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Abstract

We consider two-sided one-to-one matching problems (between men and women) and study a new requirement called "own-side singles monotonicity." Suppose that there is an agent who is not matched in a problem. Suppose for simplicity it is a woman. Now in a new problem (with the same set of agents), we improve (or leave unchanged) her ranking for each agent on the opposite side of her. *Own-side singles monotonicity* requires that each agent on her side should not be made better off (except for her). Unfortunately, no single-valued solution satisfies *own-side singles monotonicity* and *stability*. However, there is a (multi-valued) solution, the stable solution, that does. We provide two characterizations of the stable solution based on this property. It is the unique solution satisfying *weak unanimity*, *null player invariance*, *own-side singles monotonicity*, and *consistency*. The uniqueness also holds by replacing *consistency* with *Maskin invariance*. In addition, we study the impact of improving her ranking on the welfare of the agents on the opposite side of her.

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1 Introduction

We consider the problem of matching two groups of agents, men and women, in a one-to-one manner. A matching assigns each agent either an agent on the opposite side of her/him or an unmatched option. Each agent has a strict ordering over the set of agents on the opposite side and remaining unmatched. A solution is a correspondence which associates with each problem a non-empty subset of the set of all matchings.

To describe our first main property of a solution, let us take an agent, called "unmatched agent," who is not matched in a problem. Now in a new problem (with the same set of agents), we improve (or leave unchanged) the ranking of the unmatched agent for each agent on the opposite side of her. We do not change the relative ranking over the other agents and an unmatched option, that is, only the position of the unmatched agent is raised (or stays the same). Also, for each agent on the same side of the unmatched agent (including the unmatched agent), her preference remains the same. Our requirement is that no agent on the same side of the unmatched agent except the unmatched agent should be made better off.[1] We call this requirement as "own-side singles monotonicity."[2] As we consider a multi-valued solution, each agent in fact compares two sets: the set of matchings recommended by a solution "before" and "after" improving an unmatched agent. We adopt the pessimistic view.[3] That is, each agent looks at the worst matching for her in each set and compares them.

We also study the impact of improving the rank of an unmatched agent on the welfare of the agents on the opposite side. A solution is "other-side singles monotonic" if an improvement of an unmatched agent affects positively all agents on the opposite side of the unmatched agent. We require that for each of such agents,

[1]We do not require any welfare impact on the unmatched agent. Balinski and Sönmez (1999) study a property requiring that an agent (not necessarily an unmatched agent) whose ranking is improved (or unchanged) for each agent on the opposite side of her not to be worse off. We discuss details of this property and other related properties studied in the literature in the later part of this section.

[2]For notational simplicity, we use the terminology "singles monotonicity" instead of "unmatched agents monotonicity" to describe the property.

[3]We follow the literature that studies other relational properties (e.g., Toda ,2006; Klaus ,2011; and Can and Klaus ,2013).

her new match is not worse than her initial match under her new preference.[4] Here too, as we consider a multi-valued solution, each agent compares the two sets of matchings recommended by a solution before and after the improvement of an unmatched agent. We again adopt the pessimistic view. Thus each agent compares the worst matchings in the two sets.

Our underlying motivation is the "rural hospital problem," which occurred in several countries. After the introduction of a centralized matching system to assign doctors (residents) to hospitals, some rural hospitals began to face a shortage of doctors.[5] One way to tackle such shortage problem is to impose "regional caps" or to take into account "distributional constraints." Several matching systems under constraints have been proposed and the properties of the resulting matchings analyzed (Kamada and Kojima, 2012, 2015, 2017; and Fragiadakis and Troyan, 2017). We propose a different way of solving the problem. We do not impose any constraint on matchings but consider making rural hospitals more attractive to doctors (thorough subsidizing the rural hospitals, for instance).[6] We thereby intend to increase the number of doctors placed in rural hospitals.[7] When some rural hospital becomes more attractive to doctors, it is natural to require that other hospitals or doctors to be affected in the same direction.[8] If by the improvement of some rural hospital, some hospitals (or doctors) are made better off, but some other hospitals (or doctors) are made worse off, then those who become worse off will claim that the improvement was unfair. This will occur especially when the improvement is due to a government's subsidy. Since the improvement of a rural hospital is bad for the other hospitals as it increases the "competition" they face, we require that all hospitals (except the improved hospital) to become at most as well off as before (*own-side singles monotonicity*). Also, since the improvement of a rural hospital is a good thing for doctors as it increases their options, we require that for each doctor, her newly assigned hospital is not worse than her initial one under her new preference (*other-side singles monotonicity*). We analyze these properties. To do

[4]Note that for each agent on the opposite side of the unmatched agent, the rank of the unmatched agent may change after the improvement of the unmatched agent.

[5]A similar problem occurred in students placements for public schools.

[6]In the family medicine residency matching market in the United States, Agarwal (2015) estimates that an average salary of residents (doctors) is lower than the marginal product of labor by 23,000 dollars or more, which can be interpreted as the implicit tuition paid by the residents. Thus, it would be reasonable to subsidize a rural hospital so that it can give monetary or non-monetary incentives to residents.

[7]Kitahara and Okumura (2019) study the number of employed (doctors placed in hospitals) in a stable matching and an efficient matching.

[8]This solidarity idea can be found in several properties proposed in fair allocation problems, e.g., "resource monotonicity," "welfare domination under preference replacement," and "population monotonicity" (see Thomson, 2011a).

so, we focus on the most basic one-to-one matching model (we also describe the model by the usual example of matchings between men and women).

We study the implications of each of the above properties together with the requirement of "stability." A "matching is stable" if (i) it assigns each agent a match that is at least as desirable as being unmatched and (ii) there is no man-woman pair such that the woman is better for the man than his current match and the man is better for the woman than her current match. A solution is "stable" if it only assigns stable matchings. The "stable solution" selects all stable matchings for each problem. The "men-optimal stable solution" selects the stable matching which is best for the men among all stable matchings for each problem.[9] The "women-optimal stable solution" is defined in a similar way.

First, we observe that unfortunately, the men-optimal stable solution satisfies neither *own-side* nor *other-side singles monotonicity.* Moreover, we show that no single-valued solution satisfies *own-side singles monotonicity* and *stability.* Also, no single-valued solution satisfies *other-side singles monotonicity* and *stability.*

Second, however, if we restrict our attention to an improvement of an unmatched woman, then the men-optimal stable solution satisfies both *own-side* and *other-side singles monotonicity.* Also there is a single-valued solution which is different from the men-optimal stable solution and satisfies the above restricted version of *own-side* (or *other-side*) *singles monotonicity* and *stability.* We also formulate variations of *own-side* and *other-side singles monotonicity* and study these properties.

Using the fact that the men-optimal stable solution satisfies *own-side singles monotonicity* when the ranking of an unmatched woman improves, we show that the stable solution (which includes matchings recommended by the men-optimal stable solution and women-optimal stable solution) is *own-side singles monotonic.* The stable solution is not *other-side singles monotonic*, however.

Finally, we provide two characterizations of the stable solution based on *own-side singles monotonicity.* Suppose that there exists a matching at which each agent is matched with her most preferred agent (who is better than the un-matched option). A solution is "weakly unanimous" if it only selects the above matching in such situations. It is "null player invariant" if adding a new agent who is unacceptable for all agents does not affect the initial matchings (we simply assign such new agent the unmatched option). It is "consistent" if for each reduced problem obtained by imagining some agents leaving with their matches, it selects the restriction of the initial matching to the set of remaining agents. It is "Maskin invariant" if when there is a "monotonic transformation" of the preference profile

[9]Such a matching uniquely exists for each problem (see Gale and Shapley, 1962 and Roth and Sotomayor, 1990).

at the initial matching, it again selects (under the new preference profile) the matchings recommended initially. We show that the stable solution is the unique solution satisfying *weak unanimity*, *null player invariance*, *singles monotonicity*, and *consistency*. The uniqueness also holds if *consistency* is replaced by *Maskin invariance*.

In "student placement problems" (where there are sets of students and colleges, each student has a preference over colleges but each college is considered to be an object and has no preferences over students) and "college admissions problems" (here, each college is considered to be an agent and has preferences over students), Balinski and Sönmez (1999) analyze a property called "respecting improvements" (of a student's test scores). It requires that an improvement of a student's test scores not make her worse off. By contrast, Hatfield et al. (2016) focus on the improvement of a school quality and study a property called "respecting improvements of school quality." It requires that if some school improves its quality, that is, each student ranks that school not lower than before while the ordering of other schools is unchanged, then the (new) set of students assigned to the school not make the school worse off. These properties differ from ours in that we study an impact of an improvement not to herself but to agents on the same side or agents on the opposite side.[10] In addition, we only focus on an improvement of an unmatched agent. In college admission problems, Salem (2012) considers a property requiring that if a student's ranking becomes higher for at least one college (without lowering the ranking of the student at any other colleges), then both the student and the college end up at least as well off as before. Here too, the property is different from ours.

For one-to-one matching problems, Tadenuma (2011, 2013) considers another type of changes of preferences that enhance the ranking of the current partners.[11] Based on this preference changes, he studies several properties that basically require that all agents whose preferences are unchanged be affected in the same direction (either all become at least as well off as before or all become at most as well off as before).[12] These properties are obviously different from ours, but they are all expressions of the principle of solidarity.

Characterizations of the stable solution can be found under various require-

[10]Thus, Balinski and Sönmez (1999) and Hatfield et al. (2016) study "self-regarding axioms" (Thomson, 2019) while we study "other-regarding axioms" (Thomson, 2019).

[11]Such a change of preferences is called "rank-enhancements of partners" in Tadenuma (2011, 2013).

[12]One can focuses on an effect to (i) an agent whose preference is unchanged but the ranking by her initial partner becomes higher (Tadenuma, 2013), (ii) an agent whose preference is unchanged and the ranking by her initial partner is unchanged (Tadenuma, 2013), and (iii) an agent (any agent) whose preference is unchanged (Tadenuma, 2011). For (i), it is natural to require that such agent does not become worse off (Tadenuma, 2013).

ments on a solution (Sasaki and Toda, 1992; Toda, 2006; Klaus, 2011; Can and Klaus, 2013; and Nizamogullari and Özkal-Sanver, 2014). Recently, characterizations of the men-optimal stable solution ("deferred acceptance rule") are provided (Kojima and Manea, 2010; Morrill, 2013; Ehlers and Klaus, 2014; Bando and Imamura, 2016; and Chen, 2017).[13]

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 presents the results for single-valued solutions: the implications of *own-side* (or *other-side*) *singles monotonicity* and *stability* in Subsection 3.1 and some possibilities for the restricted version of *own-side* (or *other-side*) *singles monotonicity* in Subsection 3.2. Section 4 presents the results for multi-valued solutions: the properties of the stable solution in Subsection 4.1 and two characterizations of the stable solution in Subsection 4.2. Appendix A provides a proof (of one proposition) not included in the main text. Appendix B shows the logical independence of the properties listed in our characterizations.

