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Government Debt Maturity in Japan: 1965 to the Present

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Abstract

This study constructs and analyzes a dataset of Japanese government bond's maturity structure for the fiscal years 1965–2019. Using the maturity structure data at the end of each fiscal year for the past three decades, this study proposes extracting the bond supply factor from the maturity structure variables, and structurally estimates a canonical preferred-habitat term structure model. The results provide a debt maturity equation in the fiscal year cycle, and demonstrate that two yield factors (bond supply factor and short-term interest rate) can account for annual-frequency variations in Japanese bond yields. The supply factor also explains the continued decline in the long-term interest rate in a zero lower bound environment for the past two decades.

JEL Classification: E43, E52, G11, G12, H63

Keywords: maturity structure, yield curve, debt management, Japan, supply factor, bond yield

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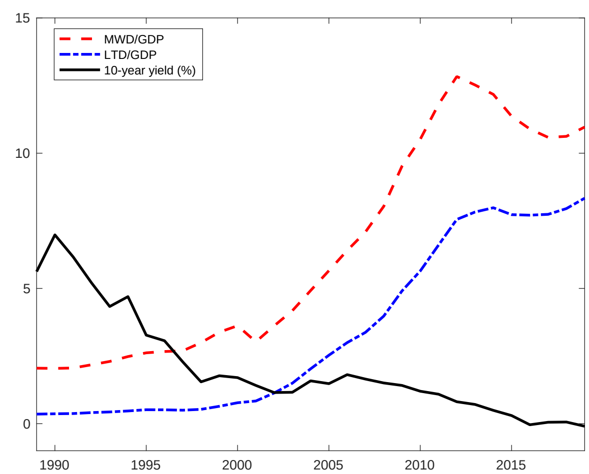
1 Introduction

Government debt has been expanding in the prevailing low-interest-rate environment worldwide. Japan is an outstanding case, where the ratio of government debt to gross domestic product (GDP) increased from over 60 percent to well-over 200 percent in the past three decades. The ratio of maturity-weighted debt to GDP, which is often used as a proxy for the bond supply factor, has continued to increase in Japan while interest rates fell (Figure 1). Does this "conundrum" indicate the supporting theories, for example, Krishnamurthy and Vissing-Jorgensen (2012) and Greenwood and Vayanos (2014), are wrong about a positive relationship between the supply factor and bond yields? Alternatively, should a different measure for the supply factor be used? We argue for the latter using a term structure framework.

Figure 1: Maturity-weighted debt to GDP and bond yield in Japan



Notes: This figure plots the ratio of maturity-weighted debt to GDP in Japan following Greenwood and Vayanos(2014). MWD/GDP and LTD/GDP stand for the ratio of maturity weighted debt to GDP and the ratio of long-term debt to GDP respectively. Ten-year bond yield is zero coupon yield (also shown in Figure 4).

Moving away from the Ricardian equivalence (Barro, 1974), the term structure literature has started to analyze the effects of the bond supply and maturity structure of government bonds on bond yields. Vayanos and Vila (2009) develop a no-arbitrage term structure model with preferred habitat investors and arbitrageurs. Greenwood and Vayanos (2014) present a special case of Vayanos and Vila (2009) assuming that the demand and supply for each maturity in the absence of arbitrageurs are price inelastic, thereby downplaying the role of preferred habitat investors in absorbing supply shocks from the government. These authors then provide empirical support for their model for the United States (US) by regressing bond yields on the ratio of maturity-weighted debt to GDP, their proxy for the supply factor. Hayashi (2018) provides an algorithm to solve a discrete version of Greenwood and Vayanos (2014) allowing for many supply factors. Hamilton and Wu (2012) estimate a discrete version of Vayanos and Vila (2009),[1] assuming the preferred habitat investors' bond demand is a decreasing affine function of the yield, and thus, at equilibrium, the supply factor is an affine function of the level, slope, and curvature factors. As a result, a bond supply shock is a combination of three-factor shocks that may be difficult to identify.

More recently, additional studies identify bond demand or supply shocks based on the preferred habitat theory formalized by Vayanos and Vila (2009). Gorodnichenko and Ray (2017) identify bond demand shocks by high-frequency changes in futures prices before and after each treasury auction, exploiting the primary market structure in the US. These authors then examine how shocks spread to other maturities. Kaminska and Zinna (2020) propose and estimate a state-space representation of Vayanos and Vila (2009) on the term structure of real rates in which the US bond supply factor is a linear function of selected observed supply factors, that is, reserves by foreign officials, large-scale bond purchases by the Fed, and Treasury supply. These authors note these supply factors better capture the low-frequency behavior of the supply factor. This study extracts the bond supply factor exploiting the bond maturity structure information at an annual frequency in Japan, and structurally estimates a discrete version of the Greenwood and Vayanos (2014) model. To the best of our knowledge, there has been no attempt in the literature to estimate the bond supply factor directly using the maturity structure information besides the maturity-weighted debt to GDP ratio. Also,

[1] Fukunaga, Kato, and Koeda (2015) and Koeda (2017) estimate a discrete version of Hamilton and Wu (2012) for Japan.

the evolution of the government debt maturity structure has not been fully analyzed for many countries.

This study constructs and analyzes a maturity structure database for Japanese government bonds (JGBs) and bills. Japan is an interesting case to study because it developed the world's second-largest government bond market (OECD, 2019), and data on issue-level government bond characteristics (e.g., coupon and maturity) have been available since fiscal year (FY) 1965 when the post-World War II (WWII) de facto debt management policy in Japan started. Moreover, the country actively implements a debt maturity policy. The Ministry of Finance (MOF) announces a detailed debt maturity plan when the budget for the upcoming fiscal year is approved by the Cabinet in December, several months before the new fiscal year begins in April. Furthermore, the Bank of Japan (BOJ), the largest government bondholder nowadays, influences the maturity structure of marketable bonds through its asset purchases under quantitative easing. The pre-announced debt maturity plan includes which specific bond maturities to issue and by how much. Thus, the plan is more detailed than other countries. For example, in the United Kingdom, the issuance plan categorizes bond maturities into three types: short, medium, and long term. In the US, the issuance plan is revised every six months. In Japan, based on the plan, government bonds are issued through "communications with the markets" until the maturity structure is finalized at the fiscal-year end. To analyze the maturity structure consistently with the fiscal year cycle, we construct the maturity structure of marketable bonds at the fiscal-year end (end-March), which is free from noises reflecting temporary changes and adjustments in the maturity structure within the year. The constructed maturity structure variable is used for model estimation via the maximum likelihood method.

This study makes several contributions. First, it proposes a novel way to extract the supply factor and show it can resolve the conundrum that interest rates are seemingly negatively correlated with the bond supply factor. Specifically, it extracts the supply factor by applying principal component analysis (PCA) to the maturity structure variables (not on bond yields). Our PCA-based supply factor chooses the loading to capture variations in the maturity structure, whereas the existing supply proxy, maturity weighted debt to GDP, prefixes the loading on debt in each maturity with increasing weights. We find that the PCA-based loading has a higher weight on 6-10 year remaining maturities than other maturities, possibly reflecting a close link with the futures markets. Furthermore, our supply factor declined for the past three decades despite the continued expansion of the government debt to GDP ratio. This decline accounts for the continued fall in the long-term interest rate in the two-decade long zero lower bound (ZLB) interest rate environment. Second, it is the first to construct maturity structure data for Japan using issue-level data since FY1965. Thus the data help us understand how debt management policy has evolved from the aspects of maturity structure. Third, the structural estimation enables us to not only identify the supply shock and its effect on bond yields without worrying about they endogeneity addressed by Greenwood and Vayanos (2014) using instrumental variable estimation, but also allows us to (i) analyze both flow and stock effects in a uniform framework,[2] where the former is captured by the bond yield responses to the shock to the lagged maturity structure and the latter by the supply-factor term of the yield equation and (ii) link the supply factor to the maturity structure which helps clarify a debt maturity policy function.

The rest of the paper is organized as follows. Section 2 documents the maturity structure data construction and discusses the evolution of the structure. Section 3 presents the model. Section 4 explains the estimation strategy and results. Section 5 conducts an impulse response exercise. Section 6 provides additional discussion. Section 7 concludes.

2 Maturity Structure and Debt Management Policy

This section explains our maturity structure data construction and documents how these variables have evolved in changing debt management policies in Japan.

[2]Existing frameworks that examine the flow and stock effects on bond prices in a unified framework include D'Amico and King (2013), who use security-level data in the US, and Sudo and Tanaka (2018), who use a dynamic stochastic general equilibrium framework for Japan.

2.1 Maturity structure data

JGBs are debt securities issued by the central government of Japan. This study constructs maturity structure data with annual frequency at the end of the fiscal year (the Japanese fiscal year starts from April and ends in March). Our sample begins from March 1966 (end of FY1965) and ends in March 2020 (end of FY2019).

As in Fukunaga, Kato, and Koeda (2015),[3] we collected data from the Japanese Bond Handbook (Ko-Shasai Binran) on every JGB issued. The handbook is published semiannually (end-March and end-September) by the Japan Securities Dealers Association (JSDA). The handbook provides data on the bond characteristics of each bond, including bond type, series number, issue date, coupon rate, maturity, direct underwriter (if any), and semi-annual observations of face value outstanding. Outstanding marketable JGBs reflect changes due to buybacks, liquidity operations, and early redemption. Since the JSDA stopped publishing this information after March 2019, the last year of the sample period was constructed by combining publicly available data on JGBs and T-bill issuance and bond holdings by the BOJ. We provide a detailed description of the data construction in Online Appendix. We break the stream of each bond's cash flows into principal and coupon payments to construct the future cash flows, as in Greenwood and Vayanos (2014).[4]

We group JGBs[5] and T-bills into marketable and non-marketable types.[6] The non-marketable bonds include the following:

- Bonds underwritten by the Trust Fund Bureau (TFB), Postal Savings, Postal Life Insurance or Pension Reserves;

[3]However, there are several differences (i) we construct data from 1965; (ii) our definitions of marketable bonds differ, and (iii) we use cash flow based calculation rather than principal based.

[4]We apply a principal based calculation for inflation-indexed bonds and flexible interest rate bonds.

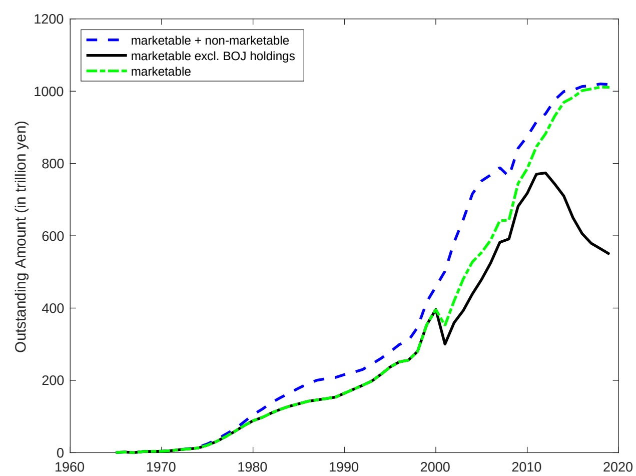
[5]Officially JGBs are government bonds issued with maturity of 1 year or longer.

