, $d \in B$ and $h \succ_d \mu_d$. By definition of GFDA, $d \in X_h^{d^*}$. Since Lemma 4 implies $X_h^{d^*} = \mu_h$, we have $d \in \mu_h$, which is a contradiction. Thus, $\mu'_h \subseteq \mu_h$, or $|\mu'_h| \leq |\mu_h|$.

Suppose that for some n, for every region $r' \in \bar{R}^{-d^*}$ with $|r| \leq n$, $\kappa_{r'}^{d^*} \geq \kappa_{r'}'^{d^*}$. Choose any $r \in \bar{R}^{-d^*}$ with $|r| = n{+}1$. If $\kappa_r^{d^*} = \bar{\kappa}_r$, since μ' is feasible, we have $\kappa_r^{d^*} = \bar{\kappa}_r \geq \kappa'_r \geq \kappa_r'^{d^*}$. Suppose that $\kappa_r^{d^*} < \bar{\kappa}_r$. Since $r \in R^-(\mu^{d^*})$, $s(r) \subseteq \bar{R}^{-d^*}$. Since $|r'| \leq n$ for $r' \in s(r)$, by assumption, $\sum_{s(r)} \kappa_{r'}^{d^*} \geq \sum_{s(r)} \kappa_{r'}'^{d^*}$. Thus, $\kappa_r^{d^*} = \max\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}^{d^*}\} \geq \max\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}'^{d^*}\} = \kappa_r'^{d^*}$.

Claim 2. For every $r \in \bar{R}^{-d^*}$, $\kappa_r^{d^*} = \kappa_r'^{d^*}$. Especially, $\mu_h^{d^*} = \mu_h'^{d^*}$ for every $h \in H^-(\mu^{d^*})$.

By Remark 11,

$$\sum_{r \in R^-(\mu^{d^*})} \delta_r'^{d^*} \leq \sum_{r \in R} \delta_r'^{d^*} = \kappa_H'^{d^*} - \sum_H \kappa_{\{h\}}'^{d^*} \leq |D| - d^*.$$

Note that for any $r \in R$, if $\delta_r^{d^*} > 0$, then $r \in R^-(\mu^{d^*})$. Thus,

$$|D| - d^* = \kappa_H^{d^*} - \sum_H \kappa_{\{h\}}^{d^*} = \sum_{r \in R} \delta_r^{d^*} = \sum_{r \in R^-(\mu^{d^*})} \delta_r^{d^*}.$$

By Claim 1 and definitions of $\kappa_r^{d^*}$ and $\kappa_r'^{d^*}$,

$$\forall r \in R^-(\mu^{d^*}), \quad \delta_r^{d^*} = \max\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}^{d^*}\} - \sum_{s(r)} \kappa_{r'}^{d^*} \leq \max\{\underline{\kappa}_r, \sum_{s(r)} \kappa_{r'}'^{d^*}\} - \sum_{s(r)} \kappa_{r'}'^{d^*} = \delta_r'^{d^*},$$

or, $\{\delta_r^{d^*}\}_{R^-(\mu^{d^*})} \leq \{\delta_r'^{d^*}\}_{R^-(\mu^{d^*})}$. Thus, $\delta_r^{d^*} = \delta_r'^{d^*}$ for every $r \in R^-(\mu^{d^*})$.

Now, we will show that $\kappa_r^{d^*} = \kappa_r'^{d^*}$ for every $r \in \bar{R}^{-d^*}$. Suppose that $\kappa_{r_0}^{d^*} \neq \kappa_{r_0}'^{d^*}$ for some $r_0 \in \bar{R}^{-d^*}$. By Claim 1, $\kappa_{r_0}^{d^*} > \kappa_{r_0}'^{d^*}$. Let $(r_n)_{n=1}^{n^*}$ be such that $r_{n-1} \in s(r_n)$ for each $n = 1, \dots, n^*$ and $r_{n^*} = H$. Suppose by induction that $r_{n-1} \in \bar{R}^{-d^*}$ and $\kappa_{r_{n-1}}^{d^*} > \kappa_{r_{n-1}}'^{d^*}$. (Note that this holds when $n = 1$.) Thus, $\delta_{r_{n-1}} = 0$. Thus, $r_n \in R^-(\mu^{d^*}) \subseteq \bar{R}^{-d^*}$. Thus, $\delta_{r_n}^{d^*} = \delta_{r_n}'^{d^*}$. Since $r_n \in R^-(\mu^{d^*})$, then $s(r_n) \subseteq \bar{R}^{-d^*}$. By Claim 1 and the inductive assumption of r, $\sum_{s(r')} \kappa_{r''}^{d^*} > \sum_{s(r')} \kappa_{r''}'^{d^*}$. By combining with $\delta_{r_n}^{d^*} = \delta_{r_n}'^{d^*}$, we have $\kappa_{r_n}^{d^*} > \kappa_{r_n}'^{d^*}$. By repeating the above argument, we obtain $H \in \bar{R}^{-d^*}$ and $\kappa_H^{d^*} \geq \underline{\kappa}_H$, which contradicts the definition of $\bar{R}^{-d^*}$. Thus, for every $\kappa_r^{d^*} = \kappa_r'^{d^*}$ for $r \in \bar{R}^{-d^*}$.

