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Abstract

Our objective is to analyze the relationship between the Shapley value and the core of cooperative games with transferable utility. We first characterize balanced games, namely, the set of games with a nonempty core, by means of geometric properties. We show that the set of balanced games generates a polyhedral cone and that a game is balanced if and only if it is a nonnegative linear combination of some simple games. Moreover, we show that the set of games whose Shapley value is in the core also yields a polyhedral cone and that a game obeys this property if and only if it is a nonnegative linear combination of some "easy" games. In addition, we also show that the number of games that correspond to the extreme rays of the polyhedron coincides with the number of minimal balanced collections.

Keywords: Cooperative games; Shapley value; Core; Minkowski-Weyl's Theorem
JEL Classification: C71

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1 Introduction

One of the objectives in cooperative game theory is to explore a "desirable" solution: how to allocate the surplus that players obtain from their cooperation. The Shapley value (Shapley, 1953b) and the core should be the most well-known solution concepts. The Shapley value is a single-valued solution, which assigns a payoff to each player based on his/her contributions to a coalition. Since the seminal study of Shapley (1953b), many studies have been devoted to analyzing the properties of the Shapley value.[1] The Shapley value not only has normative properties but also a variety of applications and strategic foundations.[2] In contrast, the core is a set-valued solution, which is a set of payoff allocations from which no groups of players have an incentive to deviate. Its axiomatic properties and strategic foundations have also been intensively studied.[3] The concept of the core is, because of its simplicity and generality, used in a wide range of fields including microeconomics, bargaining theory and matching theory.

If the Shapley value is in the core, it can be seen as a stable allocation that is free from any coalitional deviations. In this sense, the Shapley value should be an attractive core selection. However, to obtain the stable Shapley allocation, we have to face the following difficulty: The Shapley value may be outside the core for some games. In other words, what is the condition for the Shapley value to lie in the core? One of the most eminent conditions is *convexity*, introduced by Shapley (1971). He shows that if a game is convex, the Shapley value lies in the core. Inarra and Usategui (1993) and Izawa and Takahashi (1998) propose a weaker condition called *average convexity* and show that it is also a sufficient condition.[4] In addition to the

[1]See Young (1985), Casajus (2011, 2014) and Casajus and Yokote (2017).

[2]Shapley and Shubik (1954) apply the Shapley value to evaluate the distribution of power among the members of a committee system. Hart and Moore (1990) use the Shapley value as each agent's payoff to analyze the incomplete contract model. Gul (1989), Pérez-Castrillo and Wettstein (2001) and McQuillin and Sugden (2016) provide implementation procedures for obtaining the Shapely value as the subgame perfect equilibrium outcome of the game.

[3]Consistency properties play a central role in axiomatic characterizations of the core. Davis and Maschler (1965), Moulin (1985), Peleg (1986) and Tadenuma (1992) introduce different types of consistencies and axiomatize the core. Abe (2017) axiomatically characterizes the core for games with externalities. Perry and Reny (1994) offer a noncooperative game in which a core element is implemented.

[4]Average convexity is also analyzed by Sprumont (1990). He calls it *quasiconvexity* in his work. However, his approach is totally different from those of Inarra and Usategui (1993) and Izawa and Takahashi (1998). He defines the Shapley value for every subset of the grand coalition and considers an allocation scheme for all possible coalitions. He shows that an allocation scheme is population monotonic for every quasiconvex game.

sufficient conditions above, they provide some necessary and sufficient conditions. Although these necessary and sufficient conditions are important steps toward understanding the Shapley value and the core, because of their complexity, it is not straightforward to derive applicable insights from the conditions.[5] Therefore, in this paper, we attempt to provide a new necessary and sufficient condition for the Shapley value to be in the core.

To this end, we first consider a geometric property of the set of balanced games, namely, the set of games with a nonempty core.[6] Bondareva (1963) and Shapley (1967) show that a game is balanced if and only if a weighted sum of the worth of every coalition is less than that of the grand coalition. These weights are called balanced vectors, and the set of the balanced vectors is a convex set. On the basis of this result, we show that the set of balanced games yields a polar cone of a polyhedral cone that is generated from extended balanced vectors. Moreover, we obtain the explicit representation of the generating matrix for the polyhedral cone. Then, by applying Minkowski-Weyl's theorem, which is often used in the theory of convex polyhedra, we obtain the explicit characterization of the extreme rays of the set of balanced games. As a result, we also show that a game is balanced if and only if the game has a nonnegative linear combination of the games, each of which corresponds to the extreme rays: singleton unanimity games, negative singleton unanimity games, and negative standard basis games with strict subsets of the grand coalition. This result is a generalization of the decomposition result of Abe (2019), which describes the relationship between the nonemptiness of the core and the class of zero-normalized nonnegative games.

In addition to the decomposition result of the balanced games, we characterize the set of games whose Shapley value is an element of the core. We show that the set of such games also generates a certain polyhedral cone. To obtain the extreme rays of the set, we adopt the following two steps. First, by the result of Yokote, Funaki and Kamijo (2016), we decompose an arbitrary game into the sum of two classes of games: singleton unanimity games and the games whose Shapley value is a zero vector. As elaborated below, the Shapley value (of the original game) lies in the core if and only if the core of the latter class of games contains the zero vector as its element. Considering that the latter class of games can be decomposed into a nonnegative linear combination of negative standard basis games with strict subsets of the

[5]We discuss their conditions in Section 4.

[6]Shapley (1971) provides the geometric characterization of the core in convex games. Marinacci and Montrucchio (2004) provide a similar characterization by means of the Choquet integral with respect to the underlying game.

grand coalition, we identify the condition by which the Shapley value of the latter class of games coincides with the zero vector. Second, we introduce generalized balanced vectors by relaxing some constraints of balanced vectors. We show that the set of generalized balanced vectors is a polyhedral cone and that the games that correspond to the extreme rays of the set of generalized balanced vectors constitutes the extreme rays of the set of latter class of games. Combining these two steps, we conclude that the Shapley value of an arbitrary game belongs to the core if and only if it is decomposed into a nonnegative linear combination of some "easy" games. Moreover, we also show that the number of the abovementioned extreme rays coincides with the number of minimal balanced collections.

