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Shapley mapping and its axiomatization in n-person cooperative interval games

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Abstract

An interval game is an extension of the characteristic function form games, in which players are assumed to face payoff uncertainty. This characteristic function thus assigns a closed interval, instead of a real number. This paper proposes a new solution mapping of n-person interval games, called Shapley mapping and applies it to n-person interval games. It is shown that the Shapley mapping is the unique solution mapping that satisfies the axioms of: (i) efficiency, (ii) symmetry, (iii) null player property and (iv) an interval game version of additivity.

Keywords: cooperative interval games; interval uncertainty; Shapley value; solution mapping; axiomatization

1 Introduction

This paper examines cooperative game theory when players face uncertainty. One of the most familiar representations of cooperative game theory without uncertainties are the characteristic function form games with transferable utility (the so-called coalition form games or TU games) proposed by von Neumann and Morgenstern ([31]).[1] A coalition form game consists of a set N of players and a characteristic function v that gives a real number $v(S)$ (i.e., the worth of S) to every subset S of N (i.e., the coalitions). For each coalition S, $v(S)$ is the total payoff S can obtain by itself and divide among its members in any possible way. The solution concept of characteristic function form games assigns each game a (possibly empty or singleton) set of outcomes, where the outcome is represented by n-dimensional real-valued vectors. In the literature on characteristic function form games, various types of solution concepts have been proposed,

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[1]Another standard specification are the coalitional form games with nontransferable utility (the so-called NTU games) introduced by Aumann and Peleg ([10]).

including imputation, core, the von Neumann and Morgenstern stable set, the Shapley value, and the nucleolus.

However, in reality, the payoffs a coalition can entail uncertainty. For instance, under the classical "bankruptcy problem" a creditor has to decide on the amount of money to be lent, thus facing uncertainty regarding the borrower's future fiscal condition or solvency when the claim is scheduled to be paid back and cleared. Similarly, when a project is to be jointly financed by multiple investors and they need to decide whether to join, they may not be able to know (i) the exact return that would be realized by the project or (ii) the additional costs that would be incurred by them in the interim period before the project is completed. Therefore, introducing uncertainty into standard characteristic function form games is a natural and important extension. In this paper, we employ a model of interval games that consider "interval uncertainty", in that the uncertainty regarding the coalition payoff is represented by a closed interval. Specifically, we study a generalized characteristic function form game, in which a characteristic function w assigns a bounded and closed interval of real numbers to every coalition instead of real numbers as in classical characteristic function form games.[2] [3]

Interval games were initially proposed by Branzei et al. [12] as a generalization of the classical bankruptcy games. Subsequently, from the middle of the 2000s and until the beginning of the 2010s, interval game analysis was mainly driven by the research group of Tijs, Branzei, and Alparslan Gök. Alparslan Gök et al. ([7]) proposed the notion of "selection-based solution concept" as the solution concept to be applied to interval games, defined the imputation and the core sets as specifications of the selection-based solution concept, and applied them to specific interval games, mainly two-person games. Furthermore, Alparslan Gök et al. ([7]) proposed another type of solution concept, called ψ^{α}-value, which can be classified as the notion of "solution mapping" which is examined thoroughly in our paper.

As an alternative solution concept to the selection-based solution concept, Alparslan Gök et al. ([3], [4]) proposed the notion of interval solution concept. The interval core, interval stable set, and interval Shapley value are the solutions included into the interval solution concept. Among them, the interval Shapley value was characterized by axiomatizations such as those of Alparslan Gök ([2]) and Alparslan Gök et al. ([6]). As explained in the following, an interval solution is defined as a (possibly empty or singleton) set of n-dimensional interval vectors, that is, a vector whose component is a closed interval. Therefore, the notion of the interval solution concept has to solve the issue of how to use interval solutions after uncertainties are removed and one of the outcomes in the worth set is realized, as per Branzei et al. ([13]). As for applications of interval games, Palanci et al. ([25]) analyze how to introduce uncertainties into classical cooperative

[2]Throughout this paper, w denotes the characteristic function in an interval game, as to distinguish it from the characteristic function v in a characteristic function game.

