WINPEC Working Paper Series No.E2115
September 2021

Deliberation and Voting: A Matter of Truth or A Matter of Taste

Masayuki Odora

Waseda INstitute of Political EConomy
Waseda University
Tokyo, Japan

Deliberation and Voting: A Matter of Truth or A Matter of Taste[§]

Masayuki Odora[*]

September 5, 2021

Abstract This study considers strategic communication before voting. Voters have partially conflicting interests rather than common interests. That is, voters cannot tell whether a collective decision is a matter of truth, such as guilty or innocent, or a matter of taste, such as left or right. A set of imperfectly informed voters communicates before casting their votes. From a statistical perspective, truth-telling by all voters in deliberation, coupled with majority rule, may lead to desirable outcomes asymptotically as the population of voters increases. Thus, from a statistical perspective, increasing the population of voters is desirable. This study, however, shows that truthful communication is not incentive-compatible with equilibrium behavior when the size of the electorate is sufficiently large. In particular, truthful communication by all voters is inconsistent with equilibrium for any voting rule and any degree of conflict when the population of voters becomes arbitrarily large. On the other hand, truthful communication might be an equilibrium for a small population of voters. Under these circumstances, voting rules matter. This study shows that majority rule most promotes truthful communication before voting.

Keywords Information aggregation ・ Common value elections ・ Private value elections
Deliberation ・ Voting rule ・ Conflicting interests

JEL Classification C72 D71 D72

1. Introduction

Many social decisions for which are not obvious which alternative is better or that entail fundamental differences of opinions are made by voting. Typically, voters communicate before they officially cast votes. Examples include legislatures, referendums, faculty meetings,

---

[§] I am grateful for helpful comments and discussions with Yasushi Asako, Kohei Kawamura, Yoichi Hizen, Tadashi Sekiguchi, and participants of annual meetings in Japanese Economic Associations.
[*] Graduate School of Economics, Waseda University, 1-6-1, Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan. Email: odora_masa@fuji.waseda.jp

monetary policy committees, jury voting, and company boards.

There are two functions of elections: (1)preference aggregation and (2)information aggregation. Following Condorcet ([1785] 1994), the information aggregation property of elections is extensively studied (Austen-smith and Banks, 1996; Feddersen and Pesendorfer, 1997; Wit, 1998). Most of the subsequent studies that examined the information aggregation property considered binary elections with voters having common interests. In particular, there is a *correct* alternative for *all* voters, but they only have partial information about which alternative is *correct*. These studies examined whether and when the elections aggregate dispersed information and thereby identify correct alternatives. A prominent example of this common value election is jury voting. In such a jury metaphor, conflicts of opinion come from voters' information rather than the fundamental differences of ideologies.[1]

However, in real-world elections, the fundamental differences of ideologies matter in addition to the information. This study considers the collective decisions in which both the ideologies and the information matter. In particular, voters in this study have *partially conflicting interests* rather than common interests: It might be possible for voters to reach a unanimous agreement once the uncertainty is fully resolved. Still, they might exhibit a disagreement even when the uncertainty is fully resolved. In other words, collective decisions include matters of truth, such as jury voting, and matters of taste, such as ideologically driven elections. Furthermore, voters cannot tell which issues are at stake. This paper is the first to develop a voting model with voters having partially conflicting interests.

As an example of collective decision-making with voters having partially conflicting interests, consider a polity conducting a referendum to decide whether to implement a reform. Unfortunately, the electorate does not know who the beneficiaries of the reform might be due to the complex nature of the policy. The reform might benefit all voters, or it might harm them and benefit only politicians. In the former case, the beneficiaries are the entire electorate, and thus all voters would unanimously prefer to accept the reform. On the other hand, politicians are the only beneficiaries in the latter case, and hence all voters would unanimously prefer to reject. Therefore, voters have common interests in these cases, which implies that the referendum is a matter of truth. The reform, however, might only benefit some voters who engage in certain industries, while being harmful to others. Therefore, beneficiaries of the reform would prefer to accept it, while the others would not. Thus, in this case the referendum is a matter of taste. Voters, however, only have imperfect information regarding the beneficiaries of the reform due to the complex nature of the policy. This implies that voters cannot tell whether the election is a matter of truth or taste, and they cannot tell which outcome is better even when the election is a matter of truth.

As another example, consider elections with two candidates who differ in their ideologies/policy preferences and qualities (or valence). Suppose that voters evaluate candidates based on both their policy positions and their qualities. Consequently, voters decide where to vote based on their own ideologies if the difference in the candidates' qualities is small. Thus, the election is a matter of taste if the quality difference is negligible. However, when the quality difference is so large that the ideological difference is negligible, *all* voters would prefer the candidate with higher quality. Thus, the election is a matter of truth when the quality difference is significant. However, the qualities of the candidates are unknown, while their policy positions are common knowledge. Voters, therefore, cannot tell whether the election is a matter of truth or taste. In these situations where many voters participate, deliberation can be interpreted as opinion polls conducted before the voters officially cast votes.

When the quality of collective decision-making depends on the underlying state (e.g.,

[1] Other sources of conflict come from the difference in attitude toward the two types of errors: Convicting an innocent and acquitting the guilty (Coughlan, 2000; Gerardi, 2000).

the beneficiaries of the reform or the qualities of the candidates) and voters have partially conflicting interests, collective decisions under uncertainty should identify both correct alternatives and majority-preferred alternatives depending on the underlying state. In particular, society should choose the correct alternative if the election is a matter of truth. In contrast, society should choose the majority-preferred alternatives if the election is a matter of taste.

Intuitively, one might conjecture that sharing information truthfully via deliberation allows voters to tell whether the election is a matter of truth or taste, and once they can do so, voting should be able to identify both correct alternatives and majority-preferred alternatives, depending on the outcome of the deliberation. This is because each voter has imperfect information about the underlying state, and aggregating dispersed information reduces the uncertainty of the state. Thus, sharing information truthfully in deliberation and voting sincerely after such deliberation should yield a better outcome.

This conjecture depends on the assumption that voters *non-strategically* share their information in deliberation and vote based on their belief about alternatives generated by the outcomes of deliberation. This study scrutinizes this conjecture by examining whether and when strategic voters with partially conflicting interests voluntarily share their information regarding the state. The main result (Theorem 1) is that the assumption of the conjecture is inconsistent with equilibrium behavior when the number of voters is sufficiently large. Hence large elections cannot identify *correct* alternatives and majority-preferred alternatives simultaneously via truthful communication by strategic voters.

This study examines the model of a binary election with voters having partially conflicting interests. Voters collectively decide an alternative A or B under a given threshold voting rule, as in standard models of jury voting. A novel feature this model is the partially conflicting interests. In particular, the payoff from the collective decisions depends on the unknown state $\theta \in [0,1]$, where the state space $[0,1]$ is partitioned into three subsets Θ_A, Θ_B, and Θ_N. In state $\theta \in \Theta_N$, voters disagree which alternative is better; that is, the election is a matter of taste. Voters who prefer A and B in state $\theta \in \Theta_N$ are denoted as type $\mathcal{A}$ and type $\mathcal{B}$, respectively. In state $\theta \in \Theta_A \cup \Theta_B$, however, even the different types of voters have common interests. That is, the election is a matter of truth. In particular, all voters, including both types of voters, prefer alternative A in state $\theta \in \Theta_A$, while they prefer B in $\theta \in \Theta_B$. The types are private information. Although the state θ is unknown to anybody, voters receive private signals, a or b, correlated with the true state. They are allowed to communicate before they cast votes, where communication is cheap talk.

The results of this study are three-fold. The first result concerns the statistical nature of truth-telling by all voters in deliberation. In particular, this study shows that, for any voting rule, truth-telling by all voters in deliberation asymptotically identifies correct alternatives. Moreover, under majority rule, truth-telling in deliberation asymptotically identifies majority-preferred alternatives as well. Therefore, the above conjecture, saying that truth-telling with sincere voting yields better outcomes even when voters have partially conflicting interests, is true from a statistical perspective. These are the direct results of the Law of Large Numbers. Thus, from the statistical perspective, increasing the number of voters helps voters make desirable decisions.

However, the second result shows that a sufficiently large population is precisely the circumstance where the truth-telling is inconsistent with equilibrium behavior. In particular, truthful communication regarding the state by all voters never constitutes an equilibrium for *any* voting rule and any degree of conflict if the size of the electorate is sufficiently large (Theorem 1). Therefore, from a game-theoretic perspective, large elections cannot identify *correct* alternatives and alternatives preferred by the majority simultaneously *via truthful communication* when voters have partially conflicting interests.

To understand intuitively why truthful communication regarding the state is an equilibrium for small n, but not for large n, suppose that all voters truthfully reveal their private signals. Since voters have conflicting interests with positive probability, strategic voters may have incentives to misreport their signals to induce others to vote for their ex-ante biased alternatives. For example, a type $\mathcal{A}$ voter who observed a b-signal has an incentive to misreport that she has observed a-signal. However, the effect of such lying is mixed. Lying, and inducing others to vote for one's ex-ante biased alternative, may lead to undesirable outcomes. This is because lying may induce like-minded voters to vote for the wrong direction. To see this, consider a voter whose signal conflicts with her type, say type $\mathcal{A}$ voter i who observe a b-signal. Lying (i.e., reporting that she has observed a-signal) induces the other voters to vote for A only when lying makes them switch to preferring alternative A. But circumstances where this switch occur depends on types. For example, lying induce like-minded voters (i.e., type $\mathcal{A}$ voters) to vote for A when her lie makes them switch to *preferring* alternative A. However, this is exactly when voter i, whose private signal is b, perceives that B is a slightly better alternative. Thus, her lie induces an outcome that she perceives undesirable, A, by manipulating like-minded voters to vote for A. This is the only driving force for truth-telling. However, the shift in the posterior belief regarding the state, conditional on her perceiving B as slightly better, becomes smaller as the size of the electorate increases. Consequently, the only driving force for truth-telling vanishes as the population increases.

The third result concerns the effect of voting rules on information aggregation in deliberation for a small population of voters. In particular, this study shows that voting rule affects incentives for truthful communication in a small population of voters. In particular, the majority rule most promotes truthful communication among all threshold voting rules.

Before moving on, it is worth emphasizing that the main result, that for any voting rule and any degree of conflict, truth-telling in deliberation with updated sincere voting cannot be an equilibrium in a sufficiently large population, does not mean that for any voting rule, there does not exist an equilibrium in which both correct and majority-preferred outcomes are identified in the limit. There might exist an equilibrium under *some* voting rule in which both the correct and the majority-preferred outcomes are identified in the limit. This study does not examine the existence nor non-existence of such an equilibrium and voting rule. Although this is a fundamental question in a democratic society with partially conflicting interests, it is left for future work.

