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Abstract

We examine the relationship between the Shapley value and the excess. Defining the *excess of player i* by summing up the excesses of all the coalitions to which i belongs, we introduce a condition *equal excess*. Using this condition, we prove that the Shapley value is characterized as a value satisfying the equal excess with respect to a reasonable weight function. This implies that the Shapley value attains the greatest benefits of the least advantaged players.

Keywords: Shapley value; Equal excess; Difference principle; Least square values

JEL classification: C71

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1 Introduction

On the set of the preimputations, it is well known that the prenucleolus (Schmeidler, 1969) minimizes the maximum excess of the coalitions. From this, the prenucleolus is perceived as a solution which embodies the *difference principle*: "to the greatest benefit of the least advantaged (Rawls, 1999, p.266)." It is this interpretation that makes the prenucleolus more important and be a widely applied solution.

Then, is there any other solution which improves the condition of the least advantaged subjects on the set of the preimputations? If such a solution exists, that solution is substantially as important as the prenucleolus in the above sense. In this paper, we prove that the Shapley value[1] is such a solution, introducing the *excess of player* i[2]. More precisely, we show that the Shapley value is the solution which improves the condition of the least advantaged players.

Ruiz et al. (1998) examined a relationship between the Shapley value and the excess. They had the objective of 'characterizing the Shapley value, which is the representative solution of the marginal contributions world, with the excess, which is the main concept of the excesses world, in order to connect the two worlds'. They proposed that not only the excess of the least advantaged coalitions but also the excesses of all the coalitions should be considered, and defined the least square values as solutions which minimize the weighted variances of all the excesses. And they showed that the Shapley value is a special case of the least square values. By presenting this characterization, although no specific meaning is given to the weight function employed in this special case, they concluded that the distance between the two worlds, the marginal contributions world and the excesses world, is not so broad.

It can be pointed out that, on their characterization of the Shapley value via the least square values, the Shapley value is not described in the form of 'improving the condition of the least advantaged subjects'. If the Shapley value is described in this form, then the distance between the two worlds can be measured more accurately (because the prenucleolus, which is the representative solution of the excesses world and hence the counterpart of the Shapley value, is expressed in this form). Furthermore, it may be possible to conclude not only that the two are 'not far from' but also that they are 'close to' each other (because both of the representative solutions will be interpreted in relation to the difference principle).

This paper shares the objective of Ruiz et al. (1998), 'characterizing the Shapley value with the excess to connect the marginal contributions world and

[1] The original paper is Shapley (1953), and relatively recent papers include Kamijo and Kongo (2010), Casajus (2014), Yokote, Funaki, and Kamijo (2017), and McQuillin and Sugden (2018). Namekata and Driessen (1999; 2000) characterized the Shapley value via the egalitarian non-k-averaged contribution (EN^{k}AC-)value. de Clippel (2018) showed that the additivity axiom can be relaxed and replaced by the difference formula axiom. This axiom is based on the idea of dividing the set of all the coalitions into two groups: the coalitions which player i belongs to and which i does not. Our paper is influenced by this idea.

[2] As will be explained later, this is defined as the *weighted sum of the excesses of all the coalitions to which the player* i *belongs.*

the excesses world', but realizes it differently from them. In this paper, the Shapley value is characterized as 'a solution which improves the condition of the least advantaged subjects' with no reference to the weighted variances of all the excesses.

We adopt not the coalitions, but the players, as the subjects. This policy is reasonable, recalling that the players are usually adopted as the subjects not only in the cooperative game theory but also in the game theory as a whole.

This requires us to define some notion of player's excess. Redefining the excess (which is originally defined in the coalitional dimension) in another dimension has been done in the literature. For example, the maximum excess of player i against j (expressed as $s_{ij}(x)$ below) is introduced to define the kernel (Davis and Maschler, 1965). Osborne and Rubinstein (1994) summarized that, for a given V (the set of all characteristic functions v) and a given X (the set of all imputations x of $v \in V$), the kernel is defined to be "the set of imputations $x \in X$ such that for every pair (i, j) of players either $s_{ji}(x) \geq s_{ij}(x)$ or $x_j = v(\{j\})$," where for any two players i and j and any imputation x, $s_{ij}(x)$ is defined "to be the maximum excess of any coalition that contains i but not j (Osborne and Rubinstein, 1994, p.284)."

It can be pointed out that the above $s_{ij}(x)$ is somewhat complex in that it requires information about whether another player j belongs to the coalitions to which the player i belongs[3]. Can we simply define a player i's excess with no reference to any relationships with another player j?[4]

In this paper, we define the *excess of player i* as the *weighted sum of the excesses of all the coalitions to which the player i belongs*[5]. It is reasonable for any players to derive their excesses from the excesses of the coalitions to which they belong. In this sense, the above definition certainly includes all the information needed to obtain a player's excess, though the excesses of the coalitions are weighted (or possibly unweighted) by a weight function. In general, it is the number of players that makes the coalitions differ from each other. Therefore, it is quite natural for us to define the weight function as a function of the number of players belonging to the coalitions[6]. This weight function is also employed by Ruiz et al. (1998).

Now, if the excesses of all the players are equal, it means that the condi-

[3]Regarding the Shapley value, although there is no relation to the excess, Osborne and Rubinstein (1994) presented a characterization similar to that of the kernel. Introducing certain types of objections and counterobjections, they showed that the Shapley value is the unique value which satisfies "the property that for every objection of any player i against any other player j there is a counterobjection of player j (Osborne and Rubinstein, 1994, p.290)." Note that our argument on $s_{ij}(x)$ can also be applied to their definitions of objections and counterobjections because they inevitably contain information about another player j.

[4]If a simplified definition yields some notable result, then anyone would admit its usefulness. See **Section 3** and **Subsection 4.3**.

[5]Note that any relationships between the player i and the other players do not appear in this definition. Instead, we introduce a weight function, to sum up the excesses of the coalitions.

