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Abstract. We show that two strategies are Kuhn equivalent if and only if they induce the same probability measure over terminal nodes against some profile of completely mixed behaviour strategies of the other players. This result allows us to embed the equivalence classes of strategies in the probability measures over terminal nodes for various strategy concepts. This, in turn, allows a very clean statement of the relation between the various sets of strategies in games with perfect recall, linear games, and nonlinear games. It also proves useful in defining and analysing solution concepts in games without perfect recall, and, in particular, in nonlinear games.

1. Introduction

While a model of extensive form games was given by von Neumann and Morgenstern (1947), the model that is now widely used is a generalisation by Kuhn (1950b, 1953, 2003). That model was made even more general by Isbell (1954, 1957) by simply dropping the assumption of Kuhn that in any play of the game each information set occurs at most once. The possibility of games in which information sets may occur more than once in some plays of the game was addressed even earlier than Isbell by McKinsey (1952a,b), but mainly to explain why such games should be excluded from the analysis.

We shall henceforth call the more general class of games defined by Isbell nonlinear games and the subclass of games such as those Kuhn defined, where each information set occurs at most once in any play, linear games, following Isbell (1954, 1957), and later Mertens, Sorin, and Zamir (2015). Among linear games we identify an even smaller class of games, those with perfect recall, a class defined by Kuhn (1950b, 1953, 2003), which we define below in Definition 3. In games without perfect recall there may be no equilibrium in behaviour strategies, and in nonlinear games there may be no equilibria in mixed strategies either. Thus we need to consider more general classes of strategies. In

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linear games there is always an equilibrium in mixed strategies while in nonlinear games in order to guarantee the existence of equilibrium we need to consider an even more general class of strategy, namely randomisations over behaviour strategies. We call these randomisations over behaviour strategies general strategies.

We say any two strategies are Kuhn equivalent if, for any strategy profile of the other players, both induce the same probability measure on the terminal nodes of the game. While not precisely the manner that Kuhn defined mixed and behaviour strategies being equivalent it does define precisely the same equivalence relation on those strategy spaces.

For any class of strategies, we consider the set of Kuhn equivalence classes of strategies. We shall show how to embed these sets of equivalence classes into the space of probability measures over the terminal nodes and what follows the properties of these embedded sets. We also sketch some of the uses to which these structures can be put.

The work reported in this paper was motivated by our definitions of backward induction solutions in games without perfect recall, including in the nonlinear games defined by Isbell. (Hillas and Kvasov, 2020, 2021)

Finally, a note on our choice of terminology. We follow Isbell (1954, 1957) and Mertens, Sorin, and Zamir (2015) in calling the most general class of finite extensive form games that we consider nonlinear games and the subclass in which no information set occurs more than once in any play of the game linear games. Mertens, Sorin, and Zamir (2015, p. 60) suggest that the reason for calling such games linear is related to the fact that in such games any general strategy is equivalent to a mixed strategy. This itself relates to the fact that, in linear games, the set of probability measures on terminal nodes induced by the general strategies is defined by linear inequalities, while, in nonlinear games, this set may be defined by polynomial inequalities. We follow Mertens, Sorin, and Zamir (2015) in calling probability measures over the set of behaviour strategies general strategies, perhaps for no better reason than that we follow Mertens, Sorin, and Zamir. We call the equivalence notion on strategies Kuhn equivalence. This terminology or closely related terminology is used in the literature, for example by Kalai and Lehrer (1993), but not as widely as we originally thought. We used the term in Hillas and Kvasov (2020) and continue to use it here.

2. What Is a Strategy?

One of the major contributions of von Neumann and Morgenstern (1947) was the development of the concept of strategy. They describe a strategy as "a complete plan: a plan which specifies what choices he will make in every possible situation, for every possible actual information which he may possess at that moment in conformity with the pattern of information which the rules of the game provide for him at the case." (von Neumann and Morgenstern, 1947, p. 79)

Kuhn defines a pure strategy as a function that specifies at each information set a choice at that information set, even those information sets that are precluded by the strategy itself. But Kuhn regards this as a deficiency, writing "unfortunately, our definition of a pure strategy, while conceptually simple, has an inherent redundancy which we will now eliminate." (Kuhn, 1950b, p. 573) He does this elimination by defining two pure strategies as equivalent if, for any strategies of the other players, they induce the same probabilities on all plays of the game as each other. He then says that "when we speak of a pure strategy we shall mean an equivalence class under the definition just given." (Kuhn, 1950b, p. 574) A number of others have made similar points.

Rubinstein (1991) rejects this, rather supporting the "view that [an] equilibrium strategy describes a player's plan of action, as well as those considerations which support the optimality of his plan (i.e. preconceived ideas concerning the other players' plans) rather than merely a description of a 'plan of action.' ... In games which require a player to make at least two consecutive moves ... a strategy must specify his actions even after histories which are inconsistent with the player's own strategy. ... Thus, a strategy encompasses not only the player's plan but also his opponents' beliefs in the event that he does not follow that plan." Rubinstein (1991, pp. 910–911)

Rubinstein's view is internally coherent, at least for pure strategies and behaviour strategies. But we reject it in favour of the approach of Kuhn, and, at least implicitly, von Neumann and Morgenstern. We do so for two reasons which we explain in the context of an example. Consider a game like the centipede game (Rosenthal, 1981), where two players move alternately, deciding whether to stop the game (S) or continue and give the move to the other player (C), say, for concreteness, until each player has had potentially five moves, after which the game will end whatever the choice of Player 2 at his last move. The payoffs are not relevant for our purpose.

Our first reason for rejecting Rubinstein's approach is that it just, to us, seems unnatural. Suppose that Player 1 chooses to end the game on her first move. Kuhn would say that Player 1's strategy (recalling that by strategy Kuhn means an equivalence class of strategies) is just that, to end the game on her first move and Kuhn would say no more. Rubinstein, on the other hand would differentiate between all strategies that differ at the nodes that, given Player 1's initial choice, will not occur. Then when called on to move for the third time if Player 1's strategy is $SSSCC$ Player 2 will believe that Player 1 will continue on each of her remaining two moves should Player 2 decide to continue. On the other hand if Player 1's strategy is $SSSSS$ Player 2 will believe that

Player 1 will stop the game on her next opportunity should Player 2 decide to continue. This seems to us rather unnatural. When called on to move Player 2 knows that Player 1 has played neither $SSSCC$ nor $SSSSS$. We would prefer to say that at this point Player 2 has some beliefs over the strategies that start CCC.

Rubinstein suggests that the definition of strategy that distinguishes between $SSSCC$ and $SSSSS$ is "necessary for testing the rationality of a player's plan, both at the beginning of the game and at the point where he must consider the possibility of a response to an opponent's potential deviation (the subgame perfect idea)." (Rubinstein, 1991, p. 911) But while it is *possible* to use strategies interpreted in this way to define backward induction concepts such as subgame perfect equilibrium, at least in games with perfect recall, it is not necessary. Hillas and Kvasov (2020) give definitions for various backward induction concepts that do not reply on such an interpretation of strategies.

The second reason for rejecting Rubinstein's approach is, to our mind, more compelling. While the interpretation Rubinstein gives works for pure strategies and behaviour strategies it does not work for mixed strategies or for randomisations over behaviour strategies, which we are calling general strategies. Consider again the game described in the previous paragraph and consider the mixed strategy that plays $SCCCC$ and $CSSSS$, each with probability one half. Clearly on his first move Player 2 should believe that Player 1 is playing $CSSSS$ and so believe that she will play S at her next opportunity. However what should Player 2 believe if he is called on to move a fourth time? Should he retain his belief from his first move and believe that Player 1 will play S at her next move? Should he revert to his initial assessment and believe that S and C are equally likely? Perhaps, counting the number of times that Player 1 has deviated from her initial plan he should believe that she will play C. Or perhaps he should believe something else. And, since pure strategies are embedded in the set of behaviour strategies, exactly the same example shows that similar problems arise for randomisations over behaviour strategies.