The remainder of this paper is organized as follows. Section 2 provides basic definitions. In Section 3, we introduce key results for the polyhedral cone and provide a characterization result of balanced games. On the basis of the results discussed in Section 3, we provide our main result in Section 4. Section 5 offers some concluding remarks.

2 Preliminaries

2.1 TU-games

Let $N = \{1, \cdots, n\}$ be the set of players and a function $v : 2^n \to \mathbb{R}$ with $v(\emptyset) = 0$ denote a characteristic function. A *coalition* of players is a nonempty subset of the player set $S \subseteq N$. We denote the cardinality of coalition S by $|S|$. We use n to denote $|N|$. A cooperative game with transferable utility (a TU-game) is a pair (N, v). We fix the player set N throughout this paper and typically use v instead of (N, v) to denote a game. Let $\mathcal{G}_N$ be the set of all TU-games with the player set N.

For each nonempty $T \subseteq N$, a unanimity game $u_T \in \mathcal{G}_N$ is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Shapley (1953a) shows that a game $v \in \mathcal{G}_N$ is represented as a unique linear combination of unanimity games: For every game $v \in \mathcal{G}_N$, there are unique values λ^v_T, $\emptyset \neq T \subseteq N$ such that

$$v(S) = \sum_{\emptyset \neq T \subseteq N} \lambda^v_T u_T(S) = \sum_{\emptyset \neq R \subseteq S} \lambda^v_R, \tag{1}$$

where $\lambda^v_T = \sum_{\emptyset \neq R \subseteq T} (-1)^{|T|-|R|} v(R)$. For simplicity, we omit v and write λ_T instead of λ^v_T when there is no ambiguity. We use λ to denote the vector $(\lambda_T)_{\emptyset \neq T \subseteq N} \in \mathbb{R}^{2^n - 1}$.

For each nonempty $T \subseteq N$, a commander game $\bar{u}_T \in \mathcal{G}_N$ is defined as

$$\bar{u}_T(S) = \begin{cases} 1 & \text{if } |T \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Yokote, Funaki and Kamijo (2016) show that $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ is another basis of $\mathcal{G}_N$: A game v is represented as $v = \sum_{\emptyset \neq T \subseteq N} d_T \bar{u}_T$, where $d = (d_T)_{\emptyset \neq T \subseteq N}$ is the coefficient of the corresponding $\bar{u}_T$. Note that $\bar{u}_{\{i\}} = u_{\{i\}}$ for every $i \in N$.

2.2 Shapley value and core

Let σ be a permutation of N. For every game v, player i's marginal contribution in σ is $mc_{i,\sigma} = v(\rho_i^\sigma \cup \{i\}) - v(\rho_i^\sigma)$ where ρ_i^σ is the set of predecessors of player i in σ. Let Π be the set of all permutations. The Shapley value $Sh(v)$ is given as follows: For every $i \in N$,

$$Sh_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} mc_{i,\sigma}.$$

For every unanimity game, $Sh(v)$ satisfies

$$Sh_i(u_T) = \begin{cases} 1/|T| & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, in view of the linearity of Sh and (1), it follows that

$$Sh_i(v) = \sum_{T \subseteq N, i \in T} \lambda_T / |T| \tag{2}$$

where $\lambda_T/|T|$ is called *Harsanyi's dividend* to the members of T.[7] Moreover, Yokote, Funaki, and Kamijo (2016) show that, for every $i \in N$, $d_{\{i\}} = Sh_i(v)$, that is, the coefficients of singleton commander games coincide with the Shapley value and commander games $(\bar{u}_T)_{T,|T|\geq 2}$ span the null space of the Shapley value; $Sh(\bar{u}_T) = \mathbf{0}$ for every $T \subseteq N$ with $|T| \geq 2$. Hence, each game v is uniquely represented as

$$v = \sum_{i \in N} Sh_i(v) u_{\{i\}} + \sum_{\emptyset \neq T \subseteq N, |T| \geq 2} d_T \bar{u}_T. \tag{3}$$

The core $C(v)$ is the set of allocations given by

$$C(v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{j \in N} x_j \leq v(N) \text{ and } \sum_{j \in S} x_j \geq v(S) \text{ for all } S \subseteq N \right\}.$$

[7] A solution $f : \mathcal{G}_N \to \mathbb{R}^n$ is linear if for every $c, c' \in \mathbb{R}$ and $v, v' \in \mathcal{G}_N$, $f(cv + c'v') = cf(v) + c'f(v')$.

A game is said to be balanced if it has a nonempty core. Let $\mathcal{G}_N^B = \{v \in \mathcal{G}_N | C(v) \neq \emptyset\}$ be the set of balanced games. Bondareva (1963) and Shapley (1967) provide the following characterization of balanced games.

Theorem 1 (Bondareva, 1963; Shapley, 1967)**.** $v \in \mathcal{G}_N^B$ if and only if

$$\sum_{\emptyset \neq S \subsetneq N} \gamma_S v(S) \leq v(N)$$

for every $\gamma \in \mathbb{R}_+^{2^n-2}$ such that for every $i \in N$

$$\sum_{\emptyset \neq S \subsetneq N, i \in S} \gamma_S = 1. \cdots (*)$$

The condition in Theorem 1 is called the Bondareva-Shapley condition. A vector $\gamma \in \mathbb{R}_+^{2^n-2}$ that satisfies the above condition $(*)$ is called a balanced vector, and set $\mathcal{B} = \{S \subsetneq N | \gamma_S > 0\}$ is called a balanced collection. Note that the set of balanced vectors is a convex set, so that each balanced vector is a convex combination of its extreme points (see, for example, Peleg and Sudhölter, 2007). A balanced collection corresponding to some extreme point of the set of balanced vectors is called a minimal balanced collection. Let K_n be the total number of minimal balanced collections of an n-player game.[8]

3 Decomposition of balanced games

In this section, we provide a geometric characterization of the set of balanced games $\mathcal{G}_N^B$ and show that each balanced game can be decomposed into "easier" games. We first reformulate the Bondareva-Shapley condition as follows.