Especially, $\kappa_{\{h\}}^{d^*} = \kappa_{\{h\}}'^{d^*}$ for every $h \in H^-(\mu^{d^*})$. By Claim 1, $\mu_h^{d^*} = \mu_h'^{d^*}$.

Claim 3. For every $r \in \bar{R}^{-d^*}$, $\kappa_r = \kappa_r'$. Especially, $\mu_h = \mu_h'$ for every $h \in H^-(\mu^{d^*})$.

We will show this claim by induction. Suppose by induction that there exists $d \geq d^*$ such that for every $r \in \bar{R}^{-d^*}$, $\kappa_r^d = \kappa_r'^d$ and especially, $\mu_h^d = \mu_h'^d$ for every $h \in H^-(\mu^{d^*})$. Since $d \geq d^*$, $R^-(\mu^d) \subseteq R^-(\mu^{d^*})$. Thus, for every $r \in R^-(\mu^d)$, $s(r) \subseteq \bar{R}^{-d^*}$. By inductive assumption, $\sum_{R^-(\mu^d)} \delta_r'^d = \sum_{R^-(\mu^d)} \delta_r^d = |D| - d$. By Remark 11, $\delta_r'^d = 0$ for every $r \notin R^-(\mu^d)$. Thus, if $\delta_r'^d > 0$, then $r \in R^-(\mu^d)$.

If $d+1 \notin \mu_h'$ for every $h \in H$, then $\mu'^{d+1} = \mu'^d$. Thus, $\kappa_H'^{d+1} - \sum_H \kappa_{\{h\}}'^{d+1} = \kappa_H'^d - \sum_H \kappa_{\{h\}}'^d = |D| - d > |D| - (d+1)$ which contradicts Remark 11. Thus, $d+1 \in \mu_{\{h\}}'$ for some $h \in H$.

Next, if $\delta_r'^d = 0$ for every $r \in R$ with $h \in r$, by definition of $\kappa_r'^d$, then $\kappa_H'^{d+1} = \kappa_H'^d + 1$. Since $\sum_H \kappa_{\{h\}}'^{d+1} = \sum_H \kappa_{\{h\}}'^d + 1$, we have $\kappa_H'^{d+1} - \sum_H \kappa_{\{h\}}'^{d+1} = \kappa_H'^d - \sum_H \kappa_{\{h\}}'^d = |D| - d > |D| - (d+1)$, which contradicts Remark 11 again.

Let $r \in R$ be the smallest region such that $h \in r$ and $\delta_r'^d > 0$. By definition, for any $r' \subsetneq r$ with $h \in r'$, $\kappa_{r'}'^{d+1} = \kappa_{r'}'^d + 1$. In addition, since $\delta_r'^d > 0$, as shown in the first paragraph, $r \in R^-(\mu^d)$.

Consider $r' \in s(r)$ with $h \in r'$. Since $r \in R^-(\mu^d) \subseteq R^-(\mu^{d^*})$, then $r' \in \bar{R}^{-d^*}$. By the inductive assumption, $\kappa_{r'}'^d = \kappa_{r'}^d$. Thus, $\kappa_{r'}'^{d+1} = \kappa_{r'}'^d + 1 = \kappa_{r'}^d + 1$. By feasibility of μ', $\sum_{s(r')} \kappa_{r''}^d \leq \kappa_{r'}^d = \kappa_{r'}'^{d+1} - 1 \leq \kappa_{r'}' - 1 < \bar{\kappa}_{r'}$. Thus, $r' \in R^-(\mu^d)$. By repeating the above argument, we obtain that for any $r' \subseteq r$ with $h \in r'$, $r' \in \bar{R}^{-d^*}(\mu^d)$. Especially, $\{h\} \in \bar{R}^{-d^*}(\mu^d)$ means $h \in H^-(\mu^d)$. By definition of the SD procedure, $\mu_{d+1} \succeq_{d+1} h$. By definition of blocking, $d+1 \notin B$ and $h = \mu_{d+1}$. Since by assumption, $\mu_h^d = \mu_h'^d$ for every $h \in H^-(\mu^{d^*})$, we have $\mu_h^{d+1} = \mu_h'^{d+1}$ for every $H^-(\mu^{d^*})$.