[6]Legally, JGBs can be categorized into three types (i) general bonds; (ii) Fiscal Investment and Loan Program (FILP) bonds; and (iii) other bonds (MoF, 2004). General bonds are to be repaid by tax revenues. FILP bonds provide funding for government investment and lending operations under the FILP, and are to be repaid by returns from FILP operations. Other bonds include subsidy, subscription, contribution bonds. The marketable general and FILP bonds are treated as the same financial instruments in the JGB markets.

- JGBs for individual investors[7]; and
- Other small amounts (i.e., subsidy, subscription, contribution bonds).

Figure 2 shows outstanding marketable and non-marketable JGBs in trillion yen. Outstanding of non-marketable JGB is the difference between blue dashed and green dash-dotted lines. In early years, non-marketable bonds were mostly those underwritten by the TFB. The TFB is a branch of the MOF which manages funds collected by the government through postal savings, pensions and other systems in JGBs. It became active in FY1965 and was later abolished in FY2001. Bonds underwritten by postal savings, postal life insurance, and pension reserves increased particularly amid the reform of the Fiscal Loan Fund Special Account in the 2000s.[8] Since FY2013, the BOJ's purchases of long-term bonds have reduced the size of marketable bonds excluding BOJ holdings (black solid line in Figure 2).

Figure 2: Marketable vs. non-marketable JGBs, FY1965-2019



Notes: This figure plots the face-value outstanding of marketable bonds (green dash-dotted line), marketable bonds excluding BOJ holdings (black solid line), and marketable bonds plus non-marketable bonds (blue dashed line) in trillion Japanese yen.

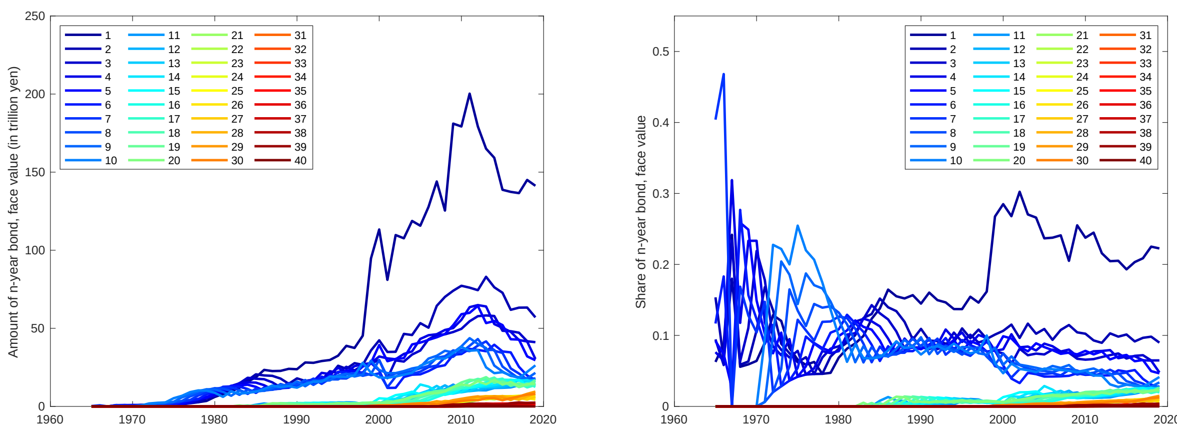
[7] JGBs for individual investors were established in 2003 to diversify JGB products.

[8] For more discussion, see for example, Cargill and Yoshino (2003).

The net face-value of bond outstanding with remaining maturity less than or equal to n years ($S^n_{FV,t}$) is shown in Figure 3a. "Net" means MOF issuance subtracting the BOJ holdings. The nominal share of the n-year bond supplied, $s^n_{FV,t}$ for $n = 1, ..., N$, is defined as $S^n_{FV,t}/\sum_i S^i_{FV,t}$. The denominator ($\sum_i S^i_{FV,t}$) is the total net face-value of bonds for all N-year maturities supplied by the government in period t, and corresponds to the black solid line in Figure 2. By construction, $s^n_{FV,t}$ adds up to one over N-year maturities. Figure 3b plots $s^n_{FV,t}$ with $N = 20$. We discuss the evolution of $s^n_{FV,t}$ and the related debt management policy changes[9] in the next subsection. It should be noted that $s^n_{FV,t}$ is constructed by aggregating cash flows across individual bonds.

Figure 3: Maturity structure

(a) Bond outstanding by maturity ($S^n_{FV,t}$, in trillion yen) (b) The share of bond supply by maturity ($s^n_{FV,t}$)



Notes: Figure 3a plots bond outstanding by maturity in trillion yen and Figure 3b plots the bond-supply share. FV in the subscript of $s^n_{FV,t}$ stands for "face value."

2.2 The evolution of debt management policy

The post-WWII debt management policy in Japan de facto started in FY1965. In that year, JGBs were issued for the first time after the war under a supplementary budget to cover a

[9] Koeda (2021) reviews the related literature on Japanese debt management policies (most of them are written in Japanese) and presents the following four phases of the debt management policy: Phase I (FY1965–1975, early market development); Phase II (FY1976–1998, stabilizing the maturity structure); Phase III (FY1999–2012, further market developments); and Phase IV (FY2013–2019, increased BOJ long-term bond purchases).

revenue shortfall (MOF, 2012). In the early years, the share of bond supply $s^n_{FV,t}$ (Figure 3b) was unstable, as there was only one type of bond (7-years). Syndication underwriting[10] was the only issuance method for marketable JGBs, and the secondary markets were underdeveloped (MoF, 2004). In the late 1970s, the market became thicker (Figure 3a) and the maturity structure gradually stabilized (Figure 3b). The development of the secondary market and the diversification of bond type gained more policy attention (MoF, 2004). Amid financial liberalization and internationalization, there were notable market developments, such as the introduction of an auction in 1978, the opening of the futures markets in 1986, and the introduction of a "partial" auction system for 10-year bonds in 1989 by Syndicate, the main underwriting body at that time. In 1999, the auction was introduced for a 1-year financial bill, and the market size notably increased for bonds and bills with remaining maturity of 1 year or less (Figure 3a). The resulting shorting of the maturity possibly reflects the introduction of BOJ's zero interest rate policy (McCauley and Ueda, 2012). "Communications with the markets" gained more importance and the TBF was finally abolished in 2001. After the abolishment of the syndication underwriting system in 2006 in response to some market participants' views that the system lacked market efficiency, the auction method became the only issuance method of marketable JGBs. Since April 2013, the BOJ has started to purchase bonds with long maturity under its qualitative and quantitative easing policy with an explicit target on the average maturity of its bond holdings. In September 2016, the BOJ introduced a yield curve control targeting the 10-year JGB yield around zero percent by committing to necessary JGB purchases. As a result, the marketable bonds outstanding excluding BOJ holdings has declined since FY2013 (Figure 2).

3 Model

The model is a discrete version of Greenwood and Vayanos (2014) which has a setting that allows us to focus on the effect of government bond supply on bond yields and risks. In the

[10] Syndication underwriting is "a means (for the government) to conclude a contract of or the handling of public offering and underwriting of government bonds with a group established for that purpose or a means to guarantee to fully digest the amount of government bonds issued" (MoF, 2012).

model, bond supply changes affect bond yields by changing the amount of interest-rate risk held by arbitrageurs. There are bonds with different maturities in the economy. Demand and supply for each maturity in the absence of arbitrageurs are assumed to be price inelastic.

As in Hamilton and Wu (2012) and Hayashi (2018), in our model, the decision problem of arbitrageurs is to maximize the risk-adjusted portfolio return subject to an adding-up constraint. The maximization problem of arbitrageurs is

$$\max_{\{z_t^n\}_{n=1}^{\bar{N}}} \left[\mathrm{E}_t \left(R_{t+1} \right) - \frac{\gamma}{2} \mathrm{Var}_t \left(R_{t+1} \right) \right] \quad \text{subject to} \quad \sum_{n=1}^{\bar{N}} z_t^n = 1, \tag{1}$$

where z_t^n is the nominal share of arbitrageurs' n-period bond holdings, γ captures the degree of risk aversion, and $\bar{N}$ is the maximum maturity that arbitrageurs hold. The holding period return on arbitrageurs' portfolio (R_{t+1}) is defined by

$$R_{t+1} \equiv \sum_{n=1}^{\bar{N}} \frac{P_{t+1}^{(n-1)} - P_t^{(n)}}{P_t^{(n)}} z_t^n. \tag{2}$$

Accordingly, the first-order condition is derived as follows

$$\mathrm{E}_t \left[\frac{P_{t+1}^{(n-1)} - P_t^{(n)}}{P_t^{(n)}} \right] - \frac{1 - P_t^{(1)}}{P_t^{(1)}} = \gamma \frac{1}{2} \frac{\partial \mathrm{Var}_t \left(R_{t+1} \right)}{\partial z_t^n}, \ n = 2, 3, \ldots, \bar{N}. \tag{3}$$

3.1 Factor dynamics

The vector of state variables X consists of the supply factor (β) and the short rate (r),

$$X_t = \begin{bmatrix} \beta_t \\ r_t \end{bmatrix},$$

It is assumed to follow a Gaussian VAR(1):

$$X_{t+1} = \boldsymbol{\mu} + \boldsymbol{\rho} X_t + \Sigma \boldsymbol{\varepsilon}_{t+1}, \ \boldsymbol{\varepsilon}_t \sim N(0, I), \tag{4}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_\beta \\ \mu_r \end{bmatrix}, \boldsymbol{\rho} = \begin{bmatrix} \rho_\beta & 0 \\ \rho_{r\beta} & \rho_{rr} \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_\beta & 0 \\ 0 & \sigma_r \end{bmatrix}, \boldsymbol{\varepsilon}_t = \begin{bmatrix} \varepsilon_t^\beta \\ \varepsilon_t^r \end{bmatrix}.$$

Thus, the supply factor dynamics is assumed to be unaffected by the short rate (as assumed by Greenwood and Vayanos, 2014) and the short rate is assumed to be unaffected by contemporaneous supply shocks implying that the central bank can control the short rate in the short run regardless of the contemporaneous maturity structure for $n = 2, 3, \ldots, \bar{N}$.[11]

This implies that the short rate r depends on its lag and the lagged vector of supply factor, as follows:

$$r_t = \mu_r + [\rho_{r\beta}, \rho_{rr}]X_{t-1} + \sigma_r \varepsilon_t^r. \tag{5}$$

3.2 Maturity structure

As in Greenwood and Vayanos (2014), the share of net bond supply with maturity n relative to total net bond supply at time t (s_t^n) is described as a factor model. The net bond supply is defined by the bond outstanding in the government bond markets subtracting bonds held by private preferred habitat investors. The net bond supply is an affine function of the vector of supply factors β_t so that the maturity structure equation is given by

$$s_t^n = \kappa_n + \psi_n \beta_t, \tag{6}$$

where $\sum_{n=1}^{\bar{N}} s_t^n = 1$.