Proposition 1. $v \in \mathcal{G}_N^B$ if and only if

$$\sum_{\emptyset \neq S \subsetneq N} \gamma_S v(S) + \gamma_N v(N) \leq 0$$

for every $\gamma \in \mathbb{R}^{2^n-1}$ such that

$$(*)_1 \cdots \begin{cases} \sum_{\emptyset \neq S \subseteq N, i \in S} \gamma_S \leq 0, \forall i \in N, \\ -\sum_{\emptyset \neq S \subseteq N, i \in S} \gamma_S \leq 0, \forall i \in N, \\ -\gamma_S \leq 0, \forall S \subsetneq N. \end{cases}$$

[8]The explicit description of each extreme point in general n-player games is still open. This is because it is generally difficult to construct extreme points of a convex polyhedron. Peleg (1965) provides an algorithm to calculate all extreme points.

Proof. The Bondareva-Shapley condition is equivalent to the following:

$$\sum_{\emptyset \neq S \subsetneq N} \gamma_S v(S) + \gamma_N v(N) \leq 0$$

for every $\gamma \in \mathbb{R}^{2^n-1}$ such that

$$(*)_2 \cdots \begin{cases} \sum_{\emptyset \neq S \subsetneq N, i \in S} \gamma_S \leq 1, \forall i \in N, \\ -\sum_{\emptyset \neq S \subsetneq N, i \in S} \gamma_S \leq -1, \forall i \in N, \\ -\gamma_S \leq 0, \forall S \subsetneq N, \\ \gamma_N \leq -1, \\ -\gamma_N \leq 1. \end{cases}$$

Since $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_2$ satisfies $(*)_1$, the set of vectors satisfying $(*)_2$ is a subset of the set of vectors satisfying $(*)_1$. Hence, if $\gamma \cdot v \leq 0$ for every $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_1$, it also holds for all $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_2$.[9]

Now, suppose that $\gamma \cdot v \leq 0$ for every $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_2$. Take any $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_1$. Note that $\gamma_N \leq 0$ because $\gamma_S \geq 0 \; \forall S \subsetneq N$. If $\gamma_N = 0$, then $\gamma = \mathbf{0}$, and $\gamma \cdot v \leq 0$ holds. If $\gamma_N < 0$, let $\gamma'_S = \frac{\gamma_S}{-\gamma_N} > 0$ for all $S \subsetneq N$ and $\gamma'_N = -1$. Then, $\gamma' = (\gamma'_S)_{S \subseteq N}$ satisfies $(*)_2$. Moreover, by the assumption,

$$\sum_{S \subsetneq N} \gamma_S v(S) + \gamma_N v(N) = (-\gamma_N)\Big(\sum_{S \subsetneq N} (\frac{\gamma_S}{-\gamma_N}) v(S) - v(N)\Big) = (-\gamma_N)(\gamma' \cdot v) \leq 0.$$

Hence, if $\gamma \cdot v \leq 0$ for every $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_2$, it also holds for all $\gamma \in \mathbb{R}^{2^n-1}$ satisfying $(*)_1$. □

The set $P = \{x \in \mathbb{R}^d | Ax \leq \mathbf{0}\}$ for some matrix $A \times \mathbb{R}^{m \times d}$ is called a polyhedral cone, and the $P^o = \{y \in \mathbb{R}^d | x \cdot y \leq 0, \forall x \in P\}$ is called a polar cone of P. Note that the set of vectors $\gamma \in \mathbb{R}^{2^n-1}$ satisfying the condition $(*)_1$ is a polyhedral cone represented as a matrix $R^t \in \mathbb{R}^{2^n+2n-2 \times 2^n-1}$ such that $P = \{\gamma \in \mathbb{R}^{2^n-1} | R^t \gamma \leq \mathbf{0}\}$. The set of balanced vectors, which is defined by $(*)$, is a cross-section of the polyhedral cone with $\gamma_N = -1$. Proposition 1 shows that we can identify the set of balanced games with a polar cone of P by enlarging the set of balanced vectors. The following result plays an important role in finding another representation of a cone.[10]

[9] For every $a, b \in \mathbb{R}^k$, $a \cdot b = \sum_{i=1}^k a_i b_i$ is a standard inner product in $\mathbb{R}^k$.

[10] For details, see Ziegler (1995).

Theorem 2 (Minkowski-Weyl's Theorem)**.** For $P \subseteq \mathbb{R}^d$, the following two statements are equivalent:

(1) There exists a matrix $A \times \mathbb{R}^{m\times d}$ for some m such that $P = \{x \in \mathbb{R}^d | Ax \leq \mathbf{0}\}$.

(2) There exists a matrix $R \times \mathbb{R}^{d\times k}$ for some k such that $P = \{x \in \mathbb{R}^d | x = R\mu, \mu \geq \mathbf{0}\}$.

A matrix A is called a generating matrix of P. The representation of cone P in the manner of (1) is its *H-representation* and that of (2) is its *V-representation*. Moreover, (1) $\Rightarrow$ (2) is known as Minkowski's Theorem and the converse, (2) $\Rightarrow$ (1), is known as Weyl's Theorem. A pair of matrices (A, R) that represents the same cone $P \subseteq \mathbb{R}^d$ is called a DD-pair (double description pair). For a DD-pair (A, R), as a corollary of Theorem 2, (R^t, A^t) is also a DD-pair, and the cone $P^o = \{y \in \mathbb{R}^d | R^t y \leq \mathbf{0}\} = \{y \in \mathbb{R}^d | y = A^t\mu, \mu \geq \mathbf{0}\}$ is the polar cone of P.