1.1. Related Literature

This study relates to a literature that investigates the effect of communication on voting (Coughlan, 2000; Austen-smith and Feddersen, 2006; Meirowitz, 2007; Schulte, 2010). Coughlan (2000) and Austen-smith and Feddersen (2006) studied the effect of the voting rule on voters' incentives to share their information truthfully in deliberation.

Coughlan (2000) studied the model of common value election in which voters differ in their evaluations towards two types of errors; convicting an innocent and acquitting the guilty. He showed that when preference is common knowledge, truthful communication constitutes an equilibrium for any voting rule, including unanimity if voters' preferences are sufficiently homogenous. Consequently, voting rules are irrelevant for truthful communication as long as preferences are common knowledge and sufficiently homogeneous. Austen-smith and Feddersen (2006) examined the effect of voting rules when preferences are private information. They showed, in contrast to Coughlan, that the voting rule matters. In particular, they showed that the unanimity rule is worst at aggregating information in deliberative committees.[2]

The most crucial difference between this study and Coughlan is the voters' preferences. The voters in his model never disagree once the uncertainty is fully resolved, while voters in my model might exhibit a disagreement even in the absence of uncertainty. Such potential disagreement enables me to considers both ideologies and information, while his study considers only information. Moreover, this study shows that allowing the potential disagreement dramatically changes the result of Coughlan. In particular, Theorem 1 shows that truthful communication in deliberation never constitutes an equilibrium for any voting rule and *any degree of conflict* if the size of the electorate is sufficiently large.

As in Austen-smith and Feddersen, the voters' preferences in my model are private information. Moreover, preferences satisfy the axioms of *Consensus* and *Monotonicity* defined in their paper. The former requires that even different types of voters might be able to reach an agreement given the full revelation of the signal profile. The latter requires that different types of voters respond to the information in the same direction. That is, voters' expected payoff over a binary alternative given the full revelation of the signal responds to the change in the information in the same direction even when their types differ.[3] However, there are three crucial differences between this study and Austen-smith and Feddersen. First, in addition to voting rule, this study examines the effect of the size of the electorate on truthful communication in deliberation, while they only focus on the voting rule. Second, their model cannot consider the probability of desirable decisions in large elections, while my model can. Third, this study shows that the majority rule is the best at promoting truthful communication in small elections, while they only showed that the unanimity rule is the worst.

Meirowitz (2007) and Schulte (2010) also studied information aggregation in pre-vote communication, provided that preferences are heterogeneous and private information. The difference between this study and Meirowitz is two-fold. The first difference is that Meirowitz focused on majority rule, while this study considers all threshold voting rules. The second difference is the heterogeneity of preferences. In Meirowitz, voters always disagree in the absence of uncertainty, while voters in this study may or may not agree depending on the underlying state once the state is fully known. Meirowitz (2007) showed that when voters have such diametrically opposed preferences, a bias that voters belong to the majority side makes truthful communication an equilibrium for any size of the electorate. Schulte (2010) considered verifiable communication under majority rule, while this study considers non-verifiable cheap talk communication under various voting rules.[4]

Like this study, Morgan and Stocken (2008) considered information aggregation in polls of ideologically diverse constituents. They showed that truth-telling by all agents regarding the state is inconsistent with equilibrium behavior when the size of the polled constituents is sufficiently large. The difference between their study and mine is the decision making process; in their model, a policy is determined by a single policymaker who uses a poll to collect information from the citizens, while the policy in this study is determined by subsequent voting. Therefore, in their model, any single citizen's report always affects the final policy regardless of the signal profile of other citizens. Consequently, lying is always beneficial for a sufficiently large sample. On the other hand, in my model, voters' reports only affect the policy when they are pivotal, and whether or not lying is beneficial depends on pivotal events.

---

[2] They show the following results: Truthful communication is not an equilibrium under unanimity rule if preferences are heterogeneous. Moreover, if truthful communication becomes an equilibrium under unanimity, it is indeed equilibrium for any other voting rule.

[3] See Meirowitz (2007) for models that violate the Monotonicity Axiom.

[4] Mathis (2011) also considered verifiable communication and studied the effect of the voting rule on voters' incentives to share information.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 introduces the variables used throughout this paper. Section 4 provides the results of this paper. Section 5 concludes. The Appendices contain all proofs omitted in the text.

2. The Model

A set of voters, $\{1,2,..,n\}$ ($n \geq 3$ odd), makes a collective decision $o \in \{A, B\}$. Voters simultaneously cast vote A or B, and the outcome is determined by a threshold voting rule $\bar{k}$, which represents a threshold for alternative A to win. That is, alternative A wins if at least $\bar{k} \in \{1..,n\}$ voters vote for it, otherwise B is chosen. For example, $\bar{k} = n$ requires a unanimous vote for alternative A being chosen, while a single vote is enough for B being chosen. As another example, $\bar{k} = (n+1)/2$ requires a majority of votes for both alternatives being chosen. Abstention is not allowed, and voting entails no cost.[5] Before they cast votes, voters are allowed to communicate, which will be explained later.

2.1. Preference

2.1.1. States and Payoff

The payoff from collective decision $o \in \{A, B\}$ depends on the unknown state θ, which is uniformly distributed on $\Theta \coloneqq [0,1]$. To describe the conflict, suppose that the set of states, Θ, is partitioned into three subsets $\Theta_A \coloneqq [0, \pi)$, $\Theta_N \coloneqq (\pi, 1-\pi)$, and $\Theta_B \coloneqq (1-\pi, 1]$, as shown in Figure 1, where $\pi \in (0, 1/2)$.

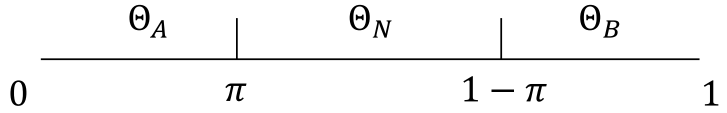

Figure 1

Voters have partially conflicting interests as follows. In state $\theta \in \Theta_A \cup \Theta_B$, voters have common interests: They unanimously prefer alternative A in state $\theta \in \Theta_A$, while they unanimously prefer B in state $\theta \in \Theta_B$. In state $\theta \in \Theta_N$, they have conflicting interests: Voters who prefer alternative A and B in state $\theta \in \Theta_N$ are denoted type $\mathcal{A}$ and type $\mathcal{B}$, respectively.

Let $u_t(o, \theta)$ be type $t \in \{\mathcal{A}, \mathcal{B}\}$ voter's payoff from alternative $o \in \{A, B\}$ in state θ. In particular, I assume that voters get a payoff of 1 from their preferred alternatives and 0 otherwise.[6] Therefore, $u_t(o, \theta)$ becomes

- $u_t(A, \theta) = 1 > 0 = u_t(B, \theta)$ for all t if $\theta \in \Theta_A$.

---

[5] Burke and Taylor (2008) studied voters' incentive to truthfully share their preference in purely private value elections with costly voting.

[6] This zero-one specification of payoff allows me to represent the expected payoff in terms of the probability. However, this assumption is not essential to my main result.

- $u_t(B,\theta) = 1 > 0 = u_t(A,\theta)$ for all t if $\theta \in \Theta_B$.
- $u_{\mathcal{A}}(A,\theta) = 1 > 0 = u_{\mathcal{A}}(B,\theta)$ if $\theta \in \Theta_N$.
- $u_{\mathcal{B}}(B,\theta) = 1 > 0 = u_{\mathcal{B}}(A,\theta)$ if $\theta \in \Theta_N$.

The payoff of each type can be summarized by the following table:

Type $\mathcal{A}$'s Payoff

	$\theta \in \Theta_A$	$\theta \in \Theta_N$	$\theta \in \Theta_B$
Outcome $= A$	1	1	0
Outcome $= B$	0	0	1

Type $\mathcal{B}$'s Payoff

	$\theta \in \Theta_A$	$\theta \in \Theta_N$	$\theta \in \Theta_B$
Outcome $= A$	1	0	0
Outcome $= B$	0	1	1

Since I focus on individual differences in fundamental ideologies, I rule out the heterogeneity of intensity of two kinds of errors; convicting an innocent and acquitting the guilty (Coughlan, 2000; Gerardi, 2000).

Since the prior on the state is uniform on the unit interval, $\Pr(\theta \in \Theta_A) = \Pr(\theta \in \Theta_B) = \pi$, and $\Pr(\theta \in \Theta_N) = 1 - 2\pi$. Thus, $1 - 2\pi$ is the prior probability of voters having conflicting interests. Suppose that types are private information and are identically and independently drawn from $\Pr(t_i = \mathcal{A}) = z \in (0,1)$. Types and the state are independent.

2.1.2. Example of Two-Candidate Elections

An example of this kind of preference structure includes a two-candidate election with candidates who differ in their ideological positions and their quality or valence. If a difference in the quality is so large that the ideological difference is negligible, then voters have common interests; they prefer candidate with higher quality. On the other hand, if the quality difference is negligible, then ideology matters, which implies that voters have conflicting interests. Since voters know little about candidates' qualities, they cannot distinguish whether the election is a matter of truth or a matter of taste.

Suppose that two candidates, L and R, who differ in their ideological positions and their qualities, compete for a single office. Voters care about ideologies and qualities of candidates. For ideology, there are two types of voters, L and R. Type L and type R voters prefer the ideological positions of candidate L and R, respectively. While the ideological positions of candidates are fully known, their qualities are unknown to voters. Let θ_L and θ_R denote the quality of each candidate and assume that $\theta := \theta_R - \theta_L$ is uniformly distributed on $[-1,1]$, which corresponds to the state in my model. Voters' ideological payoff from their preferred candidate is assumed to be $1/2$. Then, voters' payoff from each candidate can be summarized in Table 1.

Table 1: Voters' Payoff from the Candidates

	Candidate L	Candidate R
Type L voter	$1/2+\theta_L$	θ_R
Type R voter	θ_L	$1/2+\theta_R$

All voters prefer candidate L if $\theta < -1/2$, while they prefer candidate R if $1/2 < \theta$. On the other hand, if $-1/2 < \theta < 1/2$, then type L and type R prefer candidate L and R, respectively.

2.2. Information

Although the underlying state is unknown to anybody, each voter receives the private signal $s_i \in \{a, b\}$ that is correlated with the true state. The following conditional probabilities determine the distribution of the signals:

$$\Pr(s_i = b|\theta) = \theta$$

$$\Pr(s_i = a|\theta) = 1-\theta.$$

This implies that a-signal and b-signal are more likely to be observed in state $\theta \in \Theta_A$ and $\theta \in \Theta_B$, respectively, while signal distribution is relatively balanced in state $\theta \in \Theta_N$. The signals are drawn conditionally independent of state θ.