In this paper we consider a number of classes of strategy: pure strategies, behaviour strategies, mixed strategies, and general strategies. And, just as we find no meaningful difference between pure strategies that are equivalent in the sense described above, we also find no meaningful difference between any randomised strategies, behaviour, mixed, or general, that do not induce different measures on the terminal nodes for any strategy profile of the other players.

3. Extensive Form Games and Strategies

A definition of an extensive form game starts with the notion of a game tree. There are two equivalent ways of defining a game tree. The first defines a game tree as a finite partially ordered set of nodes. The second defines a game tree as a finite acyclic connected graph.

We let X be the finite set of nodes and let $x_0 \in X$ be the initial node or root. The predecessor function $p : X \to X \cup \{\emptyset\}$ specifies, for every node x, the node $p(x)$ that comes immediately before it, with the initial node x_0 being the only node with no predecessor, $p(x_0) = \emptyset$. When the game tree is interpreted as a graph the set $\{p(x), x\}$ is called a branch or edge that connects nodes $p(x)$ and x; the ordered pair $(p(x), x)$ is called the directed branch from $p(x)$ to x.

For every node x we let $p_0(x) = x$, and recursively define $p_k(x)$ as $p(p_{k-1}(x))$; thus $p_1(x) = p(x)$. If $y = p_k(x)$ for some $k \geq 1$ we say that y precedes x, or that x follows y.

A node z is a terminal node if there is no node x that follows it, that is, there is no node x such that $p(x) = z$. We partition X into the set T of terminal nodes and the set D of nonterminal or decision nodes.

A *path* from node x to node y is a list of nodes

$$(x = x_1, x_2, \ldots, x_k = y),$$

where $k > 1$ and $x_j = p(x_{j+1})$ for $j = 1, \ldots, k-1$. We let $r(z)$ be the unique path from the initial node x_0 to a terminal node z.

Definition 1. An *extensive form game* Γ is a game tree together with the following.

1. A finite set of players $N = \{1, 2, \ldots, N\}$.
2. A collection of information sets $\mathcal{H}$, where $\mathcal{H}$ is a partition of the set of decision nodes D, and a function $n : \mathcal{H} \to N \cup \{0\}$, where Player 0 is Nature and $n(h)$ is the player who controls information set h. We let $\mathcal{H}_n = \{h \in \mathcal{H} \mid n(h) = n\}$ be the information sets controlled by Player n. For convenience and notational simplicity we assume that all Natures's information sets are singletons, that is, if $h \in \mathcal{H}_0$ and $x, y \in h$ then $x = y$.
3. A set of actions A, and a labeling function $\alpha : X \backslash \{x_0\} \to A$, where $\alpha(x)$ is the action at $p(x)$ that leads to x. The function α is such that if $p(x) = p(y)$ and $x \neq y$ then $\alpha(x) \neq \alpha(y)$. We let
$$A(x) = \{a \in A \mid a = \alpha(y) \text{ for some } y \text{ with } p(y) = x\}$$
be the set of actions available at node x. If x is a terminal node then $A(x) = \emptyset$. We require that if $x, y \in h$ then $A(x) = A(y)$.
4. A function $\rho : \mathcal{H}_0 \times A \to [0, 1]$, where $\rho(a)$ is the probability that action a is taken at the information set $h \in \mathcal{H}_0$.
5. Functions $u_1, \ldots, u_N$, with $u_n : T \to \mathbb{R}$ with $u_n(z)$ being the payoff to Player n at terminal node z.

We next formally define the relevant classes of extensive form games.

Definition 2. A game is said to be *linear for Player n* if no path from x_0 to a terminal node cuts any information set of Player n more than once. If no path from x_0 to a terminal node cuts any information set more than once then we call the game a *linear game*.

We call the general class of games, where we do not assume that the game is linear, *nonlinear games*. For linear games we consider a distinguished subset, those with perfect recall. Intuitively, a player has perfect recall if, at each of her information sets, she remembers what she knew and what she did in the past. The following formal definition was given in Kuhn (1950b). A different, but equivalent, definition was given in Kuhn (1953).

Definition 3. A player is said to have *perfect recall* if, whenever that player has an information set containing nodes x and y and there is a node x' of that player that precedes node x, then there is also a node y', in the same information set as x', that precedes node y and the action of the player at y' on the path to y is the same as the action of the player at x' on the path to x. The game is said to have perfect recall if every player has perfect recall.

We now define the various notions of strategy that we shall use.

Definition 4. A *pure strategy of Player* n in an extensive form game is a function that maps each of her information sets to an action available at that information set. We denote the set of Player n's pure strategies by S_n and the set of *pure strategy profiles* by $S = \times_{n \in N} S_n$.

Definition 5. A *mixed strategy of Player* n in an extensive form game is a probability measure over her pure strategies. We denote the set of Player n's mixed strategies by Σ_n and the set of *mixed strategy profiles* by $\Sigma = \times_{n \in N} \Sigma_n$. A mixed strategy σ_n of Player n is *completely mixed* if it assigns strictly positive probability to each of her pure strategies.

Behaviour strategies made their first brief appearance, without getting any name, in von Neumann and Morgenstern (1947, pp. 192–194), as a convenient tool to simplify the solution of a version of Poker. Then, under the names of "behaviour coefficients" and "behaviour parameters," they were used to analyze two more versions of Poker by Nash and Shapley (1950) and Kuhn (1950a). The formal definition and the analysis of the general properties of behaviour strategies are due to Kuhn (1950b).

Definition 6. A *behaviour strategy of Player* n in an extensive form game is a function that maps each of her information sets to a probability measure over the actions available at that information set. We denote the set of Player n's behaviour strategies by B_n and the set of *behaviour strategy profiles* by $B = \times_{n \in N} B_n$. A behaviour strategy b_n of Player n is *completely mixed* if, at each information set, it assigns strictly positive probability to each action available at that information set. We also define the behaviour strategy of Nature as $b_0(a) = \rho(a)$.

In nonlinear games we also need to consider randomisations over behaviour strategies. Various terms have been used in the literature to refer to such strategies; we follow Mertens, Sorin, and Zamir (2015) in calling them general strategies.

Definition 7. A *general strategy of Player n* in an extensive form game is a probability measure over her behaviour strategies. We denote the set of Player n's general strategies by G_n and the set of general strategy profiles by $G = \times_{n \in N} G_n$.

We postpone the definition of completely mixed general strategies until Definition 10 in Section 5.

Remark 1. While formally Definition 1 allows the case that x_0 is a terminal node (and hence the only node) that case is trivial; each player has a single pure strategy, a single behaviour strategy, a single mixed strategy, and a single general strategy, namely, do nothing. We shall henceforth assume that x_0 is not a terminal node.

To describe the convex structure of the various strategy sets we follow the terminology of Rockafellar (1970). A *polytope* is the convex hull of finitely many points (Rockafellar (1970, p. 12)). A point of a convex set is an *extreme point* if and only if it cannot be expressed as a convex combination of any two distinct points of that set (Rockafellar (1970, p. 162)). The *relative interior* of a convex set K is the interior that results when K is regarded as a subset of its affine hull (Rockafellar (1970, p. 44)).

Any pure strategy s_n can also be viewed as the mixed strategy that puts weight 1 on s_n. And any pure strategy s_n can also be viewed as the behaviour strategy that takes each information set to the probability measure that puts weight 1 on the action that s_n selects at that information set. Thus, the set of pure strategies is naturally embedded in both the set of mixed strategies and the set of behaviour strategies. Similarly, all the other strategy sets are embedded in G_n, the set of general strategies.