[6]The most natural interpretation of this weight may be that it represents the probability that each coalition will be formed, though its functional form has not been determined yet. However, there is no limitation on interpretation so far.

tion of the least advantaged players is certainly improved (cannot be improved anymore). With this in mind, we introduced a condition *equal excess*, which reasonable solutions would satisfy. As literally expressed, this condition requires that the excesses of all the players be equivalent[7].

As an example, consider a player set $\mathcal{N} = \{A, B, C\}$[8] and calculate the excess of player A, given some characteristic function and some preimputation. We sum up the excesses of the coalitions $\{A\}$, $\{A, B\}$, and $\{A, C\}$[9] using some weight function $f_3 : \{1, 2\} \to \mathbb{R}_{++}$. The index number 3 of the function corresponds to the total number of players in this game. Consequently, the excess of player A is expressed as

$$f_3(1) \times \text{the excess of } \{A\} \\ + f_3(2) \times (\text{ the excess of } \{A, B\} + \text{the excess of } \{A, C\} \text{ }).$$

We can also calculate the excesses of the remaining players (B and C) in this manner. The equal excess requires that all of these excesses be equivalent.

This paper specifies which weight function $f_n : \{1, 2, \cdots, n-1\} \to \mathbb{R}_{++}$ does the Shapley value have on n-person games when we impose the equal excess with respect to $f_n(\cdot)$. On the set of the preimputations, we show that the Shapley value satisfies the equal excess with respect to a reasonable weight function. The fundamental part of this weight represents, as shown in **Section 3**, the probability that each coalition is formed in the context of the excess.

As a result, the Shapley value is not only shown to be 'the solution which improves the condition of the least advantaged players', but it is also found to be 'the solution which makes each player's (expected) excess equivalent'.

The structure of this paper is as follows. In **Section 2**, we present the basic notations. In **Section 3**, we introduce the equal excess and state the main result. **Section 4** presents the discussions. The proof of the main theorem is provided in **Appendix**.

2 Preliminaries

Let $\mathcal{N}$ ($|\mathcal{N}| = n \geq 2$) be the set of all *players*. A *coalition* S ($|S| = s$) is a subset of $\mathcal{N}$. The set of all coalitions is denoted as $P(\mathcal{N})$. Let $v : P(\mathcal{N}) \to \mathbb{R}$ be a *characteristic function*, which satisfies $v(\phi) = 0$. For any $S \in P(\mathcal{N})$, $v(S)$ represents the amount which the coalition S can obtain by itself. The set of all characteristic functions is denoted as V. We call any $x = (x_i)_{i \in \mathcal{N}} \in \mathbb{R}^n$ as a *payoff vector*. Let $\sigma : V \to \mathbb{R}^n$ be a *value*; that is, a value associates each

[7] While the term "equal excess" is also used in Driessen and Funaki (1991) in examining the egalitarian nonseparable contribution (ENSC-)value, there is no relation between their usage and ours.

[8] In this paper, we express the players not only using i or j but also using A, B, $\ldots$, N. This enables us to clarify the proof of the main theorem (see **Note 1.** in **Appendix**). Therefore, we express the player set not as N, but as $\mathcal{N}$ ($|\mathcal{N}| = n$).

[9] Since we restrict our attention to the set of the preimputations, the excess of the grand coalition $\{A, B, C\}$ is always equal to zero. Hence, we ignore this term in the calculation of the excess of player A.

characteristic function with a payoff vector. For each $v \in V$, we refer to a payoff vector x satisfying $\sum_{i \in \mathcal{N}} x_i = v(\mathcal{N})$ as a *preimputation*.

The main concept of the excesses world is introduced as follows: For each $v \in V$, each preimputation x, and each $S \in P(\mathcal{N})$ $(S \neq \phi,\ \mathcal{N})$, we define the *excess of coalition* S *at preimputation* x *in* v as

$$e(S, x) = v(S) - \sum_{j \in S} x_j.$$

This represents the amount by which the self-attainable gain of the coalition S exceeds its payoff (allocated by the preimputation x). A preimputation x which yields a less $e(S, x)$ is preferable for any $S \in P(\mathcal{N})$. Note that, since we restrict our attention to the set of the preimputations in this paper, the grand coalition $\mathcal{N}$'s excess is always equal to zero. Hence, we exclude the case that $S = \mathcal{N}$ in this definition.

On the marginal contributions world (and throughout this paper), we take it for granted that all the players are lined up in some order (and all the orders have the same probability). Starting from no one, the players join one by one and eventually form the grand coalition $\mathcal{N}$. Based on this, we introduce the *Shapley value* Sh, which is the representative solution of the marginal contributions world, as follows: For each $v \in V$, $Sh(v) = (Sh_i(v))_{i \in \mathcal{N}}$, where for each $i \in \mathcal{N}$,

$$\begin{aligned} Sh_i(v) &= \sum_{S \in P(\mathcal{N}) | i \in S} \frac{(s-1)!(n-s)!}{n!} (v(S) - v(S \setminus \{i\})) \\ &= \sum_{S \in P(\mathcal{N}) | i \in S} \frac{1}{s \cdot {}_nC_s} (v(S) - v(S \setminus \{i\})) \\ &= \frac{v(\mathcal{N})}{n} + \sum_{k=1}^{n-1} \frac{1}{k \cdot {}_nC_k} \sum_{S \in P(\mathcal{N}) \setminus \mathcal{N} |\ i \in S,\ s=k} v(S) \\ &\qquad - \sum_{k=1}^{n-1} \frac{1}{(k+1) \cdot {}_nC_{k+1}} \sum_{S \in P(\mathcal{N}) \setminus \mathcal{N} |\ i \notin S,\ s=k} v(S). \qquad \cdots\cdots (\alpha) \end{aligned}$$

We make three remarks on the Shapley value.