Note that number of a-signals, conditional on state θ, follows a binomial distribution with parameter $(n, 1-\theta)$, where $1-\theta$ is success probability, and n is the total number of independent trials. Conditional on observing k a-signals, the posterior distribution of θ follows a beta distribution with parameters $n-k+1$ and $k+1$. Hereafter, I denote the beta distribution with parameters α and β by Beta(α, β), and its cumulative distribution function by $G(\cdot\,|\alpha, \beta)$. That is,

$$G(x|\alpha,\beta) = \int_0^x \frac{1}{\mathcal{B}(\alpha,\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} d\theta,$$

where $\mathcal{B}(\alpha, \beta)$ is a constant chosen so that $G(1|\alpha, \beta) = 1$.

2.3. Deliberation

Before they cast votes, voters communicate as non-binding straw polls (Coughlan, 2000; Austen-smith and Feddersen, 2006; Meirowitz, 2007).[7] That is, each voter simultaneously sends a non-verifiable binary message $m_i \in \mathcal{M} \coloneqq \{a, b\}$, where sending messages entails no costs. A message profile $m = (m_1, .., m_n) \in \mathcal{M}^n$ is publicly observed after voters send messages.

[7] Mathis (2011) and Schulte (2010) considered communication with verifiable messages.

2.4. Timing and Strategies

Timing: The timing of the whole game is as follows:

1. Nature determines the state θ and the types of voters. Conditional on state θ, each voter receives the private signal $s_i \in \{a, b\}$.

2. *Communication stage*
   Each voter, privately knowing their own type and signal, sends a cheap talk message $m_i \in \mathcal{M} = \{a, b\}$ simultaneously.

3. *Voting stage*
   The message profile $m \in \mathcal{M}^n$ is publicly observed. Voters simultaneously cast their votes. The outcome is then determined by the threshold voting rule $\bar{k}$.

Strategy: Formally, a voter's strategy (μ, γ) consists of a communication strategy μ and a voting strategy γ, where

$$\mu: \{a, b\} \times \{\mathcal{A}, \mathcal{B}\} \to \mathcal{M},$$

$$\gamma: \{a, b\} \times \{\mathcal{A}, \mathcal{B}\} \times \mathcal{M}^n \to \{A, B\}.$$

The communication strategy μ specifies a message $\mu(s_i, t_i) \in \mathcal{M}$ that type t_i voter with signal s_i sends. The voting strategy γ specifies an alternative $\gamma(s_i, t_i, m) \in \{A, B\}$ that voter (t_i, s_i) who observes message profile m casts. The equilibrium concept is a Perfect Bayesian Equilibrium.

3. Type-Optimal Aggregation Rule

This section defines a type-optimal aggregation rule k_t for each type $t \in \{\mathcal{A}, \mathcal{B}\}$, which will be useful throughout the rest of the paper. This is defined as the total number of a-signals out of n required to persuade type t voters to prefer alternative A. In other words, a type t voter prefers alternative A if there are at least k_t a-signals out of n, and otherwise she prefers B.

I first define $k_{\mathcal{A}}$. Any type $\mathcal{A}$ voters prefer alternative A to B conditional on knowing signal profile $s = (s_1, .., s_n)$ containing k a-signals if and only if

$$E_\theta[u_{\mathcal{A}}(A, \theta) | k; n] \geq E_\theta[u_{\mathcal{A}}(B, \theta) | k; n],$$

which is equivalent to

$$\frac{1}{2} \leq \Pr(\theta \in \Theta_A \cup \Theta_N | k; n) = G(1 - \pi | n - k + 1, k + 1). \tag{1}$$

In other words, type $\mathcal{A}$ voters prefer alternative A to B if and only if the posterior probability that the state lies in $\Theta_A \cup \Theta_N$ is greater than or equal to $1/2$. Since the right-hand side of eq. (1) is strictly increasing in k, there uniquely exists $k_{\mathcal{A}} \in \{0,1,..,n\}$ such that

$$\Pr(\theta \in \Theta_A \cup \Theta_N | k_{\mathcal{A}} - 1; n) < \frac{1}{2} \leq \Pr(\theta \in \Theta_A \cup \Theta_N | k_{\mathcal{A}}; n),$$

which is equivalent to

$$G(1-\pi | n - k_{\mathcal{A}} + 2, k_{\mathcal{A}}) < \frac{1}{2} \leq G(1-\pi | n - k_{\mathcal{A}} + 1, k_{\mathcal{A}} + 1).$$

Thus, conditional on knowing signal profile s, type $\mathcal{A}$ voters prefer alternative A if s contains at least $k_{\mathcal{A}}$ a-signals; otherwise, they prefer B. I refer to this threshold $k_{\mathcal{A}}$ as a *type $\mathcal{A}$ optimal aggregation rule*.[8] If $k_{\mathcal{A}} = 0$, then type $\mathcal{A}$ voter is said to be purely partisan: They prefer alternative A regardless of signal profile. Voting for A is weakly dominant for them.

For type $\mathcal{B}$ voters, $k_{\mathcal{B}}$ is also defined such that they prefer alternative A if and only if there are at least $k_{\mathcal{B}}$ a-signals out of n. Conditional on k a-signals, any type $\mathcal{B}$ voter prefers alternative B if and only if

$$\frac{1}{2} \geq \Pr(\theta \in \Theta_A | k; n) = G(\pi | n - k + 1, k + 1).$$

In other words, type $\mathcal{B}$ voters prefer alternative B if and only if the posterior probability that the state lies in Θ_A is less than or equal to $1/2$. Since the right-hand side of this inequality is strictly increasing in k, there uniquely exists $k_{\mathcal{B}} \in \{1,..,n,n+1\}$ such that

$$\Pr(\theta \in \Theta_A | k_{\mathcal{B}}; n) > \frac{1}{2} \geq \Pr(\theta \in \Theta_A | k_{\mathcal{B}} - 1; n),$$

which is equivalent to

$$G(\pi | n - k_{\mathcal{B}} + 1, k_{\mathcal{B}} + 1) > \frac{1}{2} \geq G(\pi | n - k_{\mathcal{B}} + 2, k_{\mathcal{B}}).$$

Thus, conditional on knowing signal profile s, type $\mathcal{B}$ voters prefer alternative A if s contains at least $k_{\mathcal{B}}$ a-signals, otherwise they prefer B. This threshold $k_{\mathcal{B}}$ is denoted as a *type $\mathcal{B}$ optimal aggregation rule*. If $k_{\mathcal{B}} = n + 1$, then type $\mathcal{B}$ voter is said to be purely partisan: Voting for B is weakly dominant for them.

Depending on π and n, optimal aggregation rules of each type may coincide, that is, $k_{\mathcal{A}} = k_{\mathcal{B}}$. In such cases, voters are said to be *perfectly homogenous*.

Example 1: Suppose $n = 3$ and $\pi = 3/10$. Then, $k_{\mathcal{A}} = 1$ and $k_{\mathcal{B}} = 3$. ■

In example 1, a single a-signal is sufficient to persuade type $\mathcal{A}$ to prefer alternative A, while

---

[8] Although the type-optimal aggregation rule depends on n and π in general, I denote $k_{\mathcal{A}}$ instead of $k_{\mathcal{A}}(n, \pi)$ for notational convenience.

three a-signals are required to persuade type $\mathcal{B}$ voters to prefer A.

LEMMA 1: *For any π, $k_{\mathcal{A}}/n$ and $k_{\mathcal{B}}/n$ has limit*

$$\lim_{n\to\infty}\frac{k_{\mathcal{A}}}{n}=\pi$$

$$\lim_{n\to\infty}\frac{k_{\mathcal{B}}}{n}=1-\pi.$$

PROOF: See Appendix A. ■

For example, $\lim(k_{\mathcal{A}}/n)=\pi$ means that $k_{\mathcal{A}}$ (i.e., the number of a-signals required to persuade type $\mathcal{A}$ voters to prefer A) divided by n converges to $\Pr(\theta\in\Theta_A)=\pi$ (i.e., ex-ante probability that state lies in $\Theta_{\mathcal{A}}$). The symmetric argument holds for $\lim(k_{\mathcal{B}}/n)=1-\pi$.

Lemma 1 guarantees that voters are neither purely partisan nor perfectly homogenous with sufficiently large n. That is, $1\le k_{\mathcal{A}}<k_{\mathcal{B}}\le n$ for sufficiently large n.

4. Deliberation

4.1. Large Elections

The purpose of this section is to investigate the conjecture that sharing information truthfully in deliberation with sincere voting can overcome potential disagreement and hence yield desirable outcomes. To this end, this section begins by defining a strategy, that I call *fully revealing sincere voting.* Formally, the definition of *fully revealing sincere voting* strategy is as follows.[9]

DEFINITION: Fully Revealing Sincere Voting

1. *At the communication stage, type t voter i with private signal s_i truthfully reveals her signal. That is, $\mu_i(s_i,t)=s_i$ for all $t\in\{\mathcal{A},\mathcal{B}\}$.*

2. *At the voting stage, type t voter i, who observed a (truthful) message profile, votes for A if and only if $k_t\le k$, where k is the number of the a-signals in the observed message profile.*[10]

Fully revealing sincere voting requires voters to truthfully reveal their private signals

[9] Schulte (2010) referred to this voting behavior given full revelation as "Bayesian sincere voting." The concept is the same as mine.
[10] Under the non-unanimous rule, all voters voting for A regardless of the messages can constitute part of an equilibrium. However, such voting behavior is weakly dominated by sincere voting based on updated beliefs. I rule out such weaky dominated voting behavior.

and then vote based on their own optimal aggregation rule k_t. This voting behavior can be interpreted as *updated sincere voting*. This is because voters, given the full revelation of the signal profile, update their posterior belief about the state and then vote as if they alone can determine the outcome.

If voters are not strategic agents, and hence they non-strategically follow the fully revealing sincere voting profile, then the intuitive conjecture that deliberation should overcome potential disagreement and thereby yields desirable outcomes is indeed true. The following proposition states this formally.

PROPOSITION 1: *Suppose that all voters follow fully revealing sincere voting. Then for any* π, *any* z, *and any voting rule,*

$$\lim_{n\to\infty} \Pr(A \text{ is chosen}|\theta) = 1 \ if\ \theta \in \Theta_A$$

$$\lim_{n\to\infty} \Pr(B \text{ is chosen}|\theta) = 1 \ if\ \theta \in \Theta_B$$

Moreover, under majority rule,

$$\lim_{n\to\infty} \Pr(\text{majority} - \text{preferred alternatives are chosen}|\theta) = 1 \ if\ \theta \in \Theta_N.$$

PROOF: See Appendix A. ■

Proposition 1 states that, given the fully revealing sincere voting profile, deliberative society identifies, with arbitrary precision, alternatives that would be chosen if voters fully knew the underlying state in the large elections. This asymptotic property follows from the Weak Law of Large Numbers and the fact that $k_{\mathcal{A}}/n$ and $k_{\mathcal{B}}/n$ converge to π and $1-\pi$, respectively.