Applying the above discussion to the set of balanced games, we obtain the result that $v \in \mathcal{G}_N^B$ if and only if v is represented as

$$v = R\mu, \mu \geq \mathbf{0}.$$

Therefore, the column vectors of R correspond to extreme rays of $\mathcal{G}_N^B$. To be more specific, we now define a negative standard basis game as follows: For every $S \subseteq N$,

$$u_S^-(T) = \begin{cases} -1 & \text{if } T = S, \\ 0 & \text{otherwise.} \end{cases}$$

By the above discussion, we obtain the following decomposition result.

Theorem 3. $v \in \mathcal{G}_N^B$ if and only if it is a sum of a linear combination of singleton unanimity games and a positive linear combination of negative standard basis games with $S \subsetneq N$:

$$v = \sum_{i\in N} \alpha_i u_{\{i\}} + \sum_{\emptyset\neq S\subsetneq N} \alpha_S^- u_S^-$$

where $(\alpha_i)_{i\in N} \in \mathbb{R}^n$ and $(\alpha_S^-)_{S\subsetneq N} \geq \mathbf{0}$.

Corollary 1. The number of the extreme rays of $\mathcal{G}_N^B$ is $2^n + 2n - 2$. Each of them corresponds to singleton unanimity games, negative singleton unanimity games, and negative standard basis games with $S \subsetneq N$.

Table 1 shows R^t for $n = 3$ where R_i^t is the i-th row vector of R^t. This decomposition result is also useful for considering a certain subclass of balanced games. If we consider 0-normalized games,[11] $R^t_4, R^t_5, R^t_6, R^t_7, R^t_8$ and R^t_9 are excluded as extreme rays and $R^t_1 + R^t_4, R^t_2 + R^t_5, R^t_3 + R^t_6$ appear as new extreme rays. In general, it can be obtained by considering the condition for the existence of nonnegative core allocations. In the same manner as in Proposition 1, the problem reduces to

$$\sum_{\emptyset \neq S \subsetneq N, |S| \geq 2} \gamma_S v(S) + \gamma_N v(N) \leq 0$$

for every $\gamma \in \mathbb{R}^{2^n - n - 1}$ such that

$$(*)_2' \cdots \begin{cases} \sum_{\emptyset \neq S \subseteq N, |S| \geq 2, i \in S} \gamma_S \leq 0, \forall i \in N, \\ -\gamma_S \leq 0, \forall S \subsetneq N, |S| \geq 2. \end{cases}$$

Then, 0-normalized balanced games can be written as

$$v = \tilde{R}\mu, \mu \geq \mathbf{0}.$$

Table 2 shows $\tilde{R}^t$ for $n = 3$ where $\tilde{R}^t_i$ is the i-th row vector of $\tilde{R}^t$. In addition, every nonnegative 0-normalized game is represented as the nonnegative linear combination of 0-normalized simple N-monotonic veto-controlled games, which is first shown by Abe (2019).[12] We can straightforwardly prove this characterization result by using the above corollary of Theorem 3. Since the case of $v(N) = 0$ is obvious, without loss of generality, we assume that $v(N) = 1$. Then, by Theorem 3, v is balanced if and only if

$$\begin{cases} \alpha_{\{i\}} - \alpha^-_{\{i\}} = 0, \forall i \in N, \\ \sum_{i \in N} \alpha_{\{i\}} = 1, \\ \sum_{i \in S} \alpha_{\{i\}} - \alpha^-_S \in [0, 1], \forall S \subsetneq N, \\ \alpha_i, \alpha^-_S \geq 0, \forall i \in N, S \subsetneq N. \end{cases}$$

Since the set of vectors $\alpha = \left((\alpha_i)_{i \in N}, (\alpha^-_S)_{S \subsetneq N}\right) \in \mathbb{R}^{2^n + n - 1}$ satisfying the above conditions is convex, it is sufficient to consider its extreme points. Then, we can see that α is an extreme point if and only if there is $i \in N$ such that

$$\begin{cases} \alpha_{\{i\}} = \alpha^-_{\{i\}} = 1, \alpha_{\{j\}} = 0, \forall j \neq i, \\ \alpha^-_S = 0, \forall S \subseteq N \setminus \{i\}, \\ \alpha^-_S \in \{0, 1\}, \forall S \subsetneq N, \text{ with } i \in S. \end{cases}$$

[11] A game v is 0-normalized if $v(\{i\}) = 0$ for all $i \in N$.

[12] A game v is simple if $v(S) = 0$ or 1 for all $S \subseteq N$. A player $i \in N$ is a veto player in v if $v(S) = 0$ For every $S \subset N \setminus \{i\}$. A game v is veto-controlled if there is a veto player in v. A game v is N-monotonic if $v(S) \leq v(N)$ or 1 for all $S \subseteq N$.

Notice that i is a veto player in the game corresponding to such α, so that it is a veto-controlled game. The game is 0-normalized, simple and N-monotonic. Table 3 shows the extreme points in the case of $n = 3$.

$R^t \setminus S$	1	2	3	1,2	1,3	2,3,	N
R^t_1	1	0	0	1	1	0	1
R^t_2	0	1	0	1	0	1	1
R^t_3	0	0	1	0	1	1	1
R^t_4	−1	0	0	−1	−1	0	−1
R^t_5	0	−1	0	−1	0	−1	−1
R^t_6	0	0	−1	0	−1	−1	−1
R^t_7	−1	0	0	0	0	0	0
R^t_8	0	−1	0	0	0	0	0
R^t_9	0	0	−1	0	0	0	0
R^t_{10}	0	0	0	−1	0	0	0
R^t_{11}	0	0	0	0	−1	0	0
R^t_{12}	0	0	0	0	0	−1	0

Table 1: Extreme points of balanced games for $n = 3$.

4 The Shapley value and the core

Let $\mathcal{G}_N^{Sh} = \{v \in \mathcal{G}_N | Sh(v) \in C(v)\}$ be the set of games whose Shapley value is in the core. It follows that $\mathcal{G}_N^{Sh} \subsetneq \mathcal{G}_N^B$. In this section, we characterize the set $\mathcal{G}_N^{Sh}$ in view of the decomposition result of $\mathcal{G}_N^B$ discussed in Theorem 3.