First, we explain the meaning of the Shapley value using the first equation above[10]. The right-hand side of this equation has two main parts: the weight $\frac{(s-1)!(n-s)!}{n!}$ and the part $(v(S) - v(S \setminus \{i\}))$. On the one hand, since all the players are in a certain order, it follows that $(v(S) - v(S \setminus \{i\}))$ represents the marginal contribution of the player i to the coalition S, i.e., the amount which i brings to the set $S \setminus \{i\}$ of all the preceding players when i joins S as the s-th player. Hence, we can see that Sh_i takes the form of a weighted sum of the marginal contributions of i to S $(\ni i)$.

[10] Based on the explanation here, we will interpret the main theorem in **Section 3**.

On the other hand, as stated below, the weight $\frac{(s-1)!(n-s)!}{n!}$ represents the probability associated with the coalition S. Firstly, the number of ways to arrange all n players is $n!$. Secondly, the number of ways to form the coalition S while keeping i as the s-th player is $(s-1)!(n-s)!$. Consequently, the probability that S is formed when i joins S as the s-th player is expressed as $\frac{(s-1)!(n-s)!}{n!}$. In the right-hand side of the first equation, this weight is combined with the marginal contribution of i to S. As a whole, Sh_i coincides with the expected marginal contribution of i.

Second, the representation (α) of the Shapley value is based on the idea of dividing the set of all the coalitions into two groups: the coalitions which the player i belongs to and which i does not. This (α) is used in the proof of the main theorem (see **Appendix**).

Third, for each characteristic function, the Shapley value is always a preimputation; that is, for each $v \in V$,

$$\sum_{i \in \mathcal{N}} Sh_i(v) = v(\mathcal{N}). \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (\beta)$$

3 Results

In this paper, we aim to improve the condition of the least advantaged players in the context of the excess. To define the excess of player i, we introduce a function which weights the coalitions. We call any $f_n : \{1, 2, \ldots, n-1\} \to \mathbb{R}_{++}$ as a *weight function*[11]. The weight function f_n is a function from the set of the numbers of players belonging to the coalitions (except $\mathcal{N}$) to the set of positive real numbers. Hence, the coalitions which have the same number of players are assigned the same weight. As explained in **Section 1**, to calculate the excess of player i, we sum up the excesses of the coalitions to which i belongs. f_n is used in the process of this summation. The most natural interpretation of this weight may be that it represents the probability that each coalition will be formed, though its functional form has not been determined yet. However, there is no limitation on interpretation so far.

Then, we define the *excess of player i with respect to weight function f_n at preimputation x in v* as follows: For each $v \in V$, each preimputation x, each $i \in \mathcal{N}$, and a given weight function f_n,

$$e_{i,f_n}(x) = \sum_{k=1}^{n-1} \sum_{S \in P(\mathcal{N}) \setminus \mathcal{N} | \ i \in S, \ s=k} f_n(k) \ e(S,x).$$

As expressed, for a given preimputation x and a given weight function f_n, $e_{i,f_n}(x)$ denotes the weighted sum of the excesses of all the coalitions (except $\mathcal{N}$)

[11] Note that the index number n of the function corresponds to the total number of players in the game. Also, note that f_n does not depend on the characteristic function v. Therefore, for any $v \in V$, f_n assigns the same set of weights to the coalitions.

to which the player i belongs. In other words, $e_{i,f_n}(x)$ represents the weighted amount by which the self-attainable gain of each coalition S ($\ni i$) exceeds its payoff (allocated by x). Thus, in the context of the excess, since $e_{i,f_n}(x)$ contains all the information about i, it follows that $e_{i,f_n}(x)$ is a proper measurement of the condition of i. Hence, for some reasonable weight function f_n, a preimputation x which yields a less $e_{i,f_n}(x)$ is preferable for any $i \in \mathcal{N}$. Since we do not know in advance what kind of f_n is meaningful, we define the excess of player i in a way that does not specify f_n, as above.

Based on the above definition, we introduce the following condition which a value σ (assigning a preimputation to each $v \in V$) may satisfy:

Equal Excess with respect to weight function f_n (EEf):
For each $v \in V$ and a given weight function f_n,

$$e_{i,f_n}(\sigma(v)) = e_{j,f_n}(\sigma(v))$$

for any $i, j \in \mathcal{N}$.

This condition requires that, for a given weight function f_n, the excesses of all the players at $\sigma(v)$ should be equivalent. Therefore, it represents an equal treatment of the players in the context of the excess. This treatment ensures that the condition of the least advantaged players is improved (it cannot be improved anymore). As a result, if a value satisfies **EEf** with respect to some reasonable weight function, it certainly attains the greatest benefit of the least advantaged players, i.e., it certainly embodies the difference principle.

Now we proceed to characterize the Shapley value using **EEf**. Note that the weight function f_n in **EEf** has not been specified yet. The following theorem shows that f_n can be specified as a reasonable weight function if we consider the Shapley value:

Theorem 1. The following two statements are equivalent:

(i) The Shapley value satisfies **EEf**.
(ii) The weight function f_n is as follows:
For all $s \in \{1, 2, \ldots, n-1\}$,

$$f_n(s) = \frac{a}{{}_{n-2}C_{s-1}} \ (=: f_n^*(s)),$$

where $a \in \mathbb{R}_{++}$.

Proof. See **Appendix**.

To see what this theorem means, or more precisely, to see what f_n^* is about, we need some calculations. The basic part of f_n^* is rearranged as follows:

$$
\begin{aligned}
\frac{1}{{}_{n-2}C_{s-1}} &= \frac{n-1}{(n-1)_{n-2}C_{s-1}} \\
&= \frac{n-1}{\frac{(n-1)!}{(n-s-1)!(s-1)!}} \\
&= (n-1) \times \frac{(s-1)!(n-1-s)!}{(n-1)!} \\
&= (n-1) \times \frac{s!(n-1-s)!}{(n-1)!} \times \frac{1}{s}. \quad \cdots\cdots\cdots\cdots (\gamma)
\end{aligned}
$$

Based on the expression (γ), we can see that f_n^* has the following interpretation[12]:

First, we explain the meaning of $(n-1)$ appeared in the first part of (γ). Recall that, since we exclude the excess of the grand coalition $\mathcal{N}$ in this paper, f_n^* assigns numbers only to the coalitions S with $s \in \{1, 2, \ldots, n-1\}$. Thus, when we consider the excess of player i, we need to exclude one player except i, i.e., we choose one player from the set $\mathcal{N} \setminus \{i\}$. This can be done in $n-1$ ways, which is expressed in the first part of (γ).