2 Model

Let $\mathbb{M}$ and $\mathbb{W}$ be mutually disjoint sets of countably many "potential" agents, called "men" and "women," respectively. Let $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{W}}$ be the sets of all non-empty finite subsets of $\mathbb{M}$ and of $\mathbb{W}$, respectively. Let $\boldsymbol{M} \in \mathcal{M}$ and $\boldsymbol{W} \in \mathcal{W}$. For each $a \in M \cup W$, let $\boldsymbol{(M \cup W)_{-a}}$ be the set of agents on the opposite side of a, i.e., $(M \cup W)_{-a} = W$ if $a \in M$ and $(M \cup W)_{-a} = M$ if $a \in W$. Also for each $a \in M \cup W$, let $\boldsymbol{(M \cup W)_a}$ be the set of agents on the same side as a, i.e., $(M \cup W)_a = M$ if $a \in M$ and $(M \cup W)_a = W$ if $a \in W$. Each agent $a \in M \cup W$ has a strict preference ordering $\succ_a$ over the set $(M \cup W)_{-a} \cup \{\phi\}$, where ϕ represents being unmatched (or the unmatched option). We denote its associated weak ordering by $\succeq_a$, that is for each $a \in M \cup W$ and each $x, y \in (M \cup W)_{-a} \cup \{\phi\}$, $x \succeq_a y$ if and only if either $x \succ_a y$ or $x = y$. For each $a \in M \cup W$ and each $x, y \in (M \cup W)_{-a} \cup \{\phi\}$, we interpret $x \succ_a y$ as "x is better than y for agent a" and $x \succeq_a y$ as "x is at least as desirable as y for agent a." For each $a \in M \cup W$ and each $x \in (M \cup W)_{-a}$, if $\phi \succ_a x$, then x is **unacceptable** for agent a. Let $\boldsymbol{\mathcal{P}_M = \{\succ_m \mid m \in M\}}$ and $\boldsymbol{\mathcal{P}_W = \{\succ_w \mid w \in W\}}$. We write $\boldsymbol{\mathcal{P}_{M \cup W}}$ to denote a preference profile of $\mathcal{P}_M \cup \mathcal{P}_W$. By definition, for each $a \in M \cup W$, $\mathcal{P}_{M \cup W} = \mathcal{P}_{(M \cup W)_a} \cup \mathcal{P}_{(M \cup W)_{-a}}$.

A matching problem or simply a **problem** is a pair $\boldsymbol{(M \cup W, \mathcal{P}_{M \cup W})}$. Let $\boldsymbol{\mathcal{E}}$ be the set of all problems. For each $(M, W) \in \mathcal{M} \times \mathcal{W}$, a **matching** is a function $\boldsymbol{\mu}$ from $M \cup W$ into $M \cup W \cup \{\phi\}$ such that for each $a \in M \cup W$, (i)

[13]See also Ehlers and Klaus (2009), an earlier version of Ehlers and Klaus (2014). Also, Afacan (2013) and Kojima and Ünver (2014) provide characterizations of the "immediate acceptance rule."

$\mu(a) \in (M \cup W)_{-a} \cup \{\phi\}$ and (*ii*) if $\mu(a) \neq \phi$, then $\mu \circ \mu(a) = a$. We often write $(m, w) \in \mu$ to denote $\mu(m) = w$ and $a \in \mu$ to denote $\mu(a) = \phi$. Let $\boldsymbol{\mathcal{Z}(M \cup W)}$ be the set of all matchings for the agent set $M \cup W$. For each $(M, W) \in \mathcal{M} \times \mathcal{W}$, each $\mu \in \mathcal{Z}(M \cup W)$, and each $A \subseteq M \cup W$, let $\boldsymbol{\mu(A)} = \{b \in M \cup W : \mu(a) = b \text{ for some } a \in A\}$.

A **solution** is a correspondence $\boldsymbol{\varphi}$ which associates with each problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ a non-empty subset $\varphi(M \cup W, \mathcal{P}_{M\cup W})$ of $\mathcal{Z}(M \cup W)$. A solution is **single-valued** if for each problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, $|\varphi(M \cup W, \mathcal{P}_{M\cup W})| = 1$.

A matching μ is **individually rational** at $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ if for each $a \in M \cup W$, $\mu(a) \succeq_a \phi$. It is **Pareto optimal** at $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ if there is no $\mu' \in \mathcal{Z}(M \cup W)$ such that for each $a \in M \cup W$, $\mu'(a) \succeq_a \mu(a)$ and for some $a \in M \cup W$, $\mu'(a) \succ_a \mu(a)$. A pair $(m, w) \in M \times W$ **blocks** a matching μ at $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ if $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A matching μ is **stable** at $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ if it is individually rational and has no blocking pair $(m, w) \in M \times W$ at $(M \cup W, \mathcal{P}_{M\cup W})$. A matching μ is the **men-optimal (*M*-optimal) stable matching** at $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ if $\mu \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ and for each $\mu' \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ and each $m \in M$, $\mu(m) \succeq_m \mu'(m)$. Analogously, a matching μ is the **women-optimal (*W*-optimal) stable matching** at $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ if $\mu \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ and for each $\mu' \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ and each $w \in W$, $\mu(w) \succeq_w \mu'(w)$.[14] For each problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, let $\boldsymbol{\mathcal{IR}(M \cup W, \mathcal{P}_{M\cup W})}$, $\boldsymbol{\mathcal{PO}(M \cup W, \mathcal{P}_{M\cup W})}$, and $\boldsymbol{\mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})}$ be the sets of all individually rational matchings, Pareto optimal matchings, and stable matchings at $(M \cup W, \mathcal{P}_{M\cup W})$, respectively. For each problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, let $\boldsymbol{\mathcal{S}_M(M \cup W, \mathcal{P}_{M\cup W})}$ and $\boldsymbol{\mathcal{S}_W(M \cup W, \mathcal{P}_{M\cup W})}$ be the M-optimal stable matching and W-optimal stable matching at $(M \cup W, \mathcal{P}_{M\cup W})$, respectively.

The **stable solution**, denoted $\boldsymbol{\mathcal{S}}$, associates with each problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ the set $\mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$. The ***M*-optimal stable solution**, denoted $\boldsymbol{\mathcal{S}_M}$, and ***W*-optimal stable solution**, denoted $\boldsymbol{\mathcal{S}_W}$, associate with each problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ the matchings $\mathcal{S}_M(M \cup W, \mathcal{P}_{M\cup W})$ and $\mathcal{S}_W(M \cup W, \mathcal{P}_{M\cup W})$, respectively.

Next we define several properties of a solution φ. First, a solution should select stable matchings.

Stability: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, $\varphi(M \cup W, \mathcal{P}_{M\cup W}) \subseteq \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$.

We define an "improvement" of an agent's ranking. Let $h \in M \cup W$. Consider the following change of a preference ordering of an agent on the opposite side of agent h. Her preference ordering over the agents except for agent h remains the

[14]For the existence and uniqueness of the M-optimal stable matching and W-optimal stable matching, see Gale and Shapley (1962) and Roth and Sotomayor(1990).

same and agent h's position rises (or remains the same). We say that the later preference ordering is an h-improvement of the original preference ordering.

Definition 1. *For each* $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$*, each* $h \in M \cup W$*, and each* $a \in (M \cup W)_{-h}$*, a preference ordering* $\boldsymbol{\succ_a^h}$ *on* $(M \cup W)_h \cup \{\phi\}$ ***is an h-improvement of*** $\boldsymbol{\succ_a}$ *if*

(i) $\succ_a$ *and* $\succ_a^h$ *determine the same ordering over the set* $((M \cup W)_h \setminus \{h\}) \cup \{\phi\}$ *and*

(ii) *for each* $h' \in (M \cup W)_h$*,* $h \succ_a h'$ *implies* $h \succ_a^h h'$*.*[15]

Again, let $h \in M \cup W$. If for each agent on the opposite side of agent h, her preference ordering changes to an h-improvement of the original preference ordering, and for each agent on the same side of agent h, her preference ordering remains the same, then we say that the new preference profile is an h-improvement of the original preference profile.

Definition 2. *For each* $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ *and each* $h \in M \cup W$*, a preference profile* $\boldsymbol{\mathcal{P}^h_{M\cup W}} = \{\succ_a^h \mid a \in M \cup W\}$ ***is an h-improvement of*** $\boldsymbol{\mathcal{P}_{M\cup W}}$ *if*

(i) *for each* $a \in (M \cup W)_{-h}$*,* $\succ_a^h$ *is an* h*-improvement of* $\succ_a$ *and*

(ii) *for each* $a \in (M \cup W)_h$*,* $\succ_a^h = \succ_a$*.*

Now we are ready to introduce our two central properties of a solution. First, we require that if there is an agent, say agent h, who is not matched at each matching that a solution recommends in a problem, then for each h-improvement of the original preference profile, no agent on the same side of agent h (except agent h) be made better off. Since we consider a multi-valued solution, each of them compares two sets: the set of matchings recommended by a solution before and after the improvement. We adopt the pessimistic view and assume that each agent compares the worst matchings for her in the two sets.

Own-side singles monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exists $h \in M \cup W$ such that for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(h) = \phi$, then for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$ and each h-improvement $\mathcal{P}^h_{M\cup W}$ of $\mathcal{P}_{M\cup W}$, there exists $\nu \in \varphi(M \cup W, \mathcal{P}^h_{M\cup W})$ such that for each $a \in (M \cup W)_h \setminus \{h\}$, $\mu(a) \succeq_a \nu(a)$.

Second, we again consider the situation where there is an agent, say agent h, who is not matched in a problem. We require that for each h-improvement of the original preference profile, and each agent on the opposite side of agent h, her new

[15] Recall that $\mathcal{P}_{M\cup W} = \{\succ_a \mid a \in M \cup W\}$.

match is not worse than her initial match under her new preference. Here too, (as we consider a multi-valued solution,) each of them compares the worst matchings in the two sets (the set of matchings recommended by a solution before and after the improvement).

