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# Potentials and Solutions of Cooperative Games

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**Abstract**

This paper considers the solution concepts of cooperative games that admit a potential function. We say that a solution admits a potential function if the solution is given as the gradient vector of the potential function at the player set. Hart and Mas-Collel (1989) show that the Shapley value is the only solution that is efficient and admits the potential function for games with variable player sets. In this paper, first, we argue that if we remove efficiency, various solutions admit a potential function. Second, we characterize the class of the solutions that admit a potential function and provide their general functional form. Third, we define a potential function for games with a fixed player set and associate a potential function with the axioms that the Shapley value obeys. Finally, we discuss how the efficiency requirement induces the uniqueness of the Shapley value through a potential function.

Keywords: Cooperative games; Efficiency; Potential function; Shapley value

JEL Classification: C71

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# 1 Introduction

A cooperative game is a simple framework by which we formulate cooperative coalitions among players. The players obtain payoffs from their cooperation, and we use a characteristic function to denote the payoffs. The derived payoffs are distributed over the players. Such a distribution is modeled as a function, called a solution. In studying solution concepts, we have to consider how to measure and aggregate their performance in a coalition. One standard method is to consider the contributions of a player to each coalition. To see this, let $i$ denote a player and consider a coalition $S$ in which player $i$ is not, namely, $i \notin S$. The difference between the payoffs generated by $S$ and those derived by $S \cup \{i\}$ can be regarded as the contributions of player $i$, which is referred to as $i$'s marginal contribution to $S$. For each player $i$, such a marginal contribution is measured for $2^{n-1}$ different coalitions, so that a marginal contribution vector $\Delta_i \in \mathbb{R}^{2^{n-1}}$ represents $i$'s contribution in a game. A central solution concept in cooperative games that allocates the payoffs based upon the player's contributions in the above sense is the Shapley value (Shapley, 1953b). Young (1985) shows that the Shapley value is the unique monotonic solution with respect to the marginal contribution vector and satisfies other standard properties.

Another way to measure each player's contribution to the cooperation is to aggregate all information about the underlying game and define a single representative value for the game. If a solution concept is defined based on such an aggregate function, then we say that the solution allocates the payoffs on the basis of players' contributions. Hart and Mas-Collel (1989) initially introduced such an aggregate function for cooperative games and called it a potential function. Hart and Mas-Collel (1989) demonstrate that a solution admits a potential and satisfies efficiency if and only if the solution is the Shapley value (Shapley, 1953b). They prove this eminent result via a recursive approach on variable player sets: The solution is recursively calculated from the singleton player set to an arbitrary finite player set. Assuming that the solution satisfies efficiency, they show that the functional form of the solution coincides with the Shapley value.

Since Young's (1985) result shows that the Shapley value is heavily based on each player's marginal contributions, one might not regard that the above coincidence as surprising because the recursion of their potential function implicitly induces players' marginal contributions. In this paper, however, we suggest that efficiency plays a key role in the uniqueness of the Shapley value. We argue that if we remove the efficiency requirement, many solutions admit a potential function. In addition, if a solution admits a potential function, then what axiomatic properties are satisfied by the solution? This leads to the question of why efficiency singles out the Shapley value.

We first find a functional form that is common among the solutions that admit a potential function. To this end, we introduce a new concept, *interaction coalitional potential*, which is a variation of the concepts provided by Ui (2000) and Nakada (2018). An interaction coalitional potential is a family of functions, each of which is indexed by a set of coalitions, and its value only depends upon the characteristic function restricting the set of coalitions. In this sense, an interaction coalitional potential is a family of subaggregate functions, each of which summarizes the information about cooperation among the set of coalitions into a representative value. We show that a solution admits a potential if and only if there is an interaction coalitional potential such that the potential of the solution can be represented as a sum of the interaction coalitional potentials. This representation result complements the representation result offered by Casajus and Huettner (2018), which is based upon the "decomposer" of solutions. The formal relationship between our result and that of Casajus and Huettner (2018) is elaborated in Section 3.

Moreover, note that a coalitional potential function is neither a family of linear functions nor that of symmetric functions. Therefore, even if a solution admits a potential, then it is neither linear nor symmetric, which is in contrast to the Shapley value because the latter satisfies both properties. To see why efficiency singles out the Shapley value from the solutions that admit a potential function, we consider the axioms that the solutions obey. Specifically, we show that if a solution admits a potential, then it satisfies the null player property and a weaker version of symmetry on the games with a fixed player set, none of which are satisfied on the variable player set. Moreover, a solution admitting a potential also satisfies additivity if and only if it satisfies total additivity, where total additivity requires that the sum of all players' payoffs is additive. Since efficiency requires that players' allocated payoffs sum to the worth of the grand coalition, which is clearly additive, efficiency is sufficient to make the solution additive together with a potential. Therefore, efficiency and the above properties jointly imply the set of axioms that characterizes the Shapley value, which means that the Shapley value is the only efficient solution that admits a potential function. Moreover, in the absence of efficiency, we also show that a solution admits a potential function and satisfies linearity if and only if it is a generalized version of semi-values, as introduced by Weber (1988).

One of the messages of this paper is as follows: The reasons why efficiency uniquely selects the Shapley value in the class of solutions that admit a potential differ between the two domains, namely, the games with variable player sets and those with fixed player sets. If player sets are variable, then efficiency immediately induces the recursive formula of Hart and Mas-Collel (1989) and generates the Shapley value. Moreover, the potential function implies neither the null-player property nor a weaker version of symmetry. On the other hand, if a player set is fixed, then a potential function induces both properties, and additivity is also induced with the help of the two properties, which leads to the uniqueness of the Shapley value.

The rest of paper is organized as follows. In Section 2, we introduce basic concepts. In Section 3, we introduce a potential in the case of variable player sets and provide a representation result. In Section 4, we reformulate a potential in the case of fixed player sets and associate a potential with axioms of solution concepts. Section 5 is the conclusion of the paper. All proofs are provided in the Appendix.

# 2 Preliminaries

Let $\mathcal{U}$ be a countably infinite set. We consider $\mathcal{U}$ to be the universal set of players. Let $\mathcal{N}$ be the set of all finite subsets of $\mathcal{U}$. A cooperative game with transferable utility (a TU-game) on a finite player set $N \in \mathcal{N}$ is given by a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. A *coalition* of players is a nonempty subset of the player set $S \subseteq N$. We denote the cardinality of coalition $S$ by $|S|$. We use $n$ to denote $|N|$. Let $\mathcal{G}_N$ be the set of all games with the player set $N$ and $\mathcal{G}$ denote the set of all games: $\mathcal{G} = \{(N, v) | N \in \mathcal{N}, v \in \mathcal{G}_N\}$. For every game $v \in \mathcal{G}_N$ and $S \subseteq N$, let $(S, v^S) \in \mathcal{G}_S$ be the subgame of $v$ such that $v^S(T) = v(T)$ for all $T \subseteq S$. For simplicity, we use $(S, v)$ to denote subgame $(S, v^S)$.

For each nonempty $T \subseteq N$, a unanimity game $u_T \in \mathcal{G}_N$ is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Shapley (1953a) shows that every game $v \in \mathcal{G}_N$ can be represented as a unique linear combination of unanimity games: For every game $v \in \mathcal{G}_N$, there are unique values $\lambda_T^v$, $\emptyset \neq T \subseteq N$ such that

$$v(S) = \sum_{\emptyset \neq T \subseteq N} \lambda_T^v u_T(S) = \sum_{\emptyset \neq R \subseteq S} \lambda_R^v, \tag{1}$$

where $\lambda_T^v = \sum_{\emptyset \neq R \subseteq T} (-1)^{|T|-|R|} v(R)$. For simplicity, we omit $v$ and write $\lambda_T$ instead of $\lambda_T^v$ when there is no ambiguity. We use $\lambda$ to denote the vector $(\lambda_T)_{\emptyset \neq T \subseteq N} \in \mathbb{R}^{2^N - 1}$. For every $(N, v) \in \mathcal{G}$ and $i \in N$, let $\Delta_i^v = (v(S \cup \{i\}) - v(S))_{S \subseteq N \setminus \{i\}}$ denote player $i$'s marginal contribution vector. We say that $i$ is a null player in $v$ if $\Delta_i^v = \mathbf{0}$. Moreover, we say that players $i, j \in N$ are symmetric in $v$ if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for every $S \subseteq N \setminus \{i, j\}$.

A solution is a function $f$ such that $f(N, v) \in \mathbb{R}^N$ for all $(N, v) \in \mathcal{G}$. The Shapley value is a solution defined by

$$Sh_i(v) = \sum_{T \subseteq N, i \in T} \lambda_T / |T| \tag{2}$$

where $\lambda_T / |T|$ is called *Harsanyi's dividend* to the members of $T$. Shapley (1953b) shows that the Shapley value is the unique solution satisfying the following properties on $\mathcal{G}_N$ with the fixed player set $N$.