For each balanced collection $\mathcal{B}$ and a corresponding balanced vector $\gamma^{\mathcal{B}} \in \mathbb{R}_+^{2^n-2}$, define

$$v^{\mathcal{B}} = \sum_{S \subsetneq N} |S| \gamma_S u_S.$$

Then, $Sh_i(v^{\mathcal{B}}) = 1$ for every $i \in N$ because, by (2), for every $i \in N$,

$$Sh_i(v^{\mathcal{B}}) = \sum_{\emptyset \neq S \subsetneq N, i \in S} \frac{|S|\gamma_S}{|S|} = \sum_{\emptyset \neq S \subsetneq N, i \in S} \gamma_S = 1.$$

$\tilde{R}^t \setminus S$	1	2	3	1,2	1,3	2,3,	N
$\tilde{R}^t_1$	0	0	0	1	1	0	1
$\tilde{R}^t_2$	0	0	0	1	0	1	1
$\tilde{R}^t_3$	0	0	0	0	1	1	1
$\tilde{R}^t_4$	0	0	0	-1	0	0	0
$\tilde{R}^t_5$	0	0	0	0	-1	0	0
$\tilde{R}^t_6$	0	0	0	0	0	-1	0

Table 2: Extreme points of 0-normalized balanced games for $n = 3$.

$\setminus S$	1	2	3	1,2	1,3	2,3,	N
v^1	0	0	0	1	1	0	1
v^2	0	0	0	1	0	1	1
v^3	0	0	0	0	1	1	1
v^{12}	0	0	0	1	0	0	1
v^{13}	0	0	0	0	1	0	1
v^{23}	0	0	0	0	0	1	1
v^N	0	0	0	0	0	0	1

Table 3: Extreme points of 0-normalized nonnegative balanced games for $n = 3$.

In other words, the balanced collection and the balanced vector $(\mathcal{B}, \gamma^{\mathcal{B}})$ yield a game $v^{\mathcal{B}}$ in which the Shapley value assigns 1 to every player.[13] In the same vein, we construct another game that is useful for our further analysis. We say that the vector $\beta = \Big((\beta_S)_{S \subsetneq N}, \beta_N\Big) \in \mathbb{R}_+^{2^n-2} \times \mathbb{R}$ is a generalized balanced vector if for every $i \in N$,

$$\sum_{\emptyset \neq S \subsetneq N, i \in S} \beta_S + \beta_N = 0.$$

Note that this condition is equivalent to $(*)_1$: β with $\beta_N = -1$ being a balanced vector. For every generalized balanced vector $\beta = \Big((\beta_S)_{S \subsetneq N}, \beta_N\Big) \in \mathbb{R}_+^{2^n-2} \times \mathbb{R}$, define

$$v^{\beta-} = -\sum_{S \subsetneq N} |S| \beta_S u_S - n\beta_N u_N.$$

[13]In the decision theory literature, Dillenberger and Sadowski (2019) propose a similar concept, which they call generalized partition.

We write $v^{\mathcal{B}-} = v^{\beta(\mathcal{B})-}$ where $\beta(\mathcal{B}) = (\gamma^{\mathcal{B}}, -1)$, and $(\mathcal{B}, \gamma^{\mathcal{B}})$ is a balanced collection and its corresponding balanced vector.

Lemma 1. For every generalized balanced vector $\beta = \Big((\beta_S)_{S\subsetneq N}, \beta_N\Big) \in \mathbb{R}_+^{2^n-2} \times \mathbb{R}$, the game $v^{\beta-}$ has the following properties.

(i) $Sh_i(v^{\beta-}) = 0$ for every $i \in N$.

(ii) There is $(\alpha_S^-)_{S\subsetneq N} \geq \mathbf{0}$ such that $v^{\beta-} = \sum_{\emptyset\neq S\subsetneq N} \alpha_S^- u_S^-$.

(iii) There is $\mu \in \mathbb{R}_n^{K_n}$ such that $v^{\beta-} = \sum_{k=1}^{K_n} \mu_k v^{\mathcal{B}_k-}$ where $\mathcal{B}_k$ is a minimal balanced collection.

Proof. (i) It follows by construction: For every $i \in N$,

$$Sh_i(v^{\beta-}) = -\sum_{\emptyset\neq S\subseteq N, i\in S} \frac{|S|\beta_S}{|S|} = -\big(\sum_{\emptyset\neq S\subsetneq N, i\in S} \beta_S + \beta_N\big) = 0.$$

(ii) we have

$$\begin{aligned} v^{\beta-} &= -\sum_{T\subsetneq N} |T|\beta_T u_T - n\beta_N u_N \\ &= \sum_{T\subsetneq N} |T|\beta_T \big(\sum_{T\subseteq S} u_S^-\big) - n\beta_N u_N \\ &= \sum_{S\subseteq N} \Big(\sum_{T\subseteq S} |T|\beta_T\Big) u_S^- + n\beta_N u_N \\ &= \sum_{S\subsetneq N} \Big(\sum_{T\subseteq S} |T|\beta_T\Big) u_S^- \\ &= \sum_{S\subsetneq N} \alpha_S^- u_S^- \end{aligned}$$

since $-\Big(\sum_{T\subseteq N} |T|\beta_T + n\beta_N\Big) = -\sum_{i\in N}\Big(\sum_{\emptyset\neq T\subsetneq N, i\in T} \beta_T + \beta_N\Big) = 0$. Then, $\alpha_S^- = \Big(\sum_{T\subseteq S} |T|\beta_T\Big) \geq 0$ follows because $(\beta_S)_{S\subsetneq N} \in \mathbb{R}_+^{2^n-1}$.