For any $r' \subseteq r$ with $h \in r'$, since $r' \in R^-(\mu^d)$, then $s(r') \subset \bar{R}^{-d^*}$. By the inductive assumption, $\delta_{r'}^d = \delta_{r'}'^d$. Thus, r is also the smallest region such that $h \in r$ and $\delta_r^d > 0$. By Remark 7 and the inductive assumption, for any $r' \in \bar{R}^{-d^*}$, if $h \in r' \subseteq r$, then $\kappa_{r'}^{d+1} = \kappa_{r'}^d + 1 = \kappa_{r'}'^d + 1 = \kappa_{r'}'^{d+1}$, otherwise $\kappa_{r'}^{d+1} = \kappa_{r'}^d = \kappa_{r'}'^d = \kappa_{r'}'^{d+1}$. By induction, we obtain the conclusion. □

Lemma 6. For each $r \in R$, if $\{\kappa_{r'}\}_{s(r)} \neq \{\kappa_{r'}'\}_{s(r)}$, then $\{\kappa_{r'}'\}_{s(r)} \leq \{\nu_{r'}^{d^*}\}_{s(r)}$.

Proof. If $r \in R^-(m^{d^*})$, since $s(r) \in \bar{R}^{-d^*}$, by Lemma 5, $\{\kappa_{r'}\}_{s(r)} = \{\kappa_{r'}'\}_{s(r)}$. We will show that for each $r \notin R^-(m^{d^*})$, we have $\{\kappa_{r'}'\}_{s(r)} \leq \{\nu_{r'}^{d^*}\}_{s(r)}$ by induction.

Consider any $h \in H$ such that $\{h\} \notin R^-(m^{d^*})$. That is, $h \notin H^-(\mu^{d^*})$. By definition of μ and Lemma 5, for every $d' > d^*$, $d' \in \cup_{H^-(\mu^{d^*})} \mu_h = \cup_{H^-(\mu^{d^*})} \mu_h'$. Thus, $|\mu_h'| = |\mu_h'^{d^*}|$. By the definition of blocking, for any $d \leq d^*$, if $\mu_d'^{d^*} \neq \emptyset$, then $\mu_d'^{d^*} = \mu_d' \succeq_d \mu_d = \mu_d^{d^*}$ and by definition of GFDA, $(d, \mu'^{d^*}_d) \in X^{d^*}$. Thus, $|\mu'^{d^*}_h| \leq w^{d^*}_h$. Thus, $|\mu'_h| \leq w^{d^*}_h$, which is the desired conclusion when $|r| = 1$ and the induction basis.

Suppose by induction that for some n, for every $r' \in R$ with $|r'| \leq n$ such that $r' \notin R^-(m^{d^*})$, we have $\{\kappa'_{r''}\}_{s(r')} \leq \{v^{d^*}_{r''}\}_{s(r')}$. Consider any r with $|r| = n+1$ such that $r \notin R^-(m^{d^*})$. Consider any $r' \in s(r)$. If $r' \notin \bar{R}^{-d^*}$, by the inductive assumption, $\{\kappa'_{r''}\}_{s(r')} \leq \{v^{d^*}_{r''}\}_{s(r')}$. By feasibility of μ', $\kappa'_{r'} = \max\{\underline{\kappa}_{r'}, \min\{\sum_{s(r')} \kappa'_{r''}, \bar{\kappa}_{r'}\}\} \leq \max\{\underline{\kappa}_{r'}, \min\{\sum_{s(r')} v^{d^*}_{r''}, \bar{\kappa}_{r'}\}\} = v^{d^*}_{r'}$. If $r' \in \bar{R}^{-d^*}$, since $r \notin R^-(\mu^{d^*})$, then $\delta^{d^*}_{r'} > 0$. Thus, $\kappa^{d^*}_{r'} = \underline{\kappa}_{r'} \leq v^{d^*}_{r'}$. Note that for any $d \geq d^*$, since $R^-(\mu^{d^*}) \supseteq R^-(\mu^d)$, then $r \notin R^-(\mu^d)$. Since $d+1 \in \mu_h$ for some $h \in H^-(\mu^d)$, by Remark 7, $\kappa^{d+1}_{r'} = \kappa^d_{r'}$. Thus, we have $\kappa_{r'} = \kappa^{d^*}_{r'}$. In addition, since $r' \in \bar{R}^{-d^*}$, by Lemma 5, $\kappa'_{r'} = \kappa_{r'}$. Therefore, $\kappa'_{r'} = \kappa^{d^*}_{r'} = \underline{\kappa}_{r'} \leq v^{d^*}_{r'}$. Thus, $\kappa'_{r'} \leq v^{d^*}_{r'}$. By induction, we obtain the conclusion. □