**Efficiency**: For every $v \in \mathcal{G}_N$, $\sum_{i \in N} f_i(N, v) = v(N)$.

**Symmetry**: For every $v \in \mathcal{G}_N$ and $i, j \in N$, if $i$ and $j$ are symmetric in $v$, then $f_i(N, v) = f_j(N, v)$.

**Null player property**: For every $v \in \mathcal{G}_N$, if $i$ is a null player in $v$, then $f_i(N, v) = 0$.

**Additivity**: For every $v, v' \in \mathcal{G}_N$, $f(N, v + v') = f(N, v) + f(N, v')$.

# 3 Solutions and potentials

Hart and Mas-Colell (1989) introduce the following concept, which is known as the HM-potential function or simply the HM-potential.

**Definition 1.** A function $P : \mathcal{G} \to \mathbb{R}$ is the HM-potential if for every $(N, v) \in \mathcal{G}$

$$\sum_{j \in N} D^i P(N, v) = v(N)$$

where $D^i P(N, v) = P(N, v) - P(N \setminus \{i\}, v)$.

Hart and Mas-Colell (1989) show that the HM-potential uniquely exists. We denote it by $P^{HM}$. The HM-potential is obtained from the following recursive formula:

$$P^{HM}(N, v) = \frac{1}{|N|} \Big( v(N) + \sum_{i \in N} P^{HM}(N \setminus \{i\}, v) \Big) \tag{3}$$

with constant $P(\emptyset, v) \in \mathbb{R}$.[1] Moreover, they show that marginal contributions with respect to

[1]Hart and Mas-Colell (1989) define $P(\emptyset, v) = 0$ in their work. However, this difference does not matter in general: The HM-potential uniquely exists up to constant $P(\emptyset, v) = 0$. For details, see Casajus and Huettner (2018). We will briefly revisit this in Lemma 3 in Section 4.

$P^{HM}$ yield the Shapley value: For every $i \in N$,

$$D^i P^{HM}(N, v) = Sh_i(N, v).$$

Now, in view of Hart and Mas-Colell (1989), we consider the class of solutions that admit a (general) potential. The following definition is due to Calvo and Santos (1997) and Ortmann (1998).

**Definition 2.** A solution $f$ admits a potential if there is a function $P : \mathcal{G} \to \mathbb{R}$ such that

$$f_i(N, v) = P(N, v) - P(N \setminus \{i\}, v)$$

for all $i \in N$.

If a solution $f$ admits a potential, then we call $P$ a potential of $f$. Provided that the Shapley value admits a potential, the result of Hart and Mas-Colell (1989) is reformulated as follows.

**Proposition 1.** A solution $f$ admits a potential and satisfies efficiency if and only if $f(N, v) = Sh(N, v)$.

This result suggests that efficiency plays an important role in inducing the uniqueness of the Shapley value. If we remove efficiency from the proposition, many other solutions also admit a potential. To see this, we consider the following class of solutions. For every $N \in \mathcal{N}$, let $C_i = \{S \subseteq N | i \in S\}$ be the set of coalitions that contain $i$. Let $p_i : C_i \to [0, 1]$ with $\sum_{S \in C_i} p_i(S) = 1$ be a probability distribution over $C_i$. For every $i, j \in N$, $p_i$ and $p_j$ are symmetric if there is a function $\beta : \{1, \cdots, n\} \to [0, 1]$ such that $p_i(S_i) = \beta(s) = p_j(S_j)$ for any $S_i \in C_i$ and $S_j \in C_j$ with $s = |S_i| = |S_j|$ where $s = |S|$ is the number of players in $S$. Note that

$$\sum_{s=1}^{n} \beta(s) \begin{pmatrix} n \\ s \end{pmatrix} = 1. \tag{4}$$

In view of Weber (1988), a solution $f$ is a probabilistic value if there is $(p_i)_{i \in N}$ with $p_i \in \Delta(C_i)$ such that

$$f_i(N, v) = \sum_{S \in C_i} p_i(S)(v(S) - v(S \setminus \{i\})).$$

A symmetric probabilistic value is called a semi-value: A solution $f$ is a semi-value if there is a function $\beta : \{1, \cdots, n\} \to [0, 1]$ satisfying (4) such that

$$f_i(N, v) = \sum_{S \in C_i} \beta(s)(v(S) - v(S \setminus \{i\})).$$

The Shapley value and the Banzaf value (Owen, 1975) are special cases of semi-values where $\beta^{Sh}(s) = \frac{(s-1)!(n-s)!}{n!}$ and $\beta^{B}(s) = \frac{1}{2^{n}-1}$, respectively. Each semi-value $f$ has a potential that satisfies

$$P(N,v) = \sum_{S \subseteq N} \beta(s)v(S).$$

In this paper, we define the following more general solution, which we call a generalized semi-value:

$$f_i(N,v) = \sum_{S \in C_i} \gamma(S)(v(S) - v(S \setminus \{i\})) \tag{5}$$

for all $i \in N$ where $\gamma : 2^N \to \mathbb{R}$ is an arbitrary function. Each function $f$ satisfying (5) has a potential function

$$P(N,v) = \sum_{S \subseteq N} \gamma(S)v(S).$$

We now characterize the class of solutions that admit a potential. To this end, we introduce the following concept. For each $N \in \mathcal{N}$ and each $\mathcal{A} \subseteq 2^N$, let $\zeta_{\mathcal{A}} : \mathcal{G}_N \to \mathbb{R}$ be a function such that $\zeta_{\mathcal{A}}(N,v) = \zeta_{\mathcal{A}}(N,v')$ if $v(S) = v'(S)$ for all $S \in \mathcal{A}$. We call the family of functions $(\zeta_{\mathcal{A}})_{\mathcal{A} \subseteq 2^N, N \in \mathcal{N}}$ an interaction coalitional potential, which is an analog of the potential function concept studied by Ui (2000) and Nakada (2018). The following result shows that every solution that admits a potential can be represented as the sum of components of an interaction coalitional potential.

**Theorem 1.** A solution $f$ admits a potential if and only if there is an interaction coalitional potential $(\zeta_{\mathcal{A}})_{\mathcal{A} \subseteq 2^N, N \in \mathcal{N}}$ such that, for all $N \in \mathcal{N}$, $i \in N$ and $v \in \mathcal{G}_N$,

$$\begin{aligned} f_i(N,v) &= \sum_{\mathcal{A} \subseteq 2^N} \zeta_{\mathcal{A}}(N,v) - \sum_{\mathcal{A} \subseteq 2^{N \setminus \{i\}}} \zeta_{\mathcal{A}}(N \setminus \{i\}, v) \\ &= \sum_{\mathcal{A} \subseteq 2^N : \mathcal{A} \cap C_i \neq \emptyset} \zeta_{\mathcal{A}}(N,v) + \sum_{\mathcal{A} \subseteq 2^{N \setminus \{i\}}} \big(\zeta_{\mathcal{A}}(N,v) - \zeta_{\mathcal{A}}(N \setminus \{i\}, v)\big). \end{aligned}$$

A potential function is given by

$$P(N,v) = \sum_{\mathcal{A} \subseteq 2^N} \zeta_{\mathcal{A}}(N,v).$$

Theorem 1 and the following notion proposed by Casajus and Huettner (2018) complement each other.

**Definition 3.** A solution $f$ is decomposable if there is a solution $\psi$, called a decomposer, such that

$$f_i(N,v) = \psi_i(N,v) + \sum_{j \in N \setminus \{i\}} \big[\psi_j(N,v) - \psi_j(N \setminus \{i\}, v)\big]$$

for all $(N, v) \in \mathcal{G}$ and $i \in N$.

If a decomposer $\psi$ of $f$ is also decomposable, then $\psi$ is said to be a decomposable decomposer. Casajus and Huettner (2018) show that $f$ is decomposable if and only if it admits a potential. Furthermore, for each decomposable solution $f$, there is the unique decomposable decomposer. Below, we provide a new formula to obtain a decomposer of a decomposable solution $f$.

**Proposition 2.** Suppose that $f$ admits a potential and its potential function $P$ is represented as an interaction coalitional potential $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N, N\in\mathcal{N}}$. Then, a solution $\psi$ given as follows is a decomposer of $f$:

$$\psi_i(N, v) = \sum_{\mathcal{A}\subseteq 2^N : \mathcal{A}\cap C_i \neq \emptyset} \frac{\zeta_{\mathcal{A}}(N, v)}{|\{j \in N | \mathcal{A} \cap C_j \neq \emptyset\}|}$$

for all $(N, v) \in \mathcal{G}$ and $i \in N$.