However, when voters are strategic agents, they might have incentives to misreport so as to manipulate others to vote for their ex-ante biased alternative. In particular, it turns out that voters face two different pivotal events: One in which lying induces the different types of voters to vote in favor of their ex-ante biased alternatives, and the other in which lying induces like-minded voters to vote for a *wrong* direction. Thus, lying may or may not be beneficial. It turns out that the overall benefit from lying dominates the loss from it as the size of the electorate becomes arbitrarily large. Consequently, truthful communication by all voters is consistent with equilibrium behavior only with a small population of voters but not with a large one. The following Theorem 1 states this formally.

THEOREM 1: *For any* π, z, *and any voting rule* $\overline{k}$, *there exists an* n' *such that* $n' \leq n$ *implies that fully revealing sincere voting strategy is never an equilibrium.*

PROOF: See Appendix B. ■

Theorem 1 establishes that information aggregation via truth-telling by all voters is impossible when strategic voters differ in their information and their ideologies. In large elections, this is true for any voting rule, $\overline{k}$, and any degree of conflict, $\pi \in (0, 1/2)$. Therefore, Theorem 1 establishes that the assumption of the conjecture is violated with a large population of voters, although increasing the number of voters is necessary for asymptotic efficiency.

The effect of voting rules on incentives to share information via cheap talk communication (Austen-smith and Feddersen, 2006) or verifiable disclosure (Mathis, 2011) has been studied.[11] These studies showed that the voting rule matters for incentives to truthful communication when the population is small. However, they did not consider its effect asymptotically when the population becomes arbitrarily large. Theorem 1, in contrast, establishes that the voting rule is irrelevant asymptotically in large elections; truth-telling regarding the state by all voters is inconsistent with equilibrium for any voting rule when the size of the electorate is sufficiently large.

To gain some intuitions on Theorem 1, let us consider a case where voter i's *report* changes the voting outcome, i.e., is pivotal.[12] To simplify the discussion below, consider type $\mathcal{A}$ voter i with a conflicting signal $s_i = b$. Thus, lying, in this case, means that reporting that she observed the a-signal. A total number of a-signals out of $n-1$ voters is denoted as $k_{-i} \coloneqq \#\{j \neq i: s_j = a\}$.

I argue that voter i faces two pivotal events, one in which lying is harmful and the other in which it is beneficial. The voting behavior of other voters is solely determined by whether or not the number of a-signals out of n reaches their optimal aggregation rule k_t. Thus, voter i's report can manipulate the voting behavior of type t voters when the other $n-1$ voters observe $k_t - 1$ a-signals. This implies that she faces two pivotal events, one in which she can manipulate like-minded voters (i.e., type $\mathcal{A}$ voters) and the other in which she can manipulate the opposed type of voters (i.e., type $\mathcal{B}$ voters).

Lying is harmful when i's report manipulates like-minded voters, that is, when $n-1$ voters have observed $k_{\mathcal{A}} - 1$ a-signals. This is exactly when voter i, whose signal is $s_i = b$, perceives that B is a slightly better alternative. Lying, however, induces like-minded voters to vote for the alternative A, which is the *wrong* direction. Thus, lying is harmful in this case. On the other hand, lying is beneficial when her report can manipulate the different type of voters (i.e., type $\mathcal{B}$). To see why it is beneficial, note that, conditional on the event of other voters having observed $k_{\mathcal{B}} - 1$ a-signals, voter i perceives that A is better, and lying induces them to vote for A.

Theorem 1 shows that, for any voting rule, the benefit from lying dominates the loss from it as the size of the electorate becomes arbitrarily large. To gain some intuition on the effect of the size, consider again a type $\mathcal{A}$ voter i with $s_i = b$. It turns out that the loss from lying vanishes, while the benefit from lying converges to 1 as n grows large. First, consider the pivotal event in which lying is harmful, that is, $k_{-i} = k_{\mathcal{A}} - 1$. This is exactly when i's perception of the better alternative switches. Voter i, therefore, perceives that alternative B is *slightly* better than A. However, as the size of the electorate grows large, a shift in the posterior belief becomes smaller.[13] This implies that the loss from lying vanishes as n becomes arbitrarily large. Next, consider the pivotal event in which lying is beneficial, that is, $k_{-i} = k_{\mathcal{B}} - 1$. In this case, she perceives that the alternative A is *significantly* better than B. This is because $k_{\mathcal{A}}$ is smaller than $k_{\mathcal{B}}$. Voter i becomes more confident that B is better as n grows large since the difference between $k_{\mathcal{B}}$ and $k_{\mathcal{A}}$ is increasing in n. Thus, the benefit from lying converges to 1. The above discussion implies that truth-telling is not incentive compatible with sufficiently large n.

---

[11] Jackson and Tan (2003) considered the case where informed experts communicate with the voting body. Similar to this paper, they considered a committee with heterogeneous preferences in the sense that both private and common components matter.

[12] Given the full revelation at the communication stage, pivotality in the voting state does not provide any additional information about the state. Thus, I focus on pivotality at the communication stage.

[13] This is similar to the intuition behind Proposition 11 in Morgan and Stocken (2008).

4.2. Small Elections

Although truthful communication by all voters is inconsistent with equilibrium for any voting rule when the population of voters is sufficiently large, it might constitute an equilibrium with a small population of voters. The following example 2 highlights that when the number of voters is small, incentives for truthful communication depend on the voting rule.

Example 2: Suppose $\pi = 1/4$ and $z = 1/2$ so that voters are neither purely partisan nor perfectly homogenous for any n. Under majority rule, fully revealing sincere voting constitutes an equilibrium if $n \leq 9$. However, under unanimity rule, fully revealing sincere voting is not an equilibrium for any n. ■

In example 2, the circumstances in which voters are either purely partisan or perfectly homogenous are excluded. Before moving on to the analysis of small elections, it is worth emphasizing these circumstances. One can easily verify that voters might be either purely partisan or perfectly homogenous when n is small. If so, fully revealing sincere voting constitutes an equilibrium under any voting rule. For purely homogenous voters, this is because they always have unanimous agreement after full revelation of any private signals (Coughlan, 2000).

However, Lemma 1 excludes these circumstances; for sufficiently large n, voters are neither purely partisan nor perfectly homogenous. Table 2 summarizes the relation between voting rule, size of the electorate, and truthful communication.

Table 2: Voting Rule, Size of the Electorate, and Truthful Communication

	Small n	**Large n**
Perfectly homogeneous or purely partisan	Fully revealing sincere voting (FRSV) constitutes an equilibrium for any voting rule	Never occur
Neither perfectly homogeneous nor purely partisan	Voting rule matters	FRSV never constitutes an equilibrium for any voting rule (Theorem 1)

Voting rule matters for truthful communication when n is small, and voters are neither purely partisan nor purely homogenous. Proposition 2 states that, in such non-trivial cases, the unanimity rule is inferior at aggregating information, via truthful communication, for any size of the electorate.

PROPOSITION 2: *Suppose that the voting rule is unanimity rule and voters are neither purely partisan nor perfectly homogeneous. Then fully revealing sincere voting does not constitute an equilibrium for any* n.

PROOF: Suppose that unanimous agreement is required to implement alternative A. Consider type $\mathcal{A}$ voter i with $s_i = b$, and suppose that other voters follow fully revealing sincere voting. The proof is done if I show that lying yields a higher expected payoff than truth-telling.

Consider the pivotal event in which her lie induces the other type $\mathcal{A}$ voters to vote for A, which, she perceives, is the wrong direction. In this pivotal event, she can prevent alternative A from winning by exercising her veto. Thus, the pivotal event in which lying is harmful never occurs. On the other hand, the pivotal event in which lying is beneficial occurs with positive probability. To see why it is positive, it should be noted that such a pivotal event is formally given by $k_{-i} = k_{\mathcal{A}} - 1$ & $\#\{j \neq i: t_j = \mathcal{B}\} \geq 1$, where $\#\{j \neq i: t_j = \mathcal{B}\}$ is the number of types $\mathcal{B}$ voters other than i. Thus, voter i deviates from fully revealing sincere voting. An analogous argument can be applied to the voting rule in which unanimous agreement is required to implement B. ■

The intuition behind Proposition 2 is that voters can make the loss from lying zero by exercising vetoes under unanimity rule. Consider type $\mathcal{A}$ voter i with a conflicting signal $s_i = b$. By lying ($m_i' = a$), she induces like-minded voters to vote for A, which, she perceives, is the wrong direction. However, she can implement B by exercising veto even if she tells a lie. Thus, she tells a lie without worrying about the possibility that her lie induces a *wrong* alternative to win.

Austen-smith and Feddersen (2006) showed that truthful revelation under unanimity rule is never an equilibrium when voters have heterogenous preferences. Consistent with this observation, I show that truthful communication regarding the state is never an equilibrium under unanimity rule when voters have partially conflicting interest under non-trivial situations (i.e., $1 \leq k_{\mathcal{A}} < k_{\mathcal{B}} \leq n$).

Proposition 2 states that, in such non-trivial cases (i.e., voters are neither purely partisan nor perfectly homogenous), truth-telling by all voters is never an equilibrium under unanimity for any size of the electorate. To further understand how voting rules affect incentives to share private information truthfully, suppose $z = 1/2$ so that the distribution of types is symmetric. With this simplification, truth-telling by all voters is the most likely to become equilibrium under majority rule. The upper bound of the population for which truth-telling by all voters is an equilibrium, which depends on voting rule, is maximized under majority rule.

PROPOSITION 3: *Suppose $z = 1/2$ and voters are neither purely partisan nor perfectly homogenous. Let $\overline{k}$ be a super-majority voting rule. If fully revealing sincere voting constitutes an equilibrium under $\overline{k}$ and n, then it is also an equilibrium under majority rule and n.*

PROOF: See Appendix C. ■

Proposition 3 states that, in small elections with partially conflicting voters, such as a committee of experts, company boards, or faculty meetings, majority rule is the most likely to succeed at aggregating information if there is no ex-ante majority or minority (i.e., $z = 1/2$). The intuitive reason is that if voting requires a supermajority of votes to implement an alternative, say B, then type $\mathcal{B}$ voters may be reluctant to share their private information. On the other hand, the majority rule is the only voting rule that treats alternatives symmetrically. It requires a majority of approval for both alternatives to win.

5. Conclusions

This study considers situations outside the jury metaphor: an electorate with partially conflicting interests rather than common interests. Voters may or may not have common interests depending on the underlying state. In other words, the *truth*, such as guilty or innocent, and *taste*, such as left or right, can be an issue, but voters cannot tell which issues are at stake.

In this situations, this study investigates whether and when deliberation aggregates information dispersed among individuals, thereby identifying both correct alternatives and alternatives preferred by the majority simultaneously in large elections.