[11] Greenwood and Vayanos (2014) mainly focus in the case where the supply factor and the short rate are independent $\rho_{r\beta} = \sigma_{r\beta} = 0$.

3.3 Equilibrium term structure

The bond market clears for all maturities at the equilibrium. The market-clearing condition is defined by $z_t^n = s_t^n$. In equilibrium, the n-period log bond price can be expressed as

$$p_t^n = \bar{a}_n + \bar{\mathbf{b}}_n^{'} X_t,$$

where

$$\bar{\mathbf{b}}_n' = \bar{\mathbf{b}}_{n-1}' \boldsymbol{\rho} - \gamma \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \begin{bmatrix} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{\bar{N}-1} \end{bmatrix} \widetilde{\boldsymbol{\Psi}}_{\bar{N}} + \bar{\mathbf{b}}_1', \tag{7}$$

$$\bar{a}_n = \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' \boldsymbol{\mu} + \frac{1}{2} \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \bar{\mathbf{b}}_{n-1} - \gamma \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \begin{bmatrix} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{\bar{N}-1} \end{bmatrix} \boldsymbol{\kappa}_{\bar{N}} + \bar{a}_1, \tag{8}$$

where

$$\boldsymbol{\kappa}_{\bar{N}} = \begin{bmatrix} \kappa_2 \\ \vdots \\ \kappa_{\bar{N}} \end{bmatrix}, \quad \boldsymbol{\Psi}_{\bar{N}} = \begin{bmatrix} \psi_2 \\ \vdots \\ \psi_{\bar{N}} \end{bmatrix}, \quad \tilde{\boldsymbol{\Psi}}_{\bar{N}} = \begin{bmatrix} \boldsymbol{\Psi}_{\bar{N}}, & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\Omega} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}'. \tag{9}$$

Thus the n-period log bond yield is given by

$$r_t^n = a_n + \boldsymbol{b}_n^{'} X_t, \tag{10}$$

$a_n = -\bar{a}_n/n$ and $\boldsymbol{b}_n = -\bar{\boldsymbol{b}}_n/n$. Appendix C provides the derivation of Eqs. (7) and (8).

4 Estimation

This section describes the data used in the estimation and estimation strategy, and presents the results. Given the data availability of bond yields, the sample period starts from the end of FY1989 and ends at the end of FY2019 (end-March). Despite the short sample period of 31 years, we estimate the model based on annual frequency for the following reasons. First, this study analyzes the debt maturity policy consistent with the fiscal-year cycle.

Higher-frequency maturity structure data helps increase the sample size while containing noise reflecting temporary changes and adjustments in the maturity structure within the year. Second, we exploit cross-sectional information in addition to time-series information in our estimation. In the benchmark estimation, we use cross sectional information on 30 maturity-structure variables and 10 bond yield variables. Thus we use 930 data points in total, which may be sufficient. As robustness checks, in Section 6, we present the estimated results with 20 bond-yield variables. We also discuss the implied small sample behavior.

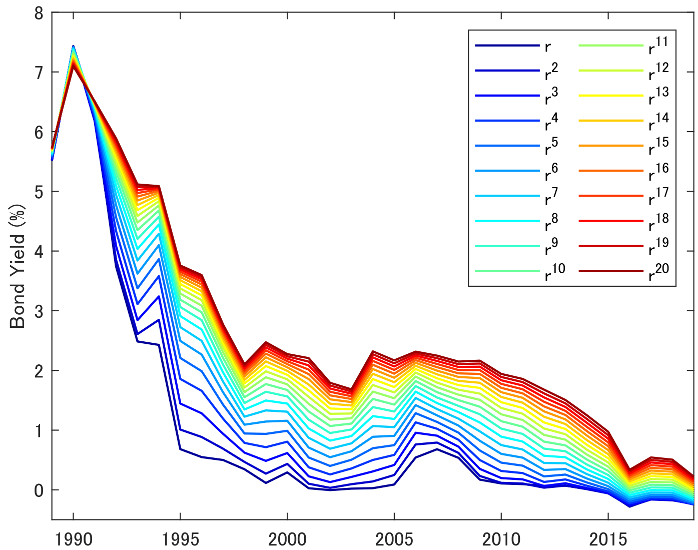
4.1 Data

To estimate the model, we need annual data on (i) bond yield with different maturities, (ii) the maturity structure of net bond supply in the market value. For (i), we use the fiscal year averages of the end-of-month Bloomberg's zero yield curve data from FY1989 to FY2019,[12] as presented in Figure 4, using the bond yields of 1, 2, .., 10-year maturities for the benchmark estimation. Thus $\bar{N} = 10$. The short rate (r) is the 1-year bond yield.

For (ii), we compute the maturity structure variable in the market values defined by $s_t^n = (P_t^n S_{FV,t}^n - H_t^n)/\sum_i (P_t^i S_{FV,t}^i - H_t^i)$, where P_t^n is the n-year bond price implied by the end of fiscal year yield curve information and $S_{FV,t}^n$ is the net face-value of bond supply as defined in Section 2.1. H_t^n is the private preferred-habitat investors' demand on n-year bond in the market value. Since full data on H_t^n is unavailable, Section 4.2 discusses how H_t^n is computed in the benchmark estimation, and Section 6.3 discusses how the computation of H_t^n affects estimation results. Table A.1 in the appendix provides summary statistics on the bond yields and maturities.

[12] Bloomberg's zero coupon bond yield data are available from April 1989. We apply Nelson-Siegel curves to zero-coupon yields obtained from Bloomberg to obtain bond yield for all maturities.

Figure 4: Bond yield data



Notes: This figure plots Japanese bond yields with different maturities in annualized rate in percent.

4.2 Estimation strategy

4.2.1 Extracting the supply factor

We apply PCA to the maturity structure variables to extract the supply factor (β) and denote the first PCA component as $\hat{\beta}$.[13] The PCA-based supply factor can address the data property of the maturity structure. To explain this point, we reproduce the maturity structure equation and the net supply factor dynamics as follows.

$$s_t^n = \kappa_n + \psi_n \beta_t, \text{ for } n = 2, ..., \bar{N}$$

$$\beta_t = \mu_\beta + \rho_\beta \beta_{t-1} + \sigma_\beta \varepsilon_t^\beta.$$

[13]The corresponding PCA score is standardized for easier interpretation of the results.

If these equations were combined, the implied s_t^n dynamics would follow the AR(1) process. However, the data property of the maturity structure indicates that s_t^n depends on the lagged maturity structure variables of other maturities, particularly s_{t-1}^{n+1}. This is because, for example, the bond with a remaining maturity of 10 years would become the bond with a remaining maturity of 9 years the following year. One way to address this data property is to apply PCA to extract the supply factor. By construction, the PCA-based supply factor $\hat{\beta}_t$ is an affine function of the maturity structure variables

$$\hat{\beta}_t = \delta_0 + \sum_{i=2}^{N} \delta_i s_t^i,$$

where N is the maximum maturity used in the PCA. We set $N = 30$ thus $\hat{\beta}$ is the first PCA component on $s_t^2, s_t^3, ..., s_t^{30}$.

Assuming $\beta_t = \hat{\beta}_t$, the following equation for s_t^n can be obtained by combining the maturity structure equation and the supply factor dynamics:

$$s_t^n = c_n + \sum_{i=2}^{N} d_{n,i} s_{t-1}^i + \omega_n \varepsilon_t^\beta, \tag{11}$$

where $c_n = \kappa_n + \psi_n \mu + \psi_n \rho_\beta \delta_0$, $d_{n,i} = \psi_n \rho_\beta \delta_i$, $\omega_n = \psi_n \sigma$, or alternatively,

$$s_t^n = c_n + D_n S_{t-1} + \omega_n \varepsilon_t^\beta, \tag{12}$$

where $S_t = [s_t^2, ..., s_t^N]'$,and $D_n = [d_{n,2}, ..., d_{n,N}]$. Eq. (12) is the transition equation for the maturity structure which is a vector of stock variables, and the shock to this transition equation turns out to be the supply shock ε_t^β.

4.2.2 Incorporating bond demand by private preferred habitat investors

In the benchmark estimation, we compute H_t^n based on recent disclosure information on life insurance companies. As Fukunaga, Kato, and Koeda (2015) indicate, insurance business law in Japan requires every insurance company to disclose the amount outstanding of JGBs and T-bills by remaining maturity at least once each business year. The amount of bond holding by insurance companies is reported mostly in the face value under the "held to maturity" purpose applying the amortized cost method. The disclosure information usually reports the end of fiscal year values.

We focus our analysis on life insurance companies because they already hold the three quarters of the JGBs held by insurance companies, according to the BOJ's flow of funds data[14]. We denote their bond holdings by the specific range of maturities over i years and less than or equal to j years as $H_{i<\tau\leq j}$, where τ is the remaining maturity in years. In the annual reports, the remaining maturities are grouped by "1 year or less," "over 1 year and less than equal to 3 years," "over 3 years and less than equal to 5 years," "over 5 years and less than equal to 7 years," "over 7 years and less than equal to 10 years," "over 10 years." We construct an unbalanced panel data on JGB holdings by maturity for about 40 life insurance companies in Japan for 2016-2019 (37, 39, 39, 40 companies respectively). The share of life insurance companies' holdings in the corresponding net bond supply is 2, 5, 8, 18, 28, 48 percent, respectively for each maturity group on the sample average. The share in each maturity year (h_t^n) is determined via cubic Hermite interpolation of the average years of each maturity group and the corresponding shares[15]. We use these shares to compute H_t^n where $H_t^n = h_t^n P_t^n S_{FV,t}^n$.

4.2.3 Estimation methodology

We estimate the model using the maximum likelihood method. The data used in the benchmark estimation are $r_t^2, r_t^3, ..., r_t^{10}$, $s_t^2, s_t^3, ..., s_t^{10}$, $\hat{\beta}$, and r. The model parameters are

[14] The flows of fund data provided by the BOJ show that over 20 % of the volume of government bonds and bills is held by private insurance companies and pension funds. These private preferred-habitat investors, especially life insurance companies, increased their holdings of JGBs with maturities over 10 years to match the duration of assets to the long duration of their liabilities under regulations and accounting standards that force them to reduce their risky asset holdings.

[15] Specifically, we choose 0.5, 2, 4, 6, 8.5, 15, 20 and 30 years for the average years and 0.02, 0.05, 0.08, 0.18, 0.28, 0.48, 0.48, and 0.48 for the shares in the benchmark estimation.