An immediate implication is that the set of pure strategies is the set of extreme points both of the set of mixed strategies and of the set of behaviour strategies, both of which are polytopes. The set of completely mixed strategies and the set of completely mixed behaviour strategies are, correspondingly, the relative interiors of the set of mixed strategies and the set of behaviour strategies.

The set of Player n's mixed strategies Σ_n is a simplex of dimension $|S_n| - 1$. The set of Player n's behaviour strategies B_n is a Cartesian product of a finite number of simplices, each of which corresponds to an information set controlled by Player n with the dimension of each simplex being the number of actions available to Player n at that information set less one. The dimension of B_n is, thus, always less than or equal to the dimension of Σ_n. If the player has at least two information sets each with at least two available actions then the dimension of B_n is strictly less than the dimension of Σ_n.

4. Kuhn Equivalence

Both von Neumann and Morgenstern (1947) and Kuhn (1950b, 1953, 2003) define a strategy as a complete contingent plan of playing the game. In giving a formal precise meaning to this, they define pure strategies, but the same idea motivates the definitions of all the other strategy sets as well. Which strategy sets we need to define depends on the class of games that the player faces and what it is that the player seeks to achieve.

In linear decision problems, that is, one-person linear games, the player always has a pure strategy that is at least as good as any more general strategy. In nonlinear decision problems this is not necessarily the case. A player may be able to do better with a behaviour strategy than she can with any pure strategy.

In two-person zero-sum games a player's optimal strategy is one that guarantees a player the highest possible payoff whatever the other player does. In this case the player may, in linear games, do better with mixed strategy than with any pure strategy and, in nonlinear games, may do better with a general strategy than with any behaviour strategy. Thus, when looking at what a player's plan can guarantee, we do need to look, at least for some cases, at general strategies.

In games with more than two players, or in non-zero sum games we typically look for some form of equilibria and usually assume that the players have some probabilistic beliefs about what the other players will do. In such cases, for any beliefs the player might have, she will always have a behaviour strategy that does at least as well as any other strategy. We might, when looking at such equilibria, restrict ourselves to at most behaviour strategies and describe the players' beliefs separately. This is the approach typically taken when looking at correlated equilibria where, for linear games, the strategies are taken to be the pure strategies and the beliefs of each player are generated from a probability measure over those pure strategies. One could take the same approach to defining Nash equilibria, but here the usual approach, again in the linear case, is to define the strategy spaces as the space of mixed strategies and to have the equilibrium strategy profile represent both the optimal plan of each player and the beliefs of each player about the plans of the others. In nonlinear games we would need to look for equilibria in general strategies. The general strategy of Player n represents both the maximising choice of behaviour strategy of Player n and the uncertainty of the other players about which behaviour strategy Player n will play.

Just as Kuhn saw the definition of pure strategies as having some redundancy so too the other definitions of strategy involve similar redundancies. We eliminate these redundancies in the same way that Kuhn did, namely, by looking at equivalence classes of strategies, and we do so for a similar reason. We also embed these equivalence classes in the set of measures over the terminal nodes. An added advantage to this construction is the clean setting it gives us to examine the relations between the various strategy sets.

Kuhn showed that in games with perfect recall one could achieve essentially the same uncertainty about the other players with behaviour strategies as one could with mixed strategies and so, in such games, one could instead look for equilibria in behaviour strategies. We can be a little more explicit and we can extend this notion of equivalence to all of the strategies that we consider. We first define the notion of the Kuhn equivalence of two strategies.

Definition 8. Two strategies of Player n, x and y in $S_n \cup B_n \cup \Sigma_n \cup G_n$ are said to be *Kuhn equivalent* if, for any general strategy profile, g_{-n} in G_{-n}, of the other players, the profiles (x, g_{-n}) and (y, g_{-n}) induce the same measure over the terminal nodes. When x and y are Kuhn equivalent we write $x \sim_K y$.

We now formally state Kuhn's Theorem.

Kuhn's Theorem. *If Player n has perfect recall then for any mixed strategy σ_n in Σ_n there is a behaviour strategy b_n in B_n that is Kuhn equivalent to σ_n. If the game is linear for Player n then, for any behaviour strategy b_n in B_n, there is a mixed strategy σ_n in Σ_n that is Kuhn equivalent to b_n.*

The first part of Kuhn's Theorem was stated and proved by Kuhn (1950b, 1953). The second part is almost implicit in Kuhn's paper and was formally stated and proved by Isbell (1954, 1957).

The definition of Kuhn equivalence given above is apparently quite demanding. In order for two strategies to be Kuhn equivalent they must induce the same measure over terminal nodes for *any* strategy profile of the other players. Now, it clearly may be that two strategies of a player induce the same measure over terminal nodes for some strategy profiles of the other players but not for others. For example, for some strategy profile of the other players it may be that the strategies of the other players simply preclude some parts of the tree from being reached. However, this is the *only* case in which two strategies of a player will for some strategy profiles of the other players induce the same measure on the terminal nodes, and for other profiles of the other players induce different measures on the terminal nodes. In particular, a profile of completely mixed behaviour strategies of the other players does not preclude any part of the tree and so any two strategies that induce the same measure over terminal nodes against such a strategy profile of the others will induce the same measure over terminal nodes for all strategy profiles of the others. For linear games the same is true for a profile of completely mixed strategies of the others. And, in general, when we define it, the same will be true for a profile of completely mixed general strategies.

Before giving a formal proof of this we shall give a less formal verbal argument. For any terminal node consider the set of branches on the path from x_0 to the terminal node. We call the set of branches on this path for which Player n moves at beginning of the branch a partial path for Player n and the set of all such partial pathes as we vary the terminal node the partial paths of Player n. (This notion of partial path could be thought of as an incomplete version of the sequence form of von Stengel (1996).) Now any strategy of Player n will induce what we will call a quasi-probability on the set of her partial paths (quasi because the "probabilities" on the partial paths will, generally, add to more than 1). Now the probability of a terminal node will be the product of the quasi-probabilities of the partial paths on the path from the initial node to that terminal node. So the strategy of Player n will impact the measure over terminal nodes only through its image in these quasi-probabilities. In other words, if two strategies induce the same quasi-probability on the partial paths they will induce the same measure over terminal nodes. Moreover, if they induce a different quasi-probability on partial paths they will, for at least some strategy profile of the other players induce a different measure over terminal nodes. We can, in fact, be a little more specific. If two strategies of Player n induce a different quasi-probability on a particular partial path then they will induce a different probability on any terminal node reached by a path that includes this partial path when combined with quasi-probabilities for the other players that put positive quasi-probability on the partial paths of the those players for that terminal node.

Any completely mixed behaviour strategy of a player will induce a positive quasi-probability on any partial path of that player. Thus two strategies of a player will induce the same measure over terminal nodes against a profile of completely mixed behaviour strategies of the other players if and only if they induce the same quasi-probabilities on the player's partial paths. This implies the result. Two strategies of a player will be Kuhn equivalent if and only if they induce the same measure over terminal nodes against some profile of completely mixed behaviour strategies of the other players.

We shall now make this argument more formally and more completely.

Definition 9. We let $Q_n = [0, 1]^T$. For every behaviour strategy b_n of Player n, we let $b_n(\alpha(x))$ be the probability of the action $\alpha(x)$ taken by Player n at $p(x)$ that leads to x. Then $q_n(b_n)$ is an element of Q_n

that associates the quasi-probability

$$\prod_{\substack{p(x) \in r(z) \\ n(p(x)) = n}} b_n(\alpha(x)) \tag{1}$$

to the terminal node z. If Player n doesn't move on the path to z then this is the empty product, which we take, as usual, to be 1. We also define $q_0(b_0, z)$ for Nature, where b_0 is Nature's only strategy.

Pure strategies are naturally embedded in the behaviour strategies so q_n is also well-defined on pure strategies. We extend q_n from pure strategies to mixed strategies by taking expectations and also from behaviour strategies to general strategies by, again, taking expectations.