Proposition 11. *B and μ' satisfy the two conditions in interval respecting stability: (i) $d' \succ_h d$ for every $d' \in \mu_h$ and $d \in \mu'_h \cap B$ and (ii) for each $h \in B \cap H$, there exist $r^* \in R_{\ni h}$ satisfying $(|\mu_{r'}|)_{s(r^*)} \neq (|\mu'_{r'}|)_{s(r^*)}$, and it follows that $|\mu_{\bar{r}}| = \bar{\kappa}_r$ for some $\bar{r} \in R$ with $\bar{r} \supseteq r^*$, and $(|\mu_{r'}|)_{s(r)} \tilde{\gtrsim}_r (|\mu'_{r'}|)_{s(r)}$ for any $r \in R$ with $r^* \subseteq r \subseteq \bar{r}$.*

Proof. Consider any $h \in B$. Then, $\mu'_h \succ_h \mu_h$. By acceptability of h, $\mu'_h \setminus \mu_h = \mu'_h \cap B$ is not empty. Choose any $d \in \mu'_h \cap B$. By definition of blocking, $\mu'_d = h \succ_d \mu_d$. For any $d' > d^*$, since $d' \in \mu_h$ for some $h \in H^-(\mu^{d'-1}) \subseteq H^-(\mu^{d^*})$, by Lemma 5, $\mu'_{d'} = \mu_{d'}$. Thus, $d \leq d^*$. Thus, $\mu_d = \mu^{d^*}_d$ and $(d, h) \in X^{d^*}$ since by definition of the GFDA algorithm, $(d, h') \in X^{d^*}$ for each $h' \succ_d \mu_d$. Since $d \notin \mu_h$, by definition of Ch, $d' \succ_h d$ for every $d' \in \mu_h$, which implies (i).

In addition, since $(d, h) \in X^{d^*}$ and $d \notin \mu_h$, by Lemma 4, there exists $r^+ \in R_{\ni h}$ such that $\bar{\kappa}_{r^+} = \kappa^{d^*}_{r^+} \leq \kappa_{r^+}$. By feasibility of μ and μ', $\mu_{r^+} = \kappa_{r^+} = \bar{\kappa}_{r^+} \leq \kappa'_{r^+} = \mu'_{r^+}$.

Suppose that $|\mu'_h| \leq |\mu_h|$. Choose any $D' \subseteq \mu_h \setminus \mu'_h$ with $|D'| = |\mu'_h \setminus \mu_h|$. By (i), for any $d' \in \mu'_h \setminus \mu_h$ and any $d \in D'$, $\{d\} \succ_h \{d'\}$. Since $\succ_h$ is strict, acceptable, and responsive, then $\mu_h \succeq_h D' \cup [\mu'_h \cap \mu_h] \succ_h \mu'$, which is a contradiction. Thus, $|\mu'_h| > |\mu_h|$ for any $h \in A_H$.

Let r^* be the largest region among all $r \in R_{\ni h} \cap 2^{r^+}$ such that $(|\mu'_{r'}|)_{s(r)} \neq (|\mu_{r'}|)_{s(r)}$.

By definition of r^*, for each $r \in R$ with $r^* \subsetneq r \subseteq r^+$, $(|\mu'_{r'}|)_{s(r)} = (|\mu_{r'}|)_{s(r)}$. Thus, $(|\mu_{r'}|)_{s(r)} \tilde{\gtrsim} (|\mu'_{r'}|)_{s(r)}$.

Finally, we will show that $(|\mu_{r'}|)_{s(r^*)} \tilde{\gtrsim} (|\mu'_{r'}|)_{s(r^*)}$.

Note that if $r^* \subsetneq r^+$, then $(|\mu_{r'}|)_{s(r^{*+})} = (|\mu'_{r'}|)_{s(r^{*+})}$ for $r^{*+} \in R$ with $r^* \in s(r^{*+})$. In particular, $\mu_{r^*} = \mu'_{r^*}$. If $r^* = r^+$, since $\mu_{r^+} \leq \mu'_{r^+}$, then $\mu_{r^*} \leq \mu'_{r^*}$. Thus, for any case, $\mu_{r^*} \leq \mu'_{r^*}$. By feasibility of μ and μ', $\kappa_{r^*} = \mu_{r^*} \leq \mu'_{r^*} = \kappa'_{r^*}$. By the same reason, $\kappa'_{s(r)} = (|\mu'_{r'}|)_{s(r)} \neq (|\mu_{r'}|)_{s(r)} = \kappa_{s(r)}$.