Second, we explain the meaning of $\frac{s!(n-1-s)!}{(n-1)!}$ in the second part of (γ). Firstly, the number of ways to arrange (all the) $n-1$ players including i is $(n-1)!$. Secondly, the number of ways to form the coalition S $(\ni i)$ is $s!(n-1-s)!$[13]. Consequently, for a given set of $n-1$ players including i, the probability that S $(\ni i)$ is formed is expressed as $\frac{s!(n-1-s)!}{(n-1)!}$. This is the second part of (γ).

Third, we explain the meaning of $\frac{1}{s}$ appeared in the third part of (γ). Recall that we sum up the excesses of coalitions S $(\ni i)$ using (γ). In the process of this summation, we divide the excess of each S by s. In other words, we sum up not the excesses, but the average excesses of coalitions S $(\ni i)$.

Consequently, we sum up the average excesses of coalitions S $(\ni i)$ using the weight $(n-1) \times \frac{s!(n-1-s)!}{(n-1)!}$. Since this weight represents the probability of the coalition S $(\ni i)$ being formed in the context of the excess, $e_{i,f_n^*}(\sigma(v))$ coincides with the expected average excess of i at $\sigma(v)$. We try to equalize $e_{i,f_n^*}(\sigma(v))$

[12]Recall that all the players are lined up in some order (and all the orders have the same probability). See also the explanation of the Shapley value in **Section 2** for comparison. There are some similarities between the Shapley value and the equalized excesses of players in our result. On the one hand, Sh_i takes the form of a weighted sum of the marginal contributions of the player i, and the employed weight represents the probability that each marginal contribution of i arises. On the other hand, the excess of player i at the Shapley value under **EEf** takes the form of a weighted sum of the excesses of the coalitions (except $\mathcal{N}$) to which i belongs. And the fundamental part of the weight f_n^* represents the probability that the excess of each coalition S $(\ni i)$ arises in the context of the excess, as will be mentioned in the main text.

[13]Recall that f_n^* is not the weight of the marginal contribution of i to S, but the weight of the excess of S $(\ni i)$. It is only necessary that i belongs to S in the process of the summation. Therefore, there is no need to keep i as the s-th player here.

among all the players, and obtain the conclusion that the Shapley value is the only value equalizing these excesses of players.

To sum up, (γ) represents $\left(\frac{1}{s}\right) \times$ the probability of the coalition $S\ (\ni i)$ being formed in the context of the excess.

From above, the meaning of the theorem becomes clear. It turns out that the Shapley value is the solution which makes the expected average excess of each player equivalent[14]. This is also clarified by the restatement of the theorem as presented below. We use the following condition which directly requires values (producing preimputations) to make the expected average excess of each player equivalent:

Equal Excess with f_n^* (EEf*):
For each $v \in V$,

$$e_{i,f_n^*}(\sigma(v)) = e_{j,f_n^*}(\sigma(v)),$$

for any $i, j \in \mathcal{N}$.

Then, **Theorem 1** is equivalent to the following theorem:

Theorem 1*. The Shapley value is the only value satisfying **EEf***.

In sum, the Shapley value is not only shown to be 'the solution which improves the condition of the least advantaged players', but it is also found to be 'the solution which makes each player's expected average excess equivalent'.

f_n^* can also be graphically explained. See the two Pascal's triangles in **Figure 1**. The right-hand side is the usual Pascal's triangle, and the left-hand side is a graphical expression of f_n^*. On the right-hand side, the number at (row m, column l) $(m, l = 0, 1, 2, \dots)$ is represented by ${}_mC_l$. On the left-hand side, we have $m = n - 2$ $(n \geq 2)$ and $l = s - 1$ $(s \geq 1)$ so that the index number of each row relates to the total number of players in each game and the index number of each column relates to the number of players in each coalition (except $\mathcal{N}$). Hence, the scalar 1 $(= {}_0C_0)$ at the top of the triangle relates to the 1-person coalitions on the 2-person games, and generally, the number at (row $n-2$, column $s-1$) is represented by ${}_{n-2}C_{s-1}$. In other words, the numbers on the left-hand side are equal to the denominators of f_n^*.

As an example, see the scalar 2 $(= {}_2C_1)$ at (row 2, column 1) corresponding to the case $n = 4$ and $s = 2$. This number implies that, on the 4-person games, the excesses of the 2-person coalitions are discounted by 2, i.e., $f_4^*(2) = \frac{a}{2},\ a \in \mathbb{R}_{++}$.

In sum, the excesses of coalitions S (except $\mathcal{N}$) in n-person games are discounted by ${}_{n-2}C_{s-1}$. This gross discount rate is illustrated on the left-hand side of **Figure 1**.

[14] It is quite natural to divide the excess of each coalition by its number of players before attributing it to each player. For this reason, one can omit the word 'average' when interpreting (γ), as we did at the end of **Section 1**.