Lemma 1. *For each n, if b_n is completely mixed then the induced quasi-probability of each z in T, $q_n(b_n, z)$, is strictly positive.*

Proof. Since b_n is completely mixed, at every information set it assigns strictly positive probability to each action available at that information set. Hence $q_n(b_n, z)$, as given in (1) is strictly positive. □

Remark 2. If the game is linear then the same is true for mixed strategies, that is, for any completely mixed strategy σ_n, $q_n(\sigma_n, z)$ is strictly positive. In nonlinear games this will not necessarily be true, though, for appropriately defined completely mixed general strategies, it will be true that for a completely mixed general strategy g_n, $q_n(g_n, z)$ is strictly positive.

Remark 3. For a profile of strategies $(x_1, x_2, \dots, x_N)$, where x_n is in

$$S_n \cup B_n \cup \Sigma_n \cup G_n,$$

the induced probability $q(x, z)$ on terminal node z is the product of the quasi-probabilities $q_n(x_n, z)$

$$\prod_{n \in N \cup \{0\}} q_n(x_n, z) = q_n(x_n, z)\ q_{-n}(x_{-n}, z), \tag{2}$$

where $q_{-n}(x_{-n}, z) = \prod_{m \neq n} q_m(x_m, z)$.

Proposition 1. *Two strategies of Player n, x_n and x'_n are Kuhn equivalent if and only if they induce the same measure over terminal nodes for some profile of completely mixed behaviour strategies b_{-n} of the other players.*

Proof. The only if follows directly from the definition of Kuhn equivalence.

Suppose that x_n and x'_n induce the same measure over terminal nodes for some profile of completely mixed behaviour strategies b_{-n} of the other players. Then by Lemma 1 for each Player m, $m \neq n$, and for each z we have $q_m(b_m, z) > 0$. Let $q(z)$ be the common probability induced on z by the strategy profiles (x_n, b_{-n}) and (x'_n, b_{-n}). Then by Remark 3 for each z

$$q_n(x_n, z) = \frac{q(z)}{q_{-n}(x_{-n}, z)} = q_n(x'_n, z).$$

And so, again by Remark 3, for any profile of strategies of the players other than n, x_{-n}, for any z in T the probability induced on z by (x_n, x_{-n}) is the same as the probability induced on z by (x'_n, x_{-n}), that is, x_n is Kuhn equivalent to x'_n. □

5. A Characterisation of Kuhn Equivalent Strategies

For each strategy concept we consider the set of Kuhn equivalence classes, that is, we consider, for each n, the quotient spaces $S_n/\sim_K$, $B_n/\sim_K$, $\Sigma_n/\sim_K$, and $G_n/\sim_K$. Proposition 1 allows us to give a very concrete embedding of the various sets of Kuhn equivalent strategies. For each of the quotient spaces we look at the image of those sets in $\Delta(T)$, the set of measures on the terminal nodes, given by taking the strategy x_n in $S_n \cup B_n \cup \Sigma_n \cup G_n$ to the point in $\Delta(T)$ generated by the strategy profile (x_n, b^0_{-n}) where, for each $m \neq n$ the behaviour strategy b^0_m is the behaviour strategy of Player m that, at each information set of Player m, puts equal probability on all the actions available at that information set. These subsets of $\Delta(T)$ we label $\hat{S}_n$, $\hat{B}_n$, $\hat{\Sigma}_n$, and $\hat{G}_n$. We let $\hat{G} = \times_{n \in N} \hat{G}_n$ and call $\hat{G}$ the Kuhn reduced space of general strategy profiles. And similarly for $\hat{S}$, $\hat{B}$, and $\hat{\Sigma}$.

We also define the completely mixed general strategies.

Definition 10. A general strategy g_n in G_n of Player n is *completely mixed* if, for any open subset O of B_n, $g_n(O) > 0$. We denote the set of all completely mixed general strategies of Player n by G^0_n and the set of completely mixed strategy profiles by G^0.

We also define the corresponding subset of $\hat{G}_n$.

Definition 11. A general strategy g_n in $\hat{G}_n$ of Player n is *completely mixed* if there is some g'_n in G^0_n such that g'_n is Kuhn equivalent to g_n. We denote the set of all such strategies of Player n by $\hat{G}^0_n$ and the corresponding set of profiles by $\hat{G}^0$.

Rather than defining completely mixed general strategies in this way one might have thought of simply requiring that the measure be over only completely mixed behaviour strategies. This however would result in an essentially different, and weaker, definition of "completely mixed." In Hillas and Kvasov (2021) we discuss how such a definition of completely mixed gives weaker definitions of quasi-perfect equilibria and perfect equilibria. If we used the weaker definition we would also not have the following convenient characterisation.

Proposition 2. *A general strategy g_n in $\hat{G}_n$ of Player n is* completely mixed *if and only if it is in the (relative) interior of $\hat{G}_n$.*

In order to guarantee the existence of equilibria in nonlinear games we need to allow not just mixtures of pure strategies but also mixtures of behaviour strategies, that is, general strategies. Thus, in some sense, in nonlinear games we might say that behaviour strategies are the analogue of pure strategies in linear games and that general strategies are the analogue of mixed strategies.

The set $\hat{G}_n$ is embedded as a compact convex subset of $\Delta(T)$. Thus, by the finite dimensional version of the Krein-Milman Theorem (see Rockafellar, 1970, Corollary 18.5.1, p. 167), it is the convex hull of its extreme points. This allows us to give a more demanding analogue in the nonlinear case of pure strategies in the linear case than simply the set of behaviour strategies; the extreme points of $\hat{G}_n$ are the natural analogue of pure strategies in the linear case. Also, if e_n is an extreme point of $\hat{G}_n$ then it is Kuhn equivalent to a behaviour strategy and each general strategy in $\hat{G}_n$ is, by the Carathéodory Theorem (see Rockafellar, 1970, Theorem 17.1, p. 155), a mixture of at most $|T|+1$ extreme points of $\hat{G}_n$. This makes the analysis of nonlinear games quite similar to the analysis of linear games. The set $\hat{E}_n$, the extreme points of $\hat{G}_n$, is the analogue of S_n and $\hat{G}_n$ the analogue of Σ_n.

In linear games $\hat{E}_n = \hat{S}_n$ and so, in such games, $\hat{E}_n$ is, except in the most trivial case, much smaller than $\hat{B}_n$. In nonlinear games $\hat{E}_n \subset \hat{B}_n$ but, in general, if the game is not linear for Player n, $\hat{E}_n$ will be much larger than $\hat{S}_n$. Indeed, we consider below an example in which $\hat{E}_n = \hat{B}_n$.

Proposition 3. *For each n, for X_n being S_n, B_n, Σ_n, or G_n we have $X_n/\sim_K$ homeomorphic to $\hat{X}_n$ with the quotient topology on $X_n/\sim_K$ and the relative topology as a subset of $\Delta(T)$ on $\hat{X}_n$.*

Proposition 4. *We have the inclusions $\hat{S}_n \subset \hat{E}_n \subset \hat{B}_n$, $\hat{S}_n \subset \hat{\Sigma}_n$, $\hat{B}_n \subset \hat{G}_n$, and $\hat{\Sigma}_n \subset \hat{G}_n$ with $\hat{\Sigma}_n$ being the convex hull of $\hat{S}_n$, and hence a polytope, and $\hat{G}_n$ being the convex hull of $\hat{B}_n$. In linear games $\hat{B}_n \subset \hat{\Sigma}_n = \hat{G}_n$. In games with perfect recall $\hat{B}_n = \hat{\Sigma}_n = \hat{G}_n$.*

Remark 4. Proposition 4 gives information about the relationships between the various sets of strategies. Proposition 1 and the definition of the embedding imply that two strategies in different strategy spaces that are Kuhn equivalent are mapped to the same point in $\Delta(T)$ by the embedding. In fact, for any X and Y in $\{\hat{S}_n, \hat{E}_n, \hat{B}_n, \hat{\Sigma}_n, \hat{G}_n\}$ if $x \in X$ and $y \in Y$ we have $x \sim_K y$ if and only if $x = y$.