Other-side singles monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exists $h \in M \cup W$ such that for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(h) = \phi$, then for each h-improvement $\mathcal{P}^h_{M\cup W}$ of $\mathcal{P}_{M\cup W}$ and each $\nu \in \varphi(M \cup W, \mathcal{P}^h_{M\cup W})$, there exists $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$ such that for each $a \in (M \cup W)_{-h}$, $\nu(a) \succeq^h_a \mu(a)$.

Next, we consider an increase in a number of agents. Before introducing the next property, we define two notions.

Given a problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, suppose that a new agent $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$ is added. Then, for each agent on the opposite side of agent h, we say that her preference ordering is an h-extension of the original preference ordering if her preference ordering over the agents except for agent h remains the same (we do not require anything about agent h's position).

Definition 3. *For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, each $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$, and each $a \in (M \cup W)_{-h}$, a preference ordering $\succ'_a$ on $(M \cup W \cup \{h\})_h \cup \{\phi\}$* ***is an h-extension of*** $\succ_a$ *if $\succ_a$ and $\succ'_a$ determine the same ordering over the set $(M \cup W)_{-a} \cup \{\phi\}$.*

Again, suppose that a new agent $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$ is added to a problem $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$. We say that the new preference profile is an h-extension of the original preference profile if for each agent on the opposite side of agent h, her preference ordering changes to an h-extension of the original preference ordering, and for each agent on the same side of agent h (except agent h), her preference ordering remains the same (we do not impose any condition on agent h's preference ordering.)

Definition 4. *For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ and each $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$, a problem* $(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ *where $\mathcal{P}'_{M\cup W\cup\{h\}} = \{\succ'_a \mid a \in M \cup W \cup \{h\}\}$* ***is an h-extension of*** $(M \cup W, \mathcal{P}_{M\cup W})$ *if*

(i) for each $a \in (M \cup W)_{-h}$, $\succ'_a$ is an h-extension of $\succ_a$ and

(ii) for each $a \in (M \cup W)_h$, $\succ'_a = \succ_a$.

An increase of an agent would negatively affect agents on the same side of her as it increases their competition. The following property says that if a new agent h is added to a problem, then for each h-extension of the original preference profile, no agent on the same side of agent h (except agent h) should be made better off (for multi-valued solution, we again adopt the pessimistic view) (Toda ,2006).

Own-side population monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, each $\mu \in \varphi(M\cup W, \mathcal{P}_{M\cup W})$, each $h \in \mathbb{M}\cup\mathbb{W} \setminus (M\cup W)$, and each h-extension $(M\cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ of $(M \cup W, \mathcal{P}_{M\cup W})$, there exists $\nu \in \varphi(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ such that for each $a \in (M \cup W \cup \{h\})_h \setminus \{h\}$, $\mu(a) \succeq_a \nu(a)$.

Given a problem and its matching, suppose that some agents leave with their matches (we also allow unmatched agents to leave by themself). A problem is called a reduced problem of the original problem if it consists of the remaining agents (with at least one agent on each side) and the preference orderings of each of them is simply the restriction of her original ranking to the set of remaining agents on the other side.

Definition 5. *For each $(M\cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, each $\mu \in \mathcal{M}(M\cup W)$, each $M' \subset M$ with $M' \neq \emptyset$, and each $W' \subset W$ with $W' \neq \emptyset$, a problem* $\boldsymbol{(M' \cup W', \mathcal{P}'_{M'\cup W'})}$ *where $\mathcal{P}'_{M'\cup W'} = \{\succ'_a \mid a \in M'\cup W'\}$* ***is the reduced problem of*** $\boldsymbol{(M \cup W, \mathcal{P}_{M\cup W})}$ ***at*** $\boldsymbol{\mu}$ *if for each $a \in M' \cup W'$,*

(*i*) *if $\mu(a) \neq \emptyset$, then $\mu(a) \in (M' \cup W')_{-a}$ and*

(*ii*) *$\succ'_a$ is the restriction of $\succ_a$ onto $(M' \cup W')_{-a} \cup \{\phi\}$.*

For each $(M, W) \in \mathcal{M} \times \mathcal{W}$ and each $\mu \in \mathcal{Z}(M \cup W)$, let $\boldsymbol{\mu_{M'\cup W'}}$ be the restriction of μ to the set $M' \cup W'$.

The following property requires a solution to select matching in a "consistent" manner when some agents leave with their matches (Sasaki and Toda ,1992). More precisely, it says that if we face with a reduced problem of an original problem at the initial matching, then restriction of the initial matching to the set of remaining agents has to be selected by the solution in the reduced problem.[16]

Consistency: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ and each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, if $(M' \cup W', \mathcal{P}'_{M'\cup W'})$ is the reduced problem of $(M\cup W, \mathcal{P}_{M\cup W})$ at μ, then $\mu_{M'\cup W'} \in \varphi(M' \cup W', \mathcal{P}'_{M'\cup W'})$.

The next property is due to Maskin (1999). It requires that if an original preference is subjected to a "monotonic transformation" at matching selected by a solution initially (defined below), all the matchings be selected again for the new preference profile.

For each $(M\cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, each $\mu \in \mathcal{Z}(M\cup W)$, and each $a \in M\cup W$, the **lower contour set of $\boldsymbol{\succ_a}$ at $\boldsymbol{\mu}$** is $\boldsymbol{L(\mu, \succ_a)} = \{b \in (M\cup W)_{-a} \cup \{\phi\} \mid \mu(a) \succeq_a b\}$.

[16]See Thomson (2011b) for a detailed survey on "consistency principle" and Thomson (2012) for its interpretations.

For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ and each $\mu \in \mathcal{Z}(M \cup W)$, a preference profile $\boldsymbol{\mathcal{P}'_{M\cup W}} = \{\succ'_a |\ a \in M \cup W\}$ **is obtained by a monotonic transformation of** $\boldsymbol{\mathcal{P}_{M\cup W}}$ **at** $\boldsymbol{\mu}$ if for each $a \in M \cup W$, $L(\mu, \succ_a) \subseteq L(\mu, \succ'_a)$.

Maskin invariance: For each $(M\cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ and each $\mu \in \varphi(M\cup W, \mathcal{P}_{M\cup W})$, if $\mathcal{P}'_{M\cup W}$ is obtained by a monotonic transformation of $\mathcal{P}_{M\cup W}$ at μ, then $\mu \in \varphi(M \cup W, \mathcal{P}'_{M\cup W})$.

Finally, we introduce some auxiliary properties of a solution. The next property requires that if there exists a matching at which each agent $a \in M\cup W$ is matched with the agent's most preferred agent who is acceptable for agent a, then only this matching should be chosen.[17]

Weak unanimity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exits a matching $\mu \in \mathcal{M}(M \cup W)$ such that for each $a \in M \cup W$ and each $b \in (M \cup W)_{-a} \cup \{\phi\}$, $\mu(a) \succ_a b$, then $\varphi(M \cup W, \mathcal{P}_{M\cup W}) = \{\mu\}$.

The next property requires a solution to select individually rational matchings.

Individual rationality: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, $\varphi(M \cup W, \mathcal{P}_{M\cup W}) \subseteq \mathcal{IR}(M \cup W, \mathcal{P}_{M\cup W})$.

Given a problem, consider matchings recommended by a solution at the problem. Suppose that we add a new agent who is unacceptable for each agent on the opposite side. Let us call such agent a "null player." The next property requires that adding her has no effect on the set of matchings that are chosen initially. That is, each initial matching augmented by leaving the null player unmatched is selected by the solution at the new problem and the solution recommends no other matching at the new problem. For each $(M, W) \in \mathcal{M} \times \mathcal{W}$, each $\mu \in \mathcal{M}(M \cup W)$, and each $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$, let $\boldsymbol{\mu_{+h} \in \mathcal{M}(M \cup W \cup \{h\})}$ be such that (i) for each $a \in M \cup W$, $\mu_{+h}(a) = \mu(a)$ and (ii) $\mu_{+h}(h) = \phi$.

Null player invariance: For each $(M\cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, each $h \in \mathbb{M}\cup\mathbb{W}\setminus(M\cup W)$, and each h-extension $(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ of $(M \cup W, \mathcal{P}_{M\cup W})$ in which h is unacceptable for each $a \in (M \cup W)_{-h}$, we have $\{\mu_{+h} \mid \mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})\} = \varphi(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$.

One can consider the following weaker notion which only requires that, after adding a null player, each initial matching augmented by leaving the null player unmatched be selected for the new problem (thus, the solution may select other matchings).

Weak null player invariance: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, each $h \in \mathbb{M}\cup\mathbb{W}\setminus(M\cup W)$, and each h-extension $(M\cup W\cup\{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$

[17] Such a matching exists only when the number of men and women are the same.

of $(M \cup W, \mathcal{P}_{M\cup W})$ in which h is unacceptable for each $a \in (M \cup W)_{-h}$, we have $\mu_{+h} \in \varphi(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$.

It is easy to see that the above requirement is equivalent with *null player invariance* when a solution satisfies *individual rationality* and *consistency*.

3 Results: Single-valued solution

3.1 Singles monotonicity and stability

We focus on a single-valued solution and study the implications of *own-side* and *other-side singles monotonicity*. Our first observation is that the M-optimal stable solution satisfies neither of them.

Example 1. *The M-optimal stable solution satisfies neither own-side nor other-side singles monotonicity.* Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\mathcal{P}_{M\cup W}$ be as following.[18]

m_1	$w_1 \succ w_2 \succ \phi \succ w_3$		w_1	$m_2 \succ m_3 \succ m_1 \succ \phi$
m_2	$w_2 \succ w_1 \succ \phi \succ w_3$		w_2	$m_1 \succ m_2 \succ \phi \succ m_3$
m_3	$w_3 \succ w_1 \succ \phi \succ w_2$		w_3	$\phi \succ m_1, m_2, m_3$

There is only one M-optimal stable matching in the problem $(M \cup W, \mathcal{P}_{M\cup W})$. It is given by,

$$\mu_F = \{(m_1, w_2), (m_2, w_1), m_3, w_3\}.$$

Notice that $\mu_F(m_3) = \phi$.

Now, let $\mathcal{P}^{m_3}_{M\cup W}$ be the m_3-improvement of $\mathcal{P}_{M\cup W}$ obtained by replacing $\mathcal{P}_W$ as follows.[19]

m_1	$w_1 \succ w_2 \succ \phi \succ w_3$		w_1	$m_2 \succ m_3 \succ m_1 \succ \phi$
m_2	$w_2 \succ w_1 \succ \phi \succ w_3$		w_2	$m_1 \succ m_2 \succ \phi \succ m_3$
m_3	$w_3 \succ w_1 \succ \phi \succ w_2$		w_3	$\boldsymbol{m_3} \succ \phi \succ m_1, m_2$

[18] At the preference ordering of m_1, for instance, "$w_1 \succ w_2 \succ \phi \succ w_3$" means that "$w_1 \succ_{m_1} w_2 \succ_{m_1} \phi \succ_{m_1} w_3$." Also, at the preference ordering of w_3, "$\phi \succ m_1, m_2, m_3$" means the order over m_1, m_2, and m_3 can be arbitrarily determined as long as they are worse than ϕ for w_3. We adopt similar representations in the rest of the paper.