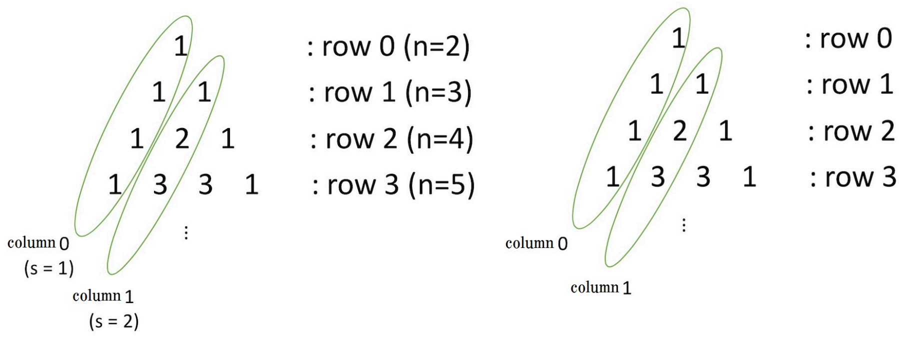

Figure 1 Graphical expression of the denominators of f_n^*

4 Discussion

1. In establishing the above theorem, we can relax the definition of the weight function f_n as follows:

$$f_n : \{1, 2, \dots, n-1\} \to \mathbb{R}_+ \quad \text{and} \quad f_n(\{1, 2, \dots, n-1\}) \neq \{0\}^{15}.$$

In other words, f_n need not assign positive real numbers to all the numbers of players belonging to the coalitions (except $\mathcal{N}$), but should assign positive real numbers to at least one number in $\{1, 2, \dots, n-1\}$, i.e., at least one type (in the cardinal sense) of the coalition S where $s \in \{1, 2, \dots, n-1\}$.

2. The characterization in this paper is still valid if we consider specific games, such as supper-additive games, simple games, or convex games. In addition, we need not restrict the definition of the excess to the set of the preimputations, though in this case some minor changes are required, i.e., the domain of f_n must be enlarged to $\{1, 2, \dots, n\}$ and the statement '$f_n(n) = b$, where $b \in \mathbb{R}_+$.' must be added to the second half of the main theorem. Thus, the equivalence of (i) and (ii) of the main theorem remains essentially the same even if we consider the entire set of the payoff vectors instead of the set of the preimputations since the term related to $\mathcal{N}$ is certainly zero in the excess at the Shapley value.

3. As noted in **Section 1**, Ruiz et al. (1998) investigated the relationship between the Shapley value and the excess by introducing the least square values. Their formation of the *least square values* LS^m is as follows: For each $v \in V$, $LS^m(v) = (LS_i^m(v))_{i \in \mathcal{N}}$, where for each $i \in \mathcal{N}$,

$$LS_i^m(v) = \frac{v(\mathcal{N})}{n} + \sum_{S: i \in S \neq N} \rho_s \frac{v(S)}{s} - \sum_{S: i \notin S} \rho_s \frac{v(S)}{n-s}.$$

Here, $\rho_s = \frac{s(n-s)}{n} \frac{m(s)}{\alpha}$, $\alpha = \sum_{l=1}^{n-1} m(l)_{n-2}C_{l-1}$, and m represents functions satisfying $m : \{1, 2, \dots, n-1\} \to \mathbb{R}_+$ and $m(\{1, 2, \dots, n-1\}) \neq \{0\}$. In this

[15] Here, $f_n(\{1, 2, \dots, n-1\})$ represents the range of f_n.

subsection, we mention the relationship between the least square values and our values presented in the right-hand side of the equation (3) in **Appendix**. Since the right-hand side of (3) represents the values satisfying **EEf**, we call it the *EEf values* hereafter.

First, if we compare the least square values with the EEf values, then we find that they are the same. And both $m(s)$ above and $f_n(s)$ in this paper are functions which weight the coalitions.

Second, on the one hand, their characterization of the Shapley value via the least square values shows that the Shapley value has the function $m^{Sh}(s) = \frac{1}{n-1}({}_{n-2}C_{s-1})^{-1}$ and $\alpha = 1$ (Ruiz et al., 1998, Remark 14). On the other hand, our characterization of the Shapley value via the EEf values shows that the Shapley value has the function $f_n(s) = f_n^*(s) = \frac{a}{{}_{n-2}C_{s-1}}$ where $a \in \mathbb{R}_{++}$. Of course, these two represent the same weight. However, since Ruiz et al. (1998) made a normalization such as $\alpha = 1$, they cannot give a meaningful interpretation of the weight $m^{Sh}(s) = \frac{1}{n-1}({}_{n-2}C_{s-1})^{-1}$, compared to our result.

We summarize the above: All the values belonging to the least square values can be characterized using the equal excess. In this sense, our approach of the equal excess has a certain generality and gives definite meanings to the least square values. Moreover, any proofs can be derived relatively easy, as **Appendix** illustrates for the case of the Shapley value. Since we have concentrated in this paper on characterizing the Shapley value as a solution which embodies the difference principle, we leave the unified characterization of the least square values as a whole to future works.

Without going into the unified characterization, our results in this paper shed light on the significance of the Shapley value in the excesses world. In a sense, the Shapley value deals with the excess more adequately and clearly than the prenucleolus. The differences (or similarities) between the two solutions can be summarized as follows:

- The prenucleolus attains the greatest benefit of the least advantaged coalitions.
- The Shapley value attains the greatest benefit of the least advantaged players.

Besides, we have the following result:

- The Shapley value is the solution which makes each player's expected average excess equivalent.

We would like to emphasize that these results are given by introducing the excess of player i.

Finally, we conclude as follows: Ruiz et al. (1998) documented that the "[pre]nucleolus treats all coalitions as equally important while the Shapley value does not", 'but,' we add, 'treats all players as equally important'.