This formula generates an explicit expression for the unique decomposable decomposer of $f$. Let $f$ be a solution and let $(N, v) \in \mathcal{G}_N$ and $S \subseteq N$. Define game $(S, v^f) \in \mathcal{G}_S$ as

$$v^f(T) = \sum_{i\in T} f_i(T, v)$$

for all $T \subseteq S$. Then, by the result of Sánchez (1997) and Calvo and Santos (1997),[2] $f$ admits a potential if and only if

$$f_i(N, v) = Sh_i(N, v^f)$$

for all $(N, v) \in \mathcal{G}$ and $i \in N$. Since the Shapley value's potential function $P^{Sh}$ is given as $P(N, v) = \sum_{T\subseteq N} \frac{\lambda^v_T}{|T|}$, $P^{Sh}$ can be represented by the following interaction coalitional potential $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N, N\in\mathcal{N}}$:

$$\zeta_{\mathcal{A}}(N, v) = \begin{cases} \frac{\lambda^v_T}{|T|} & \text{if } \mathcal{A} = \{S | S \subseteq T\} \text{ for some } T \subseteq N, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} f_i(N, v) &= Sh_i(N, v^f) \\ &= \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(N, v^f) - \sum_{\mathcal{A}\subseteq 2^{N\setminus\{i\}}} \zeta_{\mathcal{A}}(N \setminus \{i\}, v^f) \\ &= \sum_{T\subseteq N : i\in T} \frac{\lambda^{v^f}_T}{|T|}. \end{aligned}$$

[2]See Theorem 4 in Casajus and Huettner (2018).

By Proposition 2, the solution $\psi$ such that for all $(N,v) \in \mathcal{G}$ and $i \in N$

$$\psi_i(N,v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T^{v^f}}{|T|^2}$$

is a decomposable decomposer of $f$ and its potential function is given as

$$P^{\psi}(N,v) = \sum_{T \subseteq N} \frac{\lambda_T^{v^f}}{|T|^2}.$$

If $f$ is the Shapley value, by efficiency, we have $v^f(S) = v(S)$. Hence, its decomposable decomposer is given by

$$\psi_i(N,v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T^{v}}{|T|^2}$$

for all $(N,v) \in \mathcal{G}$ and $i \in N$, and its potential function is given as

$$P^{\psi}(N,v) = \sum_{T \subseteq N} \frac{\lambda_T^{v}}{|T|^2}.$$

Note that the decomposable decomposer of a solution also has its (unique) decomposable decomposer. Hence, such a decomposition process proceeds infinitely. From this observation, Casajus and Huettner (2018) consider the notion of higher order decomposability, which they call resolvability, and obtain the representation of the decomposable solution in terms of its higher order decomposer (see Proposition 10 and Theorem 12 in their paper). In this sense, Proposition 2 suggests that we can obtain a higher order decomposition of the potential $P$.

The following proposition shows that the potential of a decomposable decomposer $\psi$ of $f$ yields the potential of $f$.

**Proposition 3.** Let $\psi$ be a decomposable decomposer of $f$ and $P^{\psi}$ be its potential. Then, function $P : \mathcal{G} \to \mathbb{R}$ defined as follows is a potential of $f$: for all $(N,v) \in \mathcal{G}$

$$P(N,v) = nP^{\psi}(N,v) - \sum_{j \in N} P^{\psi}(N \setminus \{j\}, v).$$

In view of Proposition 3, we obtain a higher order decomposition of the potential $P$ in the following steps.

- Let $f^0$ be a decomposable solution.
- Let $f^k$ be a decomposable decomposer of $f^{k-1}$.
- For every decomposable solution $f$, let $P^f$ be a potential of $f$.

Fix an arbitrary game $(N, v)$. For simplicity, we write $P^f(N)$ instead of $P^f(N, v)$. Then, we have the following decomposition process.

$$\begin{aligned}
P^{f^0}(N) &= nP^{f^1}(N) - \sum_{j_1 \in N} P^{f^1}(N \setminus \{j_1\}) \\
&= n^2 P^{f^2}(N) - (2n-1) \sum_{j_1 \in N} P^{f^2}(N \setminus \{j_1\}) + \sum_{j_1 \in N} \sum_{j_2 \in N \setminus \{j_1\}} P^{f^2}(N \setminus \{j_1, j_2\}) \\
&= n^3 P^{f^3}(N) - (3n^2 - 3n + 1) \sum_{j_1 \in N} P^{f^3}(N \setminus \{j_1\}) \\
&\quad + (3n-3) \sum_{j_1 \in N} \sum_{j_2 \in N \setminus \{j_1\}} P^{f^3}(N \setminus \{j_1, j_2\}) - \sum_{j_1 \in N} \sum_{j_2 \in N \setminus \{j_1\}} \sum_{j_3 \in N \setminus \{j_1, j_2\}} P^{f^3}(N \setminus \{j_1, j_2, j_3\}) \\
&\quad \cdots .
\end{aligned}$$

In general, let the $k$-th decomposition of $P^{f^0}(N)$ denote

$$c_0^k \left[ P^{f^k}(N) \right] + c_1^k \left[ \sum_{j_1 \in N} P^{f^k}(N \setminus \{j_1\}) \right] + \cdots$$
$$+ c_k^k \left[ \sum_{j_1 \in N} \cdots \sum_{j_k \in N \setminus \{j_1, \ldots, j_{k-1}\}} P^{f^k}(N \setminus \{j_1, \ldots, j_k\}) \right].$$

for some $(c_m^k)_{m=0}^k \in \mathbb{R}^{k+1}$. Given the $k$-th decomposition, the $(k+1)$-th decomposition is

$$c_0^{k+1} \left[ P^{f^{k+1}}(N) \right] + c_1^{k+1} \left[ \sum_{j_1 \in N} P^{f^{k+1}}(N \setminus \{j_1\}) \right] + \cdots$$
$$+ c_{k+1}^{k+1} \left[ \sum_{j_1 \in N} \cdots \sum_{j_{k+1} \in N \setminus \{j_1, \ldots, j_k\}} P^{f^{k+1}}(N \setminus \{j_1, \ldots, j_{k+1}\}) \right],$$

where each coefficient is clearly given by the following recursive form:

$$c_0^{k+1} = n \cdot c_0^k = n^{k+1}, \tag{6}$$
$$c_a^{k+1} = -c_{a-1}^k + (n-a) \cdot c_a^k \text{ for each } a = 1, \ldots, k, \tag{7}$$
$$c_{k+1}^{k+1} = (-1)^{k+1}. \tag{8}$$

The following result offers a general explicit form of the decomposition.

**Proposition 4.** Let $f = f^0$ be a decomposable solution and $f^k$ be a decomposable decomposer of $f^{k-1}$. For every integer $k \geq 0$, the potential of $f$ is given as follows:

$$\begin{aligned}
P^f(N) &= c_0^k \left[ P^{f^k}(N) \right] + c_1^k \left[ \sum_{j_1 \in N} P^{f^k}(N \setminus \{j_1\}) \right] + \\
&\quad \cdots + c_k^k \left[ \sum_{j_1 \in N} \cdots \sum_{j_k \in N \setminus \{j_1, \ldots, j_{k-1}\}} P^{f^k}(N \setminus \{j_1, \ldots, j_k\}) \right],
\end{aligned}$$

where for every $a = 0, \ldots, k$,

$$c_a^k = \sum_{r=a}^{k} (-1)^r \cdot \mathrm{St}(r, a) \cdot \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r}, \tag{9}$$

and $\mathrm{St}(r, a)$ is the Stirling number of the second kind.[3]

The decomposition of a potential function given by Proposition 4 continues infinitely. Note that for the $n$-th step, the last term should be $P^{f^n}(N \setminus \{j_1, \ldots, j_n\}) = P^{f^n}(\emptyset) = 0$, while its coefficient $c_n^n$ is not zero. In general, for each $k \geq n$, some terms become zero, and hence, $P^f(N)$ is represented by the summation of $n - 1$ non-zero terms.

# 4 Potentials and axioms of solutions

As mentioned in Section 3, Proposition 1 shows that a solution $f$ admits a potential and satisfies efficiency if and only if $f = Sh$. The Shapley value is characterized in the following two different ways. On one hand, Hart and Mas-Collel (1989) prove the uniqueness of the Shapley value by using their potential function defined over all games in $\mathcal{G}$ without fixing any specific player set $N$. On the other hand, Shapley offers the axiomatic uniqueness of the Shapley value defined over the games in $\mathcal{G}_N$ when fixing a player set $N \in \mathcal{N}$.

To analyze the gap between these two uniqueness results, we first fix $N \in \mathcal{N}$ and consider potentials of solutions for games in $\mathcal{G}_N$. Hereafter, we fix a player set $N \in \mathcal{N}$ and consider $\mathcal{G}_N$. For every game $v \in \mathcal{G}_N$ and $T \subseteq N$, let $v|_T \in \mathcal{G}_N$ be a restricted game on $T$ such that

$$v|_T(S) = \begin{cases} v(T \cap S) & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $v|_T$ is a game in $\mathcal{G}_N$.[4] The definition of restricted games is due to Ui (2000). For every $v \in \mathcal{G}_N$ and $T, S \subseteq N$, we have $(v|_T)|_S = v|_{T \cup S}$. Moreover, the game $v|_T$ is a projection of $v$ into the linear subspace in the following sense.