Intuitively, one can conjecture that aggregating information helps reduce uncertainty, and voting sincerely based on the deliberative outcome can overcome potential disagreement and yield desirable outcomes. Indeed, this conjecture is statistically valid, and increasing the number of voters asymptotically identifies desirable decisions with arbitrary precision. This conjecture, however, relies on the assumption that all voters behave non-strategically, that is, truth-telling in deliberation with sincere voting.

Theorem 1 shows that this assumption is false from a game-theoretic perspective when the number of voters is sufficiently large. In particular, Theorem 1 shows that truthful communication is inconsistent with equilibrium behavior for any voting rule and any degree of conflict when the number of voters becomes arbitrarily large. Therefore, this study shows that it is impossible for large elections to simultaneously identify both correct alternatives and alternatives preferred by the majority via truthful communication when strategic voters have partially conflicting interests.

This study focuses on the existence of a fully revealing sincere voting equilibrium (Coughlan, 2000; Austen-smith and Feddersen, 2006; Schulte, 2010). Since Austen-smith and Banks (1996) pointed out that the sincere voting assumption in the Condorcet jury voting model is far from innocuous, subsequent work has investigated the information aggregation property of mixed strategy equilibria (Feddersen and Pesendorfer, 1997; Wit, 1998; Gerardi, 2000). Morgan and Stocken (2006) studied the information aggregation properties of the asymmetric pure strategy equilibrium. An exciting question unanswered in this study is whether and when allowing mixed strategies or asymmetric pure strategy profiles would simultaneously identify correct alternatives and majority-preferred alternatives in equilibrium in the limit under some voting rule. These questions are left for future research.

Appendix A: Proof of Lemma 1 and Proposition 1

Preliminaries

Let $M(\alpha,\beta)$ denote a median of Beta(α,β). That is, $M(\alpha,\beta)$ is defined implicitly by $G(M(\alpha,\beta)|\alpha,\beta)=1/2$. I first provide a useful lemma about the beta distribution and its median.

LEMMA A1 (Payton et al., 1989)

The median of Beta(α,β) is bounded by its mean and the mode. More specifically,

(1) If $1<\alpha<\beta$ *then*

$$\frac{\alpha-1}{\alpha+\beta-2}\leq M(\alpha,\beta)\leq\frac{\alpha}{\alpha+\beta}.$$

(2) If $1<\beta<\alpha$ *then*

$$\frac{\alpha - 1}{\alpha + \beta - 2} \geq M(\alpha, \beta) \geq \frac{\alpha}{\alpha + \beta}.$$

The following lemma is useful.

LEMMA A2: $k_{\mathcal{B}} = n + 1 - k_{\mathcal{A}}$.

PROOF OF LEMMA A2:
It follows from the symmetry between type $\mathcal{A}$ and type $\mathcal{B}$ voters concerning the informational environment. ■

Proof of Lemma 1

I first claim that for any n and π,

$$n\pi - 1 < k_{\mathcal{A}} < n\pi + 1. \tag{A.1}$$

Eq. (A.1) implies that $k_{\mathcal{A}}/n$ converges to π as $n \to \infty$. Moreover, the identity $n + 1 - k_{\mathcal{A}} = k_{\mathcal{B}}$ implies that eq. (A.1) is equivalent to

$$1 - \pi < \frac{k_{\mathcal{B}}}{n} < 1 - \pi + \frac{2}{n},$$

which implies that $(k_{\mathcal{B}}/n)_n$ converges to $1 - \pi$ as $n \to \infty$. Thus, if I prove eq. (A.1), then proof of Lemma 1 is done.

Eq. (A.1) is obtained as follows. By definition, $k_{\mathcal{A}}$ satisfies $\Pr(\theta \in \Theta_A \cup \Theta_N | k_{\mathcal{A}} - 1; n) < 1/2$, which is equivalent to

$$G(1 - \pi | n - k_{\mathcal{A}} + 2, k_{\mathcal{A}}) < \frac{1}{2}.$$

This implies that $1 - \pi$ is less than $M(n - k_{\mathcal{A}} + 2, k_{\mathcal{A}})$, which is the median of $\mathrm{Beta}(n - k_{\mathcal{A}} + 2, k_{\mathcal{A}})$. Moreover, from Lemma A1, $M(n - k_{\mathcal{A}} + 2, k_{\mathcal{A}})$ is bounded above by its mode,[14] that is, $M(n - k_{\mathcal{A}} + 2, k_{\mathcal{A}}) \leq (n - k_{\mathcal{A}} + 1)/n$. Thus,

$$1 - \pi < \frac{n - k_{\mathcal{A}} + 1}{n}. \tag{A.2}$$

Similarly, by definition, $k_{\mathcal{A}}$ satisfies $1/2 \leq \Pr(\theta \in \Theta_A \cup \Theta_N | k_{\mathcal{A}}; n)$, which is equivalent to

$$\frac{1}{2} \leq G(1 - \pi | n - k_{\mathcal{A}} + 1, k_{\mathcal{A}} + 1).$$

[14] To see this, note that the first parameter of $\mathrm{Beta}(n - k_{\mathcal{A}} + 2, k_{\mathcal{A}})$ is greater than the second one.

This implies that $1-\pi$ is greater than or equal to $M(n-k_{\mathcal{A}}+1, k_{\mathcal{A}}+1)$, while $M(n-k_{\mathcal{A}}+1, k_{\mathcal{A}}+1)$ is bounded below by its mean:[15]

$$\frac{n-k_{\mathcal{A}}+1}{n+2} \leq M(n-k_{\mathcal{A}}+1, k_{\mathcal{A}}+1).$$

Thus,

$$\frac{n-k_{\mathcal{A}}+1}{n+2} \leq 1-\pi. \qquad (A.3)$$

Combining eq. (A.2) and (A.3), I obtain

$$\frac{n-k_{\mathcal{A}}+1}{n+2} \leq 1-\pi < \frac{n-k_{\mathcal{A}}+1}{n},$$

which is equivalent to

$$n\pi + 2\pi - 1 \leq k_{\mathcal{A}} < n\pi + 1.$$ [16]

This implies eq. (A.1) since $\pi \in (0, 1/2)$. ■

Proof of Proposition 1

I first prove the first part of Proposition 1. First, suppose $\theta \in \Theta_A = [0, \pi)$ so that $1-\pi < 1-\theta = \Pr(s_i = a|\theta)$. Let k be the number of a-signals among n voters conditional on θ. Then k/n is random variable with mean $1-\theta$. Note that the alternative A is chosen with probability 1 in the event $k/n > k_{\mathcal{B}}/n$ since voters unanimously agree on voting for A if $k > k_{\mathcal{B}}$. I will show that $\Pr(k/n > k_{\mathcal{B}}/n\,|\theta)$ converges to 1 as $n \to \infty$ by using (1) the Weak Law of Large Numbers and (2) the fact that $(k_{\mathcal{B}}/n)_n$ converges to $1-\pi$ as $n \to \infty$. Recall that $(k/n)_n$ is a sequence of random variables whose mean is $1-\theta$ for each n, while $(k_{\mathcal{B}}/n)_n$ is a sequence of real numbers.

Concerning the sequence $(k/n)_n$, the Weak Law of Large Numbers implies that, for small enough $\varepsilon > 0$, we have[17]

$$\lim_{n\to\infty} \Pr\left(\frac{k}{n} > 1-\pi+\varepsilon|\theta\right) = 1. \qquad (A.4)$$

Concerning the sequence $(k_{\mathcal{B}}/n)_n$, from Lemma 1, it has the limit

$$\lim_{n\to\infty} \frac{k_{\mathcal{B}}}{n} = 1-\pi. \qquad (A.5)$$

Take arbitrary small $\varepsilon > 0$. Then, from eq. (A.4), there exists an $n' \in \mathbb{N}$ such that

[15] To see this, note that the first parameter of $\mathrm{Beta}(n-k_{\mathcal{A}}+1, k_{\mathcal{A}}+1)$ is greater than the second one.
[16] Note that eq. (A.2) is equivalent to $k_{\mathcal{A}} < n\pi + 1$, and eq. (A.3) is equivalent to $n\pi + 2\pi - 1 \leq k_{\mathcal{A}}$.
[17] More specifically, if ε is so small that $1-\pi+\varepsilon < 1-\theta$.

$n' \leq n$ implies

$$1-\varepsilon < \Pr\left(\frac{k}{n} > 1-\pi+\varepsilon|\theta\right). \tag{A.6}$$

From eq. (A.5), there also exists an $n'' \in \mathbb{N}$, for this ε, such that $n'' \leq n$ implies

$$\frac{k_{\mathcal{B}}}{n} < 1-\pi+\varepsilon. \tag{A.7}$$

Thus, from eqs. (A.6) and (A.7), $\max\{n', n''\} \leq n$ implies that

$$1-\varepsilon < \Pr\left(\frac{k}{n} > 1-\pi+\varepsilon|\theta\right) \leq \Pr\left(\frac{k}{n} > \frac{k_{\mathcal{B}}}{n}|\theta\right).$$

Since ε is arbitrary, I have shown that $\Pr(k/n > k_{\mathcal{B}}/n\,|\theta)$ converges to 1 if $\theta \in \Theta_A$.

The proof for $\theta \in \Theta_B = (1-\pi, 1]$ is analogous: $\Pr(k/n < k_{\mathcal{A}}/n\,|\theta)$ converges to 1 if $\theta \in \Theta_B$ due to (1) the Weak Law of Large Numbers and the fact that (2) $(k_{\mathcal{A}}/n)_n$ converges to π.

Next, I prove the second part, and hence suppose that voting rule is majority. Suppose $\theta \in \Theta_N = (\pi, 1-\pi)$ so that $\pi < 1-\theta < 1-\pi$. First, note that the alternative (ex-ante) preferred by the majority always wins in the event $k_{\mathcal{A}}/n < k/n < k_{\mathcal{B}}/n$ since the voting rule is majority and voters vote according to their type if $k_{\mathcal{A}} < k < k_{\mathcal{B}}$. I will show that $\Pr(k_{\mathcal{A}}/n < k/n < k_{\mathcal{B}}/n\,|\theta)$ converges to 1 if $\theta \in \Theta_N$.

Concerning the sequence of random variables $(k/n)_n$, the Weak Law of Large Numbers implies that, for small enough ε,[18]

$$\lim_{n\to\infty} \Pr\left(\pi+\varepsilon < \frac{k}{n} < 1-\pi-\varepsilon|\theta\right) = 1. \tag{A.8}$$

Concerning the sequences $(k_{\mathcal{A}}/n)_n$ and $(k_{\mathcal{B}}/n)_n$, they have the limits π and $1-\pi$, respectively.