(iii) For each minimal balanced collection and its corresponding balanced vector, $(\mathcal{B}, \gamma^{\mathcal{B}})$, let $\beta(\mathcal{B}) = (\gamma^{\mathcal{B}}, -1)$. Since the set of generalized balanced vectors is a polyhedral cone whose extreme rays are $\beta(\mathcal{B})$ where $\mathcal{B}$ is a minimal balanced collection, for every generalized balanced vector β, the corresponding game $v^{\beta-}$ can be represented as a nonnegative linear combination of $v^{\mathcal{B}-}$, namely, $v^{\beta-} = \sum_{k=1}^{K_n} \mu_k v^{\mathcal{B}_k-}$ where $(\mu_k)_{k=1}^{K_n} \geq \mathbf{0}$. □

Lemma 1 provides the extreme rays of the subset of games

$$\hat{\mathcal{G}}_N^- \subset \mathcal{G}_N^- = \Big\{v \in \mathbb{R}^{2^n-1} | v = \sum_{\emptyset\neq S\subsetneq N} \alpha_S^- u_S^-,\ (\alpha_S^-)_{S\subsetneq N} \geq \mathbf{0},\ Sh(v) = \mathbf{0}\Big\}.$$

By (ii) of Lemma 1, what we need to describe $\mathcal{G}_N^-$ is the constraint

$$\alpha_S = \sum_{T \subseteq S} |T| \beta_T \geq 0 \cdots (**)$$

for every $S \subsetneq N$, which is a weaker condition than $(\beta_S)_{S \subsetneq N} \in \mathbb{R}_+^{2^n-1}$. Therefore, we consider the following vectors $\beta \in \mathbb{R}^{2^n-1}$ such that

$$\begin{cases} \sum_{\emptyset \neq S \subsetneq N, i \in S} \beta_S + \beta_N = 0, \\ \beta \text{ satisfies } (**). \end{cases}$$

We call vectors $\beta \in \mathbb{R}^{2^n-1}$ that satisfy the above condition weakly generalized balanced vectors.

Lemma 2. For every $v \in \mathcal{G}_N^-$, there is $\mu_k \in \mathbb{R}_n^{K_n}$ such that $v = \sum_{k=1}^{K_n} \mu_k \tilde{v}^{\mathcal{B}_k -}$ where $\tilde{v}^{\mathcal{B}_k -} = v^{\tilde{\beta}(\mathcal{B})-}$ for some weakly generalized balanced vector $\tilde{\beta}(\mathcal{B})$ with a minimal balanced collection $\mathcal{B}$.

Proof. For every $\hat{\beta} \in \mathbb{R}^{2^n-2}$, the condition $\hat{\beta} \geq \mathbf{0}$ is equivalent to

$$E_{2^n-2} \hat{\beta} \geq \mathbf{0},$$

where $E_{2^n-2} \in \mathbb{R}^{2^n-2 \times 2^n-2}$ is the identity matrix. Similarly, the condition $(**)$ is represented by

$$A\hat{\beta} \geq \mathbf{0},$$

where $A \in \mathbb{R}^{2^n-2 \times 2^n-2}$ and it has full rank. Since an extreme ray of a polyhedral cone in $\mathbb{R}^k$ is characterized by the $k-1$ linearly independent equations of its generating matrix, an extreme ray of the set of weakly generalized balanced vectors must satisfy

$$A_I \hat{\beta} = \mathbf{0},$$

where $A_I \in \mathbb{R}^{|I| \times 2^n-2}$ is a submatrix of A for some index set $I \subseteq \{1, \cdots 2^n - 2\}$. Then, for every index set $I \subseteq \{1, \cdots 2^n - 2\}$ and $\hat{\beta}$ satisfying $(E_{2^n-2})_I \hat{\beta} = \mathbf{0}$, we have

$$\begin{aligned} \mathbf{0} &= (E_{2^n-2})_I \hat{\beta} \\ &= (AA^{-1})_I \hat{\beta} \\ &= A_I A^{-1} \hat{\beta} \\ &= A_I \tilde{\beta} \end{aligned}$$

where $\tilde{\beta} = A^{-1}\hat{\beta}$. Conversely, for every index set $I \subseteq \{1, \cdots 2^n - 2\}$ and $\tilde{\beta}$ satisfying $A_I \tilde{\beta} = \mathbf{0}$, we have

$$\begin{aligned} \mathbf{0} &= A_I \tilde{\beta} \\ &= (E_{2^n-2})_I A \tilde{\beta} \\ &= (E_{2^n-2})_I \hat{\beta} \end{aligned}$$

where $\hat{\beta} = A\tilde{\beta}$. Therefore, there is a one-to-one relationship between an extreme ray of the generalized balanced vectors, $\beta(\mathcal{B})$, and that of weakly generalized balanced vectors. Therefore, For every $v \in \mathcal{G}_N^-$, there is $\mu_k \in \mathbb{R}_n^{K_n}$ such that $v = \sum_{k=1}^{K_n} \mu_k \tilde{v}^{\mathcal{B}_k-}$ where $\tilde{v}^{\mathcal{B}_k-} = v^{\tilde{\beta}(\mathcal{B})-}$ with $(\tilde{\beta}_S(\mathcal{B}))_{S\subsetneq N} = A^{-1}(\beta_S(\mathcal{B}))_{S\subsetneq N}$. □

The following Tables 4 and 5 show $\tilde{v}^{\mathcal{B}-}$ for $n = 3, 4$ respectively.

$\mathcal{B}$	$(\beta_S)^t_{S\in\mathcal{B}}$	$\tilde{v}^{\mathcal{B}-}$
$\{1,2,3\}$	$(1,1,1)$	$u^-_{\{1\}} + u^-_{\{2\}} + u^-_{\{3\}}$
$\{1,23\}$	$(1,1)$	$u^-_{\{1\}} + u^-_{\{2,3\}}$
$\{2,13\}$	$(1,1)$	$u^-_{\{2\}} + u^-_{\{1,3\}}$
$\{3,12\}$	$(1,1)$	$u^-_{\{3\}} + u^-_{\{1,2\}}$
$\{12,13,23\}$	$(1/2,1/2,1/2)$	$u^-_{\{1,2\}} + u^-_{\{1,3\}} + u^-_{\{2,3\}}$

Table 4: $\tilde{v}^{\mathcal{B}-}$ for $n = 3$.

