WINPEC Working Paper Series No.E2101
April 2021

On the Stability of Equilibrium in the Market with Heterogeneous Investment Horizons

Takashi Nishiwaki

Waseda INstitute of Political EConomy
Waseda University
Tokyo, Japan

On the Stability of Equilibrium in the Market with Heterogeneous Investment Horizons*

Takashi Nishiwaki†

April 9, 2021

Abstract

This study incorporates the difference in agents' trading intervals into the heterogeneous agent model developed by Brock and Hommes (1998). We show that the effect of a longer investment horizon appears as either an increase or a decrease in the fraction of strategy adopted by long-term traders, depending on the values of the risk-free rate in the economy and the variance perceived by both types of traders. Specifically, when long-term traders are fundamentalists, we consider whether an increase in the intensity of choice to switch predictors can lead to market instability, which is the main result obtained by Brock and Hommes (1998). This is robust if the transaction cost borne by short-term traders is less than the training cost borne by long-term traders. Furthermore, when there is no transaction cost, to establish the stability of the fundamental steady state, the feasible short-term trend followers' belief form becomes more restrictive if the ratio of variance in the long-term investment horizon to that in the short-term investment horizon is larger than the risk-free rate in the economy.

JEL Classification: G12; G14
Keywords: Asset pricing; Heterogenous beliefs; Investment horizon

---

*The author is grateful to Ryuichi Yamamoto for helpful comments and discussions.

†Address: Graduate School of Economics, Waseda University. 1-6-1, Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan. Email: t-nishiwaki@suou.waseda.jp.

1 Introduction

While traditional financial models characterize equilibrium asset prices by describing a single, representative agent's maximization problem, several recent studies emphasize the interaction between different types of agents with heterogeneous beliefs. As Friedman (1953) argues, if irrational traders who would trade against rational traders will be driven out of the market by losing money, then adopting classical representative agent models can be rationalized to describe equilibrium asset prices in the market. Therefore, an important question is whether irrational traders can survive and influence equilibrium prices in the market. Several studies demonstrate that irrational or "noise" traders can survive in markets that consist of rational or "smart" traders. For example, DeLong et al. (1990) show that noise traders can earn a higher expected return than smart traders, and thus can survive in the market. As a result, these studies indicate that incorporating the interaction between heterogeneous types of traders will be beneficial for describing real asset markets.

In line with these studies,[1] Brock and Hommes (henceforth BH) (1998) develop an analytically tractable model incorporating heterogeneous types of traders to investigate the stability of equilibrium asset prices in the market. In the concept of an adaptively rational equilibrium that BH (1998) introduced, traders select their beliefs or predictors by comparing past profits for each type of strategy. An interesting feature of the model is that although most agents adopt a predictor that yields the highest realized profits in the past specific periods in time,[2] the actual fraction of traders of a belief type is also affected by the intensity of choice to switch predictors. That is, if it is infinite, all traders adopt the belief that yields the highest past performance, and if it is zero, all types of beliefs are eventually adopted. BH (1998) show that an increase in the intensity of choice to switch predictors can lead to market instability and the emergence of complicated dynamics for equilibrium asset prices.

Several studies have extended the BH (1998) model. The important assumptions in BH (1998) are: (1) traders are constant absolute risk averse (CARA) and asset prices are normally distributed, which implies that traders' investment decisions follow the mean-variance criteria; (2) while traders are heterogeneous in conditional expectations adopting different predictors or expectation functions,

[1]For other models of heterogeneous agent modeling, see Zeeman (1974), Haltiwanger and Waldmann (1985), Frankel and Froot (1988), DeLong et al. (1990), and Dacorogna et al. (1995).

[2]The length of periods in comparing realized performance is also determined by the strength of memory.

they are homogenous in conditional variance; and (3) all traders are assumed to have the same attitudes toward risk. Chiarella and He (2001) assume constant relative risk aversion (CRRA) utility functions and a log-normally distributed asset. While CARA traders invest a constant amount of wealth into a risky asset, CRRA traders invest a constant proportion of their wealth into the asset. This implies that under the CRRA assumption, the equilibrium price of an asset is more likely to be determined by the expectation of traders with a high probability of survival.[3] In addition, Chiarella and He (2002) introduce traders' heterogeneity in attitudes toward risk and the conditional variance of the future price of a risky asset.[4]

In contrast to the abundant research conducted after the pioneering work of BH (1998), there is surprisingly little research incorporating traders' heterogeneity in investment intervals, which is ubiquitous in the real market environment, into the model. In fact, the adaptive belief model developed by BH (1998) might be more suitable for analyzing the effect of the difference in traders' trading intervals on equilibrium asset prices than the classical expected utility model. This is because, in the classical expected utility framework, adopting a shorter investment interval never decreases an agent's expected utility since a short-term trader also has the option not to trade at any point in time; that is, a short-term trader can also act as a long-term trader. To ensure the existence of long-term traders in the classical model, therefore, short-term traders must bear some transaction costs.[5] By contrast, even when there is no cost in trading between shorter intervals, long-term traders may exist in the adaptive belief systems if the realized profits of long-term investments are similar to those of short-term investments. Considering this situation, this study assumes that under CARA utility, while traders are homogenous in their attitudes toward risk, they are heterogeneous in trading intervals as well as their beliefs or predictors of the future price of a risky asset.