Take arbitrary small $\varepsilon > 0$. Then, from eq. (A.8), there exists an n' such that $n' \leq n$ implies

$$1-\varepsilon < \Pr\left(\pi+\varepsilon < \frac{k}{n} < 1-\pi-\varepsilon|\theta\right),$$

[18] More specifically, ε is so small that $1-\theta \in (\pi+\varepsilon, 1-\pi-\varepsilon)$.

$$\frac{k_{\mathcal{A}}}{n} < \pi + \varepsilon,$$

and

$$1 - \pi - \varepsilon < \frac{k_{\mathcal{B}}}{n}.$$

Thus, we have

$$1 - \varepsilon < \Pr\left(\pi + \varepsilon < \frac{k}{n} < 1 - \pi - \varepsilon | \theta\right) \leq \Pr\left(\frac{k_{\mathcal{A}}}{n} < \frac{k}{n} < \frac{k_{\mathcal{B}}}{n} | \theta\right).$$

Since ε is arbitrary, we see that

$$\lim_{n \to \infty} \Pr\left(\frac{k_{\mathcal{A}}}{n} < \frac{k}{n} < \frac{k_{\mathcal{B}}}{n} | \theta\right) = 1 \ \ if\ \theta \in \Theta_N. \qquad \blacksquare$$

Appendix B: Proof of Theorem 1

Preliminaries

To prove Theorem 1, I first provide Lemma B1 and Lemma B2, which are used to describe incentive compatibility conditions and their asymptotic behavior. Let $\Pr(\langle n-1, k_t - 1\rangle | s_i)$ denote the conditional probability that $n-1$ voters receive $k_t - 1$ a-signals, given that voter i receives $s_i \in \{a, b\}$, where $t \in \{\mathcal{A}, \mathcal{B}\}$.

LEMMA B1:

(a) $$\Pr(\langle n-1, k_{\mathcal{A}} - 1\rangle | s_i = b) = \frac{2k_{\mathcal{B}}}{n(n+1)}$$

(b) $$\Pr(\langle n-1, k_{\mathcal{B}} - 1\rangle | s_i = b) = \frac{2k_{\mathcal{A}}}{n(n+1)}$$

(c) $$\Pr(\langle n-1, k_{\mathcal{B}} - 1\rangle | s_i = a) = \frac{2k_{\mathcal{B}}}{n(n+1)}$$

(d) $$\Pr(\langle n-1, k_{\mathcal{A}}-1\rangle | s_i = a) = \frac{2k_{\mathcal{A}}}{n(n+1)}$$

PROOF OF LEMMA B1:

I only prove (a) because the analogous argument shows the other cases.

$$\Pr(\langle n-1, k_{\mathcal{A}}-1\rangle | s_i = b)$$
$$= \int_0^1 \binom{n-1}{k_{\mathcal{A}}-1}(1-\theta)^{k_{\mathcal{A}}-1}\theta^{n-k_{\mathcal{A}}} g(\theta|2{,}1)d\theta\,, \tag{B.1}$$

where $g(\theta|2{,}1)$ is a posterior density of θ conditional on $s_i = b$. Since $g(\theta|2{,}1)$ is a density of Beta$(2{,}1)$, it has the form

$$g(\theta|2{,}1) = \frac{1}{\mathcal{B}(2{,}1)}\theta.$$

Thus, eq. (B.1) becomes

$$\Pr(\langle n-1, k_{\mathcal{A}}-1\rangle | s_i = b) = \frac{\binom{n-1}{k_{\mathcal{A}}-1}}{\mathcal{B}(2{,}1)}\int_0^1 (1-\theta)^{k_{\mathcal{A}}-1}\theta^{n-k_{\mathcal{A}}+1}d\theta$$

$$= \binom{n-1}{k_{\mathcal{A}}-1}\frac{\mathcal{B}(n-k_{\mathcal{A}}+2, k_{\mathcal{A}})}{\mathcal{B}(2{,}1)}$$

$$= \frac{2(n-k_{\mathcal{A}}+1)}{n(n+1)} = \frac{2k_{\mathcal{B}}}{n(n+1)},$$

where the last equality follows from the identity $k_{\mathcal{B}} = n - k_{\mathcal{A}} + 1$. ■

LEMMA B2 (Krishnamoorthy, 2015)

Let $G(\cdot\,|\alpha,\beta)$ *be the cumulative distribution function of Beta*(α,β)*, where* $\alpha,\beta > 1$.

(1) $$G(x|\alpha,\beta) = 1 - G(1-x|\alpha,\beta)$$

(2) $$G(x|\alpha,\beta) > G(x|\alpha+1,\beta)$$

(3) $$G(x|\alpha,\beta) > G(x|\alpha,\beta-1)$$

(4) $$G(x|\alpha,\alpha) = G(x|\alpha,\alpha) = \frac{1}{2}G\left(1-4(x-1/2)^2|\alpha,\frac{1}{2}\right)\ (x \leq 1/2).$$

Proof of Theorem 1

From Proposition 2, it suffices to consider a non-unanimous voting rule. Since n is allowed to go to infinity, it is convenient to represent the non-unanimous voting rule $\bar{k}$ as the fraction of votes needed to implement alternative A, $r \in (0,1)$. The fraction r represents $\bar{k}$ if $\bar{k} = nr$, which means that alternative A is chosen if and only if $n_A/n \geq r$, where n_A is the number of votes for A. Since I focus on large n, it is innocuous to assume $0 < k_{\mathcal{A}} < k_{\mathcal{B}} \leq n$ because this is true for sufficiently large n due to Lemma 1.

This proof has two steps. The first step provides incentive compatibility conditions in the communication stage for voters whose signals contradict with their types.[19] Then, the second step proves that at least one of these conditions is violated with sufficiently large n.

STEP 1: Incentive Compatibility Conditions

First, consider type $\mathcal{A}$ voter i with $s_i = b$. For truth-telling to be incentive compatible, it is necessary that reporting $m_i = b$ (and then following updated sincere voting) yields higher payoff than the payoff from reporting $m_i' = a$. Given that voters other than i follow fully revealing sincere voting, the events in which her report is pivotal are two-fold. One pivotal event is that voters other than i receive $k_{\mathcal{A}} - 1$ a-signals, and the other type $\mathcal{A}$ voters alone can determine the outcome. The other pivotal event is that the other voters receive $k_{\mathcal{B}} - 1$ a-signals and type $\mathcal{B}$ voters can alone determine the outcome.

To see this, note that for i's report to be pivotal, it must be that i's report changes the voting behavior of other voters. This can happen only when the number of voters other than i who receive a-signal is either $k_{\mathcal{A}} - 1$ or $k_{\mathcal{B}} - 1$. For example, conditional on other voters having received $k_{\mathcal{A}} - 1$ a-signals, reporting $m_i' = b$ induces them to vote unanimously for B, while reporting $m_i = a$ induces them to vote according their type (i.e., type $\mathcal{A}$ and type $\mathcal{B}$ voters vote for A and $\mathcal{B}$, respectively). Thus, conditional on other voters having received $k_{\mathcal{A}} - 1$ a-siganls, i's report becomes pivotal (i.e., reporting $m_i' = b$ causes alternative B to win, while $m_i = a$ induces alternative A to win) if and only if other type $\mathcal{A}$ voters alone can induce alternative A to win, that is, $\#\{j \neq i : t_j = \mathcal{A}\} \geq \bar{k} = nr$. An analogous argument holds for the case where the other voters receive $k_{\mathcal{B}} - 1$ a-signals.

Consequently, i's incentive compatibility condition is given by the following.

$$\begin{aligned}
&\Pr(\langle n-1, k_{\mathcal{A}} - 1\rangle | s_i = b) \cdot \Pr\left[\#\{j \neq i : t_j = \mathcal{A}\} \geq nr\right] \\
&\qquad \cdot \left(E_\theta[u_{\mathcal{A}}(B,\theta) | k_{\mathcal{A}} - 1; n] - E_\theta[u_{\mathcal{A}}(A,\theta) | k_{\mathcal{A}} - 1; n]\right) \\
&+ \Pr(\langle n-1, k_{\mathcal{B}}(n) - 1\rangle | s_i = b) \cdot \Pr\left[\#\{j \neq i : t_j = \mathcal{B}\} \geq n + 1 - nr\right] \\
&\qquad \cdot \left(E_\theta[u_{\mathcal{A}}(B,\theta) | k_{\mathcal{B}} - 1; n] - E_\theta[u_{\mathcal{A}}(A,\theta) | k_{\mathcal{B}} - 1; n]\right) \geq 0.
\end{aligned}$$

Note that $\Pr(\langle n-1, k_{\mathcal{A}} - 1\rangle | s_i = b) \cdot \Pr\left[\#\{j \neq i : t_j = \mathcal{A}\} \geq nr\right]$ is the probability of the pivotal event in which lying is harmful. To see why it is harmful, note that voter i, privately knowing $s_i = b$, perceives that B is a better alternative conditional on $k_{\mathcal{A}} - 1$ a-signals among n voters, including i. On the other hand, $\Pr(\langle n-1, k_{\mathcal{B}} - 1\rangle | s_i = b) \cdot \Pr\left[\#\{j \neq i : t_j = \mathcal{B}\} \geq n + 1 - nr\right]$ is the probability of the pivotal event in which lying is beneficial.

[19] Note that, given full revelation of the entire signal profile, updated sincere voting behavior is weakly dominant for all voters and for all revealed message profiles. Thus, it suffices to consider the pivotality in communication stages rather pivotality in voting stages.

To see why it is beneficial, note that i perceives alternative A as better conditional on $k_{\mathcal{B}} - 1$ a-signals among n voters.