$\mathcal{B}$	$(\beta_S)^t_{S\in\mathcal{B}}$	$\tilde{v}^{\mathcal{B}-}$
$\{12,34\}$	$(1,1)$	$u^-_{\{1,2\}} + u^-_{\{3,4\}}$
$\{123,4\}$	$(1,1)$	$u^-_{\{1,2,3\}} + u^-_{\{4\}}$
$\{12,3,4\}$	$(1,1,1)$	$2u^-_{\{1,2\}} + u^-_{\{3\}} + u^-_{\{4\}}$
$\{123,124,34\}$	$(1/2,1/2,1/2)$	$u^-_{\{1,2,3\}} + u^-_{\{1,2,4\}} + 2u^-_{\{3,4\}}$
$\{1,2,3,4\}$	$(1,1,1,1)$	$u^-_{\{1\}} + u^-_{\{2\}} + u^-_{\{3\}} + u^-_{\{4\}}$
$\{12,13,23,4\}$	$(1/2,1/2,1/2,1)$	$u^-_{\{1,2\}} + u^-_{\{1,3\}} + u^-_{\{2,3\}} + u^-_{\{4\}}$
$\{123,14,24,3\}$	$(1/2,1/2,1/2,1/2)$	$u^-_{\{1,2,3\}} + 2u^-_{\{1,4\}} + 2u^-_{\{2,4\}} + u^-_{\{3\}}$
$\{123,14,24,34\}$	$(2/3,1/3,1/3,1/3)$	$u^-_{\{1,2,3\}} + u^-_{\{1,4\}} + u^-_{\{2,4\}} + u^-_{\{3,4\}}$
$\{123,124,134,234\}$	$(1/3,1/3,1/3,1/3)$	$u^-_{\{1,2,3\}} + u^-_{\{1,2,4\}} + u^-_{\{1,3,4\}} + u^-_{\{2,3,4\}}$

Table 5: $\tilde{v}^{\mathcal{B}-}$ for $n = 4$. Each $\mathcal{B}$ is symmetric under permutation.

Now, we are ready to state our main result of the decomposition of $\mathcal{G}_N^{Sh}$.

Theorem 4. Let $v \in \mathcal{G}_N$. We have $Sh(v) \in C(v)$ if and only if it is a sum of (i) a linear combination of singleton unanimity games and (ii) a positive linear combination of $\tilde{v}^{\mathcal{B}-}$ where

$\mathcal{B}$ is a minimal balanced collection:

$$v = \sum_{i \in N} \alpha_i u_{\{i\}} + \sum_{k=1}^{K_n} \mu_k \tilde{v}^{\mathcal{B}_k -}$$

where $(\alpha_i)_{i \in N} = Sh(v)$ and $(\mu_k)_{k=1}^{K_n} \geq \mathbf{0}$.

Proof. Let us define

$$v^+ = \sum_{i \in N} Sh_i(v) u_{\{i\}}.$$

Since game v is uniquely represented as the linear combination of commander games by (3), we write

$$v = v^+ + w$$

where $Sh_i(w) = 0$ for every $i \in N$. Then, $Sh(v) \in C(v)$ if and only if $Sh(w) = \mathbf{0} \in C(w)$. Note that $\mathbf{0} \in C(w)$ implies that $w(S) \leq 0$ for all $S \subseteq N$ and $w(N) = 0$. Moreover, since $(u_S^-)_{S \subseteq N}$ is a basis of $\mathcal{G}_N$, w must be (uniquely) represented as $w = \sum_{\emptyset \neq S \subsetneq N} \alpha_S^- u_S^-$ for some $(\alpha_S^-)_{S \subsetneq N} \geq \mathbf{0}$. Therefore, the above argument shows that v is decomposed into

$$v = \sum_{i \in N} Sh_i(v) u_{\{i\}} + \sum_{\emptyset \neq S \subsetneq N} \alpha_S^- u_S^-$$

where $(\alpha_S^-)_{S \subsetneq N} \geq \mathbf{0}$ and $Sh(\sum_{\emptyset \neq S \subsetneq N} \alpha_S u_S^-) = \mathbf{0}$. Since $w = \sum_{\emptyset \neq S \subsetneq N} \alpha_S u_S^- \in \mathcal{G}_N^-$, by Lemma 2, it can be represented as

$$w = \sum_{k=1}^{K_n} \mu_k \tilde{v}^{\mathcal{B}_k -}$$

where $(\mu_k)_{k=1}^{K_n} \geq \mathbf{0}$. Therefore, v is decomposed as follows:

$$v = v^+ + w = \sum_{i \in N} Sh_i(v) u_{\{i\}} + \sum_{k=1}^{K_n} \mu_k \tilde{v}^{\mathcal{B}_k -}.$$

□

This result and Lemma 2 suggest that we can count the number of extreme rays of $\mathcal{G}_N^{Sh}$, which also implies that the number of nontrivial extreme rays coincides with the number of minimal balanced games as follows.

Corollary 2. The number of extreme rays of $\mathcal{G}_N^{Sh}$ is $2n + K_n$. Each of them corresponds to singleton unanimity games, negative singleton unanimity games, and the games $(v^{\mathcal{B}_k -})_{k=1}^{K_n}$ corresponding to minimal balanced collections $(\mathcal{B}_k)_{k=1}^{K_n}$.