[3]The Stirling number of the second kind is explicitly given as

$$\mathrm{St}(r, a) = \frac{1}{a!} \sum_{h=1}^{a} (-1)^{a-h} \begin{pmatrix} a \\ h \end{pmatrix} h^r.$$

It is known that the number coincides with the total number of ways of partitioning a set of $r$ elements into $a$ nonempty subsets.

[4]This game is also called a nullified game (Béal et al., 2014; 2016). Specifically, game $v^i$ is a nullified game with player $i$ if $v^i(S) = v(S \setminus \{i\})$ for all $S \subseteq N$. By successive elimination of players in $N \setminus T$, we can see that $v|_T = v^{N \setminus T}$.

**Lemma 1.** For any game $v \in \mathcal{G}_N$ and $S \subseteq N$, let $v' = v|_{N\setminus S}$. Then, for any $T \subseteq N$,

$$\lambda'_T = \begin{cases} \lambda_T & \text{if } S \cap T = \emptyset, \\ 0 & \text{if } S \cap T \neq \emptyset. \end{cases}$$

Therefore, $v' = \sum_{T\subseteq N\setminus S} \lambda_T u_T$.

Replacing subgames with restriction games in the definition of potentials, we obtain the following concept.

**Definition 4.** A solution $f : \mathcal{G}_N \to \mathbb{R}^N$ admits a potential if there is a function $P : \mathcal{G}_N \to \mathbb{R}$ such that

$$f_i(v) = P(v) - P(v|_{N\setminus\{i\}})$$

for all $i \in N$.

The following result shows that the existence of potentials implies the null player property if the solution is defined over games with a fixed player set, which does not hold if no player set is fixed.

**Lemma 2.** If a solution $f : \mathcal{G}_N \to \mathbb{R}^N$ admits a potential, then it satisfies the null player property.

We now provide some necessary and sufficient conditions for a solution $f : \mathcal{G}_N \to \mathbb{R}^N$ to admit a potential, which can be seen as additional conditions that are equivalent to Theorem 4 of Casajus and Huettner (2018). For each $\mathcal{A} \subseteq 2^N$, let $\zeta_{\mathcal{A}} : \mathcal{G}_N \to \mathbb{R}$ be a function such that $\zeta_{\mathcal{A}}(v) = \zeta_{\mathcal{A}}(v')$ if $v(S) = v'(S)$ for all $S \in \mathcal{A}$. We call the family of functions $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N}$ an interaction coalitional potential for games with a fixed player set.

**Theorem 2.** For every solution $f : \mathcal{G}_N \to \mathbb{R}^N$, the following properties are equivalent.

(i) $f$ admits a potential.

(ii) There is an interaction coalitional potential $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N}$ such that, for all $i \in N$,

$$f_i(v) = \sum_{\mathcal{A}\subseteq 2^N : \mathcal{A}\cap C_i \neq \emptyset} \big(\zeta_{\mathcal{A}}(v) - \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}})\big),$$

where the potential function is given by

$$P(v) = \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(v)$$

(iii) $f$ satisfies the null player property and path independence:[5] for every permutation on $N, \pi, \pi' \in \Pi$

$$\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = \sum_{i=1}^{n} f_{\pi'(i)}(v|_{N\setminus\{0,\cdots,\pi'(i-1)\}})$$

where $N \setminus \{0\} = N$.

(iv) $f$ satisfies the null player property and balanced contribution:[6] for every $v \in \mathcal{G}_N$ and $i, j \in N$,

$$f_i(v) - f_i(v|_{N\setminus\{j\}}) = f_j(v) - f_j(v|_{N\setminus\{i\}}).$$

Path independence of potentials implies the uniqueness of potential, if it exists, in the following sense.

**Lemma 3.** Suppose that $f$ admits potentials $P$ and $P'$. Then, there exists a constant $c \in \mathbb{R}$ such that $P' = P + c$.

For all games in $\mathcal{G}$, the equivalence between (i) and (iii) (or (iv)) holds without the null player property: Solution $f$ with $f(N, v) = Sh(N, v) + c$ for some $c \in \mathbb{R}$ admits a potential satisfying $P(N, v) = P^{HM}(N, v) + |N|c$ and also satisfies balanced contribution. This difference occurs because in the set of all games $\mathcal{G}$, we can make use of the size of variable player sets to construct a potential: The solution can be given as the difference in the values of the potential for games with different sizes of player sets. However, this approach cannot be applied if the class is restricted to games with a fixed player set.

The Shapley value admits a potential and satisfies symmetry and additivity. Moreover, it satisfies linearity: For any $v, v' \in \mathcal{G}_N$ and $c, c' \in \mathbb{R}$, $f(cv + c'v') = cf(v) + c'f(v')$. In contrast, without efficiency, even if a solution $f$ admits a potential, $f$ violates symmetry or additivity. To see this, consider the following examples.

**Example 1.** The following solution $f$ admits a potential, while it does not satisfy symmetry. Given $N$, fix a player $k \in N$. For every $i \in N$

$$f_i(v) \quad = \quad |\{S \subseteq N | v(S) > 0, k \in S\}| - |\{S \subseteq N | v^{-i}(S) > 0, k \in S\}|.$$

---

[5]This condition is first introduced by Hart and Mas-Collel (1989). They call it the summability condition and note that the Shapley value satisfies it. Ortmann (1998) calls this condition order-independence or path-independence, which is the same name of the corresponding concept used in Physics.

[6]This condition is introduced by Myerson (1980). Béal et al. (2016) call this axiom balanced contributions under nullification in games with a fixed player set.

Clearly, the corresponding potential function is

$$P(v) = |\{S \subseteq N | v(S) > 0, k \in S\}|.$$

However, $f$ violates symmetry. For example, let $N = \{1, 2, 3\}$ and $k = 1$. Consider $v(S) = 1$ for all $\emptyset \neq S \subseteq N$. Players $1, 2$ are symmetric in $v$. We have $v^{-1}(S) = 1$ for every $S \neq \{1\}$ and $v^{-1}(\{1\}) = 0$, and similarly $v^{-2}(S) = 1$ for every $S \neq \{2\}$ and $v^{-2}(\{2\}) = 0$. We obtain

$$\begin{aligned}
f_1(v) &= |\{S \subseteq N | v(S) > 0, k \in S\}| - |\{S \subseteq N | v^{-1}(S) > 0, 1 \in S\}| \\
&= 4 - 3 \\
&\neq 4 - 4 \\
&= |\{S \subseteq N | v(S) > 0, k \in S\}| - |\{S \subseteq N | v^{-2}(S) > 0, 1 \in S\}| \\
&= f_2(v).
\end{aligned}$$

**Example 2.** The following solution $f$ admits a potential, while it is not additive: For every $i \in N$

$$\begin{aligned}
f_i(v) &= \sum_{S \subseteq N, i \in N} \beta^{Sh}(|S|)(v(S) + v|_{N\setminus\{i\}}(S))(v(S) - v|_{N\setminus\{i\}}(S)) \\
&= \sum_{S \subseteq N, i \in N} \beta^{Sh}(|S|)(v(S)^2 - v|_{N\setminus\{i\}}(S)^2).
\end{aligned}$$

This solution $f$ has the following potential function:

$$P(v) = \sum_{S \subseteq N} \beta^{Sh}(|S|) \cdot v(S)^2.$$

However, $f$ is not additive.

Fortunately, if a solution admits a potential, then the following weaker notion of symmetry is satisfied instead of symmetry.

**Definition 5.** A solution $f$ satisfies symmetry for unanimity games if for any $T \subseteq N$, $i, j \in T$ and $c \in \mathbb{R}$, $f_i(cu_T) = f_j(cu_T)$.

**Lemma 4.** If a solution $f$ admits a potential, then it satisfies symmetry for unanimity games.

Why does efficiency imply additivity (and linearity) for solutions admitting a potential? To see this, we consider the following concepts.

**Definition 6.** A solution $f$ satisfies total additivity if for any $v, v' \in \mathcal{G}_N$, $\sum_{j \in N} f_j(v) + \sum_{j \in N} f_j(v') = \sum_{j \in N} f_j(v + v')$.

**Definition 7.** A solution $f$ satisfies total homogeneity for unanimity games if for any $c \in \mathbb{R}$ and any $T \subseteq N$, $\sum_{j \in N} f_j(cu_T) = c \sum_{j \in N} f_j(u_T)$.

Note that efficiency and additivity imply total additivity. The following result shows that additivity and total additivity are equivalent if a solution admits a potential function. Moreover, if it also satisfies total homogeneity for unanimity, then it satisfies linearity, and vice versa.

**Lemma 5.** Suppose that a solution $f$ admits a potential. Then,

(i) $f$ is additive if and only if it satisfies total additivity.