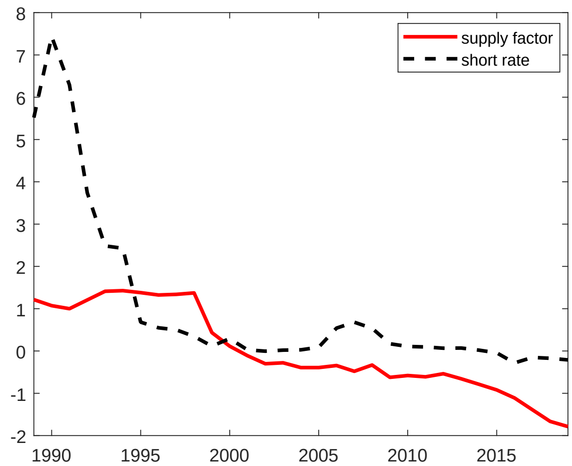
$\Theta = [\boldsymbol{\mu}, \boldsymbol{\rho}, \Sigma, \boldsymbol{\kappa}, \boldsymbol{\Psi}, \gamma, \sigma_e, \sigma_s]$ where σ_e and σ_s are measurement errors in the yield and the maturity-structure equations respectively. The corresponding likelihood function is derived in Appendix B. The initial parameter values are obtained by ordinary least squares. Standard errors are derived by numerically computing the Hessian matrix.

4.3 Benchmark results

Figure 5 shows the observed state variables in the model. The short rate (1-year bond yield, r) drops in the early 1990s and stays very low thereafter. The PCA-based supply factor ($\hat{\beta}$)[16] flips its sign from positive to negative around the year 2000. This implies that the supply factor pushes up yield curves in the 1990s, but then pushes them down in the 2000s. The supply factor notably declined in the early 2000s due to a rapid expansion of short-term government bond markets. As a result, the supply share of 1 year or less remaining maturities doubled from about 15 to 30 percent, reducing the share of other maturities (Figure 3b). Despite the rapid expansion of the maturity-weighted debt-to-GDP ratio in the 2000s (Figure 1), the supply factor is stable thanks to the diversification of bond maturity (Figure 3b). The supply factor further declined from 2013 under the quantitative and qualitative easing by the BOJ particularly through the net bond supply share of reduction of 6-10 year remaining maturities accompanied with the increase of 1 year or less remaining maturities, and this contributed to a lower $\hat{\beta}$ because the PCA loading on 6-10 year remaining maturities was positive and relatively large.

[16] The first PCA component explains 76% of variations in the maturity structure variables. By construction, the sign of scores is chosen to obtain theoretically consistent factor loadings, particularly the positive and increasing supply-factor loading shown in Figure 7.

Figure 5: Observed state variables ($\hat{\beta}$ and r)



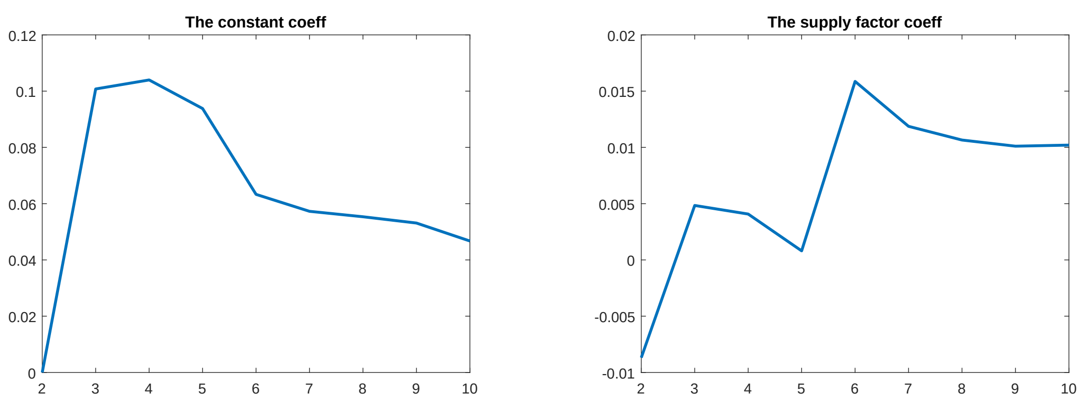
This figure plot the PCA-based supply factor (red solid line, standardized) and the short rate (1-year bond yield, annualized rate in percent, black dashed line).

Table 1 reports the estimated model parameters. Standard errors are shown in parentheses. Despite the low-frequency data, the supply factor dynamics is very persistent with ρ_β at about 0.91. An increase in supply raises expected future short rates with a positive $\rho_{r\beta}$. The maturity structure equation coefficients $\boldsymbol{\kappa}$ and $\boldsymbol{\Psi}$ are also plotted in Figure 6. The bond supply shares of 6–10 year remaining maturities were more responsive to the supply shocks than other maturities are.

Table 1: Estimated parameters

Maturity Structure Parameters									
$\boldsymbol{\kappa}$	0.1352	0.1008	0.1040	0.0938	0.0633	0.0573	0.0553	0.0531	0.0467
	(0.0021)	(0.0021)	(0.0016)	(0.0014)	(0.0020)	(0.0012)	(0.0008)	(0.0011)	(0.0012)
$\boldsymbol{\psi}$	-0.0087	0.0048	0.0041	0.0008	0.0159	0.0119	0.0107	0.0101	0.0102
	(0.0021)	(0.0022)	(0.0017)	(0.0017)	(0.0019)	(0.0011)	(0.0009)	(0.0011)	(0.00197)
Factor Dynamics Parameters									
$\boldsymbol{\mu}$	0.2064	-0.3095							
	(0.1058)	(0.1424)							
$\boldsymbol{\rho}$	0.9139	0							
	(0.0358)								
	0.1390	0.9170							
	(0.0521)	(0.0128)							
$\boldsymbol{\Sigma}$	0.3368	0							
	(0.0482)								
	0	0.3537							
		(0.0467)							
Risk Aversion									
$\boldsymbol{\gamma}$	2.4432								
	(0.4675)								
Measurement Errors									
σ_e	0.2534								
	(0.0453)								
σ_s	0.0084								
	(0.0005)								

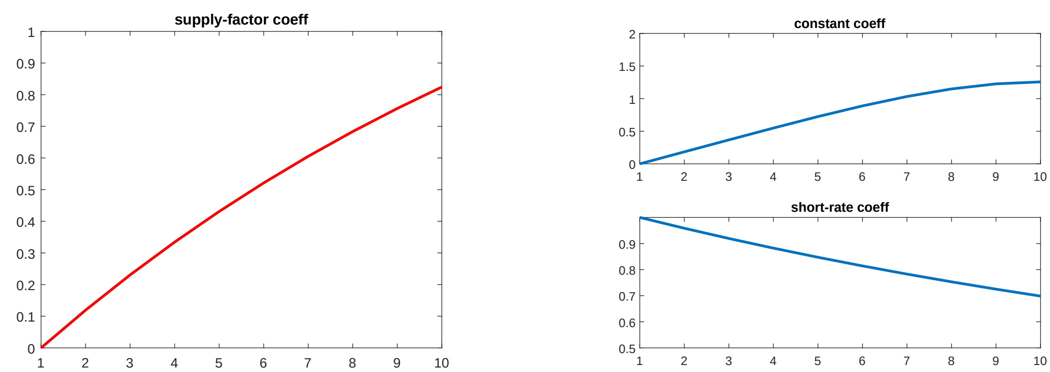
Figure 6: Estimated maturity structure



Notes: The estimated $\boldsymbol{\kappa}_{\bar{N}}$ and $\boldsymbol{\Psi}_{\bar{N}}$ in the maturity structure equation (Eq. (6)) are plotted against maturity.

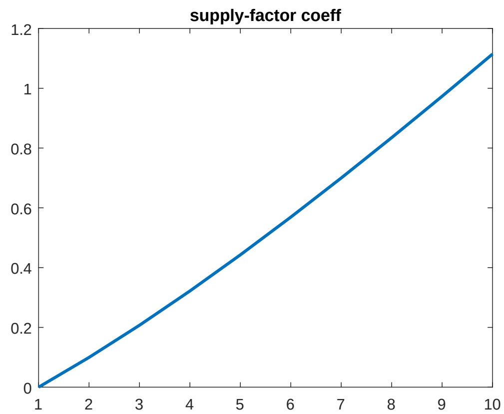
Figure 7 shows the estimated yield-curve coefficients, that is, a_n and $\boldsymbol{b}_n$ in Eq. (10). Consistent with Greenwood and Vayanos (2014), the coefficient of supply in the yield equation is increasing with maturity (Figure 7, left) and the shape of this factor loading looks like that of the slope factor. The supply coefficient in the model-implied expected one-period holding period return (i.e., $E_t(p^n_{t+1}) - p^{n+1}_t$) is computed as the first element of $\bar{\mathbf{b}}_n \boldsymbol{\rho} - \bar{\mathbf{b}}_{n+1}$ or equivalently the first element of $(n+1)\boldsymbol{b}_{n+1} - n\boldsymbol{b}_n\boldsymbol{\rho}$. This coefficient, which captures the change in the expected return with respect to one-unit change in the supply factor change (ceteris paribus), is increasing against maturity in the benchmark estimation as shown in Figure 8.

Figure 7: Estimated yield-equation coefficients



Notes: These figures plot the yield equation coefficients. The left figure corresponds to the supply-factor coefficient; the top right figure corresponds to the constant coefficient; and the bottom right figure corresponds to the short-rate coefficient.

Figure 8: Estimated supply coefficient in the return equation



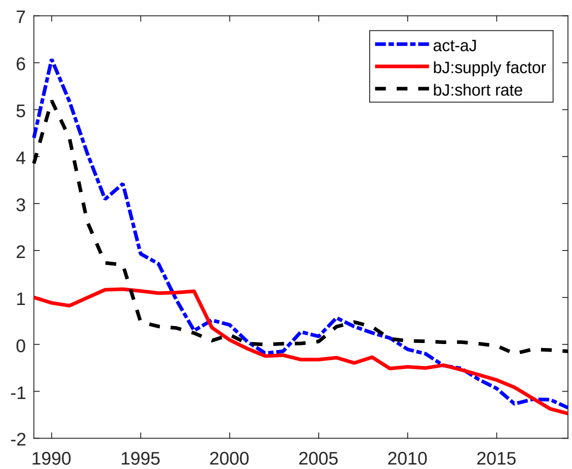
Notes: This figure plots the supply coefficient in the return equation against maturity.

Figure 9 decomposes the model-implied yield into the contributions from each factor. The figure shows how the short rate and the supply factor explain the 20-year bond fluctuations. The blue dash-dotted line shows the actual 10-year bond yield minus the constant term in the yield equation (a_n), the red solid line shows the supply factor term in the yield equation (the first element of $\bar{\mathbf{b}}_n' X_t$). The black dashed line shows the short-rate term in the yield equation (the second element of $\bar{\mathbf{b}}_n' X_t$). The supply factor term in the yield equation captures the duration effect highlighted by Greenwood and Vayanos (2014). It is positive in the 1990s pushing up the yield curves but it turns negative around FY2000 pushing down the yield curves thereafter. Since $\hat{\beta}$ is a weighted sum of maturity structure variables and is a stock variable, the magnitude of the decline through the supply factor term captures "stock" effect. It persistently pushed down the 10-year bond yield by more than 100 basis points for the last several years in the sample as shown in Figure 9. The magnitude is broadly consistent with that quantified by Sudo and Tanaka (2018).[17] In summary, a more negative supply factor

[17] Sudo and Tanaka (2018) analyze the flow and stock effects on bond yield and other variables in a dynamic stochastic general equilibrium framework using Japanese data. The former effects correspond to the size of the bond purchases in each period, and the latter effects correspond to the total amount of bonds taken away from the private sectors. Our flow effect is different from that of Sudo and Tanaka (2018) as we consider the supply shock to the maturity structure as a whole arising from net supply by the government instead of the BOJ bond purchases.

can account for the continued decline in the long-term bond yields under a ZLB environment when the short rate is stacked near the ZLB.