[19] At the preference ordering of m_1, for instance, we should write "$w_1 \succ^{m_3} w_2 \succ^{m_3} \phi \succ^{m_3} w_3$" instead of "$w_1 \succ w_2 \succ \phi \succ w_3$." However, as long as there is no confusion, we omit the detailed notations over the preference orderings.

Then, there is only one M-optimal stable matching in the problem $(M \cup W, \mathcal{P}^{m_3}_{M\cup W})$. It is given by,

$$\mu^{m_3}_F = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}.$$

But note that m_1 is made better off.[20] Therefore, the M-optimal stable solution does not satisfy *own-side singles monotonicity.*[21] Also, w_1 is made worse off.[22] Thus, the solution is not *other-side singles monotonic.*[23] □

The next two propositions (Propositions 1 and 2) state that the above unfortunate observation is inevitable for any *stable* single-valued solution.

Proposition 1. *No single-valued solution is own-side singles monotonic and stable.*

Proof. Let a single-valued solution φ satisfy the properties listed in the proposition. Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let $\mathcal{P}_{M\cup W}$ be as following.

m_1	$w_1 \succ w_4 \succ w_2 \succ \phi \succ w_3$	w_1	$m_2 \succ m_3 \succ m_1 \succ \phi \succ m_4$
m_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$	w_2	$m_1 \succ m_4 \succ m_2 \succ \phi \succ m_3$
m_3	$w_3 \succ w_1 \succ \phi \succ w_2, w_4$	w_3	$\phi \succ m_1, m_2, m_3, m_4$
m_4	$w_2 \succ \phi \succ w_1, w_3, w_4$	w_4	$m_1 \succ m_2 \succ \phi \succ m_3, m_4$

In the problem $(M \cup W, \mathcal{P}_{M\cup W})$,

$$\{(m_1, w_4), (m_2, w_1), (m_4, w_2), m_3, w_3\}$$

is the unique stable matching. Note that $\mu(m_3) = \phi$ where $\mu \equiv \varphi(M \cup W, \mathcal{P}_{M\cup W})$. Let $\mathcal{P}^{m_3}_{M\cup W}$ be the m_3-improvement of $\mathcal{P}_{M\cup W}$ obtained by replacing $\mathcal{P}_W$ as follows.

m_1	$w_1 \succ w_4 \succ w_2 \succ \phi \succ w_3$	w_1	$m_2 \succ m_3 \succ m_1 \succ \phi \succ m_4$
m_2	$w_4 \succ w_2 \succ w_1 \succ \phi \succ w_3$	w_2	$m_1 \succ m_4 \succ m_2 \succ \phi \succ m_3$
m_3	$w_3 \succ w_1 \succ \phi \succ w_2, w_4$	w_3	$\boldsymbol{m_3} \succ \phi \succ m_1, m_2, m_4$
m_4	$w_2 \succ \phi \succ w_1, w_3, w_4$	w_4	$m_1 \succ m_2 \succ \phi \succ m_3, m_4$

[20] That is, $\mu^{m_3}_M(m_1) \succ_{m_1} \mu_M(m_1)$.

[21] One can also show it by observing that m_2 is made better off. Notice that m_3 can be interpreted as an "interrupter" in Kesten (2010) and Tang and Yu (2014).

[22] That is, $\mu_M(w_1) \succ^{m_3}_{w_1} \mu^{m_3}_M(w_1)$.

[23] Here too, one can show it by observing that w_2 is made worse off.

Then, in the problem $(M \cup W, \mathcal{P}^{m_3}_{M\cup W})$,

$$\{(m_1, w_1), (m_2, w_4), (m_3, w_3), (m_4, w_2)\} \text{ and}$$
$$\{(m_1, w_4), (m_2, w_1), (m_3, w_3), (m_4, w_2)\}$$

are the only stable matchings. By *own-side singles monotonicity*,

$$\varphi(M \cup W, \mathcal{P}^{m_3}_{M\cup W}) \text{ selects } \{(m_1, w_4), (m_2, w_1), (m_3, w_3), (m_4, w_2)\}. \qquad (*)$$

Now, let $\hat{\mathcal{P}}_{M\cup W}$ be as following.

m_1	$w_1 \mathrel{\hat{\succ}} w_4 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_2, w_3$		w_1	$m_2 \mathrel{\hat{\succ}} m_3 \mathrel{\hat{\succ}} m_1 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_4$
m_2	$w_4 \mathrel{\hat{\succ}} w_2 \mathrel{\hat{\succ}} w_1 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_3$		w_2	$m_1 \mathrel{\hat{\succ}} m_4 \mathrel{\hat{\succ}} m_2 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_3$
m_3	$w_3 \mathrel{\hat{\succ}} w_1 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_2, w_4$		w_3	$m_3 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_1, m_2, m_4$
m_4	$\phi \mathrel{\hat{\succ}} w_1, w_2, w_3, w_4$		w_4	$m_1 \mathrel{\hat{\succ}} m_2 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_3, m_4$

Then, in the problem $(M \cup W, \hat{\mathcal{P}}_{M\cup W})$,

$$\{(m_1, w_1), (m_2, w_4), (m_3, w_3), m_4, w_2\}$$

is the unique stable matching. Note that $\hat{\mu}(w_2) = \phi$ where $\hat{\mu} \equiv \varphi(M \cup W, \hat{\mathcal{P}}_{M\cup W})$. Let $\hat{\mathcal{P}}^{w_2}_{M\cup W}$ be the w_2-improvement of $\hat{\mathcal{P}}_{M\cup W}$ obtained by replacing $\hat{\mathcal{P}}_M$ as follows.

m_1	$w_1 \mathrel{\hat{\succ}} w_4 \mathrel{\hat{\succ}} \boldsymbol{w_2} \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_3$		w_1	$m_2 \mathrel{\hat{\succ}} m_3 \mathrel{\hat{\succ}} m_1 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} f_4$
m_2	$w_4 \mathrel{\hat{\succ}} w_2 \mathrel{\hat{\succ}} w_1 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_3$		w_2	$m_1 \mathrel{\hat{\succ}} m_4 \mathrel{\hat{\succ}} m_2 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_3$
m_3	$w_3 \mathrel{\hat{\succ}} w_1 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_2, w_4$		w_3	$m_3 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_1, m_2, m_4$
m_4	$\boldsymbol{w_2} \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} w_1, w_3, w_4$		w_4	$m_1 \mathrel{\hat{\succ}} m_2 \mathrel{\hat{\succ}} \phi \mathrel{\hat{\succ}} m_3, m_4$

Note that $\mathcal{P}^{m_3}_{M\cup W} = \hat{\mathcal{P}}^{w_2}_{M\cup W}$. Thus by $(*)$,

$$\varphi(M \cup W, \hat{\mathcal{P}}^{w_2}_{M\cup W}) \text{ selects } \{(m_1, w_4), (m_2, w_1), (m_3, w_3), (m_4, w_2)\}.$$

However, $\hat{\mu}^{w_2}(w_1) \mathrel{\hat{\succ}_{w_1}} \hat{\mu}(w_1)$ where $\hat{\mu}^{w_2} \equiv \varphi(M \cup W, \hat{\mathcal{P}}^{w_2}_{M\cup W})$, in violation of *own-side singles monotonicity.*[24] □

[24]For more agents, one can lead to a similar contradiction by (consecutively) adding a pair of man and woman who most prefer each other.

Proposition 1 is tight. The single-valued solution that always assigns the unmatched option to each agent is *own-side singles monotonic* but not *stable*. The M-optimal stable solution is *stable* but not *own-side singles monotonic*.

How about the compatibility of *other-side singles monotonicity* and *stability*? We show that for a *stable* single-valued solution, *own-side singles monotonicity* and *other-side singles monotonicity* coincide.

Lemma 1. *Let φ be a stable single-valued solution. Then, φ is* own-side singles *monotonic if and only if it is* other-side singles monotonic*.*

Proof. First, let a single-valued solution φ be *stable* and *own-side singles monotonic*. Suppose by contradiction that φ is not *other-side singles monotonic*. Then, there exist $(M \cup W, \mathcal{P}_{M \cup W}) \in \mathcal{E}$, $\mu = \varphi(M \cup W, \mathcal{P}_{M \cup W})$, $h \in M \cup W$ such that $\mu(h) = \phi$, an h-improvement $\mathcal{P}^h_{M \cup W}$ of $\mathcal{P}_{M \cup W}$, $\mu^h = \varphi(M \cup W, \mathcal{P}^h_{M \cup W})$, $a \in (M \cup W)_{-h}$ such that $\mu(a) \succ^h_a \mu^h(a)$. Since $\mu(a) \neq \phi$, there exists $b \in (M \cup W)_h \setminus \{h\}$ such that $\mu(a) = b$. Since $b = \mu(a) \neq \mu^h(a)$, either $\mu(b) \succ_b \mu^h(b)$ or $\mu^h(b) \succ_b \mu(b)$. If $\mu(b) \succ_b \mu^h(b)$, then $\mu(b) \succ^h_b \mu^h(b)$, which contradicts the *stability* of μ^h.[25] Thus, $\mu(b)^h \succ_b \mu(b)$. But then, *own-side singles monotonicity* is violated. Thus, φ is *other-side singles monotonic*.

Second, let a single-valued solution φ be *stable* and *other-side singles monotonic*. Suppose by contradiction that φ is not *own-side singles monotonic*. Then, there exist $(M \cup W, \mathcal{P}_{M \cup W}) \in \mathcal{E}$, $\mu = \varphi(M \cup W, \mathcal{P}_{M \cup W})$, $h \in M \cup W$ such that $\mu(h) = \phi$, an h-improvement $\mathcal{P}^h_{M \cup W}$ of $\mathcal{P}_{M \cup W}$, $\mu^h = \varphi(M \cup W, \mathcal{P}^h_{M \cup W})$, $a \in (M \cup W)_h \setminus \{h\}$ such that $\mu^h(a) \succ_a \mu(a)$. Since $\mu^h(a) \neq \phi$, there exists $b \in (M \cup W)_{-h}$ such that $\mu^h(a) = b$. Since $b = \mu^h(a) \neq \mu(a)$, either $\mu^h(b) \succ^h_b \mu(b)$ or $\mu(b) \succ^h_b \mu^h(b)$. If $\mu^h(b) \succ^h_b \mu(b)$, then since $\mu^h(b) = a \neq h$ and $\mu(b) \neq h$, $\mu^h(b) \succ_b \mu(b)$, which contradicts the *stability* of μ.[26] Thus, $\mu(b) \succ^h_b \mu^h(b)$. But then, *other-side singles monotonicity* is violated. Thus, φ is *own-side singles monotonic*. □

By Proposition 1 and Lemma 1, we obtain the following.