(ii) $f$ is linear if and only if it satisfies total additivity and total homogeneity for unanimity games.

By this result, we can unify the uniqueness results of the Shapley value over $\mathcal{G}$ provided by Hart and Mas-Colell (1989) and Myerson (1980)[7] with the axiomatic characterization by Shapley (1954b) over $\mathcal{G}_N$.

**Corollary 1.** Let $f : \mathcal{G}_N \to \mathbb{R}^N$. The following properties are equivalent.

(i) $f = Sh$.

(ii) $f$ admits a potential and satisfies efficiency.

(iii) $f$ satisfies efficiency, balanced contribution and the null player property.

*Proof.* For the statement (i) implies (ii), the function $P(v) = \sum_{T \subseteq N} \frac{\lambda_T}{|T|}$ is a potential of the Shapley value. Moreover, Theorem 2 shows that the equivalence between (ii) and (iii). Then, it suffices to show that (ii) implies that (i). By Lemma 5, if $f$ admits a potential and satisfies efficiency, then it also satisfies additivity. Moreover, by Lemma 2, it satisfies null player property, and by Lemma 4, it satisfies symmetry for unanimity games. Therefore, in view of 1, $f_i(v) = \sum_{T \subseteq N} f_i(\lambda_T u_T) = \sum_{T \subseteq N, i \in T} f_i(\lambda_T u_T) = \sum_{T \subseteq N, i \in T} \frac{\lambda_T}{|T|} = Sh_i(v)$. □

Below, we provide some technical remarks on Corollary 1. Proposition 9 of Béal et al. (2016) demonstrates the equivalence between (i) and (iii) by directly showing that the set of axioms in (iii) characterizes the Shapley value. In their proof, they assume that at least one null

[7]The former result shows that the unique efficient solution that admits a potential is the Shapley value, and the latter shows that the unique efficient solution that satisfies balanced contribution is the Shapley value.

player exists. For the equivalence between (i) and (ii), they provide the concept of a nullified potential, which is equivalently defined as $\sum_{i\in N} P(v) - P(v|_{N\setminus\{i\}}) = v(N)$ in our notion.[8] In this sense, their definition requires efficiency as a primitive in the same way as Hart and Mas-Colell (1989). They show that such a function $P$ uniquely exists and satisfies $P(v)-P(v|_{N\setminus\{i\}}) = Sh_i(v)$. In contrast, our definition of potentials does not require efficiency, which enables us to consider more solutions.

We now consider the class of solutions that admit a potential and satisfy linearity. Note that the Shapley value is an element of the class. Suppose that a solution $f$ is linear. Then, by Lemma 1, order independence and the null player property, a potential of $f$ is defined as

$$\begin{aligned}
P(v) &= \sum_{i=1}^{n} f_i(v|_{N\setminus\{0,\cdots,i-1\}}) \\
&= \sum_{i=1}^{n} \sum_{T\subseteq N} \lambda_T(v|_{N\setminus\{0,\cdots,i-1\}}) f_i(u_T) \\
&= \sum_{T\subseteq N;1\in T} \lambda_T f_1(u_T) + \sum_{T\subseteq N;1\notin T,2\in T} \lambda_T f_2(u_T) + \cdots + \sum_{T\subseteq N;1,2,\cdots n-1\notin T,n\in T} \lambda_T f_n(u_T).
\end{aligned}$$

We define $w_T \in \mathbb{R}$ such that $w_T = f_i(u_T)$ for each $T \subseteq N$ with $i \in T$ and $0,\cdots,i-1 \notin T$. Then, for $(w_T)_{T\subseteq N}$, we have

$$P(v) = \sum_{T\subseteq N} w_T \lambda_T = \sum_{T\subseteq N} \tilde{w}_T v(T)$$

where $\tilde{w}_T = w_T U^{-1}$, and $U \in \mathbb{R}^{2^n-1\times 2^n-1}$ is the matrix whose column vectors correspond to unanimity games. Therefore, this is the potential of a generalized semi-value defined in (5). These arguments generate the following characterization of generalized semi-values.

**Theorem 3.** A solution $f$ admits a potential and satisfies linearity if and only if it is a generalized semi-value.

# 5 Concluding remarks

We consider a general class of solutions of TU-games that admits a potential function. We provide a representation result for the solutions in that class for both $\mathcal{G}$, i.e., the class of games with a variable player set, and $\mathcal{G}_N$, i.e., the class of games with a fixed player set.

[8] Béal et.al. (2016) additionally impose that $P(\mathbf{0})$ as a normalization and argue that this property is related to the null player property. However, our Lemma 3 suggests that these restrictions are unnecessary and the null player property holds for every potential with $P(\mathbf{0})$. See the proof of Lemma 2.

In both cases, the combination of efficiency and potentials yields the uniqueness of the Shapley value. However, the structures of the uniqueness differ between the classes. In the former case $\mathcal{G}$, efficiency ensures the recursive formula (3), and it directly coincides with the Shapley value. However, the potential does not imply either the null player property or (even the weaker notion of) symmetry. On the other hand, in the latter case, a potential generates a solution that satisfies the null player property and (a weaker notion) of symmetry. Together with these properties, efficiency ensures the additivity of the solution, which leads to the uniqueness of the Shapley value. Moreover, a solution is a generalized semi-value if and only if the solution is linear and has a potential.

# A. Proofs in Section 3

*Proof of Theorem 1.* The proof is similar to Ui (2000) and Nakada (2018). Suppose that a solution $f$ admits a potential $P$. For each $N \in \mathcal{N}$, let us define $\zeta_{\mathcal{A}} : \mathcal{G}_N \to \mathbb{R}$ as

$$\zeta_{\mathcal{A}}(N,v) = \begin{cases} P(N,v) + \sum_{i\in N} P(N\backslash\{i\},v) & \text{if } \mathcal{A} = 2^N, \\ -P_i(N\backslash\{i\},v) & \text{if } \mathcal{A} = 2^{N\backslash\{i\}} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, for each $i \in N$,

$$\begin{aligned} \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(N,v) &= P(N,v) + \sum_{i\in N} P(N\backslash\{i\},v) - \sum_{i\in N} P(N\backslash\{i\},v) \\ &= P(N,v) \end{aligned}$$

and, for each $i \in N$,

$$\begin{aligned} \sum_{\mathcal{A}\subseteq 2^{N\backslash\{i\}}} \zeta_{\mathcal{A}}(N\backslash\{i\},v) &= P(N\backslash\{i\},v) + \sum_{j\in N\backslash\{i\}} P(N\backslash\{i,j\},v) - \sum_{j\in N\backslash\{i\}} P(N\backslash\{i,j\},v) \\ &= P(N\backslash\{i\},v). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(N,v) - \sum_{\mathcal{A}\subseteq 2^{N\backslash\{i\}}} \zeta_{\mathcal{A}}(N \setminus \{i\},v) =& P(N,v) - P(N\backslash\{i\},v) \\ =& f_i(N,v). \end{aligned}$$

Conversely, suppose that there is an interaction coalitional potential $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N, N\in\mathcal{N}}$ such that

$$f_i(N,v) = \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(N,v) - \sum_{\mathcal{A}\subseteq 2^{N\setminus\{i\}}} \zeta_{\mathcal{A}}(N\setminus\{i\},v)$$

for each $N \in \mathcal{N}$, $i \in N$ and $v \in \mathcal{G}_N$. Then, it is easily verified that a solution $f$ admits a potential $P(N,v) = \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(N,v)$ for each $(N,v) \in \mathcal{G}$. □

*Proof of Proposition 2.* It is easy to see that

$$\begin{aligned}
\sum_{i\in N}\psi_i(N,v) &= \sum_{i\in N}\sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset} \frac{\zeta_{\mathcal{A}}(N,v)}{|\{j\in N|\mathcal{A}\cap C_j\neq\emptyset\}|} \\
&= \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(N,v) \\
&= P(N,v),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j\in N\setminus\{i\}}\psi_j(N\setminus\{i\},v) &= \sum_{j\in N\setminus\{i\}}\sum_{\mathcal{A}\subseteq 2^{N\setminus\{i\}}:\mathcal{A}\cap C_j\neq\emptyset} \frac{\zeta_{\mathcal{A}}(N\setminus\{i\},v)}{|\{j\in N\setminus\{i\}|\mathcal{A}\cap C_j\neq\emptyset\}|} \\
&= \sum_{\mathcal{A}\subseteq 2^{N\setminus\{i\}}} \zeta_{\mathcal{A}}(N\setminus\{i\},v) \\
&= P(N\setminus\{i\},v).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_i(N,v) &= P(N,v) - P(N\setminus\{i\},v) \\
&= \sum_{i\in N}\psi_i(N,v) - \sum_{j\in N\setminus\{i\}}\psi_j(N\setminus\{i\},v) \\
&= \psi_i(N,v) + \sum_{j\in N\setminus\{i\}}\left[\psi_j(N,v) - \psi_j(N\setminus\{i\},v)\right],
\end{aligned}$$

which means that $\psi$ is a decomposer of $f$. □

*Proof of Proposition 3.* We first offer the following basic properties for combination and the Stirling number of the second kind:

$$\begin{pmatrix} k+1 \\ r \end{pmatrix} = \begin{pmatrix} k \\ r \end{pmatrix} + \begin{pmatrix} k \\ r-1 \end{pmatrix}, \tag{10}$$

$$\mathrm{St}(r+1,a) = \mathrm{St}(r,a-1) + a\cdot\mathrm{St}(r,a), \tag{11}$$

$$\mathrm{St}(r,0) = \begin{cases} 1 & \text{if } r=0, \\ 0 & \text{otherwise } (r=1,2,3,\ldots), \end{cases} \tag{12}$$

$$\mathrm{St}(r,r) = 1 \text{ for any } r\geq 0. \tag{13}$$

We show that $c_0^k = n^k$ and $c_k^k = (-1)^k$, which becomes the induction basis. In view of (9), we have

$$c_0^k = \sum_{r=0}^{k} (-1)^r \cdot \mathrm{St}(r,0) \cdot \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \overset{(12)}{=} n^k.$$