In nonlinear games the sets may all be different, even in the simplest cases. The various sets for the one-person absent-minded driver game of Piccione and Rubinstein (1997), shown in Figure 1, are given in

Figure 2.[1] Since the set of pure strategies $\hat{S}_1$ is a subset of both $\hat{B}_1$ and $\hat{\Sigma}_1$ these sets must have a nonempty intersection. But in this game $\hat{S}_1$ *is* the intersection of $\hat{B}_1$ and $\hat{\Sigma}_1$. Thus the various strategy sets differ by as much as is permitted by Proposition 4, except that $\hat{E}_n = \hat{B}_n$.

In the absent-minded driver game the set $\hat{B}_1$ is the set of extreme points of $\hat{G}_1$ and so $\hat{E}_n$ is much bigger than $\hat{S}_n$. In linear games $\hat{E}_n = \hat{S}_n$ and so, in such games, $\hat{E}_n$ is, except in the most trivial case, much smaller than $\hat{B}_n$. We could construct an example in which $\hat{E}_n$ was both bigger than $\hat{S}_n$ and smaller than $\hat{B}_n$, for example in a game where Nature first decided whether the players played a linear game or Player 1 played the absent-minded driver game.

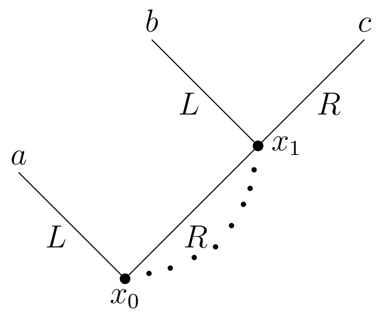

FIGURE 1. The Absent-Minded Driver Game

The Hasse diagrams in Figure 3 illustrate the set inclusions we have described. The partial orders given for the three classes of games are $X \succ Y$ if and only if Y is a subset of X for all games in the specified class and Y is a proper subset of X for at least one game in the specified class, where $X, Y \in \{\hat{S}_n, \hat{E}_n, \hat{B}_n, \hat{\Sigma}_n, \hat{G}_n\}$. There is no claim that the given partial order necessarily represents the partial order defined by set inclusion for a particular game in the class. (Indeed, this is impossible, since the classes of games are nested while the partial orders are different.)

[1]This game is quite well known since the paper of Piccione and Rubinstein (1997) and a number of papers commenting on that paper. A very similar game was considered much earlier by Isbell (1954) who tells a rather grim story about a player about to be hanged. In the game form we consider here he is saved at outcome b and hanged at each of the other outcomes. We discuss the example as Isbell gave it in Section 6.5 below. Even earlier than that, McKinsey (1952b, pp. 604–606) examines the same game, though he does not define it to be a game since he uses Kuhn's definition of an extensive form game, telling a similarly grim story of two policemen who are attempting to shoot an insane criminal. (What was it about 1950's game theorists and their grim stories?)

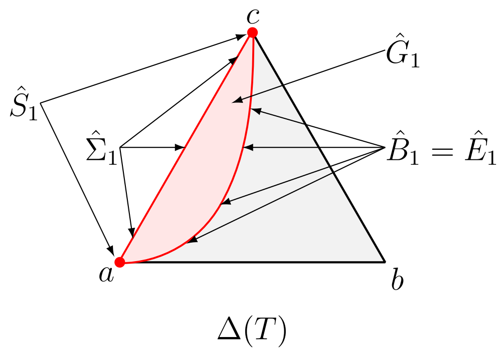

FIGURE 2. The Strategy Spaces Embedded in $\Delta(T)$

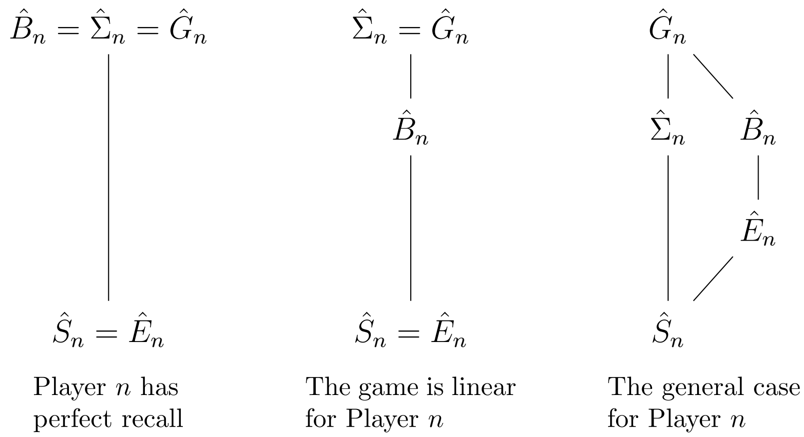

FIGURE 3. Hasse Diagrams

We see in examples in the next section that $\hat{\Sigma}_n$ is typically of lower dimension than Σ_n, even in cases where $\hat{S}_n$ is the "same" as S_n, that is, has the same cardinality. However, while $\hat{B}_n$ may differ from B_n, as it does in Example 2 of the following section where some behaviour strategies are Kuhn equivalent to each other, it is necessarily of the same dimension. On the interior of B_n, where all actions are taken with strictly positive probability, no two different behaviour strategies are Kuhn equivalent to each other. Thus there is a bijection between the interior of B_n and its image in $\hat{B}_n$. And it is clear that this map and its inverse are continuous. Thus these sets have the same dimension. Also all the sets are semialgebraic and so the closures have the same dimension as the interiors. Thus $\hat{B}_n$ and B_n have the same dimension.

Since there are an infinite number of behaviour strategies, the space of general strategies, G_n, is infinite dimensional. However, since we can embed the space of Kuhn equivalence classes of G_n in $\Delta(T)$, that embedded space, $\hat{G}_n$, is a subset of $\mathbb{R}^T$ and so Carathéodory's Theorem implies the following result. This result was stated by Isbell (1954, 1957, p. 83) and proved by Alpern (1988).[2]

Proposition 5 (Isbell, 1957, Alpern 1988)**.** *For any player n in N there is a finite number K_n such that for any general strategy g_n of Player n there is a general strategy g'_n putting probability on only K_n elements of B_n that is Kuhn equivalent to g_n.*

Remark 5. For any g_{-n} in $\hat{G}_{-n}$ the measure over $\Delta(T)$ that results from (g_n, g_{-n}) is a linear function of g_n. Hence, for any fixed assignment of payoffs to the terminal nodes, the set of best replies to g_n is convex. It is equally standard that the graph of the equilibrium correspondence on $\hat{G}$ is closed. And $\hat{G}$ is a compact and convex subset of $\mathbb{R}^T$. Thus, the existence of an equilibrium on $\hat{G}$ is completely standard. Then Proposition 4 and Remark 4 imply the standard results that for linear games an equilibrium in mixed strategies exists and for games with perfect recall an equilibrium in behaviour strategies exists.

6. Examples

We gave one example in the previous section that illustrated how even in a simple one-person nonlinear game the various strategy spaces could differ. In this section we give a number of additional simple examples that illustrate how the various strategy spaces behave. In each case, but the last, there are at most four terminal nodes so that at most there is a three-dimensional simplex on the terminal nodes so that it is not even beyond our ability to draw the diagrams. In the last case we can project the strategy space for the player in whom we are interested into two dimensions. Three of the games are two-person games and three of the games are one-person games.