Before closing this section, we compare our result with other conditions studied in the literature. Inarra and Usategui (1993) show that $v \in \mathcal{G}_N^{Sh}$ if and only if for every $T \subseteq N$,

$$\sum_{\emptyset \neq S \subseteq N} \frac{(n-s)!(s-1)!}{n!} h_T(S)(v(S) - v(S \setminus T) - v(S \cap T)) \geq 0,$$

where

$$h_T(S) = \begin{cases} |S| \cdot \left(\frac{|S \cap T|}{|S|} - \frac{|T \setminus S|}{|N \setminus S|}\right) & \text{if } S \neq N, \\ |T| & \text{if } S = N. \end{cases}$$

Izawa and Takahashi (1998) show that $v \in \mathcal{G}_N^{Sh}$ if and only if for every $T \subseteq N$,

$$\sum_{S \subset N} \sum_{i \in S \cap T} \frac{(n-s)!(s-1)!}{n!} (v^i(S) - v^i(S \cap T)) \geq 0,$$

where $v^i(S) = v(S) - v(S \setminus \{i\})$. Since both conditions are written as a linear transformation of v, there is $A \in \mathbb{R}^{m \times 2^n - 1}$ such that $\mathcal{G}_N^{SC} = \{v \in \mathcal{G}_N | Av \geq \mathbf{0}\}$. Hence, these conditions also show that $\mathcal{G}_N^{Sh}$ is a polyhedral cone. However, both conditions neither provide any information about its extreme rays nor any reduced expression of the generating matrix. In contrast, our approach has the following three advantages. First, we can construct each extreme ray in an explicit way. Second, we can decompose every game in the class $\mathcal{G}_N^{SC}$ into simple games, each of which corresponds to an extreme ray. Finally, and more important, our method is applicable to other linear solutions, whereas the two conditions above are applicable only for the Shapley value.

In the next section, we conclude our results and elaborate the third advantage, i.e., applicability, mentioned above.

5 Concluding remarks

In this paper, we provide the geometric characterization of balanced games and a new necessary and sufficient condition for the Shapley value to be in the core. To be more specific, we show that all balanced games and all games whose Shapley value lies in the core are decomposed into some "easy" games. This result shows that (i) different classes of games may have some common geometric properties and (ii) we can exploit the common properties to analyze solution concepts in the classes. Our approach can be a powerful method when we consider the relationship between the core and any other linear solutions including the Shapley value.[14] We elaborate this point below.

If a solution f is linear, in view of (1), $f_i(v) = \sum_{\emptyset \neq T \subseteq N} \lambda_T f_i(u_T)$ for every $i \in N$. If we can identify a new basis $\big((u_{\{i\}})_{i \in N}, (u_S^f)_{S \subseteq N, |S| \geq 2}\big)$ such that $f(u_S^f) = \mathbf{0}$ for every $S \subseteq N$ with $|S| \geq 2$; then, by the same argument as in Theorem 4, we have $f(v) \in C(v)$ if and only if $f(w^f) = \mathbf{0} \in C(w^f)$ where $v = \sum_{i \in N} f_i(v) u_{\{i\}} + w^f$. To find the condition for $\mathbf{0} \in C(w^f)$, suppose that $f_i(u_T)$ is written as $f_i(u_T) = \frac{f_{i,T}}{f_T}$ for some $f_{i,T} \in \mathbb{R}$ and $f_T > 0$. For instance, if $f = Sh$,

$$f_{i,T} = \begin{cases} 1 \text{ if } i \in T, \\ 0 \text{ otherwise.} \end{cases} \qquad f_T = |T|.$$

Now, consider the following constraints: For every $\beta^f = \Big((\beta_S^f)_{S \subsetneq N}, \beta_N^f\Big) \in \mathbb{R}_+^{2^n - 2} \times \mathbb{R}$ and every $i \in N$,

$$\sum_{S \subsetneq N} f_{i,S} \beta_S^f + f_{i,N} \beta_N^f = 0.$$

A vector β^f satisfying the above condition can be seen as a weighted generalized balanced vector. Now, define a game

$$\begin{aligned} v^{\beta^f -} &= -\sum_{S \subsetneq N} f_T \beta_S u_S - f_T \beta_N u_N \\ &= \sum_{S \subsetneq N} \Big(\sum_{T \subseteq S} f_T \beta_T^f\Big) u_S^-. \end{aligned}$$

In the same manner as Lemma 1, $f_i(v^{\beta^f -}) = 0$ for all $i \in N$. Moreover, each $v^{\beta^f -}$ can be decomposed into a nonnegative linear combination of the games derived from the extreme rays of weighted generalized balanced vectors. Hence, the remained step is to consider the extreme rays of the following polyhedral cone

$$\begin{cases} \sum_{S \subsetneq N} f_{i,S} \beta_S^f + f_{i,N} \beta_N^f = 0, \forall i \in N, \\ \sum_{T \subseteq S} f_T \beta_T^f \geq 0, \forall S \subsetneq N. \end{cases}$$

[14]Various linear solutions are intensively studied as a complement to or a counterpart of the Shapley value: for example, weighted Shapley values (Shapley, 1953a; Chun, 1988, 1991; Kalai and Samet, 1987; Nowak and Radzik, 1995; Yokote, 2015), egalitarian Shapley values and their generalization (Joosten, 1996; Casajus and Huettner, 2013, 2014; van den Brink, Funaki and Ju, 2013; Abe and Nakada, 2019; Yokote, Kongo and Funaki, 2018), and the CIS/ENSC value (Driessen and Funaki, 1991). See also Yokote and Funaki (2017) for other solutions.

Since the matrix that constitutes the inequality constraints has full rank (i.e, ignoring β_N and consider the constraints in dimension $\mathbb{R}^{2^n-2}$) by $f_T > 0$ for all $T \subseteq N$, the same argument as in Lemma 2 generates the extreme rays as desired.

As a special case, Yokote, Funaki, and Kamijo (2016) propose a basis $\big((u_{\{i\}})_{i\in N}, (u_S^f)_{S\subseteq N,|S|\geq 2}\big)$ when f is a weighted Shapley value and a (extended version of) discounted Shapley value. We can straightforwardly apply the above procedure even to these cases. In addition, since our method is applicable for every linear solution as long as a suitable basis is obtained, we can similarly obtain the core selection result for every linear solution.

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