The remainder of this paper is organized as follows. Section 2 describes our model of heterogeneous investment horizons by introducing several assumptions and defining the market equilibrium in the model. Section 3 analyzes the stability of the fundamental steady state in some simple linear belief types with different investment horizons. In particular, we consider the case in which short-term traders are trend-followers and long-term traders are fundamentalists and compare it with BH's (1998) results. Finally, Section 4 concludes.

---

[3]Recall that an agent with a logarithmic utility function has the highest probability of surviving in the classical homogeneous expectation model, which implies that agents with a correct belief may be driven out of the market, see Blume and Easley (1992) and Sandroni (2000).

[4]For empirical analysis of BH (1998), see, e.g., Boswijk et al (2007).

[5]For an example see, Duffie and Sun (1990).

2 The Model

2.1 Economic environment

Suppose that there are only two types of investment horizons for analytical tractability, and there is a one-to-one correspondence between an agent's investment horizon and his/her belief type. A trader is said to be a short-term (long-term) trader, which is denoted by the subscript or superscript S (L), if the investment horizon is 1 (2)[6]. In addition, we say that a trader is active at t if the trader participates in the market at t. Let W_t denote a trader's wealth at t. The dynamics of wealth are described by:

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t, \tag{1}$$

where p_t and p_{t+1} denote the ex-dividend price per share of the risky asset at t and $t+1$, respectively, y_{t+1} denotes the dividend of the risky asset at $t+1$, and z_t denotes the trader's investment amount for the risky asset at t. If the trader is a CARA-type myopic mean-variance maximizer, the maximization problem for a short-term trader can be written as

$$\underset{z_t}{\text{Max}} \quad E_{St}W_{t+1} - \frac{\alpha_S}{2}V_{St}W_{t+1}, \tag{2}$$

where E_{St} and V_{St} denote the conditional expectation and variance for the short-term trader, respectively, and α_S denotes the short-term trader's risk aversion index. For a long-term trader, the maximization problem becomes

$$\underset{z_t}{\text{Max}} \quad E_{Lt}W_{t+2} - \frac{\alpha_L}{2}V_{Lt}W_{t+2}, \tag{3}$$

where E_{Lt} and V_{Lt} denote the conditional expectation and variance for the long-term trader, respectively, and $W_{t+2} = R^2W_t + (p_{t+1} + y_{t+1} - Rp_t)Rz_t + (p_{t+2} + y_{t+2} - Rp_{t+1})z_t$.

Let z^*_{St} and z^*_{Lt} denote optimal investment amounts of the short- and long-term traders, respectively. A direct calculation gives

$$z^*_{St} = \frac{E_{St}(p_{t+1} + y_{t+1} - Rp_t)}{\alpha_S V_{St}(p_{t+1} + y_{t+1})} \quad \text{and} \quad z^*_{Lt} = \frac{E_{Lt}(p_{t+2} + Ry_{t+1} + y_{t+2} - R^2p_t)}{\alpha_L V_{Lt}(p_{t+2} + Ry_{t+1} + y_{t+2})}. \tag{4}$$

Further, let us impose the following assumptions:

[6] For the case when long-term traders' investment horizons are n, see Appendix B.

Assumption 1. *The risk aversion is equal for all traders, that is,* $\alpha_S = \alpha_L$.

Assumption 2. $V_{Lt}(p_{t+2} + y_{t+1} + y_{t+2}) = \psi V_{St}(p_{t+1} + y_{t+1}) = \psi\sigma^2$.

Assumption 2 states that the beliefs about the conditional variance depend only on the length of the investment horizon. This assumption is a slight extension of the assumption adopted in BH (1998): while traders have a distinct way of constructing expectations about the excess return of the asset, they are assumed to perceive the same variance. Merton (1980) and Nelson (1992) provide theoretical foundations to rationalize why the variance of risky assets is more predictable than its expectation.

2.2 *Equilibrium*

Let $N_{S,t-1}$ denote the number of active short-term traders at $t-1$, and $N_{L,t-1}$ denote the number of active long-term traders at $t-1$. The market equilibrium condition is described as

$$N_{S,t-1}\frac{E_{St}[p_{t+1} + y_{t+1} - Rp_t]}{\alpha\sigma^2} + N_{L,t-1}\frac{E_{Lt}[p_{t+2} + y_{t+2} + Ry_{t+1} - R^2 p_t]}{\alpha\psi\sigma^2} = z_{st}, \tag{5}$$

where z_{st} denotes the supply of the risky asset at t. For mathematical tractability, suppose that the risky asset has zero net supply, that is, $z_{st} = 0$ for any t.[7] Then, rearranging (5) yields

$$\begin{aligned} Rp_t =& \frac{1}{N_{S,t-1} + \frac{N_{L,t-1}R}{\psi}}(N_{S,t-1}E_{St}[p_{t+1} + y_{t+1}] \\ &+ \frac{N_{L,t-1}R}{\psi}\frac{1}{R}E_{Lt}[p_{t+2} + y_{t+2} + Ry_{t+1}]). \end{aligned} \tag{6}$$

For a further discussion of the model, it is convenient to express the equilibrium condition in terms of the fractions of beliefs; thus, we define

$$\hat{n}_{S,t-1} = \frac{N_{S,t-1}}{N_{S,t-1} + \frac{N_{L,t-1}R}{\psi}} \quad \text{and} \quad \hat{n}_{L,t-1} = \frac{\frac{N_{L,t-1}R}{\psi}}{N_{S,t-1} + \frac{N_{L,t-1}R}{\psi}}, \tag{7}$$

which implies that (6) can be rewritten as

$$Rp_t = \hat{n}_{S,t-1}E_{St}[p_{t+1} + y_{t+1}] + \frac{\hat{n}_{L,t-1}}{R}E_{Lt}[p_{t+2} + y_{t+2} + Ry_{t+1}]. \tag{8}$$

[7]For non-zero supply of the risky asset, see, for example, Chiarella and He (2001).