Proposition 2. *No single-valued solution is* other-side singles monotonic *and stable.*

3.2 Restricted singles monotonicity

In Example 1, we consider an m_3-improvement of $\mathcal{P}_{M \cup W}$ and observe that the M-optimal stable solution satisfies neither *own-side* nor *other-side singles monotonicity*. Interestingly, if instead we adopt the W-optimal stable solution and

[25]Notice that $\mu(a) = b \succ^h_a \mu^h(a)$ and $\mu(b) = a \succ^h_b \mu^h(b)$.

[26]Notice that $\mu(a)^h = b \succ_a \mu(a)$ and $\mu(b)^h = a \succ_b \mu(b)$.

consider an m_3-improvement, the conclusion drastically changes, as shown in the next example.

Example 2. *The W-optimal stable solution satisfies own-side and other-side singles monotonicity for the m_3-improvement in Example 1.* Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\mathcal{P}_{M\cup W}$ be given as in Example 1.

The W-optimal stable matching in the problem $(M \cup W, \mathcal{P}_{M\cup W})$ is

$$\mu_W = \{(m_1, w_2), (m_2, w_1), m_3, w_3\},$$

which is the same as the M-optimal stable matching. Note that $\mu_W(m_3) = \phi$.

Now, let $\mathcal{P}^{m_3}_{M\cup W}$ be the m_3-improvement of $\mathcal{P}_{M\cup W}$ of Example 1.

Then, the W-optimal stable matching in the problem $(M \cup W, \mathcal{P}^{m_3}_{M\cup W})$ is

$$\mu^{m_3}_W = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}.$$

Then, for each $m \in M \setminus \{m_3\}$, $\mu_W(m) \succeq_m \mu^{m_3}_W(m)$, and for each $w \in W$, $\mu^{m_3}_W(w) \succeq^{m_3}_w \mu_W(w)$. Thus, *own-side* and *other-side singles monotonicity* are satisfied. □

The above observation is generally true, namely that the W-optimal stable solution satisfies *own-side* and *other-side singles monotonicity* for each h-improvement as long as $h \in M$. Symmetrically, the M-optimal stable solution satisfies these properties for each h-improvement as long as $h \in W$ (Propositions 3 and 4).

We begin by providing the formal definitions of those "restricted singles monotonicity."[27] The first two properties restrict attention to an improvement of an unmatched man.

Own-side $\boldsymbol{M}$-singles monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exists $m \in M$ such that for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(m) = \phi$, then for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$ and each m-improvement $\mathcal{P}^m_{M\cup W}$ of $\mathcal{P}_{M\cup W}$, there exists $\nu \in \varphi(M \cup W, \mathcal{P}^m_{M\cup W})$ such that for each $a \in M \setminus \{m\}$, $\mu(a) \succeq_a \nu(a)$.

Other-side $\boldsymbol{M}$-singles monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exists $m \in M$ such that for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(m) = \phi$, then for each m-improvement $\mathcal{P}^m_{M\cup W}$ of $\mathcal{P}_{M\cup W}$ and each $\nu \in \varphi(M \cup W, \mathcal{P}^m_{M\cup W})$, there exists $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$ such that for each $a \in W$, $\nu(a) \succeq^m_a \mu(a)$.

The next two properties restrict attention to an improvement of an unmatched woman.

[27] We define them in terms of multi-valued solutions as we study some of them in the next section.

Own-side W-singles monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exists $w \in W$ such that for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(w) = \phi$, then for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$ and each w-improvement $\mathcal{P}^w_{M\cup W}$ of $\mathcal{P}_{M\cup W}$, there exists $\nu \in \varphi(M \cup W, \mathcal{P}^w_{M\cup W})$ such that for each $a \in W \setminus \{w\}$, $\mu(a) \succeq_a \nu(a)$.

Other-side W-singles monotonicity: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, if there exists $w \in W$ such that for each $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(w) = \phi$, then for each w-improvement $\mathcal{P}^w_{M\cup W}$ of $\mathcal{P}_{M\cup W}$ and each $\nu \in \varphi(M \cup W, \mathcal{P}^w_{M\cup W})$, there exists $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$ such that for each $a \in M$, $\nu(a) \succeq^w_a \mu(a)$.

Similar to Lemma 1, we have the following equivalences between the properties under single-valued *stable* solution. We omit its proof as we can prove it in the same way as Lemma 1.

Lemma 2. *Let φ be a stable single-valued solution. Then, φ is own-side M-singles monotonic if and only if it is other-side M-singles monotonic. Also, φ is own-side W-singles monotonic if and only if it is other-side W-singles monotonic.*

Now we are ready to state two important propositions.

Proposition 3. *The M-optimal stable solution is own-side W-singles monotonic.*

Proposition 4. *The M-optimal stable solution is other-side W-singles monotonic.*

To show Propositions 3 and 4, because we have Lemma 2, it is enough to show Proposition 4. We first refer to the well-known "Blocking Lemma."[28]

Lemma 3 (The Blocking Lemma)**.** *Let $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, $\mu_M = \mathcal{S}_M(M \cup W, \mathcal{P}_{M\cup W})$, and $\mu \in \mathcal{IR}(M \cup W, \mathcal{P}_{M\cup W})$. If*

$$M' \equiv \{m' \in M \mid \mu(m') \succ_{m'} \mu_M(m')\} \neq \emptyset,$$

there exists a blocking pair (m, w') of μ at $(M \cup W, \mathcal{P}_{M\cup W})$ such that $m \in M \setminus M'$ and $w' \in \mu(M')$.

Proof of Proposition 4. Let $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ and $\mu_M = \mathcal{S}_M(M \cup W, \mathcal{P}_{M\cup W})$. Suppose that there exists $w \in W$ such that $\mu_M(w) = \phi$. Let $\mathcal{P}^w_{M\cup W}$ be a w-improvement of $\mathcal{P}_{M\cup W}$ and $\mu^w_M = \mathcal{S}_M(M \cup W, \mathcal{P}^w_{M\cup W})$. Note that μ_M is individually rational at $(M \cup W, \mathcal{P}^w_{M\cup W})$. Suppose

$$M' \equiv \{m' \in F \mid \mu_M(m') \succ^w_{m'} \mu^w_M(m')\} \neq \emptyset.$$

[28]The statement of the Blocking Lemma is due to J. S. Hwang. See Roth and Sotomayor (1990). For the proof of the Blocking Lemma, see Gale and Sotomayor (1985).

	OWN.S.MON		*OTHER.S.MON*	
	OWN. *M-S.MON*	*OWN.* *W-S.MON*	*OTHER.* *M-S.MON*	*OTHER.* *W-S.MON*
$\mathcal{S}_M$	−		−	
	−	+	−	+
$\mathcal{S}_W$	−		−	
	+	−	+	−

Table 1: *Own-side* and *other-side singles monotonicity* and other variations. *Own-side singles monotonicity* (*OWN.S.MON*) consists of *own-side M-singles monotonicity* (*OWN.M-S.MON*) and *own-side W-singles monotonicity* (*OWN.W-S.MON*). Similarly, *other-side singles monotonicity* (*OTHER.S.MON*) consists of *other-side M-singles monotonicity* (*OTHER.M-S.MON*) and *other-side W-singles monotonicity* (*OTHER.W-S.MON*). The symbol "+" (respectively, "−") means the corresponding solution satisfies (respectively, does not satisfy) the corresponding property.

By the Blocking Lemma, there exist a pair (m, w') such that $m \in M \setminus M'$ and $w' \in \mu_M(M')$, and $w' \succ^w_m \mu_M(m)$ and $m \succ^w_{w'} \mu_M(w')$. Since $w' \in \mu_M(M')$, $w' \neq w$. Also, $\mu_M(m) \neq w$. Therefore,

$$w' \succ_m \mu_M(m). \tag{1}$$

Since $\succ^w_{w'} = \succ_{w'}$, we have

$$m \succ_{w'} \mu_M(w'). \tag{2}$$

By (1) and (2), there exists a blocking pair $(m, w') \in M \times W$ of μ_M at $(M \cup W, \mathcal{P}_{M \cup W})$, which is a contradiction. Hence, $M' = \emptyset$ and for each $m \in M$, $\mu^w_M(m) \succeq^w_m \mu_M(m)$. □

As a corollary of Proposition 3, we obtain the following.

Corollary 1. *The W-optimal stable solution is* own-side *M*-singles monotonic.

The next is a corollary of Proposition 4.

Corollary 2. *The W-optimal stable solution is* other-side *M*-singles monotonic.

Table 1 summarizes our findings regarding versions of the restricted singles monotonicity.

Let a solution be ***M*-singles monotonic** if it is *own-side* and *other-side M-singles monotonic*. That is, we require that for each improvement of an unmatched man, each man (own-side) except the improved one should not be made better off under his original preference **and** each woman (other-side) should not be assigned a match that is worse than the initial one under her new preference. Also, let a solution be ***W*-singles monotonic** if it is *own-side* and *other-side W-singles monotonic*.

By the previous argument, we have the followings.

Remark 1. *The M-optimal stable solution is W-singles monotonic.*

Remark 2. *The W-optimal stable solution is M-singles monotonic.*

Given Remark 1, is natural to ask whether the M-optimal stable solution is the only single-valued *stable* solution that satisfies *W-singles monotonicity*. The answer is no.

Proposition 5. *Let $|M| = |W| \geq 2$. There exists a single-valued stable solution satisfying W-singles monotonicity that is different from the M-optimal stable solution.*

The proof is in Appendix A.

4 Results: Multi-valued solution

4.1 Properties of the stable solution

We consider a solution that may select multiple matchings for a problem. First, we observe that the stable solution is not *other-side singles monotonic*.

Example 3. *The stable solution is not other-side singles monotonic.* Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\mathcal{P}_{M\cup W}$ be given as in Example 1.