Moreover,

$$c_k^k = \sum_{r=k}^{k} (-1)^r \cdot \mathrm{St}(r,k) \cdot \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \overset{(13)}{=} (-1)^k.$$

Now we fix $k$ and $a$ ($k \geq a$) and assume that $c_{a-1}^k$ and $c_a^k$ satisfy (9). In view of the recursive relationship (7), we show that $c_a^{k+1}$ satisfies (9). Note that

$$\begin{aligned} c_a^{k+1} &\overset{(7)}{=} -c_{a-1}^k + (n-a) \cdot c_a^k \\ &\overset{(9)}{=} -\left[ \sum_{r=a-1}^{k} (-1)^r \mathrm{St}(r,a-1) \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \right] \\ &\quad +(n-a) \left[ \sum_{r=a}^{k} (-1)^r \mathrm{St}(r,a) \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \right] \end{aligned}$$

Extracting the second term, we have

$$\begin{aligned} &-\left[ \sum_{r=a-1}^{k} (-1)^r \mathrm{St}(r,a-1) \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \right] \\ &+\left[ \sum_{r=a}^{k} (-1)^r \mathrm{St}(r,a) \begin{pmatrix} k \\ r \end{pmatrix} n^{k+1-r} \right] \\ &-\left[ \sum_{r=a}^{k} (-1)^r a \cdot \mathrm{St}(r,a) \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \right]. \end{aligned}$$

Extracting $r = a-1$ from the first term, we have

$$\begin{aligned} &-(-1)^{a-1} \mathrm{St}(a-1,a-1) \begin{pmatrix} k \\ a-1 \end{pmatrix} n^{k-(a-1)} \\ &-\left[ \sum_{r=a}^{k} (-1)^r \mathrm{St}(r,a-1) \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \right] \\ &+\left[ \sum_{r=a}^{k} (-1)^r \mathrm{St}(r,a) \begin{pmatrix} k \\ r \end{pmatrix} n^{k+1-r} \right] \\ &-\left[ \sum_{r=a}^{k} (-1)^r a \cdot \mathrm{St}(r,a) \begin{pmatrix} k \\ r \end{pmatrix} n^{k-r} \right]. \end{aligned}$$

In view of (11), combining the second term and the forth term and applying (13) to the first term, we obtain

$$
\begin{aligned}
&(-1)^a \mathrm{St}(a,a)\begin{pmatrix} k \\ a-1 \end{pmatrix} n^{k+1-a} \\
&-\left[\sum_{r=a}^{k}(-1)^r \mathrm{St}(r+1,a)\begin{pmatrix} k \\ r \end{pmatrix} n^{k-r}\right] \\
&+\left[\sum_{r=a}^{k}(-1)^r \mathrm{St}(r,a)\begin{pmatrix} k \\ r \end{pmatrix} n^{k+1-r}\right].
\end{aligned} \tag{14}
$$

Extracting $r = k$ from the second term of (14), the second term is equal to

$$
\begin{aligned}
&-\left[\sum_{r=a}^{k-1}(-1)^r \mathrm{St}(r+1,a)\begin{pmatrix} k \\ r \end{pmatrix} n^{k-r}\right] \\
&-(-1)^k \mathrm{St}(k+1,a)\begin{pmatrix} k \\ k \end{pmatrix} n^{k-k}.
\end{aligned} \tag{15}
$$

Extracting $r = a$ from the third term of (14), the third term is equal to

$$
\begin{aligned}
&(-1)^a \mathrm{St}(a,a)\begin{pmatrix} k \\ a \end{pmatrix} n^{k+1-a} \\
&+\left[\sum_{r=a+1}^{k}(-1)^r \mathrm{St}(r,a)\begin{pmatrix} k \\ r \end{pmatrix} n^{k+1-r}\right].
\end{aligned} \tag{16}
$$

In view of (10), summing up the first term of (14) and the first term of (16) generates

$$
(-1)^a \mathrm{St}(a,a)\begin{pmatrix} k+1 \\ a \end{pmatrix} n^{k+1-a}. \tag{17}
$$

In view of (10), summing up the first term of (15) and the second term of (16) yields

$$
\sum_{r=a+1}^{k}(-1)^r \mathrm{St}(r,a)\begin{pmatrix} k+1 \\ r \end{pmatrix} n^{k+1-r}. \tag{18}
$$

The second term of (15) is readily equal to

$$
(-1)^{k+1} \mathrm{St}(k+1,a)\begin{pmatrix} k+1 \\ k+1 \end{pmatrix} n^{k+1-(k+1)}. \tag{19}
$$

Hence, formula (14) is equal to the summation of (17), (18), and (19), namely

$$
\sum_{r=a}^{k+1}(-1)^r \mathrm{St}(r,a)\begin{pmatrix} k+1 \\ r \end{pmatrix} n^{k+1-r}.
$$

Thus, $c_a^{k+1}$ satisfies (9). This completes the proof. □

# B. Proofs in Section 4

*Proof of Lemma 1.* First, suppose that $S \cap T = \emptyset$. Then,

$$\begin{aligned}\lambda'_T &= \sum_{R \subseteq T} (-1)^{|T|-|R|} v'(R) \\ &= \sum_{R \subseteq T} (-1)^{|T|-|R|} v(R) \\ &= \lambda_T.\end{aligned}$$

Second, suppose that $S \cap T \neq \emptyset$. Then,

$$\begin{aligned}\lambda_T &= \sum_{R \subseteq T} (-1)^{|T|-|R|} v'(R) \\ &= \sum_{R \subseteq T : R \cap S \neq \emptyset} (-1)^{|T|-|R|} v'(R) + \sum_{R \subseteq T : R \cap S = \emptyset} (-1)^{|T|-|R|} v'(R) \\ &= \sum_{R \subseteq T : R \cap S \neq \emptyset} (-1)^{|T|-|R|} v(R \setminus S) + \sum_{R \subseteq T : R \cap S = \emptyset} (-1)^{|T|-|R|} v(R) \\ &= \sum_{R \subseteq T : R \cap S = \emptyset} \Big( \sum_{k=1}^{|T \cap S|} {}_{|T \cap S|}C_k (-1)^{|T|-|R|-k} + (-1)^{|T|-|R|} \Big) v(R) \\ &= \sum_{R \subseteq T : R \cap S = \emptyset} \Big( (-1)^{|T|-|R|} \Big( \sum_{k=0}^{|T \cap S|} {}_{|T \cap S|}C_k (-1)^k - 1 \Big) + (-1)^{|T|-|R|} \Big) v(R) \\ &= \sum_{R \subseteq T : R \cap S = \emptyset} \Big( -(-1)^{|T|-|R|} + (-1)^{|T|-|R|} \Big) v(R) \\ &= 0\end{aligned}$$

where the sixth equality holds by the binomial theorem $\sum_{k=0}^{|T \cap S|} {}_{|T \cap S|}C_k (-1)^k = (1-1)^{|T \cap S|} = 0$. □

*Proof of Lemma 2.* Note that, for all $v \in \mathcal{G}_N$ and $i \in N$, $i$ is a null-player in $v|_{N \setminus \{i\}}$. Therefore, if $f$ admits a potential $P$ and $i \in N$ is a null-player in $v$, then

$$f_i(v) = P(v) - P(v|_{N \setminus \{i\}}) = P(v) - P(v) = 0.$$

Hence, $f$ satisfies null-player axiom. □

*Proof of Theorem 2.* (i) $\Rightarrow$ (ii): The proof is almost same as of Theorem 1, but slight modification is needed. For completeness, we offer the proof as follows.