Note that $\hat{n}_{S,t-1}$ and $\hat{n}_{L,t-1}$ differ from the fractions of active short- and long-term traders at $t-1$, which are defined as

$$n_{S,t-1} = \frac{n_{S,t-1}}{n_{S,t-1} + n_{L,t-1}} \quad \text{and} \quad n_{L,t-1} = \frac{n_{L,t-1}}{n_{S,t-1} + n_{L,t-1}}. \tag{9}$$

Using the definition of (9), (7) can be written as

$$\hat{n}_{S,t-1} = \frac{n_{S,t-1}}{n_{S,t-1} + \frac{n_{L,t-1}R}{\psi}} \quad \text{and} \quad \hat{n}_{L,t-1} = \frac{\frac{n_{L,t-1}R}{\psi}}{n_{S,t-1} + \frac{n_{L,t-1}R}{\psi}}. \tag{10}$$

Let N_{t-1} denote the total number of traders at $t-1$. Notice that in our heterogeneous trading interval model, $N_{t-1} = N_{S,t-1} + N_{L,t-1}$ is not necessary to hold since non-active traders may also exist. To apply the concept of an adoptively rational equilibrium in our context, we assume that the updated fractions of active traders, n_{St} and n_{Lt}, are determined by

$$n_{S,t-1} = \frac{\exp[\beta\pi_{S,t-1}]}{Z_{t-1}} \quad \text{and} \quad n_{L,t-1} = \frac{\exp[\beta\pi_{L,t-1}]}{Z_{t-1}}, \tag{11}$$

where β is the intensity of choice, $Z_{t-1} = \exp[\beta\pi_{S,t-1}] + \exp[\beta\pi_{L,t-1}]$, and $\pi_{h,t-1}, h \in \{S, L\}$ is the realized profit for type h defined below.

Let p_t^* be the fundamental price of the asset, which is defined as

$$Rp_t^* = E_t(p_{t+1}^* + y_{t+1}), \tag{12}$$

and let x_t denote the deviation from the fundamental price of the asset, which is defined as

$$x_t = p_t - p_t^*. \tag{13}$$

This study assumes that all beliefs are of the form,

$$E_{St}[p_{t+1} + y_{t+1}] = E_t[p_{t+1}^* + y_{t+1}] + f_S(x_{t-1}, ..., x_{t-L}); \tag{14}$$

$$E_{Lt}[p_{t+2} + y_{t+2} + Ry_{t+1}] = E_t[p_{t+2}^* + y_{t+2} + Ry_{t+1}] + Rf_L(x_{t-1}, ..., x_{t-L}), \tag{15}$$

where f_S and f_L denote some deterministic functions used by the short- and long-term traders, respectively. Noting that

$$E_t[p_{t+2}^* + y_{t+2} + Ry_{t+1}] = E_t[E_{t+1}[p_{t+2}^* + y_{t+2}] + Ry_{t+1}], \tag{16}$$

and using (12) yields

$$E_t[p^*_{t+2} + y_{t+2} + Ry_{t+1}] = E_t[Rp^*_{t+1} + Ry_{t+1}] = R^2 p^*_t. \tag{17}$$

As a result, (14) and (15) are equivalent to

$$E_{St}[p_{t+1} + y_{t+1}] = Rp^*_t + f_S(x_{t-1}, ..., x_{t-L_S}); \tag{18}$$

$$E_{Lt}[p_{t+2} + y_{t+2} + Ry_{t+1}] = R^2 p^*_t + Rf_L(x_{t-1}, ..., x_{t-L_L}). \tag{19}$$

By using (18) and (19), the equilibrium condition (8) becomes

$$Rx_t = \hat{n}_{S,t-1} f_{St} + \hat{n}_{L,t-1} f_{Lt}, \tag{20}$$

where $f_{St} = f_S(x_{t-1}, ..., x_{t-L_S})$, $f_{Lt} = f_L(x_{t-1}, ..., x_{t-L_L})$. (20) indicates that Rx_t is determined by the weighted average of the beliefs of both types of traders, as in BH (1998). The difference between the homogenous trading interval model of BH (1998) and this study's heterogeneous trading interval model appears in the fractions of each type of trader; that is, if $R > \psi$ ($R < \psi$), the long-term investment increases (decreases) the fraction of the strategy adopted by long-term traders.

2.3 Selection of belief type

Using (11), (10) can be written as

$$\hat{n}_{S,t-1} = \frac{\frac{e^{\beta\pi_{S,t-1}}}{Z_{t-1}}}{\frac{e^{\beta\pi_{S,t-1}}}{Z_{t-1}} + \frac{Re^{\beta\pi_{L,t-1}}}{\psi Z_{t-1}}} = \frac{e^{\beta\pi_{S,t-1}}}{e^{\beta\pi_{S,t-1}} + \frac{Re^{\beta\pi_{L,t-1}}}{\psi}}; \tag{21}$$

$$\hat{n}_{L,t-1} = \frac{\frac{Re^{\beta\pi_{L,t-1}}}{\psi Z_{t-1}}}{\frac{e^{\beta\pi_{S,t-1}}}{Z_{t-1}} + \frac{Re^{\beta\pi_{L,t-1}}}{\psi Z_{t-1}}} = \frac{\frac{Re^{\beta\pi_{L,t-1}}}{\psi}}{e^{\beta\pi_{S,t-1}} + \frac{Re^{\beta\pi_{L,t-1}}}{\psi}}. \tag{22}$$