By definition of κ, $\underline{\kappa}^*_{s(r)} \leq \kappa'_{s(r)}$ and $\kappa_r \geq \kappa'_r \geq \sum_{s(r)} \kappa'_{r'}$. In addition, since $\kappa_{s(r)} \neq \kappa'_{s(r)}$, by Lemma 6, $\kappa'_{s(r)} \leq v^{d^*}_{s(r)}$. Since $\tilde{Ch}_r(\underline{\kappa}^*_{s(r)}, v^{d^*}_{s(r)}; \kappa_r) = (\kappa^{d^*})_{s(r)}$, then $(\kappa^{d^*})_{s(r)} \tilde{\gtrsim}_r (\kappa'_{r'})_{s(r)}$. Since $\kappa_{s(r)} \geq \kappa^{d^*}_{s(r)}$, by monotonicity of $\tilde{\gtrsim}_r$, $\kappa_{s(r)} \tilde{\gtrsim}_r \kappa'_{s(r)}$. That is, $(|\mu'_{r'}|)_{s(r)} \tilde{\gtrsim}_r (|\mu_{r'}|)_{s(r)}$ □

6.6.3 Proof of Proposition 9

Proof. By definition, $\kappa_H \leq \min\{\bar{\kappa}_H, |D|\}$. Thus, if $\kappa_H - |\mu| > 0$, then $|\mu| < |D|$. That is, there exists $d \notin \mu_D$. Since Step $|D|$ is terminated, by acceptability, $Ac_{\succeq_d} = H = X_d$. In addition, by Lemma 1 and 4, $H^-(\mu) \neq \emptyset$ and $h \in H^-(\mu)$ satisfies $\mu_h = X_h$. Thus, $d \in \mu_h$, which is a contradiction. Thus, $\kappa_H - |\mu| = 0$. By Remarks 3 and 5, μ is feasible. □

6.7 Proof of Proposition 10

Proof. (i)⇒ (ii): By Proposition 9, if the matching μ obtained by the GFDA with ceiling $|D|$ is the same as that obtained by the GFDA, then μ is feasible.

(ii)⇒ (i): Suppose that the matching μ generated by the GFDA algorithm is feasible. Thus, $\kappa_H - |\mu| = 0$. Since $|\mu| \leq |D|$, $\kappa_H \leq |D|$. By Lemma 4 (ii), $\kappa_H^{d,n} \leq \kappa_H$ for any substep (d, n). By definition of the reserved quota allocation, $v_H^{d,n} = \bar{C}h_H(X^{d,n}) = \kappa_H^{d,n} \leq \kappa_H \leq |D|$ for any substep (d, n).

Let the matching μ' be the GFDA with ceiling $|D|$ matching. We show by induction that $\mu^{d,n} = \mu'^{d,n}$ for each substep (d, n). Note that $\mu^0 = \mu'^0$ by definition. Suppose that $\mu^{d,n-1} = \mu'^{d,n-1}$ for some $d \in D$. Thus, $X^{d,n-1} = X'^{d,n-1}$. Since $v_H^{d,n-1} \leq |D|$, $v_H^{|D|,d,n-1} = \min\{|D|, v_H^{d,n-1}\} = v_H^{d,n-1}$. That is, $(\bar{C}h(X^{d,n-1}), Ch(X^{d,n-1})) = (\bar{C}h^{|D|}(X^{d,n-1}), Ch^{|D|}(X^{d,n-1})) = (\bar{C}h^{|D|}(X'^{d,n-1}), Ch^{|D|}(X'^{d,n-1}))$. Thus, $\mu^{d,n} = \mu'^{d,n}$. By induction, we have $\mu = \mu'$.

(iii)→ (ii): This is obvious by definition.

(ii)→ (iii): Suppose that the matching μ generated by the GFDA algorithm is feasible. Thus, $\kappa_H - |\mu| = 0 = |D| - |D|$.

Note that the proof of Theorem 1 depend only on the facts that $\kappa_H^{d^*} - |\mu^{d^*}| = |D| - d^*$ as the property of Step d^* of the GFDA phase. In addition, in that proof, acceptability is used only in the SD phase. Thus, if we regard Step $|D|$ of this GFDA algorithm as Step d^* in the that proof, we have $\kappa_H^{d^*} - |\mu^{d^*}| = |D| - d^*$ and we do not need acceptability since there is no step in the SD phase. Thus, by the same proof, μ is interval respecting stable. □