Figure 9: Decomposing the model-implied 10-year yield



Notes: This figure plots the estimated short-rate and supply-factor terms (black dashed and red solid lines, respectively) and the actual 10-year bond yield minus the estimated constant term (blue dash dotted line) in Eq. (10).

5 Impulse Responses to the Supply Shock

How do bond yields and the maturity structure respond to a supply shock? The impulse response of a model-implied variable from the term structure model, y_t to a yield-factor shock can be defined by the difference between the following conditional expectations:

$$E_t\left[y_{t+k} \mid \hat{X}_t + \nu_t; \Theta\right] - E_t\left[y_{t+k} \mid \hat{X}_t; \Theta\right], \tag{13}$$

where ν_t represents the vector of shocks. We numerically compute Eq. (13) given the model parameter estimates. The error bands are obtained by drawing parameter vectors from the asymptotic distribution and picking the 84th and 16th percentiles.

Figure 10 shows the impulse responses of the model-implied variables to a positive supply shock that increases the supply factor by 1 upon impact. It is expected to take nearly a decade for the effect on the supply factor to be halved as the supply factor is persistent. The model-implied 10-year bond yield jumps up by about 80 basis points upon impact with relatively large risk of further interest hikes. The short rate (1-year bond yield) is likely to increase. One caveat regarding this impulse response exercise is that it assumes that the shock is fully absorbed by arbitrageurs. If the shock were fully absorbed by preferred-habitat investors, there will be no change in the maturity structure of the net bond supply that arbitrageurs face in the model. Since the supply shock ε_t^β is the shock to the transition equation for the maturity structure as a whole (Eq. (12)), the impulse responses to the supply shock can be interpreted as a "flow" effect. This should be distinguished from the effect of the shock arising from asset purchases by a central bank, which is commonly examined in the literature.

6 Discussion

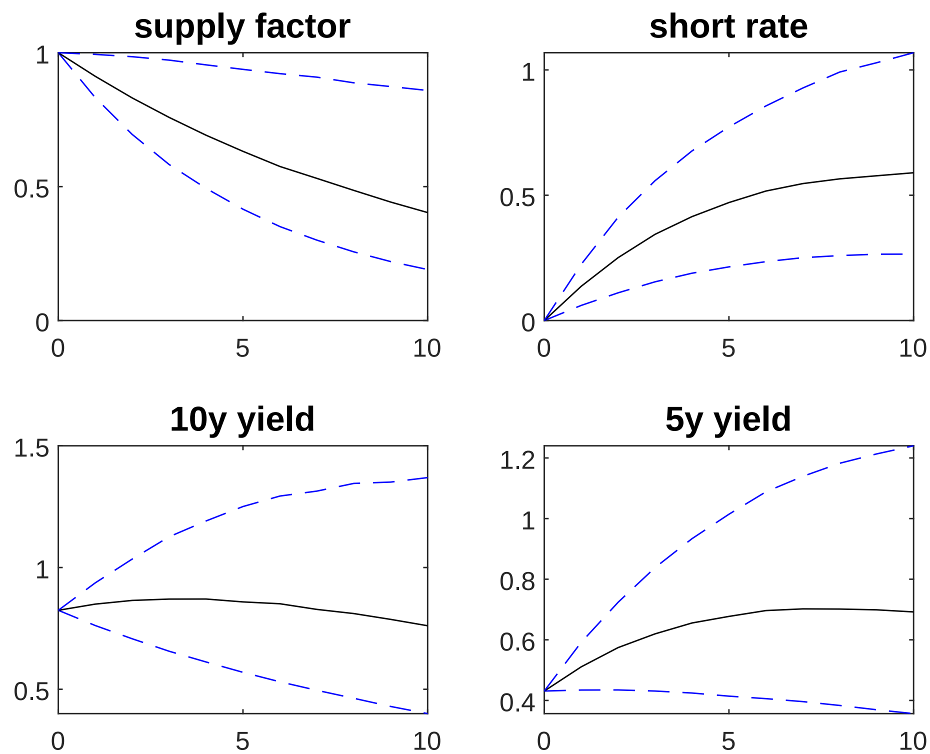
6.1 Our supply measure and alternative measures

To examine how well our supply measure ($\hat{\beta}$) accounts for bond yield fluctuations[18] compared with alternative measures, we conducted simple model-free regression of a 10-year bond yield on $\hat{\beta}$ and an alternative supply measure controlling for the short rate. Table 2 includes the existing proxies (MWD/GDP or LTD/GDP) as the alternative supply measure, and shows that $\hat{\beta}$ outstandingly explains the 10-year bond yield fluctuations.

One important model assumption, as in Hamilton and Wu (2012); Hayashi (2018), was

[18] While there are only two yield-curve factors in our model, that is the supply factor and the short rate, there may be other factors that explain bond yields, for example, inflation. To address this concern, Appendix Table A.2 regresses the 10-year bond yields on the two yield-curve factors and inflation. The coefficient for inflation is positive without control variables. However, the sign flips to weakly negative when the short rate is included in the regression. It may be also useful to examine the role of other possible theories, for example, demand for Japanese government bonds as safe assets (Krishnamurthy and Vissing-Jorgensen (2012)). We could, however, note that Japanese financial institutions were less severely affected by the global financial crisis compared to those in the US or the Euro area. Moreover, Japanese government debt to GDP has more than tripled in the past three decades, which in theory would reduce the convenience yield, unless, say, foreign investors' demand for Japanese safe assets had increased (Ichiue et al. (2012)).

Figure 10: Impulse responses to a supply shock



Notes: This figure plots the impulse responses of supply factor (top left), short rate (top right, annualized rate in percent), 10-year yield (bottom left, annualized rate in percent), and 5-year yield (bottom right, annualized rate in percent).

that the model is silent about the size effect of government debt on bond yields, because $\hat{\beta}$ is a linear function of the shares of bond supply by maturity and thus captures the composition effect. Table 3 adds the size of government debt, more precisely, total face-value bond outstanding to GDP (D/GDP) as the additional supply measure. The estimated results show that the coefficient for $\hat{\beta}$ was statistically significantly positive, whereas the coefficient for D/GDP was statistically insignificant. Thus only the composition effect was confirmed in the regression. Going forward, however, increasing the size of the outstanding JGBs while keeping the same maturity structure may significantly impact bond yields, particularly in light of shrinking country's JGB absorbing capacity (Hoshi and Ito (2014)).

Table 2: Bond Yields and Alternative Supply Factor

The table below regresses 10-year bond yield on the variables of interest, the supply factor and the short rate using our annual data from FY1989 to FY2019. The variable of interest is an alternative supply factor measure, where MWD/GDP stands for the fraction of maturity weighted debt to GDP and LTD/GDP stands for the fraction of long-term debt to GDP. We use two supply factors: $\hat{\beta}_{MV}$ stands for supply factor computed using maturity structure based on market value and $\hat{\beta}_{MV}$ stands for supply factor based on face value. The columns of 2SLS estimation show the estimation results from second-stage regressions. We instrument for the alternative supply factors by the debt-to-GDP, D/GDP, and for $\hat{\beta}$ by lagged $\hat{\beta}$. Both the first- and second-stage regressions include the short-rate as a control. Newey-West standard errors with 1-year lag are shown in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

Dependent variable:	r^{10}											
Alternative measure:	MWD/GDP						LTD/GDP					
Estimator:	OLS			2SLS			OLS			2SLS		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Alternative measure	-0.150	-0.036	-0.039	-0.139	-0.020	-0.017	-0.195*	-0.0561	-0.0667	-0.187*	-0.035	-0.028
	(0.221)	(0.029)	(0.027)	(0.095)	(0.026)	(0.026)	(0.105)	(0.045)	(0.043)	(0.099)	(0.042)	(0.040)
$\hat{\beta}_{MV}$		0.6817***			0.655***			0.564***			0.623***	
		(0.214)			(0.1964)			(0.251)			(0.179)	
$\hat{\beta}_{FV}$			0.659***			0.739***			0.597**			0.713***
			(0.241)			(0.189)			(0.273)			(0.200)
r	0.764***	0.703***	0.659***	0.776***	0.731***	0.677***	0.7173***	0.710***	0.670***	0.779***	0.735***	0.682***
	(0.107)	(0.055)	(0.072)	(0.091)	(0.053)	(0.066)	(0.080)	(0.057)	(0.075)	(0.087)	(0.053)	(0.067)
Constant	2.251**	1.587***	1.655***	2.169***	1.479***	1.509***	1.944***	1.558***	1.619***	1.911***	1.464***	1.493***
	(1.100)	(0.180)	(0.194)	(0.694)	(0.216)	(0.234)	(0.504)	(0.153)	(0.171)	(0.558)	(0.197)	(0.208)
Observations	31	31	31	31	30	30	31	31	31	31	30	30
Adj. R^2	0.943	0.970	0.969	0.943	0.969	0.967	0.952	0.971	0.971	0.952	0.970	0.9568

Table 3: Size versus Composition

The table below regresses 10-year bond yield on the variables of interest, the supply factor and the short rate using our annual data from FY1989 to FY2019. The variable of interest is a debt to GDP, D/GDP. We use two supply factors: $\hat{\beta}_{MV}$ stands for supply factor computed using maturity structure based on market value and $\hat{\beta}_{FV}$ stands for supply factor based on face value. The columns of 2SLS estimation show the estimation results from second-stage regressions. We instrument for the debt-to-GDP by the lagged debt-to-GDP, D/GDP, and for $\hat{\beta}$ by lagged $\hat{\beta}$. Both the first- and second-stage regressions include the short-rate as a control. Newey-West standard errors with 1-year lag are shown in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

Dependent variable:	r^{10}					
Estimator:		OLS			2SLS	
	(1)	(2)	(3)	(4)	(5)	(6)
D/GDP	-1.249	-0.127	-0.137	-1.394	-0.358	-0.352
	(1.466)	(0.199)	(0.195)	(2.403)	(0.304)	(0.292)
$\hat{\beta}_{MV}$		0.692***			0.604***	
		(0.174)			(0.179)	
$\hat{\beta}_{FV}$			0.749***			0.664***
			(0.202)			(0.221)
r	0.761***	0.702***	0.652***	0.652***	0.767***	0.675***
	(0.096)	(0.055)	(0.069)	(0.069)	(0.076)	(0.070)
Constant	2.424***	1.474***	1.534***	2.570*	1.682***	1.723***
	(0.844)	(0.215)	(0.234)	(1.322)	(0.284)	(0.297)
Observations	31	31	31	30	30	30
Adj. R^2	0.928	0.968	0.967	0.9021	0.966	0.965

6.2 Are private preferred-habitat investors price elastic?

Another important model assumption is that preferred-habitat investors are assumed to be price inelastic as in Greenwood and Vayanos (2014). We investigated this assumption's validity via a simple regression analysis using the panel data for life insurance companies constructed in Section 4.2. Table 4 reports the firm-fixed effect regression of the log of their bond holdings by specific range of maturities ($logH_{i<\tau\leq j}$) on the average of bond yields over the corresponding maturity range ($r_{i<\tau\leq j}$), where τ is the remaining maturity in years. The estimated results show that all the coefficients were statistically insignificant, supporting the model assumption that the private preferred-habitat investors were price inelastic in Japan.