Now, let us introduce the following notation, which is a generalization of the difference in fractions introduced in BH (1998):[8]

$$\hat{m}_{t-1} = \hat{n}_{S,t-1} - \hat{n}_{L,t-1} = \frac{1 - \frac{Re^{-\beta(\pi_{S,t-2} - \pi_{L,t-2})}}{\psi}}{1 + \frac{Re^{-\beta(\pi_{S,t-2} - \pi_{L,t-2})}}{\psi}}. \tag{23}$$

[8]We use $\hat{m}$ because the difference in fractions is denoted by m in BH (1998).

Using (23), (20) can be rewritten as

$$Rx_t = \frac{1+\hat{m}_{t-1}}{2} f_{St} + \frac{1-\hat{m}_{t-1}}{2} f_{Lt}. \tag{24}$$

In our multiple investment interval setting, it is reasonable to assume that traders use the realized profits in the past two periods when selecting the next period's investment interval (i.e., strategy). That is, the difference in realized performance between the short-term and the long-term traders at $t-1$ is given by

$$\begin{aligned}\pi_{S,t-2} - \pi_{L,t-2} =& (x_{t-1} - Rx_{t-2}) \frac{f_{S,t-2} - Rx_{t-2}}{\alpha\sigma^2} + (x_{t-2} - Rx_{t-3}) \frac{f_{S,t-3} - Rx_{t-3}}{\alpha\sigma^2} \\ & - C - \frac{1}{\psi\alpha\sigma^2}(x_{t-1} - R^2 x_{t-3})(Rf_{L,t-3} - R^2 x_{t-3}),\end{aligned} \tag{25}$$

where C denotes the relative cost borne by the short-term traders. If there is no cost borne by long-term traders, then it is reasonable to assume the non-negativity of C (i.e., $C \geq 0$) as a transaction cost for an additional trade in the same period.[9]

3 Stability of the fundamental steady state

By the definition of x, BH (1998) refers to $x = 0$ as a fundamental steady state. This section examines its stability under specific belief types.

3.1 Short- and long-term traders are both trend followers

Let us assume that traders' beliefs have the form:

[9] BH (1997) proposed a risk-adjusted performance measure, which is given by

$$\begin{aligned}\pi_{S,t-2} - \pi_{L,t-2} =& (x_{t-1} - Rx_{t-2}) \frac{f_{S,t-2} - Rx_{t-2}}{\alpha\sigma^2} + (x_{t-2} - Rx_{t-3}) \frac{f_{S,t-3} - Rx_{t-3}}{\alpha\sigma^2} \\ & - \frac{\alpha}{2}\sigma^2 \left(\frac{f_{S,t-2} - Rx_{t-2}}{\alpha\sigma^2}\right)^2 - \frac{\alpha}{2}\sigma^2 \left(\frac{f_{S,t-3} - Rx_{t-3}}{\alpha\sigma^2}\right)^2 - C \\ & - \frac{1}{\psi\alpha\sigma^2}(x_{t-1} - R^2 x_{t-3})(f_{L,t-3} - R^2 x_{t-3}) + \frac{\alpha}{2}\psi\sigma^2 \left(\frac{Rf_{L,t-3} - R^2 x_{t-3}}{\psi\alpha\sigma^2}\right)^2.\end{aligned}$$

The additional negative terms express the negative aspects of taking risks consistent with the mean-variance maximizer.

$$f_{St} = g_S x_{t-1} \quad \text{and} \quad f_{Lt} = g_L x_{t-1}. \tag{26}$$

BH (1998) says that trader h is a pure trend chaser if $g_h > 0, h \in \{S, L\}$, and a contrarian if $g_h < 0, h \in \{S, L\}$. In addition, trader h is said to be a fundamentalist if $g_h = 0, h \in \{S, L\}$ because trader h believes that prices will return to the fundamental value. In this linear belief type, while $x = 0$ always satisfies (20), non-fundamental steady states, $x \neq 0$, can also exist depending on other parametric values. First, we explore the condition for the existence of non-fundamental steady states.

If there is a non-fundamental steady state, $x^* \neq 0$, using (24), we have

$$R = \frac{1+\hat{m}^*}{2} g_S + \frac{1-\hat{m}^*}{2} g_L, \tag{27}$$

where $\hat{m}^*$ is (23) evaluated at this non-fundamental steady state, that is,

$$\frac{1 - \frac{Re^{-\beta(\pi_S - \pi_L)}}{\psi}}{1 + \frac{Re^{-\beta(\pi_S - \pi_L)}}{\psi}} = \hat{m}^*, \tag{28}$$

where

$$\pi_S - \pi_L = \frac{1-R}{\alpha\sigma^2}[2(g_S - R) - \frac{1+R}{\psi}(Rg_L - R^2)]x^{*2} - C. \tag{29}$$

In addition, rearranging (27), we obtain

$$\hat{m}^* = \frac{2R - g_S - g_L}{g_S - g_L}. \tag{30}$$

This leads to the following proposition.