Appendix: Proof of Theorem 1

We introduce some preliminaries. First, for any $k \in \{1, 2, \cdots, n-1\}$ and $i \in \mathcal{N}$, we define the following three subsets of $P(\mathcal{N})$:

$$\begin{aligned}
P(\mathcal{N})_k &:= \{S \in P(\mathcal{N}) \setminus \mathcal{N} \mid s = k\}.\\
P(\mathcal{N})_k^i &:= \{S \in P(\mathcal{N}) \setminus \mathcal{N} \mid i \in S,\ s = k\}.\\
P(\mathcal{N})_k^{-i} &:= \{S \in P(\mathcal{N}) \setminus \mathcal{N} \mid i \notin S,\ s = k\}.
\end{aligned}$$

$P(\mathcal{N})_k$ is the set of coalitions where the number of players is k. According to whether player i is included or not, $P(\mathcal{N})_k$ is divided into two subsets: $P(\mathcal{N})_k^i$ and $P(\mathcal{N})_k^{-i}$. Here, none of these three subsets of $P(\mathcal{N})$ includes the grand coalition $\mathcal{N}$.

Second, for any $k \in \{1, 2, \cdots, n-1\}$, we list some results of calculations without proofs (Hereafter, ${}_{n-2}C_{-1} \equiv 0$.):

Calculation 1 (C1):

$$-{}_{n-1}C_{k-1} - (-{}_{n-2}C_{k-2}) = -{}_{n-2}C_{k-1}.$$

Calculation 2 (C2):

$$\frac{k}{n}{}_{n-1}C_{k-1} - {}_{n-2}C_{k-2} = \frac{1}{n}{}_{n-2}C_{k-1}.$$

Calculation 3 (C3):

$$(k+1){}_nC_{k+1} = (n-k){}_nC_k.$$

Calculation 4 (C4):

$$\frac{(n-k)k}{n}{}_nC_k = (n-1){}_{n-2}C_{k-1}.$$

Then, we can present the proof of the main theorem as follows:

(Proof of Theorem 1.)
Note that the summation of the excesses of all the players with respect to f_n at a preimputation x is represented as follows:

$$\begin{aligned}
\sum_{i \in \mathcal{N}} e_{i,f_n}(x) &= \sum_{i \in \mathcal{N}} \sum_{k=1}^{n-1} \sum_{S \in P(\mathcal{N})_k^i} f_n(k)\ e(S,x)\\
&= \sum_{k=1}^{n-1} \sum_{S \in P(\mathcal{N})_k} f_n(k)k\ e(S,x).
\end{aligned}$$

Then, "The Shapley value satisfies **EEf**" equals the following: For all $i \in \mathcal{N}$,

$$
\begin{aligned}
& \frac{1}{n}\sum_{j\in\mathcal{N}} e_{j,f_n}(Sh(v)) \;= e_{i,f_n}(Sh(v)) \\
\Leftrightarrow\quad & \frac{1}{n}\sum_{k=1}^{n-1}\sum_{S\in P(\mathcal{N})_k} f_n(k)k\ e(S,Sh(v)) \\
& \qquad = \sum_{k=1}^{n-1}\sum_{S\in P(\mathcal{N})_k^i} f_n(k)\ e(S,Sh(v)) \\
\Leftrightarrow\quad & \frac{1}{n}\sum_{k=1}^{n-1}\sum_{S\in P(\mathcal{N})_k} f_n(k)k\ (v(S)-\sum_{j\in S} Sh_j(v)) \\
& \qquad = \sum_{k=1}^{n-1}\sum_{S\in P(\mathcal{N})_k^i} f_n(k)\ (v(S)-\sum_{j\in S} Sh_j(v)) \cdots\cdots\cdots (1) \\
\Leftrightarrow\quad & \frac{1}{n}\sum_{k=1}^{n-1} f_n(k)k\ (\sum_{S\in P(\mathcal{N})_k} v(S) - {}_{n-1}C_{k-1}\sum_{j\in\mathcal{N}} Sh_j(v)) \\
& \qquad = \sum_{k=1}^{n-1} f_n(k)(\sum_{S\in P(\mathcal{N})_k^i} v(S) - {}_{n-1}C_{k-1}\ Sh_i(v) - {}_{n-2}C_{k-2}\sum_{j\in\mathcal{N}\setminus\{i\}} Sh_j(v)) \cdot\cdot (2) \\
& \qquad = \sum_{k=1}^{n-1} f_n(k)(\sum_{S\in P(\mathcal{N})_k^i} v(S) - {}_{n-2}C_{k-1}\ Sh_i(v) - {}_{n-2}C_{k-2}\sum_{j\in\mathcal{N}} Sh_j(v)) \quad (\because \mathbf{C1}) \\
\Leftrightarrow\quad & \frac{1}{n}\sum_{k=1}^{n-1} f_n(k)k\ (\sum_{S\in P(\mathcal{N})_k} v(S) - {}_{n-1}C_{k-1}\ v(\mathcal{N})) \\
& \qquad = \sum_{k=1}^{n-1} f_n(k)(\sum_{S\in P(\mathcal{N})_k^i} v(S) - {}_{n-2}C_{k-1}\ Sh_i(v) - {}_{n-2}C_{k-2}\ v(\mathcal{N})) \qquad (\because (\beta)) \\
\Leftrightarrow\quad & \sum_{k=1}^{n-1} f_n(k)_{n-2}C_{k-1}\ Sh_i(v) \\
& \qquad = (\frac{1}{n}\sum_{k=1}^{n-1} f_n(k)k_{n-1}C_{k-1}\ v(\mathcal{N}) - \sum_{k=1}^{n-1} f_n(k)_{n-2}C_{k-2}\ v(\mathcal{N})) \\
& \qquad\quad + \sum_{k=1}^{n-1} f_n(k)\sum_{S\in P(\mathcal{N})_k^i} v(S) - \frac{1}{n}\sum_{k=1}^{n-1} f_n(k)k\sum_{S\in P(\mathcal{N})_k} v(S) \\
& \qquad = v(\mathcal{N})\sum_{k=1}^{n-1} f_n(k)\ (\frac{k}{n}{}_{n-1}C_{k-1} - {}_{n-2}C_{k-2}\ ) \\
& \qquad\quad + \sum_{k=1}^{n-1} f_n(k)\ (1-\frac{k}{n})\sum_{S\in P(\mathcal{N})_k^i} v(S) - \frac{1}{n}\sum_{k=1}^{n-1} f_n(k)k\sum_{S\in P(\mathcal{N})_k^{-i}} v(S)
\end{aligned}
$$