6.1. **Example 1.**

The first example is a two-person game with the game shown in Figure 4 and the strategy spaces shown in Figure 5. This is a game of perfect information, and hence of perfect recall. Thus $\hat{G}_n = \hat{\Sigma}_n = \hat{B}_n$ for $n = 1, 2$. There are two distinct Kuhn equivalent pure strategies of Player 1 and four distinct Kuhn equivalent pure strategies of Player 2. In spite of Player 2 having four different elements in $\hat{S}_2$ the dimension of $\hat{\Sigma}_2$ is only 2. The four pure strategies of Player 2, LW, LE, RW, and RE all lie in the same two-dimensional linear subspace. For Player 1 the situation is quite simple, with $\hat{\Sigma}_1$ just being the line between the two elements of $\hat{S}_1$ in $\Delta(T)$.

[2] Isbell (1954, 1957) says the result follows from a theorem of Fenchel (1929), but the theorem in the paper of Fenchel (in German) he cites appears to be Carathéodory's Theorem, with Fenchel correctly attributing it. In Hillas and Kvasov (2021) we give a proof along the same lines as Alpern (1988) who proves the result using Carathéodory's Theorem without explicitly naming it.

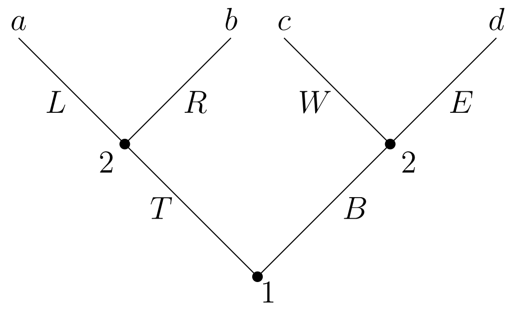

FIGURE 4. Example 1

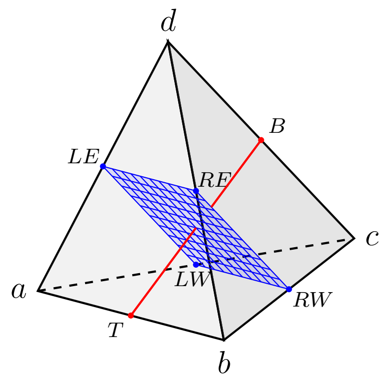

FIGURE 5. Strategy Spaces in $\Delta(T)$ for Example 1

6.2. **Example 2.** The second example is a one-person game shown in Figure 6. This is a game of perfect information, and hence of perfect recall. Thus $\hat{\Sigma}_1 = \hat{B}_1 = \hat{G}_1$. Player 1 has four pure strategies TM, TB, $\neg TM$, and $\neg TB$. But TM and TB are Kuhn equivalent, so we label the three elements of $\hat{S}_1$ as T, M, and B. The images of T, M, and B in $\Delta(T)$ are, respectively, the points a, b, and c. $\hat{\Sigma}_1$ is the convex hull of T, M, and B, that is, the whole of $\Delta(T)$.

The behaviour strategy spaces B_1 and $\hat{B}_1$ are shown in Figure 7. A behaviour strategy of Player 1 is (p, q), where p is the probability that the player chooses $\neg T$ and q is the probability that the player chooses B. The behaviour strategy space B_1 is $[0, 1] \times [0, 1]$, that is a Cartesian product of two simplices. However, all the behaviour strategies in $\{0\} \times [0, 1]$ are Kuhn equivalent, and so all these strategies are identified under $\sim_K$ and $\hat{B}_1$ is the simplex, $\Delta(T)$.

6.3. **Example 3.** The third example is a two-person game with the game shown in Figure 8 and the strategy spaces shown in Figure 9. (With the appropriate payoffs it would be a matching pennies game.) In this example both players have perfect recall. Both $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$

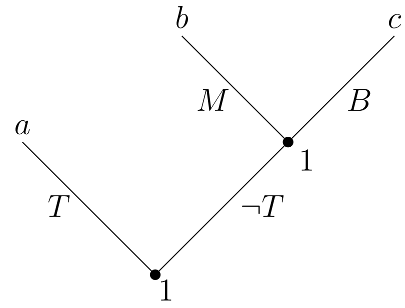

Figure 6. Example 2

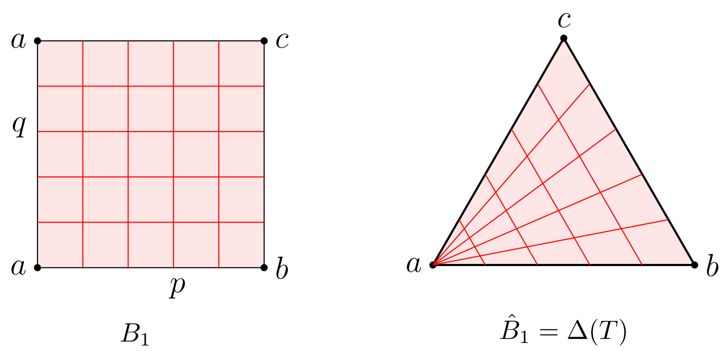

Figure 7. Behaviour strategies of Example 2

are one-dimensional while $\Delta(T)$ is three-dimensional. Thus we see an example where the dimension of $\Delta(T)$ is a strict upper bound on the dimension of $\hat{G}$.

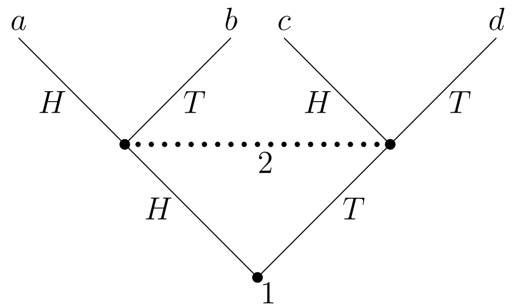

Figure 8. Example 3

6.4. **Example 4.** The fourth example is a one-person linear game without perfect recall. The game is shown in Figure 10 and B_1 and $\hat{B}_1$ in Figure 11. It has a similar structure to the previous game but

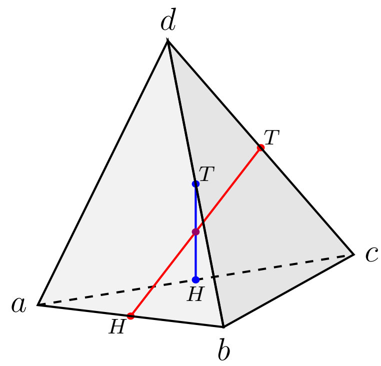

FIGURE 9. Strategy Spaces in $\Delta(T)$ for Example 3

with both information sets owned by the same player. In this game none of the behaviour strategies are Kuhn equivalent to each other. Thus B_1 and $\hat{B}_1$ are homeomorphic to each other. However the linear structure of B_1 is not the same linear structure as on $\Delta(T)$ and $\hat{B}_1$ is not a convex set and $\hat{B}_1$ is a strict subset of $\hat{\Sigma}_1$, which is, of course, a convex subset of $\Delta(T)$; in this case it actually *is* $\Delta(T)$.

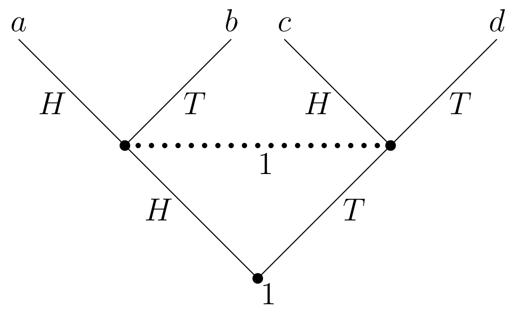

FIGURE 10. Example 4

6.5. **Example 5.** The fifth example is a one-person game similar to the absent-minded driver game except that the player has a move after each of her first choices. A game with this structure is given by Isbell (1954). In the game he considers the payoffs to outcomes a and b are the same so the game really does have the same structure as the absent-minded driver game, though the payoffs he gives are not exactly the same.