Proposition 1. *(Existence of the steady states of Eqs. (27), (28), (29), and (30))* *Let* $\hat{m}^{eq} = \frac{1-\frac{Re^{\beta C}}{\psi}}{1+\frac{Re^{\beta C}}{\psi}}$, $\hat{m}^* = 1 - 2\frac{g_S - R}{g_S - g_L}$, *and* x^* *be the positive solution (if it exists) of*

$$\frac{1 - \frac{Re^{-\beta(\pi_S - \pi_L)}}{\psi}}{1 + \frac{Re^{-\beta(\pi_S - \pi_L)}}{\psi}} = \hat{m}^*, \tag{31}$$

where

$$\pi_S - \pi_L = \frac{1-R}{\alpha\sigma^2}[2(g_S - R) - \frac{1+R}{\psi}(g_L - R^2)]x^{*2} - C. \tag{32}$$

Then, we have

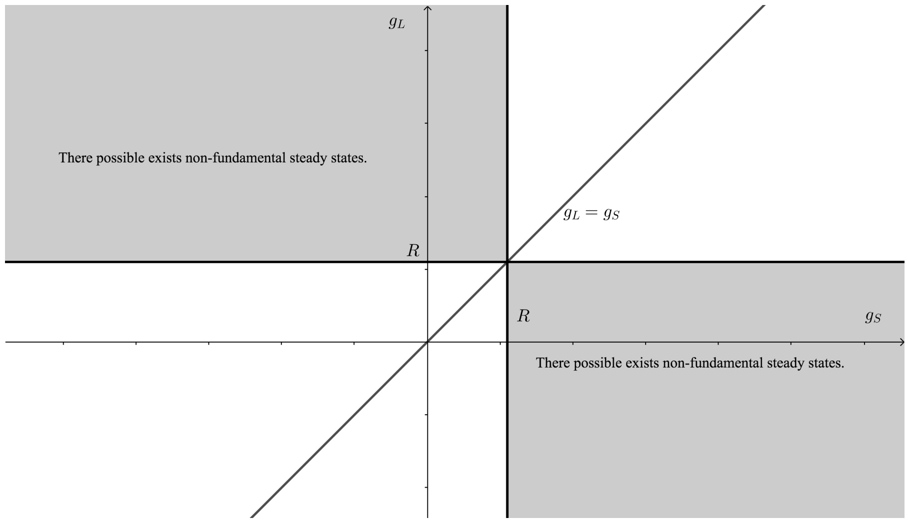

Figure. 1 The region for the uniqueness of the steady state

- *For* $g_S < g_L < R$*,* $R < g_S < g_L$*,* $R < g_L < g_S$*, and* $g_L < g_S < R$*, (*$0,\hat{m}^{eq}$*) is the unique steady state.*
- *For* $g_S < R < g_L$ *and* $g_L < R < g_S$*, (*$0,\hat{m}^{eq}$*) is the unique steady state if* $(\hat{m}^* - \hat{m}^{eq})[(1+R)(Rg_L - R^2) - 2\psi(g_S - R)] < 0$ *and there are three steady states* $(0, \hat{m}^{eq})$ *and* $(\pm x^*, \hat{m}^*)$ *or a steady state and a period two cycle* $(0, \hat{m}^{eq})$ $\{(x^*, \hat{m}^*), (-x^*, \hat{m}^*)\}$ *if* $(\hat{m}^* - \hat{m}^{eq})[(1+R)(Rg_L - R^2) - 2\psi(g_S - R)] > 0$*.*

Proof. See Appendix A. □

Figure 1 illustrates the region to ensure that the fundamental price of the asset $x = 0$ is the unique steady state of the system (27)–(30).

3.2 Stability of the fundamental steady state

This subsection derives the condition to ensure the stability of the fundamental steady state under the belief types described in Section 3.1.

First, by substituting (25) into (23), we obtain $\hat{m}_{t-1}$ as a nonlinear function of x_{t-1}, x_{t-2}, x_{t-3}, and x_{t-4}, which is denoted by $\hat{m}_{t-1}(x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4})$.

Using this, (24) can be rewritten as:

$$
\begin{aligned}
0 = & R x_t - \frac{1 + \hat{m}_{t-1}(x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4})}{2} g_S x_{t-1} \\
& - \frac{1 - \hat{m}_{t-1}(x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4})}{2} g_L x_{t-1}.
\end{aligned} \tag{33}
$$

Denoting the right-hand side of the above equation by $F(x_t, x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4})$ yields

$$
F(x_t, x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4}) = 0. \tag{34}
$$

The characteristic function for (34) is defined as

$$
F_1 \lambda^4 + F_2 \lambda^3 + F_3 \lambda^2 + F_4 \lambda + F_5 = 0, \tag{35}
$$

where each F_i denotes the partial derivative of the i-th argument evaluated at a particular steady state. To consider the stability of the fundamental steady state, we should evaluate F_i at $x = 0$. Direct calculation yields:

$$
F_1 = R, F_2 = -\frac{1 + \hat{m}^{eq}}{2} g_S, F_3 = -\frac{1 - \hat{m}^{eq}}{2} g_L, F_4 = F_5 = 0. \tag{36}
$$

Thus, the characteristic function for the fundamental steady state is described as

$$
R \lambda^4 - \frac{1 + \hat{m}^{eq}}{2} g_S \lambda^3 - \frac{1 - \hat{m}^{eq}}{2} g_L \lambda^3 = 0. \tag{37}
$$

The above equation has triple eigenvalues 0 and an eigenvalue that should be examined. The non-zero eigenvalue is obtained as

$$
\lambda = \frac{1 + \hat{m}^{eq}}{2} \frac{g_S}{R} + \frac{1 - \hat{m}^{eq}}{2} \frac{g_L}{R}. \tag{38}
$$

Let us define $\phi = Re^{\beta C} / \psi$; then, (38) can be written as

$$
\lambda = \frac{1}{1 + \phi} \frac{g_S}{R} + \frac{\phi}{1 + \phi} \frac{g_L}{R}. \tag{39}
$$

A direct calculation yields the following lemma.