$$= \frac{v(\mathcal{N})}{n} \sum_{k=1}^{n-1} f_n(k)_{n-2}C_{k-1}$$

$$+ \sum_{k=1}^{n-1} f_n(k) \frac{n-k}{n} \sum_{S \in P(\mathcal{N})_k^i} v(S) - \sum_{k=1}^{n-1} f_n(k) \frac{k}{n} \sum_{S \in P(\mathcal{N})_k^{-i}} v(S) \qquad (\because \textbf{C2})$$

$$\Leftrightarrow \quad Sh_i(v) = \frac{v(\mathcal{N})}{n} + \sum_{k=1}^{n-1} \left(\frac{f_n(k)\frac{n-k}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} \right) \sum_{S \in P(\mathcal{N})_k^i} v(S)$$

$$- \sum_{k=1}^{n-1} \left(\frac{f_n(k)\frac{k}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} \right) \sum_{S \in P(\mathcal{N})_k^{-i}} v(S). \quad (\because f_n \text{ is a positive real function.}) \cdots (3)$$

See **Note 1.** at the end of this **Appendix** to make sure that the two equations (1) and (2) are equivalent. In terms of the equation (3), note that the representation (3) of the Shapley value certainly satisfies the equation (β).

Since the two representations (α) and (3) of the Shapley value must be equal, we have for all $i \in \mathcal{N}$,

$$\sum_{k=1}^{n-1} \left(\frac{f_n(k)\frac{n-k}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} - \frac{1}{k_n C_k} \right) \sum_{S \in P(\mathcal{N})_k^i} v(S)$$

$$- \sum_{k=1}^{n-1} \left(\frac{f_n(k)\frac{k}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} - \frac{1}{(k+1)_n C_{k+1}} \right) \sum_{S \in P(\mathcal{N})_k^{-i}} v(S) = 0$$

$$\Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{1}{k} \left(\frac{f_n(k)\frac{(n-k)k}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} - \frac{1}{{}_nC_k} \right) \sum_{S \in P(\mathcal{N})_k^i} v(S)$$

$$- \sum_{k=1}^{n-1} \frac{1}{n-k} \left(\frac{f_n(k)\frac{(n-k)k}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} - \frac{1}{{}_nC_k} \right) \sum_{S \in P(\mathcal{N})_k^{-i}} v(S) = 0 \qquad (\because \textbf{C3})$$

$$\Leftrightarrow \quad \frac{f_n(s)\frac{(n-s)s}{n}}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1}} - \frac{1}{{}_nC_s} = 0 \ , \qquad s = 1, 2, \ldots, n-1$$

$$\Leftrightarrow \quad \frac{1}{\sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1} \ {}_nC_s} \left(f_n(s) \frac{(n-s)s}{n} {}_nC_s - \sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1} \right) = 0 \ , \quad s = 1, 2, \ldots, n-1$$

$$\Leftrightarrow \quad f_n(s) \frac{(n-s)s}{n} {}_nC_s = \sum_{l=1}^{n-1} f_n(l)_{n-2}C_{l-1} \ , \qquad s = 1, 2, \ldots, n-1$$

$$\Leftrightarrow \quad (n-1)_{n-2}C_{s-1} f_n(s) = \sum_{l=1}^{n-1} {}_{n-2}C_{l-1} f_n(l) \ , \qquad s = 1, 2, \ldots, n-1 \ (\because \textbf{C4})$$

$$\Leftrightarrow \quad \begin{cases} (n-1)_{n-2}C_0 f_n(1) &= {}_{n-2}C_0 f_n(1) + {}_{n-2}C_1 f_n(2) + \cdots + {}_{n-2}C_{n-2} f_n(n-1) \\ (n-1)_{n-2}C_1 f_n(2) &= {}_{n-2}C_0 f_n(1) + {}_{n-2}C_1 f_n(2) + \cdots + {}_{n-2}C_{n-2} f_n(n-1) \\ & \vdots \\ (n-1)_{n-2}C_{n-2} f_n(n-1) &= {}_{n-2}C_0 f_n(1) + {}_{n-2}C_1 f_n(2) + \cdots + {}_{n-2}C_{n-2} f_n(n-1) \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 0 = (2-n)_{n-2}C_0 f_n(1) + {}_{n-2}C_1 f_n(2) + \cdots + {}_{n-2}C_{n-2} f_n(n-1) \\ 0 = {}_{n-2}C_0 f_n(1) + (2-n)_{n-2}C_1 f_n(2) + \cdots + {}_{n-2}C_{n-2} f_n(n-1) \\ \quad \vdots \\ 0 = {}_{n-2}C_0 f_n(1) + {}_{n-2}C_1 f_n(2) + \cdots + (2-n)_{n-2}C_{n-2} f_n(n-1) \end{cases}$$

$$\Leftrightarrow \quad \mathbf{0} = \mathbf{B} \begin{pmatrix} f_n(1) \\ f_n(2) \\ \vdots \\ f_n(n-1) \end{pmatrix}, \quad \cdots\cdots\cdots\cdots\cdots\cdots (4)$$

where

$$\mathbf{B} = \begin{pmatrix} (2-n)_{n-2}C_0 & {}_{n-2}C_1 & \cdots & {}_{n-2}C_{n-2} \\ {}_{n-2}C_0 & (2-n)_{n-2}C_1 & \cdots & {}_{n-2}C_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{n-2}C_0 & {}_{n-2}C_1 & \cdots & (2-n)_{n-2}C_{n-2} \end{pmatrix}.$$