Suppose that a solution $f$ admits a potential $P$. Let us define

$$\zeta_{\mathcal{A}}(v) = \begin{cases} P(v) + \sum_{i\in N} P(v|_{N\setminus\{i\}}) & \text{if } \mathcal{A} = 2^N, \\ -P_i(v|_{N\setminus\{i\}}) & \text{if } \mathcal{A} = 2^{N\setminus\{i\}} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, the family of functions $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N}$ is an interaction coalitional potential. Note that $\zeta_{\mathcal{A}}(v) = \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}})$ if $\mathcal{A} \cap C_i = \emptyset$ because $v(S) = v|_{N\setminus\{i\}}(S)$ for all $S \subseteq N \setminus \{i\}$. Moreover, by construction, for each $i \in N$,

$$\sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset} \zeta_{\mathcal{A}}(v) = P(v) + \sum_{i\in N} P(v|_{N\setminus\{i\}}) - \sum_{j\neq i} P(v|_{N\setminus\{i\}}) = P(v) + P(v|_{N\setminus\{i\}})$$

and

$$\sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset} \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}}) = P(v|_{N\setminus\{i\}}) + \sum_{k\in N} P(v|_{N\setminus\{k,i\}}) - \sum_{j\neq i} P(v|_{N\setminus\{i,j\}}) = 2P(v|_{N\setminus\{i\}}).$$

Therefore,

$$\begin{aligned} \sum_{\mathcal{A}\subseteq 2^N} \left(\zeta_{\mathcal{A}}(v) - \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}})\right) = & \sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset} \left(\zeta_{\mathcal{A}}(v) - \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}})\right) \\ = & P(v) - P(v|_{N\setminus\{i\}}) \\ = & f_i(v). \end{aligned}$$

(ii) $\Rightarrow$ (i): Conversely, suppose that there is an interaction coalitional potential $(\zeta_{\mathcal{A}})_{\mathcal{A}\subseteq 2^N}$ such that $f_i(v) = \sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset}\left(\zeta_{\mathcal{A}}(v) - \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}})\right)$. Let $P(v) = \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(v)$. Then,

$$\begin{aligned} P(v) - P(v|_{N\setminus\{i\}}) = & \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(v) - \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}}) \\ = & \sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset} \zeta_{\mathcal{A}}(v) - \sum_{\mathcal{A}\subseteq 2^N:\mathcal{A}\cap C_i\neq\emptyset} \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}}) \\ = & f_i(v) \end{aligned}$$

where the second equality holds because, again, $\zeta_{\mathcal{A}}(v) = \zeta_{\mathcal{A}}(v|_{N\setminus\{i\}})$ if $\mathcal{A} \cap C_i = \emptyset$. Therefore, a solution $f$ admits a potential $P(v) = \sum_{\mathcal{A}\subseteq 2^N} \zeta_{\mathcal{A}}(v)$.

(i) $\Rightarrow$ (iii): Suppose that a solution $f$ admits a potential $P$. It suffices to show that $f$ satisfies path independence. Then, for any order $\pi \in \Pi$,

$$\begin{aligned} f_{\pi(1)}(v) &= P(v) - P(v|_{N\setminus\{\pi(1)\}}) \\ f_{\pi(2)}(v|_{N\setminus\{\pi(1)\}}) &= P(v|_{N\setminus\{\pi(1)\}}) - P(v|_{N\setminus\{\pi(1),\pi(2)\}}) \\ &\vdots \\ f_{\pi(n)}(v|_{N\setminus\{\pi(1),\pi(2),\cdots,\pi(n-1)\}}) &= P(v|_{N\setminus\{\pi(1),\pi(2),\cdots,\pi(n-1)\}}) - P(\mathbf{0}). \end{aligned}$$

Hence, $\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = P(v) - P(\mathbf{0})$, which implies path independence.

(iii) $\Rightarrow$ (i): Suppose that $f$ satisfies path independence and null player axiom. Then, for each $i$, define

$$P(v) = f_i(v) + \sum_{i=2}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}})$$

where $\pi = (i, \pi(2), \cdots, \pi(n)) \in \Pi$. By order independence, $P(v)$ is well-defined independent of $\pi$. By null-player axiom, $f_i(v|_{N\setminus\{i\}}) = 0$. Hence,

$$\begin{aligned} P(v) - P(v|_{N\setminus\{i\}}) &= f_i(v) - f_i(v|_{N\setminus\{i\}}) \\ &\quad + \sum_{j=2}^{n} (f_{\pi(j)}(v|_{N\setminus\{0,\cdots,\pi(j-1)\}}) - f_{\pi(j)}(v|_{N\setminus\{0,\cdots,\pi(j-1),i\}}) \\ &= f_i(v) - f_i(v|_{N\setminus\{i\}}) \\ &= f_i(v), \end{aligned}$$

which implies that $f$ admits a potential $P$.

To show the equivalence between (iii) and (iv), we need to show the equivalence between path independence and balanced contribution.

(iii) $\Rightarrow$ (iv): Suppose that $f$ is path independent. Take any $v \in \mathcal{G}_N$ and $i, j \in N$. Then, consider the following two permutation

$$\pi : i, j, i_3, \cdots, i_n,$$

$$\pi' : j, i, i_3, \cdots, i_n.$$

Since $f$ is path independent, we have

$$\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = \sum_{i=1}^{n} f_{\pi'(i)}(v|_{N\setminus\{0,\cdots,\pi'(i-1)\}})$$

Moreover, by construction,

$$\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = f_i(v) + f_j(v|_{N\setminus\{i\}}) + \sum_{k=1}^{n} f_{i_k}(v|_{N\setminus\{i,j,\cdots,i_{k-1})\}}),$$

$$\sum_{i=1}^{n} f_{\pi'(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = f_j(v) + f_i(v|_{N\setminus\{j\}}) + \sum_{k=1}^{n} f_{i_k}(v|_{N\setminus\{i,j,\cdots,i_{k-1})\}}),$$

which implies that $f_i(v) - f_i(v|_{N\setminus\{j\}}) = f_j(v) - f_j(v|_{N\setminus\{i\}})$. Hence, $f$ satisfies balanced contribution.

(iv) $\Rightarrow$ (iii): Suppose that $f$ satisfies balanced contribution. Take any $\pi, \pi' \in \Pi$ and we write $\pi = (i_1, i_2, \cdots, i_n), \pi = (i'_1, i'_2, \cdots, i'_n)$. Note that any permutation is a one-to-one and onto mapping $\tau : N \to N$ and it can be represented as a product of adjacent transposition. Hence, there are finite sequence of adjacent transpositions $\tau_1, \cdots \tau_k$ such that $\pi' = \tau_k \circ \cdots \tau_1 \circ \pi$.[9] Therefore, it suffices to show that

$$\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = \sum_{i=1}^{n} f_{\pi''(i)}(v|_{N\setminus\{0,\cdots,\pi''(i-1)\}})$$

where $\pi'' = (i_1, i_2, \cdots i_{k+1}, i_k, \cdots i_n)$. By construction,

$$\begin{aligned}
\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) &= \sum_{j=1}^{k-1} f_{i_j}(v|_{N\setminus\{0,\cdots,i_{j-1})\}}) + f_{i_k}(v|_{N\setminus\{i_1,\cdots,i_{k-1}\}}) \\
&\quad + f_{i_{k+1}}(v|_{N\setminus\{i_1,\cdots,i_k\}}) + \sum_{j=k+2}^{n} f_{i_j}(v|_{N\setminus\{i_1,\cdots,i_{j-1})\}}) \\
\sum_{i=1}^{n} f_{\pi'(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) &= \sum_{j=1}^{k-1} f_{i_j}(v|_{N\setminus\{0,\cdots,i_{j-1})\}}) + f_{i_{k+1}}(v|_{N\setminus\{i_1,\cdots,i_{k-1}\}}), \\
&\quad + f_{i_k}(v|_{N\setminus\{i_1,\cdots,i_{k-1},i_{k+1}\}}) + \sum_{j=k+2}^{n} f_{i_j}(v|_{N\setminus\{i_1,\cdots,i_{j-1})\}}).
\end{aligned}$$

Let $\tilde{v} = v|_{N\setminus\{i_1,\cdots,i_{k-1}\}}$ Then, by balanced contribution, we have

$$f_{i_k}(\tilde{v}) + f_{i_{k+1}}(\tilde{v}|_{N\setminus\{i_k\}}) = f_{i_{k+1}}(\tilde{v}) + f_{i_k}(\tilde{v}|_{N\setminus\{i_{k+1}\}}),$$

which implies that

$$\sum_{i=1}^{n} f_{\pi(i)}(v|_{N\setminus\{0,\cdots,\pi(i-1)\}}) = \sum_{i=1}^{n} f_{\pi''(i)}(v|_{N\setminus\{0,\cdots,\pi''(i-1)\}}).$$

Hence, $f$ satisfies path independence. □

*Proof of Lemma 3.* By order independence,

$$\sum_{i=1}^{n} f_i(v|_{N\setminus\{0,\cdots,i-1\}}) = P(v) - P(\mathbf{0}) = P'(v) - P'(\mathbf{0})$$

Define $c = P'(\mathbf{0}) - P(\mathbf{0})$. Then, we can see that $P'(v) = P(v) + c$. □

[9]A permutation $\tau : N \to N$ is a adjacent transposition if there is $i, i+1 \in N$ such that $\tau(i) = j, \tau(j) = i$ and $\tau(k) = k$ for any $k \neq i, j$. It is known that for any permutation $\tau : N \to N$, there are finite sequence of adjacent transpositions $\tau_1, \cdots \tau_k$ such that $\tau = \tau_k \circ \cdots \tau_1$.