Table 4: Demand Elasticity of Life Insurance Companies

The table below regresses life insurance companies' bond holdings for specific maturities on the corresponding bond yields. All specifications use firm fixed effects. Standard errors are shown in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

Dependent Variable:	$\log H_{\tau\leq 1}$	$\log H_{1<\tau\leq 3}$	$\log H_{3<\tau\leq 5}$	$\log H_{5<\tau\leq 7}$	$\log H_{7<\tau\leq 10}$	$\log H_{10<\tau}$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
r	-0.026						
	(1.454)						
$r_{1<\tau\leq 3}$		-0.007					
		(1.669)					
$r_{3<\tau\leq 5}$			-1.001				
			(1.478)				
$r_{5<\tau\leq 7}$				1.181			
				(0.914)			
$r_{7<\tau\leq 10}$					-0.518		
					(0.635)		
r_{10}						-0.135	
						(0.355)	
r_{20}							-0.024
							(0.176)
Observations	111	121	122	124	130	139	139
Adj. R^2	0.854	0.844	0.833	0.886	0.945	0.989	0.989

6.3 Changing the setting on the super-long maturities

6.3.1 Reestimating the model with $\bar{N} = 20$

As a robustness check, this subsection re-estimated the model by setting $\bar{N} = 20$, where the range of maturities used for estimation was extended from 2, 3, ..., 10 years to 2, 3, ..., 20 years for both the maturity structure and bond yield variables including super-long (greater than 10 years) maturity information. The estimated results, reported in Table A.3 and Figures A.1-A.5, still support the paper's main findings, such as, the supply-factor coefficient in the yield equation increases with maturity (Figure A.2), and the supply-factor coefficient in the maturity structure equation is more positive on the 6-10 year maturities (Figure A.1). Yet there were a few new important results. First, the supply coefficient in the excess return equation now decreased with maturity (Figure A.3, the left figure).[19] Recall

[19]When $\boldsymbol{\rho}$ is diagonal, the supply coefficient in the return equation increases against maturity given that the supply coefficient in the yield equation is upward sloping. However, when $\boldsymbol{\rho}$ is lower triangular as in our estimation, this is not necessarily the case.

that the supply coefficient in the model-implied expected one-period holding period return is computed as the first element of $(n+1)\boldsymbol{b}_{n+1} - n\boldsymbol{b}_n\boldsymbol{\rho}$. This expression can be negative if the supply factor is sufficiently persistent or the correlation coefficient between the supply factor and the short rate is sufficiently positive or both. When $\bar{N} = 20$, both ρ_β and $\rho_{r\beta}$ were estimated to be more positive than the benchmark estimates (Table A.3). Further, the loading of the short rate on bond yield was estimated to be positive and downward sloping (Figure A.2). As a result, the supply coefficient in the excess return equation decreased with maturity. Second, regarding the maturity-structure equation, the supply-factor coefficients ($\boldsymbol{\Psi}$) were estimated as somewhat negative over the 10-year remaining maturities (Figure A.1, the right figure) indicating that an increase in the supply of super-long bonds was associated with lowering the bond supply factor. This result may reflect excess demand for super-long bonds during the investigated period. Third, the impulse responses to the supply shock indicated a higher risk of large and persistent interest rate hikes, when arbitrageurs were allowed to trade super-long bonds (Figure A.4).

6.3.2 A higher holding share of private preferred habitat investors

Our assumption on the share of private preferred habitat investors' bond holding relative to the net bond supply (see Section 4.2.2 for details) may be somewhat underestimated. This is because there are other private preferred habitat investors, such as pension funds, the postal-saving bank, other insurance companies, credit unions, labor banks, regional banks, who buy and hold super-long bonds. With a higher share of private preferred habitat investors, say 70 percent instead of 48 percent on super-long bonds,[20] the supply coefficient becomes upward sloping against maturity (Figure A.3, the right figure). Here, we found that the supply factor becames less persistent (ρ_β decreases from 0.95 to 0.92) when the share of super-long bonds held by private preferred habitat investors increases. In this case (i) the PCA weights decrease on super long maturities, and (ii) the maturity-structure variables on the

[20]Here, the only change in the setting is in footnote 15 is that 0.48 is replaced with 0.7 for the shares.

super long maturities were persistently increasing. Since the loading from these persistent maturity-structure variables decreased, the supply factor became less persistent.

6.4 Short sample simulations

To examine the implied small sample behavior of the model coefficients, we used the estimated model to generate 10,000 samples of the same length as our sample period (31 observations) using a similar approach to that applied by Ang, Piazzesi, and Wei (2006). In each simulated sample, we re-estimated the model via maximum likelihood. We found that none of the simulation results suggested that the yield curve coefficient regarding the supply factor was negative, thereby supporting Proposition 1 in Greenwood and Vayanos (2014). We also examined the small sample distributions of ρ_β from the re-estimated model. The population coefficient from the estimated model ("truth") was 0.91. The average and median coefficients across all the simulations from re-estimating the model were both around 0.90, thus they were reasonably close to the truth, with standard deviations of the coefficients across the simulations of 0.04.

7 Conclusions

Using the constructed maturity structure data, we analyzed the evolution of the maturity structure for JGB markets over the past 5 decades. We also proposed a novel measure of bond supply factor which focuses on the composition rather than the size of the outstanding JGB for the past 3 decades. Despite the continued expansion of government debt to GDP ratio, our supply factor has been stable reflecting the public debt managers' effort, and it has declined due to the expansion of money markets in the 2000s and the BOJ's quantitative and qualitative easing policy since 2013.

The estimation results from the preferred-habitat term structure model indicated the supply-factor effect on bond yields was significant. For example, the stock effect of the declined supply factor has been pushing down the 10-year bond yield since 2000, and by more than 100 basis points for the last several years of the sample. Going forward, however, a positive supply shock could persistently raise bond yields, heightening risks in JGB markets that have the rollover size of about twenty percent of the GDP each year. In future research, it may be useful to formally incorporate the size effect which links this paper's analysis to the debt sustainability literature. It may be also important to endogenize the behavior of both public and private preferred-habitat investors in the model.

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Appendix

A Appendix Tables and Figures

Table A.1: Summary Statistics

The table below shows the summary statistics of our data. Our sample is 31 years with cross-sectional information on the different maturities of bond yields and maturity structure. Bond yields are zero coupon rates obtained from Bloomberg. The maturity structure variable s^n is the shares of bonds with remaining maturity less than or equal to n years but greater than $n-1$ years divided by the total net value of bonds for all government bonds. Supply factor $\hat{\beta}$ is the first principal component derived from the maturity structure variables. The maturity-weighted debt to GDP ratio, MWD/GDP, and the long-term debt to GDP ratio, LTD/GDP, are computed following Greenwood and Vayanos (2014) and Krishnamurthy and Vissing-Jorgensen (2012), respectively.

Variable	Mean	SD	Min	Median	Max	Variable	Mean	SD	Min	Median	Max
Bond Yield (%)						Maturity Structure					
r	1.058	2.038	-0.284	0.141	7.523	s^1	0.226	0.044	0.147	0.233	0.305
r^2	1.118	1.949	-0.241	0.259	7.267	s^2	0.108	0.009	0.096	0.108	0.138
r^3	1.232	1.948	-0.213	0.384	7.238	s^3	0.083	0.012	0.060	0.080	0.111
r^4	1.374	1.964	-0.217	0.559	7.180	s^4	0.086	0.012	0.055	0.084	0.114
r^5	1.524	2.011	-0.215	0.751	7.295	s^5	0.081	0.010	0.054	0.082	0.099
r^6	1.671	2.081	-0.223	0.905	7.687	s^6	0.060	0.020	0.031	0.049	0.095
r^7	1.793	2.080	-0.220	1.081	7.750	s^7	0.057	0.016	0.031	0.053	0.089
r^8	1.904	2.034	-0.198	1.233	7.765	s^8	0.058	0.016	0.031	0.054	0.085
r^9	1.998	1.970	-0.151	1.393	7.362	s^9	0.058	0.015	0.032	0.057	0.083
r^{10}	2.078	1.915	-0.094	1.506	6.980	s^{10}	0.055	0.016	0.030	0.053	0.098
Supply Factor											
$\hat{\beta}_{MV}$	0.000	1.000	-1.737	-0.358	1.326						
MWD/GDP	6.381	3.983	2.044	4.922	12.833						
LTD/GDP	3.363	3.145	0.354	2.029	8.335						

Table A.2: Bond Yield, Inflation Rate and Supply Factors

The table below regresses 10-year bond yield on the variables of interest, the inflation rate, supply factor, the short rate using our annual data from FY1989 to FY2019. We use two supply factors: $\hat{\beta}_{MV}$ stands for supply factor computed using maturity structure based on market value and $\hat{\beta}_{MV}$ stands for supply factor based on face value. The columns of 2SLS estimation show the estimation results from second-stage regressions. We instrument for the inflation rate by the lagged inflation rate and for $\hat{\beta}$ by lagged $\hat{\beta}$. Both the first- and second-stage regressions for columns 2-4 and 6-8 include the short-rate as a control. Newey-West standard errors with 1-year lag are shown in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

Dependent Variable:	r^{10}							
Estimator:	OLS				2SLS			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Inflation Rate (π)	0.979*	-0.267**	-0.170***	-0.166***	1.711***	-0.637	-0.604*	-0.648*
	(0.513)	(0.105)	(0.037)	(0.038)	(0.413)	(0.455)	(0.332)	(0.343)
r		1.037***	0.788***	0.735***		1.211***	1.018***	0.990***
		(0.111)	(0.050)	(0.060)		(0.188)	(0.140)	(0.149)
$\hat{\beta}_{MV}$			0.707***				0.602***	
			(0.127)				(0.121)	
$\hat{\beta}_{FV}$				0.769***				0.658***
				(0.144)				(0.134)
Constant	1.556***	1.157***	1.362***	1.415***	1.177**	1.190***	1.369***	1.420***
	(0.554)	(0.341)	(0.113)	(0.128)	(0.586)	(0.278)	(0.146)	(0.156)
Observations	31	31	31	31	30	30	30	30
Adj. R^2	0.336	0.887	0.974	0.972	0.019	0.843	0.929	0.918