Note that arbitrary $n-2$ column vectors of the $(n-1)$-by-$(n-1)$ matrix $\mathbf{B}$ are linearly independent. Also, note that

$$\begin{pmatrix} f_n(1) \\ f_n(2) \\ \vdots \\ f_n(n-1) \end{pmatrix} = \begin{pmatrix} \frac{1}{{}_{n-2}C_0} \\ \frac{1}{{}_{n-2}C_1} \\ \vdots \\ \frac{1}{{}_{n-2}C_{n-2}} \end{pmatrix} \quad \cdots\cdots\cdots\cdots (5)$$

is a non-zero solution of the equation (4), which implies that all the $n-1$ column vectors of the matrix $\mathbf{B}$ are linearly dependent. Thus, we obtain that $rank(\mathbf{B})$ (the maximum number of linearly independent column vectors of $\mathbf{B}$) is equal to $n-2$. This implies that the number of basic solutions of (4) is equal to 1 $(= (n-1)-(n-2))$, and the vector (5) is itself the basic solution of (4). Hence, for arbitrary positive d,

$$\begin{pmatrix} f_n(1) \\ f_n(2) \\ \vdots \\ f_n(n-1) \end{pmatrix} = d \begin{pmatrix} \frac{1}{{}_{n-2}C_0} \\ \frac{1}{{}_{n-2}C_1} \\ \vdots \\ \frac{1}{{}_{n-2}C_{n-2}} \end{pmatrix}$$

represents the set of all solutions of the equation (4) (This "positive d" is derived from the fact that every f_n is a positive real function.). Therefore, (4) $\Leftrightarrow f_n(s) = \frac{a}{{}_{n-2}C_{s-1}}$ for some $a \in \mathbb{R}_{++}$ and for any $s \in \{1, 2, \ldots, n-1\}$. ■

Finally, we make a note about the two equations (1) and (2) in the proof:

(Note 1.) To verify that (1) and (2) in the proof of **Theorem 1** are equivalent, let $\mathcal{N} = \{A, B, \ldots, N\}$ and, without loss of generality, let $i = A$. Then,

$$
\begin{aligned}
(1) \Leftrightarrow \frac{1}{n} [\quad & f_n(1) \cdot 1 \cdot \{(v(\{A\}) - Sh_A(v)) + (v(\{B\}) - Sh_B(v)) + \cdots + (v(\{N\}) - Sh_N(v))\} \\
& + f_n(2) \cdot 2 \cdot \{(v(\{A, B\}) - Sh_A(v) - Sh_B(v)) + \cdots + (v(\{M, N\}) - Sh_M(v) - Sh_N(v))\} \\
& + f_n(3) \cdot 3 \cdot \{(v(\{A, B, C\}) - Sh_A(v) - Sh_B(v) - Sh_C(v)) + \cdots \\
& \qquad + (v(\{L, M, N\}) - Sh_L(v) - Sh_M(v) - Sh_N(v))\} \\
& + \cdots \\
& + f_n(n-1) \cdot (n-1) \cdot \{\cdots\} \ ] \\
= \ & f_n(1) \cdot \{(v(\{A\}) - Sh_A(v))\} \\
& + f_n(2) \cdot \{(v(\{A, B\}) - Sh_A(v) - Sh_B(v)) + \cdots + (v(\{A, N\}) - Sh_A(v) - \sigma_N(v))\} \\
& + f_n(3) \cdot \{(v(\{A, B, C\}) - Sh_A(v) - Sh_B(v) - Sh_C(v)) + \cdots \\
& \qquad + (v(\{A, M, N\}) - Sh_A(v) - Sh_M(v) - Sh_N(v))\} \\
& + \cdots \\
& + f_n(n-1) \cdot \{(v(\mathcal{N} \setminus \{N\}) - \sum_{j \neq N} Sh_j(v)) + \cdots + (v(\mathcal{N} \setminus \{B\}) - \sum_{j \neq B} Sh_j(v))\} \\
\Leftrightarrow \frac{1}{n} [\quad & f_n(1) \cdot 1 \cdot \{ \sum_{S \in P(\mathcal{N})_1} v(S) - \ {}_{n-1}C_0 \sum_{j \in \mathcal{N}} Sh_j(v)\} \\
& + f_n(2) \cdot 2 \cdot \{ \sum_{S \in P(\mathcal{N})_2} v(S) - \ {}_{n-1}C_1 \sum_{j \in \mathcal{N}} Sh_j(v)\} \\
& + f_n(3) \cdot 3 \cdot \{ \sum_{S \in P(\mathcal{N})_3} v(S) - \ {}_{n-1}C_2 \sum_{j \in \mathcal{N}} Sh_j(v)\} \\
& + \cdots \\
& + f_n(n-1) \cdot (n-1) \cdot \{ \sum_{S \in P(\mathcal{N})_{n-1}} v(S) - \ {}_{n-1}C_{n-2} \sum_{j \in \mathcal{N}} Sh_j(v)\} \ ] \\
= \ & f_n(1) \cdot \{ \sum_{S \in P(\mathcal{N})_1^A} v(S) - \ {}_{n-1}C_0 \ Sh_A(v)\} \\
& + f_n(2) \cdot \{ \sum_{S \in P(\mathcal{N})_2^A} v(S) - \ {}_{n-1}C_1 \ Sh_A(v) - \ {}_{n-2}C_0 \sum_{j \in \mathcal{N} \setminus \{A\}} Sh_j(v)\} \\
& + f_n(3) \cdot \{ \sum_{S \in P(\mathcal{N})_3^A} v(S) - \ {}_{n-1}C_2 \ Sh_A(v) - \ {}_{n-2}C_1 \sum_{j \in \mathcal{N} \setminus \{A\}} Sh_j(v)\} \\
& + \cdots \\
& + f_n(n-1) \cdot \{ \sum_{S \in P(\mathcal{N})_{n-1}^A} v(S) - \ {}_{n-1}C_{n-2} \ Sh_A(v) - \ {}_{n-2}C_{n-3} \sum_{j \in \mathcal{N} \setminus \{A\}} Sh_j(v)\} \\
\Leftrightarrow \ & (2).
\end{aligned}
$$

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