*Proof of Lemma 4.* For each unanimity game $u_T$ and $S \subseteq N$ and $c \in \mathbb{R}$, note that

$$cu_T|_S = \begin{cases} cu_T & \text{if } T \subseteq S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Note also that $f_i(cu_T) = 0$ if $i \notin T$ because $f$ satisfies null player axiom. Then, for any $\pi \in \Pi$,

$$f_i(cu_T) = \begin{cases} P(cu_T) - P(\mathbf{0}) & \text{if } \{i\} = \text{argmin}_{j \in T} \pi(j), \\ 0 & \text{otherwise.} \end{cases}$$

Since $f$ is order independent, by considering other orders, we have

$$f_i(cu_T) = f_j(cu_T)$$

for all $i, j \in T$. □

*Proof of Lemma 5.* Since $f$ admits a potential, we have $f_i(v) = P(v) - P(v|_{N \setminus \{i\}})$ for any $i \in N$. Moreover, by Lemma 3, we can choose $P(\mathbf{0}) = 0$. Hence,

$$\begin{aligned} \sum_{j \in N} f_j(v) &= \sum_{j \in N} [P(v) - P(v|_{N \setminus \{j\}})] \\ &= nP(v) - \sum_{j \in N} P(v|_{N \setminus \{j\}}). \end{aligned}$$

For simplicity, we write $\Sigma f(v) := \sum_{j \in N} f_j(v)$. Then, we obtain

$$\begin{aligned} P(v) &= \frac{1}{n}\Sigma f(v) + \frac{1}{n} \sum_{j_1 \in N} P(v|_{N \setminus \{j_1\}}) \\ &= \frac{1}{n}\Sigma f(v) + \frac{1}{n} \sum_{j_1 \in N} \left[ \frac{1}{n}\Sigma f(v|_{N \setminus \{j_1\}}) + \frac{1}{n} \sum_{j_2 \in N} P(v|_{N \setminus \{j_1, j_2\}}) \right] \\ &= \frac{1}{n}\Sigma f(v) + \frac{1}{n^2} \sum_{j_1 \in N} \Sigma f(v|_{N \setminus \{j_1\}}) + \frac{1}{n^2} \sum_{j_1 \in N} \sum_{j_2 \in N} P(v|_{N \setminus \{j_1, j_2\}}) \\ &= \frac{1}{n}\Sigma f(v) + \frac{1}{n^2} \sum_{j_1 \in N} \Sigma f(v|_{N \setminus \{j_1\}}) + \frac{1}{n^3} \sum_{j_1 \in N} \sum_{j_2 \in N} \Sigma f(v|_{N \setminus \{j_1, j_2\}}) \\ &\quad + \frac{1}{n^3} \sum_{j_1 \in N} \sum_{j_2 \in N} \sum_{j_3 \in N} P(v|_{N \setminus \{j_1, j_2, j_3\}}) \\ &= \dots \\ &= \frac{1}{n}\Sigma f(v) + \frac{1}{n^2} \sum_{j_1 \in N} \Sigma f(v|_{N \setminus \{j_1\}}) + \frac{1}{n^3} \sum_{j_1 \in N} \sum_{j_2 \in N} \Sigma f(v|_{N \setminus \{j_1, j_2\}}) + \dots \\ &\quad + \frac{1}{n^n} \sum_{j_1 \in N} \cdots \sum_{j_{n-1} \in N} \Sigma f(v|_{N \setminus \{j_1, \dots, j_{n-1}\}}) + \left[ \frac{1}{n^n} \sum_{j_1 \in N} \cdots \sum_{j_n \in N} P(\mathbf{0}) \right] \\ &= \frac{1}{n}\Sigma f(v) + \frac{1}{n^2} \sum_{j_1 \in N} \Sigma f(v|_{N \setminus \{j_1\}}) + \frac{1}{n^3} \sum_{j_1 \in N} \sum_{j_2 \in N} \Sigma f(v|_{N \setminus \{j_1, j_2\}}) + \dots \\ &\quad + \frac{1}{n^n} \sum_{j_1 \in N} \cdots \sum_{j_{n-1} \in N} \Sigma f(v|_{N \setminus \{j_1, \dots, j_{n-1}\}}). \end{aligned}$$

For any $T \subseteq N$, it readily follows from its definition that

$$u_T|_{N\setminus\{i\}} = \begin{cases} u_T & \text{if } i \notin T, \\ \mathbf{0} & \text{if } i \in T. \end{cases}$$

Hence, setting

$$\begin{aligned} Q^k(\Sigma f(u_T)) : &= \frac{1}{n^k}\left[\sum_{j_1 \in N\setminus T} \cdots \sum_{j_{k-1} \in N\setminus T} \Sigma f(u_T) + \sum_{j_1 \in T} \cdots \sum_{j_{k-1} \in T} \Sigma f(\mathbf{0})\right] \\ &= \frac{1}{n^k}\left[\sum_{j_1 \in N\setminus T} \cdots \sum_{j_{k-1} \in N\setminus T} \Sigma f(u_T)\right] \end{aligned}$$

for $k = 2, \ldots, n$ (note that $f_i(\mathbf{0}) = P(\mathbf{0}) - P(\mathbf{0}|_{N\setminus\{i\}}) = 0 - 0 = 0$), we have

$$P(u_T) = \frac{1}{n}\Sigma f(u_T) + \sum_{k=2}^{n} Q^k(\Sigma f(u_T)).$$

for any $T \subseteq N$. Hence, it follows from total efficiency that $P(cu_T) + P(c'u_{T'}) = P(cu_T + c'u_{T'})$ for any $T, T \subseteq N$ and $c, c' \in \mathbb{R}$. For any $i \in N$, we have

$$\begin{aligned} f_i(cu_T) + f_i(c'u_{T'}) &= (P(cu_T) - P(cu_T|_{N\setminus\{i\}})) + (P(c'u_{T'}) - P(c'u_{T'}|_{N\setminus\{i\}})) \\ &= P(cu_T + c'u_{T'}) - P((cu_T + c'u_{T'})|_{N\setminus\{i\}}) \\ &= f_i(cu_T + c'u_{T'}). \end{aligned} \tag{20}$$

Therefore, we have, for $v, v' \in \mathcal{G}_N$

$$\begin{aligned} f_i(v + v') &= f(\sum_{T \subseteq N} \lambda_T^v u^T + \lambda_T^v u^T) \\ &= \sum_{T \subseteq N} f(\lambda_T^v u^T + \lambda_T^v u^T) \\ &= \sum_{T \subseteq N} f(\lambda_T^v u^T) + \sum_{T \subseteq N} f(\lambda_T^v u^T) \\ &= f(\sum_{T \subseteq N} \lambda_T^v u^T) + f(\sum_{T \subseteq N} \lambda_T^v u^T) \\ &= f(v) + f(v') \end{aligned}$$

which implies that $f$ is additive.

If we additional impose total homogeneity for unanimity games, it follows that $\Sigma f(cu_T) = c\Sigma f(u_T)$ for any $c \in \mathbb{R}$ and hence $Q^k(\Sigma f(cu_T)) = cQ^k(\Sigma f(u_T))$. Hence, we have $P(cu_T) = cP(u_T)$, which implies that for any $i \in N$, $T \subseteq N$, and $c \in \mathbb{R}$

$$f_i(cu_T) = P(cu_T) - P(cu_T|_{N\setminus\{i\}}) = cP(u_T) - cP(u_T|_{N\setminus\{i\}}) = cf_i(u_T). \tag{21}$$

Finally, for $v, v' \in \mathcal{G}_N$ and $c, c' \in \mathbb{R}$, we have

$$\begin{aligned} f(cv + c'v') &= f(\sum_{T \subseteq N} c\lambda_T^v u^T + c'\lambda_T^v u^T) \\ &\overset{(20)}{=} \sum_{T \subseteq N} f(c\lambda_T^v u^T) + \sum_{T \subseteq N} f(c'\lambda_T^{v'} u^T) \\ &\overset{(21)}{=} c\sum_{T \subseteq N} f(\lambda_T^v u^T) + c'\sum_{T \subseteq N} f(\lambda_T^{v'} u^T) \\ &\overset{(20)}{=} cf(v) + c'f(v'). \end{aligned}$$

which implies that $f$ is linear. □

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