Table A.3: Estimated parameters when when $\bar{N} = 20$

Maturity Structure Parameters										
$\boldsymbol{\kappa}$	$n = 2, \ldots, 10$									
	0.1235	0.0924	0.0954	0.0860	0.0585	0.0529	0.0511	0.0491	0.0431	
	(0.0018)	(0.0016)	(0.0013)	(0.0013)	(0.0017)	(0.0011)	(0.0008)	(0.0011)	(0.0012)	
	$n = 11, \ldots, 20$									
	0.0085	0.0086	0.0087	0.0093	0.0092	0.0075	0.0078	0.0082	0.0087	0.0084
	(0.0006)	(0.0006)	(0.0005)	(0.0006)	(0.0004)	(0.0006)	(0.0006)	(0.0005)	(0.0004)	(0.0004)
$\boldsymbol{\psi}$	$n = 2, \ldots, 10$									
	-0.0016	0.0094	0.0086	0.0049	0.0178	0.0136	0.0122	0.0116	0.0115	
	(0.0020)	(0.0017)	(0.0014)	(0.0016)	(0.0016)	(0.0010)	(0.0009)	(0.0010)	(0.0015)	
	$n = 11, \ldots, 20$									
	-0.0043	-0.0045	-0.0045	-0.0048	-0.0045	-0.0040	-0.0039	-0.0042	-0.0045	-0.0044
	(0.0006)	(0.0007)	(0.0006)	(0.00046)	(0.0005)	(0.0006)	(0.0005)	(0.0004)	(0.0004)	(0.0004)
Factor Dynamics Parameters										
$\boldsymbol{\mu}$	0.1930	-0.3433								
	(0.0723)	(0.1416)								
$\boldsymbol{\rho}$	0.9476	0								
	(0.0231)									
	0.2202	0.9287								
	(0.0471)	(0.0107)								
$\boldsymbol{\Sigma}$	0.1929	0								
	(0.0243)									
	0	0.2789								
		(0.0360)								
Risk Aversion										
$\boldsymbol{\gamma}$	3.0175									
	(0.5328)									
Measurement Errors										
σ_e	0.3327									
	(0.0565)									
σ_s	0.0055									
	(0.0003)									

Figure A.1: Estimated maturity structure when $\bar{N} = N = 20$

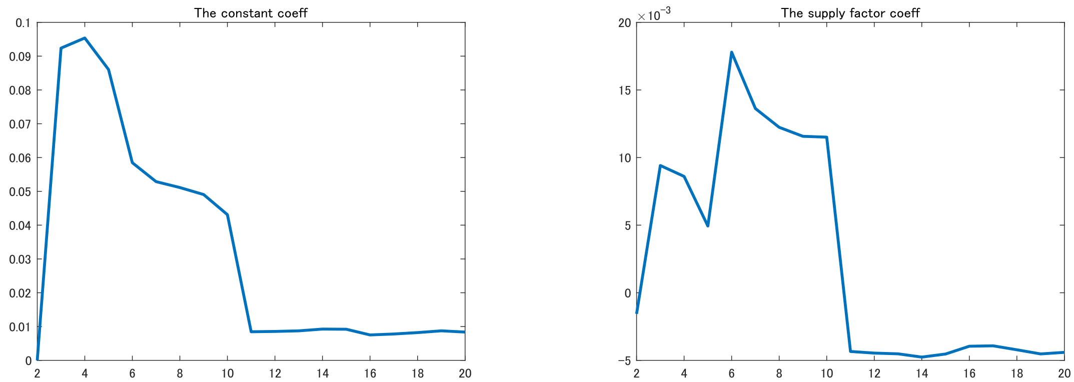


Figure A.2: Estimated yield-equation coefficients with $\bar{N} = 20$

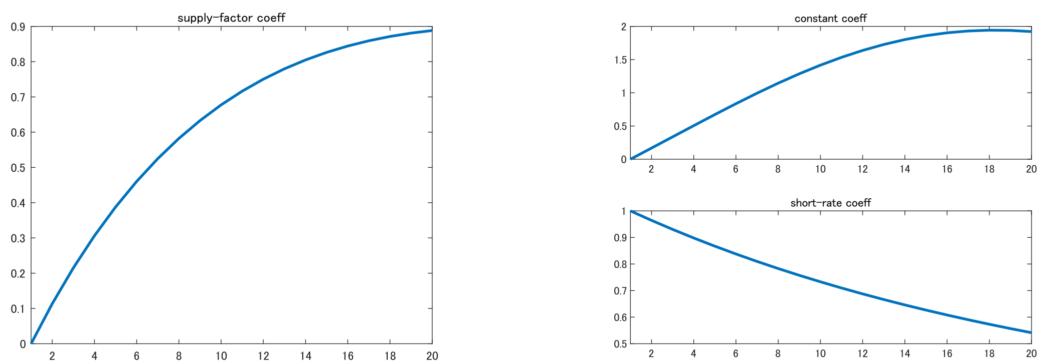
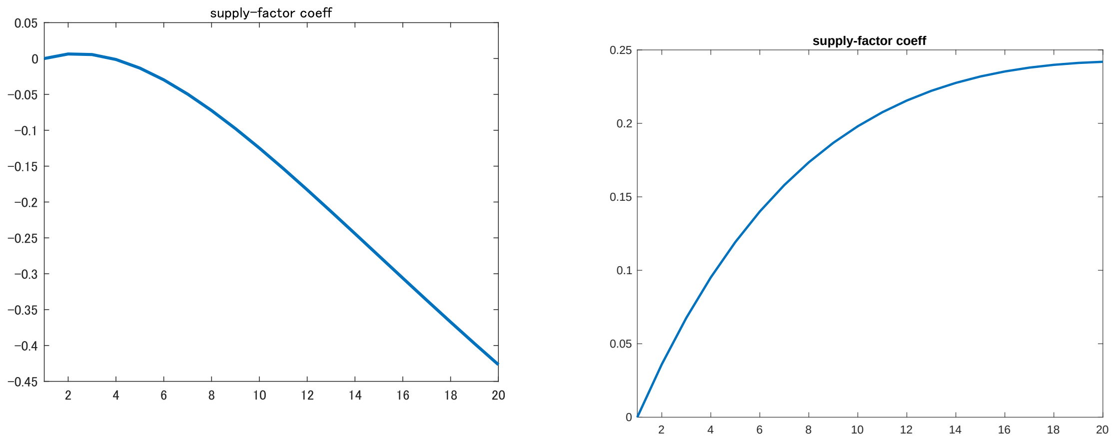


Figure A.3: Estimated supply coefficient in the return equation with $\bar{N} = 20$

(a) $\bar{N} = 20$

(b) $\bar{N} = 20$ with a higher share of private preferred habitat investors



Notes: This figure plots the supply coefficient in the return equation against maturity with (a) N=20 and (b) N=20 with a higher share of private preferred habitat investors as described in Section

Figure A.4: Impulse responses to a supply shock with $\bar{N} = 20$

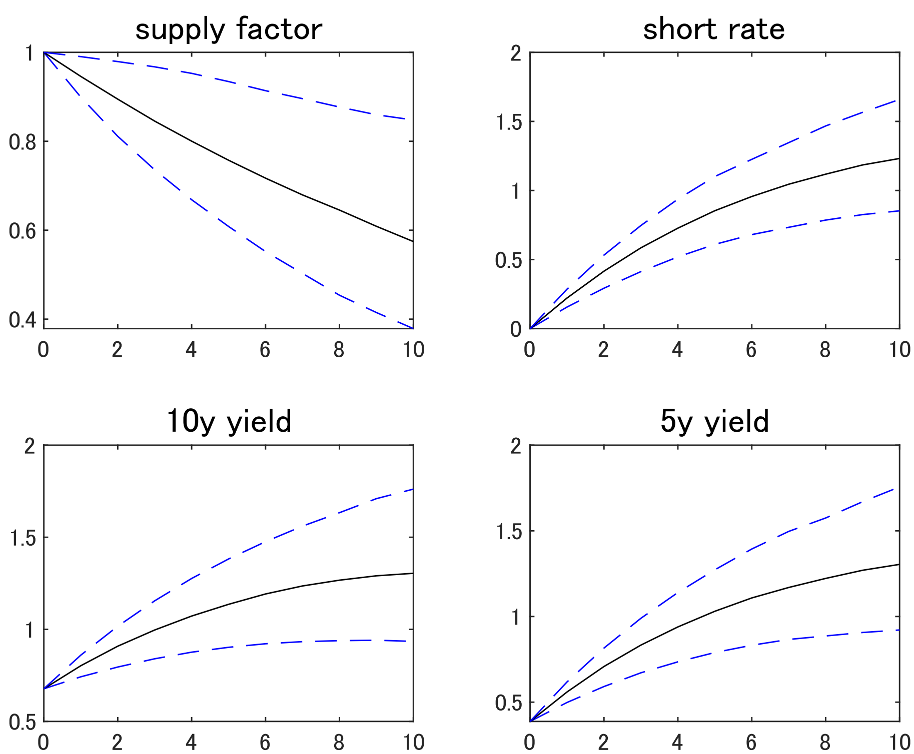
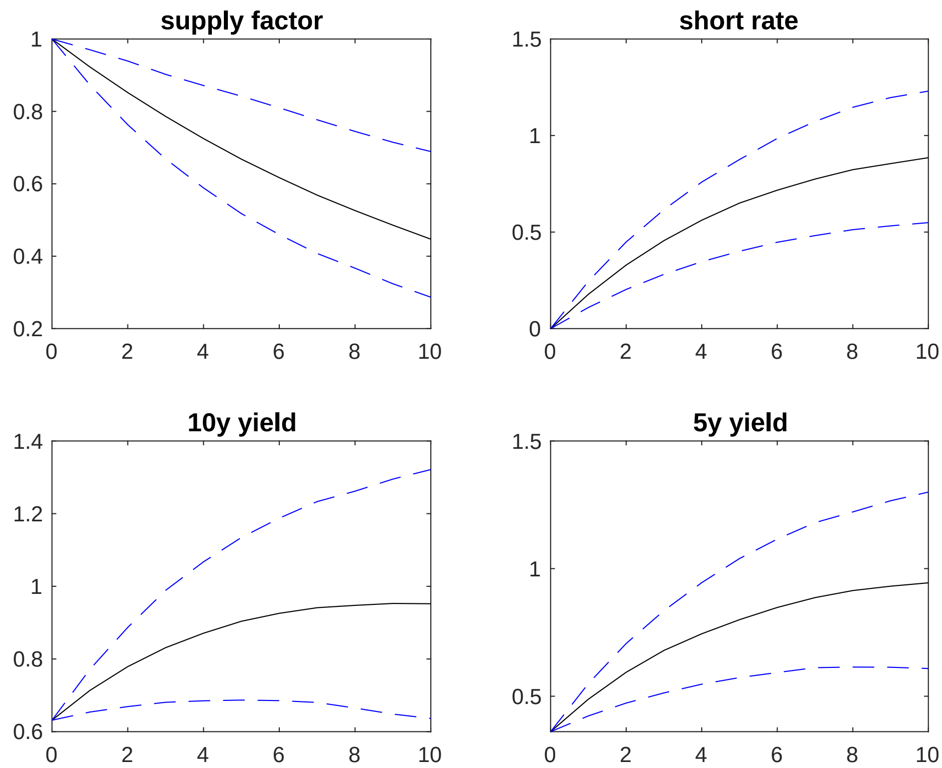