Using the distribution function of the beta distribution, $G(\cdot \mid \cdot,\cdot)$, I obtain

$$E_\theta[u_{\mathcal{A}}(B,\theta)|k_{\mathcal{A}}-1;n] - E_\theta[u_{\mathcal{A}}(A,\theta)|k_{\mathcal{A}}-1;n] = \underbrace{1-2G(1-\pi|n-k_{\mathcal{A}}+2,k_{\mathcal{A}})}_{positive},$$

and

$$E_\theta[u_{\mathcal{A}}(B,\theta)|k_{\mathcal{B}}-1;n] - E_\theta[u_{\mathcal{A}}(A,\theta)|k_{\mathcal{B}}-1;n] = \underbrace{1-2G(1-\pi|n-k_{\mathcal{B}}+2,k_{\mathcal{B}})}_{negative}.$$

Therefore, i's incentive compatibility condition becomes

$$\Pr(\langle n-1,k_{\mathcal{A}}-1\rangle|s_i=b)\Pr\left[\#\{j\neq i: t_j=\mathcal{A}\}\geq nr\right]\left(1-2G(1-\pi|n-k_{\mathcal{A}}+2,k_{\mathcal{A}})\right)+$$
$$\Pr(\langle n-1,k_{\mathcal{B}}-1\rangle|s_i=b)\Pr\left[\#\{j\neq i: t_j=\mathcal{B}\}\geq n+1-nr\right]\left(1-2G(1-\pi|n-k_{\mathcal{B}}+2,k_{\mathcal{B}})\right)$$
$$\geq 0.$$

Moreover, using the identity $k_{\mathcal{A}} = n+1-k_{\mathcal{B}}$ and Lemma B1, this is equivalent to

$$\frac{k_{\mathcal{B}}}{n}\cdot\Pr\left[\#\{j\neq i: t_j=\mathcal{A}\}\geq nr\right]\cdot\left(1-2G(1-\pi|n-k_{\mathcal{A}}+2,k_{\mathcal{A}})\right)+$$
$$\frac{k_{\mathcal{A}}}{n}\Pr\left[\#\{j\neq i: t_j=\mathcal{B}\}\geq n+1-nr\right]\left(1-2G(1-\pi|k_{\mathcal{A}}+1,n-k_{\mathcal{A}}+1)\right)\geq 0. \quad (B.2)$$

Similarly, the incentive compatibility condition for type $\mathcal{B}$ voter i' with $s_{i'} = a$ is given by

$$\frac{k_{\mathcal{B}}}{n}\Pr\left[\#\{j\neq i: t_j=\mathcal{B}\}\geq n+1-nr\right]\cdot(2G(\pi|n-k_{\mathcal{B}}+1,k_{\mathcal{B}}+1)-1)+$$
$$\frac{k_{\mathcal{A}}}{n}\Pr\left[\#\{j\neq i: t_j=\mathcal{A}\}\geq nr\right]\cdot(2G(\pi|n-k_{\mathcal{A}}+1,k_{\mathcal{A}}+1)-1)\geq 0.$$

Using the identity $k_{\mathcal{A}} = n+1-k_{\mathcal{B}}$ and $G(x|\alpha,\beta) = 1 - G(1-x|\alpha,\beta)$ in Lemma B2 (1), this is equivalent to

$$\frac{k_{\mathcal{B}}}{n}\Pr\left[\#\{j\neq i: t_j=\mathcal{B}\}\geq n+1-nr\right]\cdot\left(1-2G(1-\pi|n-k_{\mathcal{A}}+2,k_{\mathcal{A}})\right)+$$
$$\frac{k_{\mathcal{A}}}{n}\Pr\left[\#\{j\neq i: t_j=\mathcal{A}\}\geq nr\right]\left(1-2G(1-\pi|k_{\mathcal{A}}+1,n-k_{\mathcal{A}}+1)\right)\geq 0. \quad (B.3)$$

STEP 2: Asymptotic Behavior of Incentive Compatibility Conditions

The second step contains the following three claims whose proofs are shown later.
Here, I make the following claims:

***CLAIM* 1**: $$\lim_{n\to\infty} G(1-\pi|k_{\mathcal{A}}+1,n-k_{\mathcal{A}}+1)=1$$

***CLAIM* 2**: $$\lim_{n\to\infty} G(1-\pi|n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) = \frac{1}{2}$$

CLAIM 3: *If* $z = r$, *then*

$$\lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq nr\right] = \lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-nr\right] = \frac{1}{2}.$$

Otherwise, exactly one of the followings must be true.

(1) $\lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq nr\right] = 1$ *and* $\lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-nr\right] = 0$

(2) $\lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq nr\right] = 0$ *and* $\lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-nr\right] = 1$

These claims together imply that there is at least one type of voter whose incentive compatibility condition is violated for sufficiently large n. For instance, suppose that the following in Claim 3 is true.

$$\lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq nr\right] = 0 \textit{ and } \lim_{n\to\infty} \Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-nr\right] = 1$$

Then the incentive compatibility condition for a type $\mathcal{A}$ voter whose signal conflicts, eq. (B.2), is violated for sufficiently large n. In particular, the left-hand side of eq. (B.2) converges to $-\pi$. To see this, it should be noted that Claims 1 and 2 state that (1) the loss from lying conditional on pivotality converges to 0 and (2) the benefit from lying converges to 1. Moreover, by Claim 3, the probability that the like-minded voters can determine the outcome and the probability that the opposed type can do so converge to 0 and 1, respectively. Thus, due to lemma 1, the left-hand side of eq. (B.2) converges to $-\pi$. ■

Proof of Claim 1

Since $k_{\mathcal{A}} + 1 \leq n + 1 - k_{\mathcal{A}}$, we obtain

$$G(1-\pi|k_{\mathcal{A}}+1, n+1-k_{\mathcal{A}}) \geq G(1-\pi|k_{\mathcal{A}}+1, k_{\mathcal{A}}+1) = 1 - G(\pi|k_{\mathcal{A}}+1, k_{\mathcal{A}}+1),$$

where the first inequality comes from Lemma B2 (3) and the second equality comes from the identity $G(x|\alpha,\beta) = 1 - G(1-x|\alpha,\beta)$ in Lemma B2 (1). Thus, it suffices to show that

$$\lim_{n\to\infty} G(\pi|k_{\mathcal{A}}+1, k_{\mathcal{A}}+1) = 0.$$

Here, from Lemma B2 (4), the distribution function of the symmetric beta distribution with parameter $k_{\mathcal{A}} + 1$, $G(\pi|k_{\mathcal{A}}+1, k_{\mathcal{A}}+1)$, can be expressed as follows.

$$G(\pi|k_{\mathcal{A}}+1, k_{\mathcal{A}}+1) = \frac{1}{2} G\left(4\pi(1-\pi)|k_{\mathcal{A}}+1, \frac{1}{2}\right).$$

Therefore, it suffices to show the following eq. (B.4).

$$\lim_{n\to\infty} G\left(4\pi(1-\pi)|k_{\mathcal{A}}+1,\frac{1}{2}\right)=0. \qquad (B.4)$$

To prove eq. (B.4), I use an ε-quantile function of $\mathrm{Beta}(\alpha,\beta)$, which is denoted by $q_\varepsilon(\cdot)$ (Askitis, 2021). To define this function, I first define an ε-quantile of $\mathrm{Beta}(\alpha,\beta)$: A value $q_\varepsilon \in [0,1]$ is an ε-quantile of $\mathrm{Beta}(\alpha,\beta)$ if $G(q_\varepsilon|\alpha,\beta)=\varepsilon$.[20] An ε-quantile function of $\mathrm{Beta}(\alpha,\beta)$ is defined as follows: Let $\varepsilon\in(0,1)$ and the second parameter β be fixed. A function $q_\varepsilon(\cdot)$ assigns to the first parameter α an ε-quantile of $\mathrm{Beta}(\alpha,\beta)$, $q_\varepsilon(\alpha)$. That is, $q_\varepsilon(\alpha)$ is defined implicitly as follows.

$$G(q_\varepsilon(\alpha)|\alpha,\beta)=\varepsilon.$$

Askitis (2021) studied the asymptotic behavior of the function $q_\varepsilon(\cdot)$ when the input (i.e., first parameter α) goes to infinity.

LEMMA B3 (Askitis, 2021) *Let $\varepsilon\in(0,1)$ and the second parameter $\beta>0$ be fixed. Then function $q_\varepsilon(\cdot)$ has the limit*

$$\lim_{\alpha\to\infty} q_\varepsilon(\alpha)=1.$$

To prove eq. (B.4), I use this Lemma B3 and the fact that $k_{\mathcal{A}}+1$ diverges as $n\to\infty$. Take $\varepsilon\in(0,1)$ arbitrarily. Due to Lemma B3 and the fact that $k_{\mathcal{A}}+1\to\infty$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty} q_\varepsilon(k_{\mathcal{A}}+1)=1.$$

Therefore, since $4\pi(1-\pi)<1$, there exists an n' such that $n'\le n$ implies

$$4\pi(1-\pi)<q_\varepsilon(k_{\mathcal{A}}+1).$$

Thus, $n'\le n$ implies

$$G\left(4\pi(1-\pi)|k_{\mathcal{A}}+1,\frac{1}{2}\right)<G\left(q_\varepsilon(k_A+1)|k_{\mathcal{A}}+1,\frac{1}{2}\right)=\varepsilon.$$

Since $\varepsilon\in(0,1)$ is arbitrary, I have shown eq. (B.4). ■

Proof of Claim 2

In $\mathrm{Beta}(n-k_{\mathcal{A}}+2,k_{\mathcal{A}})$, the first parameter, $n-k_{\mathcal{A}}+2$, is greater than the second, $k_{\mathcal{A}}$. Therefore, from Lemma A1 (2), the median of $\mathrm{Beta}(n-k_{\mathcal{A}}+2,k_{\mathcal{A}})$, $M(n-k_{\mathcal{A}}+2,k_{\mathcal{A}})$, satisfies

[20] $1/2$-quantile is called median.

$$\frac{n-k_{\mathcal{A}}+2}{n+2} \le M(n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) \le \frac{n-1+k_{\mathcal{A}}}{n}.$$

From the identity $k_{\mathcal{B}} = n+1-k_{\mathcal{A}}$ in Lemma A2, this can be rearranged as follows.

$$\frac{k_{\mathcal{B}}+1}{n+2} \le M(n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) \le \frac{k_{\mathcal{B}}}{n}. \qquad (B.5)$$

Moreover, the right-hand side of eq. (B.5), $k_{\mathcal{B}}/n$, is bounded above as follows.

$$\frac{k_{\mathcal{B}}}{n} < 1-\pi+\frac{2}{n}. \qquad (B.6)$$

This is due to the fact that $n\pi - 1 < k_{\mathcal{A}} < n\pi + 1$ and the identity $k_{\mathcal{B}} = n+1-k_{\mathcal{A}}$.

Concerning the left-hand side of eq. (B.5), $(k_{\mathcal{B}}+1)/(n+2)$, it is bounded below as follows.

$$1-\pi-\frac{1}{n+2} < \frac{k_{\mathcal{B}}+1}{n+2}. \qquad (B.7)$$

Note that eq. (B.7) can be derived from the following algebraic manipulation.

$$1-\pi < \frac{k_{\mathcal{B}}}{n} \Leftrightarrow \frac{n(1-t)+1}{n+2} < \frac{k_{\mathcal{B}}+1}{n+2}$$

$$\Leftrightarrow \frac{n(1-\pi)+1+2(1-\pi)-2(1-\pi)}{n+2} < \frac{k_{\mathcal{B}}+1}{n+2}$$

$$\Leftrightarrow 1-\pi-\frac{1-2\pi}{n+2} < \frac{k_{\mathcal{B}}+1}{n+2},$$

where $1-\pi < k_{\mathcal{B}}/n$ is true by Lemma 1 and $\pi < 1/2$.