Since the M-optimal and W-optimal stable matchings coincide in the problem $(M \cup W, \mathcal{P}_{M\cup W})$,

$$\mu_S = \{(m_1, w_2), (m_2, w_1), m_3, w_3\}$$

is the unique stable matching. Note that $\mu_S(m_3) = \phi$.

Now, let $\mathcal{P}^{m_3}_{M\cup W}$ be a m_3-improvement of $\mathcal{P}_{M\cup W}$ as in Example 1.

Then, in the problem $(M \cup W, \mathcal{P}^{m_3}_{M\cup W})$,

$$\mu^{m_3}_S = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$$

is a stable matching.

Then, for $w_1 \in (M \cup W)_{-m_3}$, $\mu_S(w_1) \succ_{w_1}^{m_3} \mu_S^{m_3}(w_1)$. But then, for $\mu_S^{m_3} \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W}^{m_3})$, there is no $\mu \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ such that $\mu_S^{m_3}(w_1) \succeq_{w_1}^{m_3} \mu(w_1)$, in violation of *other-side singles monotonicity.* □

From the above example, we learn that the stable solution is not even *other-side M-singles monotonic.* By changing the role of men and women in the same example, we can show that it is not *other-side W-singles monotonic*, either. Hence, the stable solution is neither *M-singles* nor *W-singles monotonic.*

Next, for *own-side singles monotonicity*, by contrast to the impossibility obtained for a single-valued solution (Proposition 1), we have a positive result for a multi-valued solution.

Theorem 1. *The stable solution is own-side singles monotonic.*

Proof of Theorem 1. Let $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$. Suppose that there exists $h \in M \cup W$ such that for each $\mu \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$, $\mu(h) = \phi$. Let $\mathcal{P}_{M\cup W}^h$ be an h-improvement of $\mathcal{P}_{M\cup W}$. Let $\mu_M = \mathcal{S}_M(M \cup W, \mathcal{P}_{M\cup W})$ and $\mu_M^h = \mathcal{S}_M(M \cup W, \mathcal{P}_{M\cup W}^h)$.

If $h \in W$, by Proposition 3, for each $w \in W \setminus \{h\}$, $\mu_M(w) \succeq_w \mu_M^h(w)$. Since the M-optimal stable matching is the worst stable matching for the women (Knuth, 1976),[29] then for each $\mu \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ and each $w \in W$, $\mu(w) \succeq_w \mu_M(w)$. Overall, for each $\mu \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W})$ and each $w \in W \setminus \{h\}$, there exists $\mu_M^h \in \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W}^h)$ such that $\mu(w) \succeq_w \mu_M^h(w)$.

By a symmetric argument, if $h \in M$, then Corollary 1 gives us the required conclusion. □

4.2 Characterizations of the stable solution

We provide two characterizations of the stable solution based on *own-side singles monotonicity.* The following relationship among the properties of a solution is key to proving our characterizations.

Proposition 6. *If a solution is null player invariant and own-side singles monotonic, then it is own-side population monotonic.*

Proof. Let a solution φ satisfy the hypotheses of the proposition. Let $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$ and $\mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})$. Let $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$ and $(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ be the h-extension of $(M \cup W, \mathcal{P}_{M\cup W})$. Now, let $(M \cup W \cup \{h\}, \mathcal{P}''_{M\cup W\cup\{h\}})$ be the problem obtained from $\mathcal{P}'_{M\cup W\cup\{h\}}$ in the following way:

[29] See also Roth and Sotomayor (1990).

- $\mathcal{P}''_{(M\cup W\cup\{h\})_h} = \mathcal{P}'_{(M\cup W\cup\{h\})_h}$.
- For each $a \in (M \cup W \cup \{h\})_{-h}$, her preference ordering in $\mathcal{P}''_{M\cup W\cup\{h\}}$ is identical with the one in $\mathcal{P}'_{M\cup W\cup\{h\}}$ on the set $(M \cup W \cup \{h\})_h \setminus \{h\}$.
- For each $a \in (M\cup W\cup\{h\})_{-h}$ and each $x \in (M\cup W\cup\{h\})_h\setminus\{h\}$, (i) $x \succ_a h$ and (ii) h is unacceptable for agent a.[30]

Since $(M\cup W\cup\{h\}, \mathcal{P}''_{M\cup W\cup\{h\}})$ is an h-extension of $(M\cup W, \mathcal{P}_{M\cup W})$ in which h is unacceptable for each $a \in (M\cup W)_{-h}$, by *null player invariance*, we have

$$\{\nu_{+h} \mid \nu \in \varphi(M\cup W, \mathcal{P}_{M\cup W})\} = \varphi(M\cup W\cup\{h\}, \mathcal{P}''_{M\cup W\cup\{h\}}).$$

Thus,

$$\mu_{+h} \in \varphi(M\cup W\cup\{h\}, \mathcal{P}''_{M\cup W\cup\{h\}}), \tag{3}$$

and for each $\nu \in \varphi(M\cup W\cup\{h\}, \mathcal{P}''_{M\cup W\cup\{h\}})$, $\nu(h) = \phi$. By construction, $\mathcal{P}'_{M\cup W\cup\{h\}}$ is an h-improvement of $\mathcal{P}''_{M\cup W\cup\{h\}}$. Then, by *own-side singles monotonicity* and (3), there exists $\mu' \in \varphi(M\cup W\cup\{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ such that for each $a \in (M\cup W\cup \{h\})_h \setminus \{h\}$, $\mu_{+h}(a) \succeq_a \mu'(a)$.

Since for each $a \in M\cup W$, $\mu(a) = \mu_{+h}(a)$, we conclude that there exists $\mu' \in \varphi(M\cup W\cup\{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$ such that for each $a \in (M\cup W\cup\{h\})_h \setminus \{h\}$, $\mu(a) \succeq_a \mu'(a)$, as desired. □

Now we are ready to provide our first characterization.

Theorem 2. *The stable solution is the unique solution satisfying* weak unanimity, null player invariance, own-side singles monotonicity, *and* consistency.

Proof. Let a solution φ satisfy the above properties. By Proposition 6, φ is *own-side population monotonic*. Since the stable solution is the unique solution satisfying *weak unanimity*, *own-side population monotonicity*, and *consistency* (Toda, 2006), φ is the stable solution. By Theorem 1, the stable solution satisfies *own-side singles monotonicity*, and obviously it also satisfies *null player invariance*. □

The logical independence of the properties listed in Theorem 2 is shown in Appendix B.

In our second characterization, we replace *consistency* in Theorem 2 with *Maskin invariance*.

[30]That is, for each $a \in (M\cup W\cup\{h\})_{-h}$, h is the worst option for agent a.

Theorem 3. *The stable solution is the unique solution satisfying weak unanimity, null player invariance, own-side singles monotonicity, and Maskin invariance.*

Proof. Since the stable solution is the unique solution satisfying *weak unanimity*, *own-side population monotonicity*, and *Maskin invariance* (Toda, 2006), we can adopt the similar argument as in the proof of Theorem 2 (by referring to Proposition 6 and Theorem 1). □

We show the logical independence of the properties in Theorem 3 in Appendix B.

Appendix A

We prove Proposition 5.

Proposition 5 *Let* $|M| = |W| \geq 2$. *There exists a single-valued stable solution satisfying* W*-singles monotonicity that is different from the* M*-optimal stable solution.*

Proof. Let $|M| = |W| = n \geq 2$. We distinguish two cases.

Case 1: $\boldsymbol{n = 2}$**.**

Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let $\hat{\mathcal{P}}_{M\cup W}$ be as following.

m_1	$w_2 \hat{\succ} w_1 \hat{\succ} \phi$
m_2	$w_1 \hat{\succ} w_2 \hat{\succ} \phi$

w_1	$m_1 \hat{\succ} m_2 \hat{\succ} \phi$
w_2	$m_2 \hat{\succ} m_1 \hat{\succ} \phi$

In the problem $(M \cup W, \hat{\mathcal{P}}_{M\cup W})$, the M-optimal and W-optimal stable matchings are given by,

$$\hat{\mu}_M = \{(m_1, w_2), (m_2, w_1)\} \text{ and}$$
$$\hat{\mu}_W = \{(m_1, w_1), (m_2, w_2)\},$$

respectively.

Now consider a single-valued solution that chooses the W-optimal stable matching in the problem $(M\cup W, \hat{\mathcal{P}}_{M\cup W})$, and the M-optimal stable matching in the other problems. Let us call the solution "solution A^*." Obviously, A^* always chooses a stable matching, and it differs from the M-optimal stable solution. We will show that A^* is W*-singles monotonic*. Since A^* is single-valued and *stable*, by Lemma 2, it is enough to show that it is *own-side* W*-singles monotonic*. Since the M-optimal stable solution satisfies *own-side* W*-singles monotonicity*, it is enough to show that the requirement is satisfied for any pair of problems involving $(M \cup W, \hat{\mathcal{P}}_{M\cup W})$.

First, for each $w \in W$ and each w-improvement of $\hat{\mathcal{P}}_{M \cup W}$, *own-side W-singles monotonicity* is trivially satisfied since no worker is single in the problem $(M \cup W, \hat{\mathcal{P}}_{M \cup W})$.

Next, consider $\mathcal{P}_M$ such that $\mathcal{P}_M^w = \hat{\mathcal{P}}_M$ for some $w \in W$. Let μ_{A^*} and $\hat{\mu}_{A^*}$ be the matchings given by A^* in the problems $(M \cup W, \mathcal{P}_{M \cup W})$ and $(M \cup W, \hat{\mathcal{P}}_{M \cup W})$, respectively.[31] If $\mu_{A^*}(w) \neq \phi$, then *own-side W-singles monotonicity* is trivially satisfied. Thus, suppose that $\mu_{A^*}(w) = \phi$.

Subcase 1.1: If $w = w_1$.

Then, $\mu_{A*}(w_1) = \phi$. If $\mu_{A*}(w_2) \neq m_2$, then $\mu_{A*}(m_2) = \phi$. But then, (m_2, w_2) blocks μ_{A*}. This is a contradiction since μ_{A*} is stable. Thus, $\mu_{A*}(w_2) = m_2$. Then, for each $w \in W \setminus w_1$,

$$\mu_{A*}(w) \succeq_w \hat{\mu}_{A*}(w).$$

Subcase 1.2: If $w = w_2$.

Then, $\mu_{A*}(w_2) = \phi$. By a similar argument as in Subcase 1.1, $\mu_{A*}(w_1) = m_1$. Thus, for each $w \in W \setminus w_2$,

$$\mu_{A*}(w) \succeq_w \hat{\mu}_{A*}(w).$$

By Subcases 1.1 and 1.2, *own-side W-singles monotonicity* is satisfied.