Figure A.5: Impulse responses to a supply shock with a higher share of private preferred habitat investors on super-long bonds and $\bar{N} = 20$



B Likelihood function

Define the vector of state variables as $\hat{X}_t = [\hat{\beta}_t; r_t]$ where $\hat{\beta}_t$ is the PCA-based supply factor and r is the short rate, the vector of observed bond yields as $R_{\bar{N},t} = [r_t^2, r_t^3, ..., r_t^{10}]$', and the vector of observed maturity structure as $S_{\bar{N},t} = [s_t^2, s_t^3, ..., s_t^{10}]$'. The vector of measurement errors, $\boldsymbol{u}_t \sim N(\mathbf{0}, I_{18})$ is added to the yield and maturity structure equations as follows.

$$
\begin{bmatrix} R_{\bar{N},t} \\ S_{\bar{N},t} \end{bmatrix} = \begin{bmatrix} a_2 \\ \vdots \\ a_{10} \\ \kappa_2 \\ \vdots \\ \kappa_{10} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_2^{'} \\ \vdots \\ \boldsymbol{b}_{10}^{'} \\ [\psi_2, 0] \\ \vdots \\ [\psi_{10}, 0] \end{bmatrix} \hat{X}_t + \begin{bmatrix} \sigma_e I_9 & \mathbf{0} \\ \mathbf{0} & \sigma_s I_9 \end{bmatrix} \boldsymbol{u}_t, \tag{B.1}
$$

where recursive equations correspond to eq(7) and eq(8) and the factor dynamics correspond to eq(4). The likelihood of the data is given by

$$
L = p(R_{\bar{N},1}, ..., R_{\bar{N},T}, S_{\bar{N},1}, ..., S_{\bar{N},T}, r_1, ..., r_T \hat{\beta}_1, ..., \hat{\beta}_T | R_{\bar{N},0}, S_{\bar{N},0}, \hat{X}_0; \Theta)
$$

$$
= \prod_{t=0} p(R_{\bar{N},t}, S_{\bar{N},t}, r_t, \hat{\beta}_t | \hat{X}_{t-1}; \Theta) = \prod_{t=0} p(R_{\bar{N},t}, S_{\bar{N},t} | \hat{X}_t; \Theta) p(\hat{X}_t | \hat{X}_{t-1}; \Theta)
$$

where the second equality holds by the Markov property of factor dynamics and the third equality holds by the sequential factorization. The parameter vector consists of $\Theta = [\boldsymbol{\mu}, \boldsymbol{\rho}, \Sigma, \boldsymbol{\kappa}, \Psi, \gamma, \sigma_e, \sigma_s]$.

C Derivation of factor-loadings equations

The FOCs for n-period bonds are

$$
E_t \left[\frac{P_{t+1}^{n-1} - P_t^n}{P_t^n} \right] - \frac{1 - P_t^1}{P_t^1} = \gamma \frac{1}{2} \frac{\partial \text{Var}\left(R_{t+1}\right)}{\partial z_t^n}.
$$

Although the left-hand side is approximated by the same form as Eq. (A1.3) of Hayashi (2018), the right-hand side is slightly different because of the constant term in maturity structure equations:

$$
S_{\bar{N},t} = \boldsymbol{\kappa}_{\bar{N}} + \boldsymbol{\Psi}_{\bar{N}} \beta_t = \boldsymbol{\kappa}_{\bar{N}} + \widetilde{\boldsymbol{\Psi}}_{\bar{N}} X_t
$$

Assume

$$\log P_t^{(n)} = \bar{a}_n + \bar{\mathbf{b}}_n' X_t$$

$$\begin{aligned}
E_t\left[\frac{P_{t+1}^{(n-1)} - P_t^{(n)}}{P_t^{(n)}}\right] &= E_t\left[\exp\left(\log P_{t+1}^{(n-1)} - \log P_t^{(n)}\right)\right] - 1 \\
&\approx E_t\left(\log P_{t+1}^{(n-1)} - \log P_t^{(n)}\right) + \frac{1}{2} Var_t\left(\log P_{t+1}^{(n-1)} - \log P_t^{(n)}\right) \\
&= E_t\left(\bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' X_{t+1} - \bar{a}_n + \bar{\mathbf{b}}_n' X_t\right) + \frac{1}{2} Var\left(\bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' X_{t+1} - \bar{a}_n + \bar{\mathbf{b}}_n' X_t\right) \\
&= \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' \left(\mu + \rho X_t\right) - \bar{a}_n + \bar{\mathbf{b}}_n' X_t + \frac{1}{2} Var\left(\bar{\mathbf{b}}_{n-1}' \Sigma \varepsilon_{t+1}\right) \\
&= \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' \mu - \bar{a}_n + \left(\bar{\mathbf{b}}_{n-1}' \rho + \bar{\mathbf{b}}_n'\right) X_t + \frac{1}{2} \bar{\mathbf{b}}_{n-1}' \Omega \bar{\mathbf{b}}_{n-1}
\end{aligned}$$

where $\Omega = \Sigma\Sigma'$. The second term of the left-hand side is approximated by

$$\frac{1 - P_t^{(1)}}{P_t^{(1)}} \approx -\log P_t^{(1)} = \bar{a}_1 + \bar{\mathbf{b}}_1' X_t.$$

Thus, the left-hand side becomes

$$\begin{aligned}
E_t\left[\frac{P_{t+1}^{(n-1)} - P_t^{(n)}}{P_t^{(n)}}\right] - \frac{1 - P_t^{(1)}}{P_t^{(1)}} &= \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' \mu - \bar{a}_n + \left(\bar{\mathbf{b}}_{n-1}' \rho + \bar{\mathbf{b}}_n'\right) X_t + \frac{1}{2} \bar{\mathbf{b}}_{n-1}' \Omega \bar{\mathbf{b}}_{n-1} - \bar{a}_1 + \bar{\mathbf{b}}_1' X_t \\
&= \left(\bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' \boldsymbol{\mu} - \bar{a}_n + \frac{1}{2} \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \bar{\mathbf{b}}_{n-1} + \bar{a}_1\right) + \left(\bar{\mathbf{b}}_{n-1}' \boldsymbol{\mu} - \bar{\mathbf{b}}_n' + \bar{\mathbf{b}}_1'\right) X_t.
\end{aligned}$$

Next, we approximate the portfolio return to derive the variance in the right-hand side as follows:

$$
\begin{aligned}
R_{t+1} &\equiv \sum_{n=1}^{\bar{N}} \frac{P_{t+1}^{(n-1)} - P_t^{(n)}}{P_t^{(n)}} z_t^n \approx \sum_{n=1}^{\bar{N}} \left(\log P_{t+1}^{(n-1)} - \log P_t^n \right) z_t^n \\
&= -\log P_t^{(1)} z_t^1 + \sum_{n=2}^{\bar{N}} \left(\log P_{t+1}^{(n-1)} - \log P_t^n \right) z_t^n \\
&= -\log P_t^{(1)} z_t^1 + \sum_{n=2}^{\bar{N}} \left(\bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}^{'} \left(\boldsymbol{\mu} + \boldsymbol{\rho} X_t \right) - \bar{a}_n + \bar{\mathbf{b}}_n X_t \right) z_t^n + \left(\sum_{n=2}^{\bar{N}} \bar{\mathbf{b}}_{n-1}^{'} z_t^n \right) \varepsilon_{t+1} \\
&= B_t + \left(\sum_{n=2}^{\bar{N}} \bar{\mathbf{b}}_{n-1}^{'} z_t^n \right) \varepsilon_{t+1} \\
&= B_t + \mathbf{d}_t^{'} \varepsilon_{t+1}
\end{aligned}
$$

where $\mathbf{d}_t \equiv \sum_{n=2}^{\bar{N}} \bar{\mathbf{b}}_{n-1}^{'} z_t^n$. Thus, the variance of arbitrager's portfolio return is approximated by $Var(R_{t+1}) \approx \mathbf{d}_t^{'} \Omega \mathbf{d}_t$. The right-hand side is

$$
\begin{aligned}
\frac{1}{2} \frac{\partial \mathrm{Var}(R_{t+1})}{\partial z_t^n} &\approx \bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Omega} \left(\bar{\mathbf{b}}_1 s_t^2 + \cdots + \bar{\mathbf{b}}_{N-1} s_t^N \right) \\
&= \bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Omega} \left(\bar{\mathbf{b}}_1 z_t^2 + \cdots + \bar{\mathbf{b}}_{\bar{N}-1} z_t^{\bar{N}} \right) \qquad \text{(By market clearing conditions)} \\
&= \bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Omega} \begin{bmatrix} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{\bar{N}-1} \end{bmatrix} \begin{bmatrix} s_t^2 \\ s_t^3 \\ \vdots \\ s_t^{\bar{N}} \end{bmatrix} \\
&= \bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Omega} \begin{bmatrix} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{\bar{N}-1} \end{bmatrix} \left(\boldsymbol{\kappa}_{\bar{N}} + \tilde{\boldsymbol{\Psi}}_{\bar{N}} X_t \right)
\end{aligned}
$$

Using the above expressions, the FOCs for n-period bonds can be approximated as:

$$
\begin{aligned}
&\left(\bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}^{\prime} \mu - \bar{a}_n + \frac{1}{2} \bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Omega} \bar{\mathbf{b}}_{n-1} + \bar{a}_1 \right) + \left(\bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Phi} - \bar{\mathbf{b}}_n^{\prime} + \bar{\mathbf{b}}_1^{\prime} \right) X_t \\
&\qquad\qquad = \gamma \bar{\mathbf{b}}_{n-1}^{\prime} \mathbf{\Omega} \begin{bmatrix} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{N-1} \end{bmatrix} \left(\boldsymbol{\kappa}_{\bar{N}} + \tilde{\boldsymbol{\Psi}}_{\bar{N}} X_t \right)
\end{aligned}
$$

Accordingly, we can derive the factor loading equations by comparing the coefficients.

$$\bar{\mathbf{b}}'_n = \bar{\mathbf{b}}'_{n-1}\boldsymbol{\mu} - \gamma\bar{\mathbf{b}}'_{n-1}\boldsymbol{\Omega}\left[\begin{array}{cccc} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{\bar{N}-1} \end{array}\right]\tilde{\boldsymbol{\Psi}}_{\bar{N}} + \bar{\mathbf{b}}'_1$$

$$\bar{a}_n = \bar{a}_{n-1} + \bar{\mathbf{b}}'_{n-1}\boldsymbol{\mu} + \frac{1}{2}\bar{\mathbf{b}}'_{n-1}\boldsymbol{\Omega}\bar{\mathbf{b}}_{n-1} - \gamma\bar{\mathbf{b}}'_{n-1}\boldsymbol{\Omega}\left[\begin{array}{cccc} \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \cdots & \bar{\mathbf{b}}_{\bar{N}-1} \end{array}\right]\boldsymbol{\kappa} + \bar{a}_1$$