Lemma 1. $|\lambda| < 1$ *is equivalent to*

$$
-\frac{1 + \phi}{\phi} R - \frac{g_S}{\phi} < g_L < \frac{1 + \phi}{\phi} R - \frac{g_S}{\phi}. \tag{40}
$$

According to the theory of nonlinear dynamic systems, the fundamental steady state is stable if the absolute values of all eigenvalues are smaller than unity. Therefore, the fundamental steady state is stable if the absolute value of the non-zero eigenvalue is smaller than unity. Combining this with Lemma 1 yields the following result.

Proposition 2. *The fundamental steady state is stable if*

$$-\frac{1+\phi}{\phi}R - \frac{g_S}{\phi} < g_L < \frac{1+\phi}{\phi}R - \frac{g_S}{\phi}. \tag{41}$$

Rearranging (41), we obtain

$$-R < \frac{g_S + \phi g_L}{1+\phi} < R, \tag{42}$$

which indicates that to establish the stability of the fundamental steady state, the absolute value of the average of g_S and g_L weighted by $1/1+\phi$ and $\phi/1+\phi$ must be less than R. The weight given to g_L is increasing in R and C and decreasing in ψ. In addition, for $C > 0$ ($C < 0$), it is increasing (decreasing) in β because ϕ is increasing in R and C and decreasing in ψ. Furthermore, if $C = 0$, that is, the relative cost borne by short-term traders is zero, then, (41) is reduced to

$$-R < \frac{g_S + \frac{R}{\psi} g_L}{1+\frac{R}{\psi}} < R. \tag{43}$$

This is intuitive: similar to the result obtained in Section 2, (43) implies that the weight given to the strategy adopted by long-term traders increases (decreases) if $R > \psi$ ($R < \psi$) in this stability analysis.

3.3 When long-term traders are fundamentalist

When there are two types of traders in the market, BH (1998) considers two cases: fundamentalists versus trend chasers and fundamentalists versus contrarians. To investigate the effect of heterogeneous investment horizons on the stability analysis of the steady states by comparison with the result under BHs (1998) homogenous trading intervals model, suppose that long-term traders are fundamentalists (i.e., $g_L = 0$). There are two reasons for adopting this assumption.

First, DeBond and Thaler (1985) document that stock returns exhibit a return reversal effect in the long term. That is, “winners” in shorter horizons become "losers" in longer horizons. They argue that a possible explanation for this phenomenon might be that traders "overreact" to unexpected news events in short-term intervals. A possible way to describe this characteristic is to impose that $g_S > 0$ and $g_L = 0$, that is, short-term traders are trend-chasers, and long-term traders are fundamentalists. Second, although it is plausible to say that long-term traders will utilize longer past price deviations as their information set, there is no consensus on its exact length.

In BH's (1998) homogeneous investment horizon model, to ensure the stability of the fundamental price of the asset, we need

$$-(1 + e^{-\beta C})R < g_S < (1 + e^{-\beta C})R, \tag{44}$$

where $C > 0$ is the cost borne by the fundamentalists. A key feature of BH's model is that the additional cost of obtaining access to the belief system is borne by fundamentalists as the training cost for understanding the fundamental tenets of the efficient market hypothesis (EMH). BH (1998) rationalize the existence of this cost by writing "This cost C may be positive because 'training' costs must be borne to obtain enough 'understanding' of how markets work in order to believe that they should price according to the EMH fundamental." As a result, $C > 0$ is crucial to deriving their conclusion that an increase in the intensity of choice to switch predictors can lead to instability of the fundamental steady state. That is, as the intensity of choice β increases, the bound for g_S to establish the stability of the fundamental price of the asset becomes more restrictive. This is because an increase in the intensity to switch predictors has the same effect as an increase in the cost to become fundamentalist, which in turn makes the trend-following strategy more attractive.

As already discussed, in our model of heterogeneous trading intervals, it is reasonable to suppose that there is also a non-negative transaction cost as well as this training cost. When long-term traders are fundamentalists, they bear a training cost to understand the EMH fundamentals, while short-term traders bear a transaction cost for additional trading in the same period of time. Let C_f and C_t denote the training cost borne by long-term fundamentalists and the transaction cost borne by short-term trend followers, respectively. Then, as shown in the previous sections, to establish the stability of the fundamental steady state in our model, we need

$$-(1 + \frac{R}{\psi} e^{\beta(C_t - C_f)})R < g_S < (1 + \frac{R}{\psi} e^{\beta(C_t - C_f)})R. \tag{45}$$

This indicates that as the intensity of choice β increases, while the required bound for g_S becomes tightened if $C_t - C_f < 0$ (i.e., the transaction cost is smaller than the training cost), it becomes loosened if $C_t - C_f > 0$ (i.e., the transaction cost is larger than the training cost). This is intuitive: when $C_t - C_f > 0$ ($C_t - C_f < 0$), adopting beliefs of trend-followers (fundamentalists) becomes more and more costly as the intensity of choice β increases. Thus, if $C_t - C_f > 0$, that is, the transaction cost is larger than the training cost, an increase in β yields a reduction in the fraction of trend-followers, which in turn can lead to the stability of the fundamental steady state.