Thus, from eqs. (B.5), (B.6), and (B.7), we obtain

$$1-\pi-\frac{1}{n+2} < M(n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) < 1-\pi+\frac{2}{n},$$

which can be rearranged as follows.

$$M(n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) - \frac{2}{n} < 1-\pi < M(n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) + \frac{1}{n+2}.$$

This implies that

$$\lim_{n\to\infty} G\left(M(n-k_{\mathcal{A}}+2, k_{\mathcal{A}}) - \frac{2}{n} \middle| n-k_{\mathcal{A}}+2, k_{\mathcal{A}}\right) = \frac{1}{2},$$

and

$$\lim_{n\to\infty} G\left(M(n-k_{\mathcal{A}}+2,k_{\mathcal{A}})+\frac{1}{n+2}|n-k_{\mathcal{A}}+2,k_{\mathcal{A}}\right)=\frac{1}{2}.$$

Thus, we obtain

$$\lim_{n\to\infty} G(1-\pi|n-k_{\mathcal{A}}+2,k_{\mathcal{A}})=\frac{1}{2}.$$ ■

Proof of Claim 3

Case 1: $z \neq r$.

Suppose $z > r$. I will show that

$$\lim_{n\to\infty} \Pr\left[\frac{\#\{j\neq i: t_j=\mathcal{A}\}}{n}\geq r\right]=1, \tag{B.8}$$

and

$$\lim_{n\to\infty} \Pr\left[\frac{\#\{j\neq i: t_j=\mathcal{B}\}}{n}\geq 1+\frac{1}{n}-r\right]=0. \tag{B.9}$$

First, I will show eq. (B.8). Let X_n denote a random variable $\#\{j\neq i: t_j=\mathcal{A}\}/n$, and its expectation and variance are denoted by μ_n and σ_n^2, respectively. Take $\delta>0$ such that $r<z-\delta$. Then it suffices to show that

$$\lim_{n\to\infty}\Pr\big[X_n\in[z-\delta,z+\delta]\big]=1.$$

Take arbitrary small $\varepsilon>0$. I will argue that there exists an n^* such that $n^*\leq n$ implies that

$$1-\varepsilon<\Pr\big[X_n\in[z-\delta,z+\delta]\big].$$

First, note that μ_n and σ_n^2 have the limits[21]

$$\lim_{n\to\infty}\mu_n=z, \tag{B.10}$$

$$\lim_{n\to\infty}\sigma_n^2=0. \tag{B.11}$$

By Chebyshev's inequality, we obtain

$$1-\frac{\sigma_n^2}{(\delta/2)^2}\leq\Pr\left[X_n\in\left[\mu_n-\frac{\delta}{2},\mu_n+\frac{\delta}{2}\right]\right]\text{ for any } n, \tag{B.12}$$

where left-hand side of eq. (B.12) converges to 1 by eq. (B.11). Therefore, there exists an n_1

[21] To see this, note that $\mu_n=(1-1/n)z$, and $\sigma_n^2=(1/n-1/n^2)z(1-z)$.

such that $n_1 \leq n$ implies that

$$1-\varepsilon < 1-\frac{\sigma_n^2}{(\delta/2)^2} \leq \Pr\left[X_n \in \left[\mu_n - \frac{\delta}{2}, \mu_n + \frac{\delta}{2}\right]\right].$$

Moreover, by eq. (B.10), there exists an n_2 such that $n_2 \leq n$ implies $\mu_n \in [z-\delta/2\,, z+\delta/2]$. Thus, for $n \geq n_2$, $X_n \in [\mu_n - \delta/2\,, \mu_n + \delta/2]$ implies $X_n \in [z-\delta, z+\delta]$.

Therefore, for $n \geq n^* = \max\{n_1, n_2\}$, we obtain

$$1-\varepsilon < \Pr\left[X_n \in \left[\mu_n - \frac{\delta}{2}, \mu_n + \frac{\delta}{2}\right]\right] \leq \Pr\big[X_n \in [z-\delta, z+\delta]\big].$$

This completes the proof of eq. (B.8). An analogous argument shows the eq. (B.9).

Next, suppose $z < r$. An analogous argument shows that

$$\lim_{n\to\infty} \Pr\left[\frac{\#\{j \neq i : t_j = \mathcal{A}\}}{n} \geq r\right] = 0,$$

and

$$\lim_{n\to\infty} \Pr\left[\frac{\#\{j \neq i : t_j = \mathcal{B}\}}{n} \geq 1 + \frac{1}{n} - r\right] = 1.$$

Case 2: $z = r$.

This follows from the symmetry. ■

Appendix C: Proof of Proposition 3

Suppose $z = 1/2$.Consider type $\mathcal{A}$ voter i with private signal $s_i = b$ and type $\mathcal{B}$ voter j with $s_j = a$. From the proof of Theorem 1, the incentive compatibility conditions for i and j are given by $I(n, \pi, \bar{k}) \geq 0$ and $J(n, \pi, \bar{k}) \geq 0$, respectively, where

$$I(n,\pi,\bar{k}) = k_{\mathcal{B}} \cdot \Pr\big[\#\{j \neq i : t_j = \mathcal{A}\} \geq \bar{k}\big] \cdot \Big(\underbrace{1 - 2G(1-\pi | n - k_{\mathcal{A}} + 2, k_{\mathcal{A}})}_{positive}\Big)$$
$$+ k_{\mathcal{A}} \cdot \Pr\big[\#\{j \neq i : t_j = \mathcal{B}\} \geq n + 1 - \bar{k}\big] \Big(\underbrace{1 - 2G(1-\pi | k_{\mathcal{A}} + 1, n - k_{\mathcal{A}} + 1)}_{negative}\Big),$$

and

$$J(n,\pi,\bar{k})$$

$$= k_{\mathcal{B}} \Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-\overline{k}\right]\left(\underbrace{1-2G(1-\pi|n-k_{\mathcal{A}}+2, k_{\mathcal{A}})}_{positive}\right)$$

$$+k_{\mathcal{A}} \cdot \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq \overline{k}\right]\left(\underbrace{1-2G(1-\pi|k_{\mathcal{A}}+1, n-k_{\mathcal{A}}+1)}_{negative}\right).$$

For fully revealing sincere voting to be an equilibrium, both $I(n,\pi,\overline{k}) \geq 0$ and $J(n,\pi,\overline{k}) \geq 0$ must be satisfied. That is, $\min\{I(n,\pi,\overline{k}), J(n,\pi,\overline{k})\} \geq 0$ must be satisfied. Fix π. For each $\overline{k}$, let $n^*_{\overline{k}}$ be the largest odd integer such that $\min\{I(n,\pi,\overline{k}), J(n,\pi,\overline{k})\} \geq 0$ is true. That is, $n^*_{\overline{k}}$ is the largest size of the population in which fully revealing sincere voting is an equilibrium under voting rule $\overline{k}$.

I will argue that

$$\text{If } \overline{k} \text{ is supermajority, then } \min\left\{I\left(n^*_{\overline{k}}, \tfrac{n+1}{2}\right), J\left(n, \tfrac{n+1}{2}\right)\right\} \geq 0.$$

This means that if fully revealing sincere voting is an equilibrium under supermajority $\overline{k}$ when there are $n^*_{\overline{k}}$ voters, then it is also an equilibrium under majority rule when there are $n^*_{\overline{k}}$ voters. Therefore, if this is proved, then the proof of Proposition 3 is complete.

First, I will show this to be true when $\overline{k} < (n+1)/2$ (i.e., supermajority is required for alternative B to win). To show this, the following facts are useful, where these facts are consequences of the assumption $z = 1/2$.

Facts:

(i) $\Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-\overline{k}\right] = \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq \overline{k}\right]$ if $\overline{k}$ is the majority rule.

(ii) $\Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-\overline{k}\right] > \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq \overline{k}\right]$ if $\overline{k} > (n+1)/2$.

(iii) $\Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-\overline{k}\right] < \Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq \overline{k}\right]$ if $\overline{k} < (n+1)/2$.

(iv) $\Pr\left[\#\{j \neq i: t_j = \mathcal{A}\} \geq \overline{k}\right]$ is strictly decreasing in $\overline{k}$

(v) $\Pr\left[\#\{j \neq i: t_j = \mathcal{B}\} \geq n+1-\overline{k}\right]$ is strictly increasing in $\overline{k}$.

Suppose $\overline{k} < (n+1)/2$. Then, from Fact (v) and (i), I obtain

$$\min\left\{I\left(n^*_{\overline{k}}, \frac{n+1}{2}\right), J\left(n^*_{\overline{k}}, \frac{n+1}{2}\right)\right\} = J\left(n^*_{\overline{k}}, \frac{n+1}{2}\right) > J\left(n^*_{\overline{k}}, \overline{k}\right) \geq 0,$$

where the first equality is due to Fact (i), second inequality is due to Fact (v), and the last inequality is due to the definition of $n^*_{\overline{k}}$.

An analogous argument shows the case $\overline{k} > (n+1)/2$. ■

References

Askitis, D. (2021). Logarithmic Concavity of the Inverse Incomplete Beta Function with Respect to the First Parameter. *Mathematica Scandinavica* 127(1), 111-130.

Austen-Smith, D., and Banks, J. S. (1996). Information Aggregation, Rationality, and the Condorcet Jury Theorem. *American Political Science Review* 90(1), 34-45.

Austen-Smith, D., and Feddersen, T. J. (2006). Deliberation, Preference Uncertainty, and Voting Rules. *American Political Science Review* 100(2), 209-217.

Burke, J., and Taylor, C.R. (2008). What's in a Poll? Incentives for Truthful Reporting in Preelection Opinion Surveys. *Public Choice* 137(1-2), 221-244.

Condorcet, Marquis de. [1785] 1994. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Trans. lain McLean and Fiona Hewitt. Paris

Coughlan, P. J. (2000). In Defense of Unanimous Jury Verdicts: Mistrials, Communication, and Strategic Voting. *American Political Science Review* 94(2), 375-393.

Feddersen, T., and Pesendorfer, W. (1997). Voting Behavior and Information Aggregation in Elections with Private Information. *Econometrica* 65(5), 1029-1058.

Gerardi, D. (2000). Jury Verdicts and Preference Diversity. *American Political Science Review 94*(2), 395-406.

Jackson, M, O., and Tan X. (2013) Deliberation, Disclosure of Information, and Voting. *Journal of Economic Theory*. 148(1), 2-30

Krishnamoorthy, K. (2015). *Handbook of Statistical Distributions with Applications.*(2nd ed). CRC Press.

Mathis, J. (2011). Deliberation with Evidence. *American Political Science Review* 105(3), 516-529

Meirowitz, A. (2007). In Defense of Exclusionary Deliberation: Communication and Voting with Private Beliefs and Values. *Journal of Theoretical Politics* 19(3), 301327.

Morgan, J., and Stocken, P. C. (2008). Information Aggregation in Polls. *American Economic Review 98*(3), 864-896.

Payton, M.E., Young, L.J. and Young, J.H. (1989). Bounds for the Difference Between Median and Mean of Beta and Negative Binomial Distributions. *Metrika* 36(1), 347-354

Schulte, E. (2010). Information Aggregation and Preference Heterogeneity in Committees. *Theory and Decision* 69(1), 97-118.

Wit, J. (1998). Rational Choice and the Condorcet Jury Theorem. *Games and Economic*

Behavior. 22(2), 364-376.