Case 2: $n \geq 3$.

Let $M = \{m_1, m_2, \ldots, m_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. For each $k \in \{3, 4, \ldots, n\}$, let $\underline{k}[m]$ be the m-th lowest number in $\{3, 4, \ldots, n\} \setminus k$. Let $\hat{\mathcal{P}}_{M \cup W}$ be as following.

m_1	$w_2 \hat{\succ} w_1 \hat{\succ} \phi \hat{\succ} w_3 \hat{\succ} w_4 \hat{\succ} \cdots\cdots \hat{\succ} w_n$
m_2	$w_1 \hat{\succ} w_2 \hat{\succ} \phi \hat{\succ} w_3 \hat{\succ} w_4 \hat{\succ} \cdots\cdots \hat{\succ} w_n$
m_k $(k=3,4,\ldots,n)$	$w_k \hat{\succ} w_1 \hat{\succ} \phi \hat{\succ} w_2 \hat{\succ} w_{\underline{k}[1]} \hat{\succ} w_{\underline{k}[2]} \hat{\succ} \cdots \hat{\succ} w_{\underline{k}[n-3]}$

w_1	$m_1 \hat{\succ} m_n \hat{\succ} m_{n-1} \hat{\succ} \cdots\cdots \hat{\succ} m_3 \hat{\succ} m_2 \hat{\succ} \phi$
w_2	$m_2 \hat{\succ} m_1 \hat{\succ} \phi \hat{\succ} m_3 \hat{\succ} m_4 \hat{\succ} \cdots\cdots \hat{\succ} m_n$
w_k $(k=3,4,\ldots,n)$	$m_k \hat{\succ} \phi \hat{\succ} m_1 \hat{\succ} m_2 \hat{\succ} m_{\underline{k}[1]} \hat{\succ} m_{\underline{k}[2]} \hat{\succ} \cdots \hat{\succ} m_{\underline{k}[n-3]}$

In the problem $(M \cup W, \hat{\mathcal{P}}_{M \cup W})$, the M-optimal and W-optimal stable matchings are given by

$$\hat{\mu}_M = \{(m_1, w_2), (m_2, w_1), \{(m_k, w_k)\}_{k \in \{3,4,\ldots,n\}}\} \text{ and}$$
$$\hat{\mu}_W = \{(m_1, w_1), (m_2, w_2), \{(m_k, w_k)\}_{k \in \{3,4,\ldots,n\}}\},$$

[31] By definition, $\hat{\mu}_{A^*} = \hat{\mu}_W$.

respectively.

As in Case 1, consider a single-valued solution that chooses the W-optimal stable matching in the problem $(M \cup W, \hat{\mathcal{P}}_{M\cup W})$, and the M-optimal stable matching in the other problems. Let us call this solution "solution $\tilde{A}$." By by Lemma 2, it is enough to show that $\tilde{A}$ is *own-side W-singles monotonic*. Now consider $\mathcal{P}_M$ such that $\mathcal{P}_M^w = \hat{\mathcal{P}}_M$ for some $w \in W$. Let $\mu_{\tilde{A}}$ and $\hat{\mu}_{\tilde{A}}$ be the matchings given by $\tilde{A}$ in the problems $(M \cup W, \mathcal{P}_{M\cup W})$ and $(M \cup W, \hat{\mathcal{P}}_{M\cup W})$, respectively. As argued previously, it is enough to investigate the case $\mu_{\tilde{A}}(w) = \phi$.

Subcase 2.1: If $w = w_1$.

Then, $\mu_{\tilde{A}}(w_1) = \phi$. If $\mu_{\tilde{A}}(w_k) \neq m_k$ for some $k \in \{3, 4, \ldots, n\}$, then (m_k, w_k) blocks $\mu_{\tilde{A}}$. This is a contradiction since $\mu_{\tilde{A}}$ is stable. Thus, for each $k \in \{3, 4, \ldots, n\}$, $\mu_{\tilde{A}}(w_k) = m_k$. If $\mu_{\tilde{A}}(w_2) \neq m_2$, then (since $\mu_{\tilde{A}}(w_1) = \phi$ and for each $k \in \{3, 4, \ldots, n\}$, $\mu_{\tilde{A}}(w_k) = m_k$), $\mu_{\tilde{A}}(m_2) = \phi$. But then, (m_2, w_2) blocks $\mu_{\tilde{A}}$, a contradiction. Thus, $\mu_{\tilde{A}}(w_2) = m_2$. Overall, for each $w \in W \setminus w_1$,

$$\mu_{\tilde{A}}(w) \succeq_w \hat{\mu}_{\tilde{A}}(w).$$

Subcase 2.2: If $w = w_2$.

Then, $\mu_{\tilde{A}}(w_2) = \phi$. By a similar argument as in Subcase 2.1, for each $k \in \{3, 4, \ldots, n\}$, $\mu_{\tilde{A}}(w_k) = m_k$. If $\mu_{\tilde{A}}(w_1) \neq m_1$, then, $\mu_{\tilde{A}}(m_1) = \phi$. But then, (m_1, w_1) blocks $\mu_{\tilde{A}}$, a contradiction. Thus, $\mu_{\tilde{A}}(w_1) = m_1$. Overall, for each $w \in W \setminus w_2$,

$$\mu_{\tilde{A}}(w) \succeq_w \hat{\mu}_{\tilde{A}}(w).$$

Subcase 2.3: Let $k^* \in \{3, 4, \ldots, n\}$. If $w = w_{k^*}$.

Then, $\mu_{\tilde{A}}(w_{k^*}) = \phi$.

If $\mu_{\tilde{A}}(w_k) \neq m_k$ for some $k \in \{3, 4, \ldots, n\} \setminus k^*$, then (m_k, w_k) blocks $\mu_{\tilde{A}}$, a contradiction. Thus, for each $k \in \{3, 4, \ldots, n\} \setminus k^*$, $\mu_{\tilde{A}}(w_k) = m_k$.

If $\mu_{\tilde{A}}(w_2) \neq m_2$, then $\mu_{\tilde{A}}(m_2) = w_1$ (if not, m_2 and w_2 blocks $\mu_{\tilde{A}}$). But then, $\mu_{\tilde{A}}(m_1) = w_2$ (if not, m_1 and w_1 blocks $\mu_{\tilde{A}}$). Then, (since $\mu_{\tilde{A}}(w_1) = m_2$, $\mu_{\tilde{A}}(w_2) = m_1$, $\mu_{\tilde{A}}(w_{k^*}) = \phi$ and for each $k \in \{3, 4, \ldots, n\} \setminus k^*$, $\mu_{\tilde{A}}(w_k) = m_k$), $\mu_{\tilde{A}}(m_{k^*}) = \phi$. But then, (m_{k^*}, w_1) blocks $\mu_{\tilde{A}}$, a contradiction. Thus, $\mu_{\tilde{A}}(w_2) = m_2$.

If $\mu_{\tilde{A}}(w_1) \neq m_1$, then (since $\mu_{\tilde{A}}(w_2) = m_2$, $\mu_{\tilde{A}}(w_{k^*}) = \phi$ and for each $k \in \{3, 4, \ldots, n\} \setminus k^*$, $\mu_{\tilde{A}}(w_k) = m_k$), $\mu_{\tilde{A}}(m_1) = \phi$. But then, (m_1, w_1) blocks $\mu_{\tilde{A}}$, a contradiction. Thus, $\mu_{\tilde{A}}(w_1) = m_1$.

Overall, for each $w \in W \setminus w_{k^*}$,

$$\mu_{\tilde{A}}(w) \succeq_w \hat{\mu}_{\tilde{A}}(w).$$

By Subcases 2.1-2.3, *own-side W-singles monotonicity* is satisfied. □

Appendix B

We show the logical independence of the properties listed in Theorems 2 and 3. We define the following four solutions.

Solution 1, φ_1 : For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$,

$$\varphi_1(M \cup W, \mathcal{P}_{M\cup W}) = \mathcal{IR}(M \cup W, \mathcal{P}_{M\cup W}).$$

Obviously, φ_1 satisfies all properties in Theorems 2 and 3 but *weak unanimity*.

Solution 2, φ_2 : For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$,

$$\varphi_2(M\cup W, \mathcal{P}_{M\cup W}) = \begin{cases} \{\hat{\nu}, \nu\} & \text{if } (M \cup W, \mathcal{P}_{M\cup W}) = (M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1}) \\ \mathcal{S}(M \cup W, \mathcal{P}_{M\cup W}) & \text{otherwise,} \end{cases}$$

where $M_1 = \{m_1\}$, $W_1 = \{w_1, w_2\}$, $\hat{\nu}$ is such that $\hat{\nu}(m_1) = w_1$ and $\hat{\nu}(w_2) = \phi$, ν is such $\nu(m_1) = w_2$ and $\nu(w_1) = \phi$, and $\mathcal{P}_{M_1\cup W_1}$ is defined as below.

m_1	$w_1 \succ w_2 \succ \phi$

w_1	$m_1 \succ \phi$
w_2	$m_1 \succ \phi$

Note note that $\hat{\nu}$ is the unique stable matching in $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$. Thus, φ_2 always includes all stable matchings and it differs from the stable solution only at $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$.

Since $|M_1| < |W_1|$ and φ_2 differs from the stable solution only at $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$, *weak unanimity* is obviously satisfied.

In the problem $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$, since no agent is unmatched at both $\hat{\nu}$ and ν, *own-side singles monotonicity* is trivially satisfied. If $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$ is obtained by an improvement of an unmatched agent at a stable matching of another problem, the *own-side singles monotonicity* of the stable solution ensures that φ_2 also satisfies the property.

The solution φ_2 is *consistent* as well. Indeed, for each problem having $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$ as its sub-problem at a stable matching, and since the stable solution is *consistent*, the restriction of the stable matching to $M_1 \cup W_1$ is stable and hence equal to $\hat{\nu}$. Again, since the stable solution is *consistent*, the property is satisfied for each proper sub-problem of $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$ at $\hat{\nu}$. A proper sub-problem of $(M_1 \cup W_1, \mathcal{P}_{M_1\cup W_1})$ at ν consists of either m_1 and w_2 or w_1 only. In each case, the restriction of ν to the sub-problem is stable.