Finally, to investigate the effect of heterogeneity in trading intervals, suppose that there is only a training cost to become a fundamentalist, as in BH (1998). Then, (45) becomes

$$-(1+\frac{R}{\psi}e^{-\beta C_f})R < g_S < (1+\frac{R}{\psi}e^{-\beta C_f})R. \tag{46}$$

Comparing (44) and (46) indicates that when fundamentalists trade between longer periods in time, the bound for g_S to establish the stability of the fundamental steady state becomes restrictive if and only if $\psi > R$; that is, when the constant multiplier of variance of the excess return of the asset is greater than the risk-free rate in the economy. The results are summarized as follows.

Proposition 3. *Under heterogeneous investment horizons, an increase in the intensity of choice can lead to market instability if the transaction cost borne by short-term traders is less than the training cost borne by fundamentalists (i.e., long-term traders). In addition, when there is no transaction cost, the bound of g_S to establish the stability of the fundamental steady state becomes restrictive if $\psi > R$.*

While the reason behind the first part of the proposition is already explained, the second part of the proposition is also intuitive: because the effect of the long-term investment appears as a reduction (an increase) in the fraction of the strategy adopted by the long-term traders if $R < \psi$ ($R > \psi$), when $R < \psi$, we need a more restrictive bound for g_S to establish the stability of the fundamental steady state.

4 Conclusion

This study incorporates the difference in agents' trading intervals into the heterogeneous agent model developed by BH (1998). Since there is a one-to-one correspondence between the investment horizon and the agent's belief type in our model, traders select their trading intervals and belief types simultaneously by comparing past realized profits. We show that the effect of the longer investment horizon appears as an increase or a decrease in the fraction of strategy adopted by long-term traders, depending on the other parametric values. Specifically, the fraction of strategy adopted by long-term traders increases if and only if $R < \psi$, that is, the risk-free rate is less than the multiplier of the variance. Using this observation, we consider the case when long-term traders are fundamentalist and short-term traders are trend followers to compare with the results obtained in BH (1998). We show that when short-term traders need to pay a transaction cost for an additional trade, an increase in the intensity of choice to switch beliefs can lead to market instability (consistent with BH's (1998) results) is robust if and only if the transaction cost is less than the training cost borne by fundamentalists. This is because an increase in the intensity of choice increases the attractiveness of the trend-follower's strategy if and only if the cost borne by trend-followers (i.e., the transaction cost) is less than the cost borne by fundamentalists (i.e., the training cost). Furthermore, when there is no transaction cost, the feasible trend followers' strategy for establishing the stability of the fundamental steady state becomes more (less) restrictive if $R > \psi$ ($R < \psi$), since if $R > \psi$, short-term trend chasers have a greater effect on the market equilibrium.

Though this study derives some analytical conditions for establishing the stability of the fundamental steady state in the multiple investment horizons setting, our results are based on several simplified assumptions: two types of trading intervals, linearity in the variance, and CARA traders. A possible extension of this study would be to loosen these restrictions by assuming more than two types of trading intervals or CRRA traders. Specifically, considering CRRA utility (and log-normally distributed asset price) is informative because, as discussed in the Introduction, the CRRA assumption can shed light on the issue of what types of traders can survive in the market. These problems will be addressed in future studies.

References

BLUME, L. AND D. EASLEY (1992): "Evolution and market behavior," *Journal of Economic theory*, 58, 9–40.

BOSWIJK, H. P., C. H. HOMMES, AND S. MANZAN (2007): "Behavioral heterogeneity in stock prices," *Journal of Economic dynamics and control*, 31, 1938–1970.

BROCK, W. A. AND C. H. HOMMES (1997): "Adaptive beliefs and the emergence of complex dynamics in asset pricing models." *Working Paper, Department of Economics, University of Wisconsin, in preparation.*

——— (1998): "Heterogeneous beliefs and routes to chaos in a simple asset pricing model," *Journal of Economic dynamics and Control*, 22, 1235–1274.

CHIARELLA, C. AND X.-Z. HE (2001): "Asset price and wealth dynamics under heterogeneous expectations," *Quantitative Finance*, 1, 509–526.

——— (2002): "Heterogeneous beliefs, risk and learning in a simple asset pricing model," *Computational Economics*, 19, 95–132.

DACOROGNA, M. M., U. A. MÜLLER, C. JOST, O. V. PICTET, AND J. WARD (1995): "Heterogeneous real-time trading strategies in the foreign exchange market," *The European Journal of Finance*, 1, 383–403.

DE BONDT, W. F. AND R. THALER (1985): "Does the stock market overreact?" *The Journal of finance*, 40, 793–805.

DE LONG, J. B., A. SHLEIFER, L. H. SUMMERS, AND R. J. WALDMANN (1990): "Noise trader risk in financial markets," *Journal of political Economy*, 98, 703–738.

DUFFIE, D. AND T.-S. SUN (1990): "Transactions costs and portfolio choice in a discrete-continuous-time setting," *Journal of Economic dynamics and control*, 14, 35–51.

FRANKEL, J. A. AND K. FROOT (1991): "Chartists, fundamentalists and the demand for dollars," *NBER Working Paper*.

FRIEDMAN, M. ET AL. (1953): "The case for flexible exchange rates," *Essays in positive economics*, 157, 33.