In this game $\Delta(T)$ is three dimensional. But the strategy spaces are the same "shape" as the strategy spaces we found for the absent-minded driver game in Section 5, in the sense that they are homeomorphic under a linear homeomorphism.

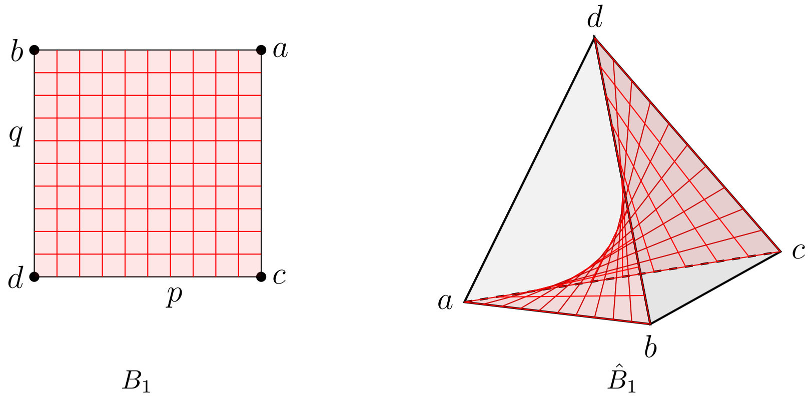

FIGURE 11. Behaviour strategies for Example 4

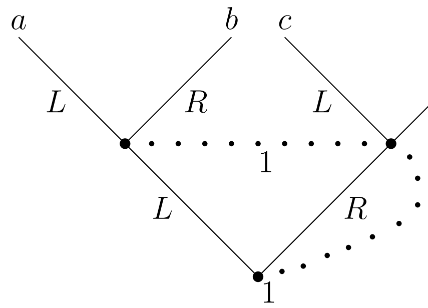

FIGURE 12. Example 5

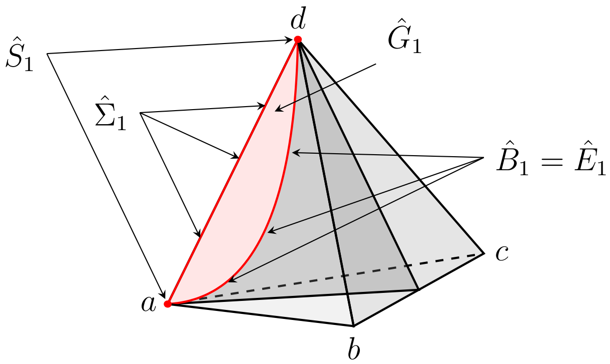

FIGURE 13. Strategy Spaces For Example 5

6.6. **Example 6.** Our final example adds another player to the absent-minded driver game we examined in the previous section. The idea of this game is that it is a modified version of Rock-Paper-Scissors with the choices of Player 1 generated by an absent-minded driver game and the choices of Player 2 chosen directly and the payoffs adjusted so that the optimal choice of Player 1 is in the interior of $\hat{G}_1$ The extensive form of the game is given in Figure 14.

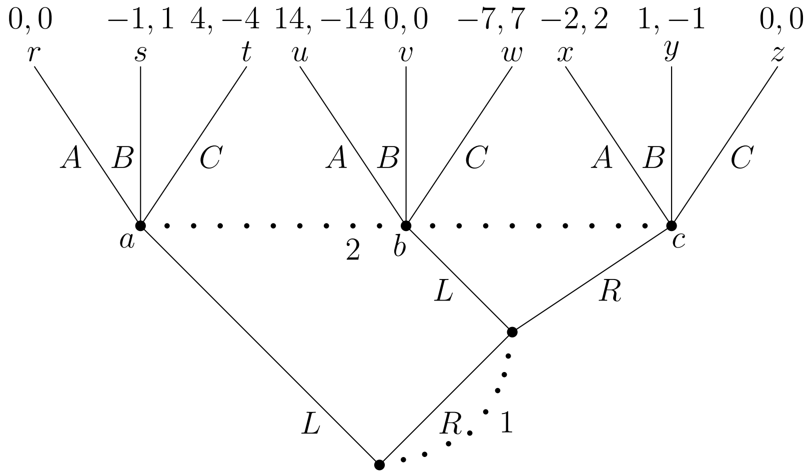

FIGURE 14. Example 6

		Player 2 A	B	C
Player 1	a	$0, 0$	$-1, 1$	$4, -4$
	b	$14, -14$	$0, 0$	$-7, 7$
	c	$-2, 2$	$1, -1$	$0, 0$

FIGURE 15. A "Similar" Normal Form Game

In Figure 15 we give the normal form of the game in which Player 1 chooses between a, b, and c. Of course, as we have seen, Player 1 cannot achieve all randomisations over a, b, and c through her choices of L and R. However, if the game shown in Figure 15 has an equilibrium in which the randomisation over a, b, and c does lie in $\hat{G}_1$ then that strategy, together with the strategy for Player 2 in the equilibrium of the "similar" game, is an equilibrium of the actual game given in Figure 14. In fact, the unique equilibrium of the game given in Figure 15 is $((\frac{7}{16}, \frac{1}{8}, \frac{7}{16}), (\frac{7}{101}, \frac{78}{101}, \frac{16}{101}))$, and that randomisation over a, b, and c, Player 1's equilibrium strategy in the "similar" game, does lie in $\hat{G}_1$ in the actual game. We show that value, which we label $\hat{g}_1^*$, and the strategy spaces of Player 1, in Figure 16.

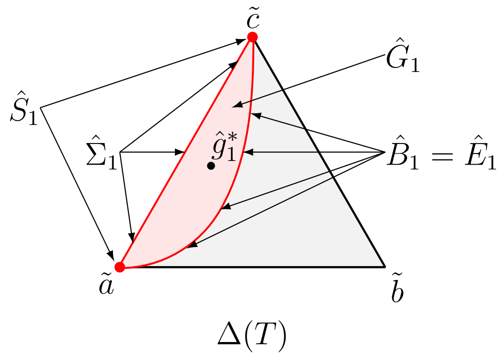

FIGURE 16. Strategy Spaces of Player 1 and the Equilibrium Value of $\hat{g}_1^*$ for Example 6

7. APPLICATIONS

We have been concerned fairly abstractly with the structure of the various strategy spaces and the relations between them. However our motivation was originally concerned with various backward induction solution concepts for games without perfect recall, and, in particular for nonlinear extensive form games. A number of results in this paper result in useful applications in defining and analysing solution concepts for such games.

7.1. Subgame Perfect Equilibrium in Nonlinear Games.

The notion of a subgame (Selten, 1965, 1975) does not depend on whether or not the game has perfect recall, or indeed even whether it is linear. The requirement that all information sets be either completely in or completely outside the potential subgame means that the subgame can be examined independently of the part of the game outside the subgame to exactly the same extent as in games with perfect recall. However, the notion of subgame perfect equilibrium, as usually defined, is more problematic.

The standard definition of a subgame perfect equilibrium in a game with perfect recall is as a profile of behaviour strategies that induces an equilibrium in any subgame. As we discussed earlier, in games without perfect recall there may not be equilibria in behaviour strategies. And in nonlinear games it may even be necessary to look for equilibria in general strategies. And when one considers either mixed strategies or general strategies it is not immediately clear how to, from a profile of strategies, induce behaviour onto unreached subgames, and, in particular, on how to specify the behaviour of a player whose own strategy excludes the subgame.

However, even in a game with perfect recall, the way that we find subgame perfect equilibria is not, typically, to find all the equilibria in behaviour strategies, and then check whether these equilibria are indeed also equilibria in the subgames. Rather we proceed by backward induction, looking first at those subgames with no proper subgames, choosing some equilibrium for that subgame and then replacing the subgame with a terminal node with payoff being the expected payoff in the subgame to the chosen equilibrium. And we continue until we are left with just a single terminal node. At each point at which we eliminate a subgame we specify a behaviour at each of the eliminated information sets. Thus, when we are done we can patch together all of the behaviours at the various information sets to construct a profile of behaviour strategies and that profile will be a subgame perfect equilibrium. Moreover, all subgame perfect equilibria can be found in this way for some choice of the equilibria of the eliminated subgames.