For *Maskin invariance*, consider a problem $(M_1\cup W_1, \mathcal{P}^*_{M_1\cup W_1}) \in \mathcal{E}$ having $(M_1\cup W_1, \mathcal{P}_{M_1\cup W_1})$ as a result of a monotonic transformation of $\mathcal{P}^*_{M_1\cup W_1}$ at $\mu \in \varphi_2(M_1 \cup W_1, \mathcal{P}^*_{M_1\cup W_1})$. Since the stable solution is *Maskin invariant* (Tadenuma, 1993), μ

is stable at $(M_1 \cup W_1, \mathcal{P}_{M_1 \cup W_1})$ and hence equal to $\hat{\nu}$. For $(M_1 \cup W_1, \mathcal{P}_{M_1 \cup W_1})$, if $\mathcal{P}'_{M_1 \cup W_1}$ is obtained by a monotonic transformation of $\mathcal{P}_{M_1 \cup W_1}$ at $\hat{\nu}$, then again since the stable solution is *Maskin invariant*, $\hat{\nu}$ is stable at $(M_1 \cup W_1, \mathcal{P}'_{M_1 \cup W_1})$. On the other hand, if $\mathcal{P}'_{M_1 \cup W_1}$ is obtained by a monotonic transformation of $\mathcal{P}_{M_1 \cup W_1}$ at ν, then we have four possibilities for $\mathcal{P}'_{M_1 \cup W_1}$: (i) $\mathcal{P}'_{M_1 \cup W_1} = \mathcal{P}_{M_1 \cup W_1}$, (ii) for each $a \in \{w_1, \phi\}$, $w_2 \succ'_{m_1} a$ and $\mathcal{P}'_{W_1} = \mathcal{P}_{W_1}$, (iii) $\phi \succ'_{w_1} m_1$ and $\mathcal{P}'_{\{m_1\} \cup \{w_2\}} = \mathcal{P}_{\{m_1\} \cup \{w_2\}}$, and (iv) for each $a \in \{w_1, \phi\}$, $w_2 \succ'_{m_1} a$, $\phi \succ'_{w_1} m_1$, and $\mathcal{P}'_{\{w_2\}} = \mathcal{P}_{\{w_2\}}$. For each possibility, ν is stable at $(M_1 \cup W_1, \mathcal{P}'_{M_1 \cup W_1})$.

The solution φ_2 is not *null-player invariant.* Let us consider the problem in which $M'_1 = \{m_1, m_2\}$ and $W_1 = \{w_1, w_2\}$ in which the preference profile $\mathcal{P}'_{M'_1 \cup W_1}$ is the following:

m_1	$w_1 \succ' w_2 \succ' \phi$		w_1	$m_1 \succ' \phi \succ' m_2$
m_2	$\phi \succ' w_1 \succ' w_2$		w_2	$m_1 \succ' \phi \succ' m_2$

This problem is obtained by adding m_2 to $(M_1 \cup W_1, \mathcal{P}_{M_1 \cup W_1})$, with m_2 not being acceptable for each womam. By definition, $\varphi_2(M'_1 \cup W_1, \mathcal{P}'_{M'_1 \cup W_1}) = \{\hat{\nu}_{+m_2}\}$ but ν_{+m_2} is not contained in the solution set. This shows that φ_2 is not *null-player invariant*.

Solution 3, $\boldsymbol{\varphi_3}$: For each $(M \cup W, \mathcal{P}_{M \cup W}) \in \mathcal{E}$,

$$\varphi_3(M \cup W, \mathcal{P}_{M \cup W}) = \mathcal{IR}(M \cup W, \mathcal{P}_{M \cup W}) \cap \mathcal{PO}(M \cup W, \mathcal{P}_{M \cup W}).^{32}$$

Obviously, φ_3 is *weakly unanimous.*

It is also *null player invariant.* Indeed, let $(M \cup W, \mathcal{P}_{M \cup W}) \in \mathcal{E}$, $h \in \mathbb{M} \cup \mathbb{W} \setminus (M \cup W)$, and h-extension $(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}})$ of $(M \cup W, \mathcal{P}_{M \cup W})$ in which h is unacceptable for each $a \in (M \cup W)_{-h}$. Let $\mu \in \varphi_3(M \cup W \mathcal{P}_{M \cup W})$. Since μ is individually rational at $(M \cup W, \mathcal{P}_{M \cup W})$, for each $a \in (M \cup W)_{-h}$, $\mu_{+h}(a) \succeq'_a \phi \succ'_a h$. Thus μ_{+h} is individually rational at $(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}})$. Since for each $a \in (M \cup W)_{-h}$, $\mu_{+h}(a) \succ'_a h$ and μ is Pareto optimal at $(M \cup W, \mathcal{P}_{M \cup W})$, μ_{+h} is Pareto optimal at $(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}})$. Thus, $\mu_{+h} \in \mathcal{IR}(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}}) \cap \mathcal{PO}(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}})$. This shows that

$$\{\mu_{+h} \mid \mu \in \varphi(M \cup W, \mathcal{P}_{M \cup W})\} \subseteq \varphi(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}}).$$

Suppose that there exists $\nu \in \varphi_3(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}})$ such that $\nu \notin \{\mu_{+h} \mid \mu \in \varphi_3(M \cup W, \mathcal{P}_{M \cup W})\}$. Since ν is individually rational at $(M \cup W \cup \{h\}, \mathcal{P}'_{M \cup W \cup \{h\}})$,

[32] For each $(M \cup W, \mathcal{P}_{M \cup W}) \in \mathcal{E}$, $\varphi_3(M \cup W, \mathcal{P}_{M \cup W}) \neq \emptyset$, e.g., $\mathcal{S}_F(M \cup W, \mathcal{P}_{M \cup W}) \in \mathcal{IR}(M \cup W, \mathcal{P}_{M \cup W}) \cap \mathcal{PO}(M \cup W, \mathcal{P}_{M \cup W})$.

$\nu(h) = \phi$. Then, let $\nu^* \in \mathcal{M}(M \cup W, \mathcal{P}_{M\cup W})$ be such that $\nu^*_{+h} = \nu$. By construction, ν^* is not Pareto optimal at $(M \cup W, \mathcal{P}_{M\cup W})$. But then, ν^* is not Pareto optimal at $(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}})$, a contradiction. This shows that

$$\{\mu_{+h} \mid \mu \in \varphi(M \cup W, \mathcal{P}_{M\cup W})\} \supseteq \varphi(M \cup W \cup \{h\}, \mathcal{P}'_{M\cup W\cup\{h\}}).$$

Let the Pareto optimal solution (φ_{PO}) be such for each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$, $\varphi_{PO}(M \cup W, \mathcal{P}_{M\cup W}) = \mathcal{PO}(M \cup W, \mathcal{P}_{M\cup W})$. Then φ_{PO} is *consistent* and *Maskin invariant* (Toda, 2006). Together with the fact that φ_1 is *consistent* and *Maskin invariant*, one can verify that φ_3 satisfies those properties.

The solution φ_3 is not *own-side singles monotonic*. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\mathcal{P}_{F\cup W}$ be as following.

m_1	$w_1 \succ w_2 \succ \phi \succ w_3$		w_1	$m_2 \succ m_1 \succ \phi \succ m_3$
m_2	$w_1 \succ w_2 \succ \phi \succ w_3$		w_2	$m_3 \succ m_2 \succ \phi \succ m_1$
m_3	$w_2 \succ \phi \succ w_1, w_3$		w_3	$m_1 \succ \phi \succ m_2, m_3$

Then,

$$\mu = \{(m_1, w_1), (m_2, w_2), m_3, w_3\} \in \mathcal{IR}(M \cup W, \mathcal{P}_{M\cup W}) \cap \mathcal{PO}(M \cup W, \mathcal{P}_{M\cup W}).$$

Now, let $\mathcal{P}^{w_3}_{M\cup W}$ be a w_3-improvement of $\mathcal{P}_{M\cup W}$ obtained by replacing $\mathcal{P}_M$ as follows.

m_1	$\boldsymbol{w_3} \succ w_1 \succ w_2 \succ \phi$		w_1	$m_2 \succ m_1 \succ \phi \succ m_3$
m_2	$w_1 \succ w_2 \succ \phi \succ w_3$		w_2	$m_3 \succ m_2 \succ \phi \succ m_1$
m_3	$w_2 \succ \phi \succ w_1, w_3$		w_3	$m_1 \succ \phi \succ m_2, m_3$

Then, the unique Pareto optimal (and individually rational) matching in the problem $(M \cup W, \mathcal{P}^{w_3}_{M\cup W})$ is given by

$$\mu^{w_3} = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\} = \mathcal{PO}(M \cup W, \mathcal{P}^{w_3}_{M\cup W}).$$

But then, since $\mu^{w_3}(w_1) \succ_{w_1} \mu_(w_1)$, w_1 is made better off by the w_3-improvement of $\mathcal{P}_{M\cup W}$ in each Pareto optimal matching at $(M \cup W, \mathcal{P}^{w_3}_{M\cup W})$. Therefore, φ_3 is not *own-side singles monotonic*.

Solution 4, $\boldsymbol{\varphi_4}$: For each $(M \cup W, \mathcal{P}_{M\cup W}) \in \mathcal{E}$,

$$\varphi_4(M \cup W, \mathcal{P}_{M\cup W}) = \mathcal{S}_F(M \cup W, \mathcal{P}_{M\cup W}) \cup \mathcal{S}_W(M \cup W, \mathcal{P}_{M\cup W}).$$

By the fact that both the M-optimal and W-optimal stable solutions satisfy *weak unanimity*, *individual rationality*, and *null player invariance*, one can easily

	$W.U$	$I.R$	$N.P.INV$	$OWN.S.MON$	$CONS$	$M.INV$
φ_1	$-$	$+$	$+$	$+$	$+$	$+$
φ_2	$+$	$+$	$-$	$+$	$+$	$+$
φ_3	$+$	$+$	$+$	$-$	$+$	$+$
φ_4	$+$	$+$	$+$	$+$	$-$	$-$

Table 2: The independence of the properties in Theorem 2 and Theorem 3. The symbols "+" and "−" have the same meanings as in Table 1.

see that φ_4 satisfies those properties. By applying the argument used in the proof of Theorem 1, we can show that φ_4 is *own-side singles monotonic*. The solution φ_4 is neither *consistent* nor *Maskin invariant* (Toda, 2006).

Table 2 summarizes the properties of the Solutions 1-4.[33] One can easily see from the table that for Theorem 2, there are four examples of solutions, each of which violates exactly one property listed in the theorem. A similar statement holds for Theorem 3.

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[33] In Table 2, "*W.U*," "*I.R*," "*N.P.INV*," "*OWN.S.MON*," "*CONS*" and "*M.INV*" stand for "*weak unanimity*," "*individual rationality*," "*null player invariance*," "*own-side singles monotonicity*," "*consistency*" and "*Maskin invariance*," respectively.

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