HALTIWANGER, J. AND M. WALDMAN (1985): "Rational expectations and the limits of rationality: An analysis of heterogeneity," *The American Economic Review*, 75, 326–340.

MERTON, R. C. (1980): “On estimating the expected return on the market: An exploratory investigation,” *Journal of financial economics*, 8, 323–361.

NELSON, D. B. (1992): “Filtering and forecasting with misspecified ARCH models I: Getting the right variance with the wrong model,” *Journal of econometrics*, 52, 61–90.

SANDRONI, A. (2000): “Do markets favor agents able to make accurate predictions?” *Econometrica*, 68, 1303–1341.

ZEEMAN, E. C. (1974): “On the unstable behaviour of stock exchanges,” *Journal of mathematical economics*, 1, 39–49.

Appendix A: Proof of Proposition 1

Notice that

$$\hat{m}^*(z) = \frac{1 - \frac{Re^{-\beta z}}{\psi}}{1 + \frac{Re^{-\beta z}}{\psi}} \tag{A.1}$$

is increasing in z and taking the limit $z \to \pm\infty$ we have

$$\lim_{z\to\infty} \hat{m}^*(z) = 1, \quad \text{and,} \quad \lim_{z\to-\infty} \hat{m}^*(z) = -1. \tag{A.2}$$

Thus, there is no steady state other than $x = 0$ if $\hat{m}^* < -1$ or $\hat{m}^* > 1$. By using (30), $x = 0$ is the unique steady state if $g_S < g_L < R$, $R < g_S < g_L$, $R < g_L < g_S$, and $g_L < g_S < R$. In other words, non-fundamental steady states can exist only if $g_S < R < g_L$ or $g_L < R < g_S$.

In addition, because $\pi_S - \pi_L$ is increasing in x^* if and only if $(1-R)[2(g_S - R) - (1+R)(Rg_L - R^2)/\psi]/\alpha\sigma^2 > 0 \Leftrightarrow (1+R)(Rg_L - R^2) - 2\psi(g_S - R) > 0$, non-fundamental steady states exist if $\hat{m}^* > \hat{m}^{eq}$ and $(1+R)(Rg_L - R^2) - 2\psi(g_S - R) > 0$, or $\hat{m}^* < \hat{m}^{eq}$ and $(1+R)(Rg_L - R^2) - 2\psi(g_S - R) < 0$.

Appendix B: The case when the investment horizon for long-term traders is n

Let us suppose that the investment horizon for long-term traders is n, which implies that their maximization problem can be rewritten as

$$\underset{z_t}{\text{Max}} \quad E_{Lt}W_{t+n} - \frac{\alpha_L}{2} V_{Lt} W_{t+n}. \tag{B.1}$$

Similar to (4), their optimal investment amount is given by

$$z_{Lt}^* = \frac{E_{Lt}(p_{t+n} + R^{n-1}y_{t+1} + R^{n-2}y_{t+2} + \ldots + y_{t+n} - R^n P_t)}{V_{Lt}(p_{t+n} + R^{n-1}y_{t+1} + R^{n-2}y_{t+2} + \ldots + y_{t+n} - R^n P_t)}. \tag{B.2}$$

For further investigation, let us impose the following assumptions, which are extensions of the assumption adopted in Section 2 in this n-period environment:

Assumption 3.

$$\begin{aligned} E_{Lt}(p_{t+n} + R^{n-1}y_{t+1} + R^{n-2}y_{t+2} + \ldots + y_{t+n}) =& E_t[p_{t+n}^* + R^{n-1}y_{t+1} + R^{n-2}y_{t+2} \\ &+ \ldots + y_{t+n}] + R^{n-1}f_L, \end{aligned} \tag{B.3}$$

Assumption 4.

$$V_{Lt}(p_{t+n} + R^{n-1}y_{t+1} + R^{n-2}y_{t+2} + \ldots + y_{t+n} - R^n P_t) = (n-1)\psi\sigma^2. \tag{B.4}$$

Note that Assumption 4 implies that the volatility for the excess return of the asset increases linearly in the investment horizon. For instance, Assumption 4 can be rationalized if the excess return of the asset is normally distributed.

Using these assumptions, calculations similar to Section 2 yield the equilibrium condition in this n-period environment:

$$Rx_t = \frac{N_{S,t-1}}{N_{S,t-1} + \frac{R^{n-1}}{(n-1)\psi}N_{L,t-1}} f_{St} + \frac{\frac{R^{n-1}}{(n-1)\psi}N_{L,t-1}}{N_{S,t-1} + \frac{R^{n-1}}{(n-1)\psi}N_{L,t-1}} f_{Lt}. \tag{B.5}$$

(B.5) indicates that the effect of long-term investment appears as an increase (decrease) in the fraction of strategy adopted by long-term traders if $R^{n-1} > (n-1)\psi$ ($R^{n-1} < (n-1)\psi$) in this n-period environment.

An interesting feature is that the effect of long-term trading can have an adverse effect depending on the investment horizon adopted by long-term traders because while $(n-1)\psi$ increases linearly in n, R^{n-1} increases exponentially in n, the effect of long-term trading also depends on n as well as R and ψ. In particular, if the excess return is normally distributed, then it is reasonable to suppose that $\psi = 2$ by the square root of time rule in the Wiener process. Thus, because $R < \psi$ is reasonable, more-than-one-period longer trading may appear as a decrease in the fraction of strategy adopted by the long-term traders. Nevertheless, the effect will be reversed when the investment horizon for long-term traders is extremely long so that $R^{n-1} > (n-1)\psi$ holds.