Now, we can carry out a similar construction for any nonlinear game. Of course, rather than choosing an equilibrium in behaviour strategies, we look for an equilibrium in general strategies. But, finding such an equilibrium, we can then replace the subgame with a single terminal node with payoff being the expected payoff in the subgame to the chosen equilibrium. The only difficulty is in patching the various equilibria in general strategies of the truncated subgames together to form a profile of general strategies of the original game. And this is tedious rather than fundamentally hard.

However, if one looks, instead of general strategies at the space $\hat{G}$ of the subgames, the process of patching together the strategies becomes completely straightforward. To keep the explanation simple let us consider a game that has just one proper subgame, which we shall call Γ' and let us choose an equilibrium g' in $\hat{G}' \subset \Delta(T')$ where T' is the set of terminal nodes of Γ'. Recall that, for each Player n, g'_n is the image in $\Delta(T')$ of an equivalence class of general strategies given by the induced outcome on T' of such a strategy against the completely mixed behaviour strategies of the other players in Γ' that put equal weight on all actions at all of their information sets in Γ'. We now replace the subgame Γ' with a new terminal node, z' and look at the truncated game Γ'' with terminal nodes and choose an equilibrium $g''$of Γ'' in $\hat{G}''$. We can now patch together g' and g'' to form a profile g for the original game. For each Player n the strategy g''_n gives a measure over the terminal nodes outside the subgame Γ' together with z'. And g'_n gives a measure over the terminal nodes of Γ'. We define g_n to be that measure over T where for each z outside of T' the probability assigned by g_n is that assigned by g''_n and for each z in T' it is the probability assigned to z' by g''_n times the probability assigned by g'_n to z.

This proof is sketched in Hillas and Kvasov (2020) and is covered in somewhat more detail in Hillas and Kvasov (2021).

7.2. **A Definition of Proper Equilibrium in Nonlinear Games.** We now sketch how proper equilibrium, defined for normal form games by Myerson (1978), might be defined for nonlinear games. As we indicated earlier the appropriate analogue of pure strategies in linear games is not behaviour strategies but rather the extreme points of the set of general strategies. Now even the set $\hat{E}_n$ may be infinite. There are no doubt a number of ways of defining proper equilibria. We give one straightforward manner.

Let us consider some nonlinear game. For each n we define, $\hat{E}_n^\varepsilon$ an ε-approximation of $\hat{E}_n$ to be a finite subset of $\hat{E}_n$ such that each element of $\hat{E}_n$ is within ε of $\hat{E}_n^\varepsilon$. We now let e^ε be a proper equilibrium of a standard finite normal form game with finite strategies $(\hat{E}_n^\varepsilon)_{n\in N}$, with the natural payoffs. Now, of course, e^ε "is" an element of $\hat{G}$ and since $\hat{G}$ is compact any sequence of e^ε has a convergent subsequence converging to a limit. Such a limit we call a proper equilibrium. In a linear game $\hat{E}_n$ will coincide with $\hat{S}_n$ and so, for all sufficiently small ε, the set $\hat{E}_n^\varepsilon$ will coincide with $\hat{S}_n$. Thus the definition of proper equilibrium will coincide with the usual definition.

In Hillas and Kvasov (2021) we give definitions of quasi-perfect equilibria, originally defined for normal form games by van Damme (1984), for nonlinear games. We conjecture that, with these definitions for nonlinear games, a proper equilibrium is a quasi-perfect equilibrium. Since we know that adding arbitrary behaviour strategies as new pure strategies changes the set of proper equilibria it seems unlikely that defining proper equilibrium using all behaviour strategies rather than just the extreme points will give the same result.

7.3. **Strategic Stability in Nonlinear Games.** Mertens (2004, 1989, 1991) gives a number of definitions of small perturbations of a game. We give here just one of them. (For more details see Mertens (1989, pp 584–585), of which we give a brief summary.)

The perturbations smaller than $\delta \leq 1$ are defined to be

$$P_\delta = \{\eta = (\varepsilon_n \sigma_n) \mid 0 \leq \varepsilon_n \leq \delta,\ \sigma_n \in \Sigma_n\}.$$

The boundary of P_δ is denoted ∂P_δ and its interior P_δ^i. We let $P = P_1$.

For any perturbation $\eta \in P_\delta$ it is straightforward to define the perturbed game $\Gamma(\eta)$ and also the equilibria of $\Gamma(\eta)$, which we denote $Eq(\eta)$. We consider the graph of the equilibrium correspondence on the interior of P, that is,

$$\mathcal{E} = \left\{(\sigma, \eta) \in \Sigma \times P^i \mid \sigma \in Eq(\eta)\right\}.$$

We consider subsets S^i of $\mathcal{E}$ such that S^i is closed and semialgebraic and let S be the closure of S^i in $\Sigma \times P$. We also define S_δ^i, S_δ, and ∂S_δ, as the inverse images in S of P_δ^i, P_δ, and ∂P_δ. The stability requirement on the sets S that define stable sets are that, for all sufficiently small δ,

S^i_δ is connected and the projection map from S_δ to P_δ is nontrivial. (The form of nontriviality that Mertens uses is that the projection map from $(S_\delta, \partial S_\delta)$ to $(P_\delta, \partial P_\delta)$ is homologically nontrivial, and he actually considers homologies with different coefficient modules, but this need not concern us here.) The stable sets are then S_0 the limits at 0 of such sets, that is the sets $\{(\sigma, \eta) \in S \mid \eta = 0\}$, and the Hausdorff limits of such sets. (Looking at the Hausdorff limits just "undoes" the restriction of S to semialgebraic sets.)

We can define stable sets for nonlinear games by simply replacing Σ_n in the definition of the perturbations with $\hat{G}_n$. We define perturbed games, $\Gamma(\eta)$, and the equilibria of the perturbed games and the graph of the equilibrium correspondence on the interior of the space of perturbations then becomes

$$\mathcal{E} = \left\{(g, \eta) \in \hat{G} \times P^i \mid g \in Eq(\eta)\right\}$$

and the rest of the definition of stable sets is as described above for normal form games, that is, for the linear case.

There is a lot to check, but much of the work in Mertens (2004, 1989, 1991) is to show that nothing in the construction of the stable sets depends on the topological or linear structure of the various sets, so there is much reason to be optimistic that the construction will go through. Of course, some of the properties, such as the admissibility and backward induction properties, will use different definitions and so these properties will also require careful checking. The most challenging task would seem to be finding the appropriate analogue to Theorem 1 of Kohlberg and Mertens (1986), and proving it.

A less challenging, and less satisfactory, approach would be to define stable sets using perturbations to the best reply correspondence on $\hat{G}$, following the approach of Hillas (1990).

8. Conclusion

Kuhn's Theorem tells us that, for games with perfect recall, behaviour strategies and mixed strategies are equivalent in the sense that, as a function of the strategies of the other players, they induce the same measures over the terminal nodes. We have given here a systematic treatment of such equivalence between all classes of strategies.

All of the results about the equivalence and nonequivalence of various strategy sets for different classes of games are either previously known or are immediate and trivial consequences of the known results. The result in Proposition 1, that two strategies are equivalent if and only if they induce the same measure over terminal nodes against some profile of completely mixed strategies of the other players, has not, as far as we can tell, appeared in the literature. It is, however, quite straightforward. Its implication, that all the strategy sets can be embedded in the set of measures over the terminal nodes, that is, the T-simplex, is, we believe, new and interesting. This embedding allows a very clean expression of the relation between the various strategy sets in the different classes of games. And, in Section 7, we sketched a few other interesting applications of this embedding.

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