[3]As for the related literature on the introduction of uncertainty to cooperative game-theoretic analysis, Habis and Herings's ([17]) TUU game is a two-stage game, in which the first stage consists of multiple coalition form games and, in the second stage, one of them is realized and played by the players. Suijs et al. ([29], [30]) considered a stochastic cooperative game that assigns a random variable to a coalition as its coalition value, rather than a real number. Aubin ([8]) formalized fuzzy games in which the uncertainties regarding coalition formation are introduced.

transportation games and formalize them as interval games. Alparslan Gök et al.[5] examine the Shapley value and Baker–Thompson rule in the interval-game version of the airport game. Those theoretical and applied analyses of interval games are summarized in Alparslan Gök ([1]), Branzei et al. ([11]), and Ishihara and Shino ([19]).

Recently, interval game analyses have progressed further. In particular, as a main topic that relates to the interval Shapley value, researchers have focused on how to define interval subtractions between closed intervals. Similar to the Shapley value in coalition form games, the interval Shapley value is defined based on the marginal contribution of each player. Since the marginal contribution is defined as a gap between two different coalition values and since, in interval games, the values are within a closed interval rather than real numbers, we need to define a subtraction operator regarding those closed intervals. As examined in the following sections, Alparslan Gök et al. ([6]), which originally proposed the interval Shapley value, employed a partial subtraction operator. However, this operator is only defined for an "ordered interval pairs," namely two closed intervals $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ satisfying $\overline{J} - \underline{J} \leq \overline{I} - \underline{I}$, which substantially reduces the applicability of the interval Shapley value to interval games. On this issue, Alparslan Gök et al. ([4]) restricted the analysis to size monotonic interval games, in which the subtraction operator is always well-defined; thus, the interval Shapley value can be computed. Furthermore, they provided an axiomatic characterization of the interval Shapley value on a special subclass of size monotonic interval games, called the convex cone. However, this restriction of coverage seems too strong, and they argued that "*to characterize the interval Shapley value on the class of size monotonic games is a topic for further research,...*"(page 138 of Alparslan Gök et al. [6]).

Moreover, to address the interval subtraction problem, Han et al. ([18]) introduced Moore's subtraction operator, applied it in defining a player's marginal contribution and the interval Shapley value, and attempted to axiomatize the value. However, while Moore's operator is always defined for any pair of closed intervals, the interval Shapley value does not satisfy efficiency (i.e., efficiency of the worth set of the grand coalition; see footnote 7). Meng et al. ([23]) employed the concept of "imaginary" interval numbers and imaginary differences, such that $[a, b]$ satisfying $a > b$, and defined the interval Shapley value by using these concepts. However, as a natural consequence, the interval Shapley value based on imaginary differences can assign imaginary interval numbers to a player, which provides no "guideline" about the desirable allocation and is thus impossible to interpret.

Moreover, Fei et al. ([15]) proposed the interval-valued discounted Shapley value, which is defined without using interval subtraction and examined the properties of the value. For a parameter called "attitude factor" $\alpha \in [0, 1]$, they introduced α-cooperative games associated with interval games, which are categorized as coalition form games. Then, they set the discounted Shapley values of α-cooperative games when $\alpha = 0$ and $\alpha = 1$ as the lower and upper bounds of the solution in the original interval game, respectively. In this case, for this solution, defined as an n-dimensional interval vector, to be well-defined, the following condition has to be satisfied: for every player, the payoff given by the discounted Shapley value when $\alpha = 0$ must not be strictly larger than the one when $\alpha = 1$. Fei et al. ([15]) showed that, for any interval games, there exists a discount factor $\delta \in [0,1]$ associated with the definition of the discounted Shapley value, so that the above condition is satisfied. However, this solution concept, called the interval discounted Shapley value, guarantees its existence only when $\delta = 0$, which corresponds to the perfect egalitarian rule. However, in case of $\delta = 1$, which corresponds to the (non-discounted and standard) Shapley value, this solution may not exist. Under a similar framework, Lian and Li ([22]) proposed and examined the interval Banzhaf value. However, they just restricted the coverage of analysis to size monotonic interval games rather than searching for a condition on the existence of the solution, such as Fei et al. ([15]). Furthermore, this does not satisfy efficiency. Finally, Li et al. ([21]) studied the interval solidarity value but also restricted the coverage of games to be analyzed.

In this study, we define a new solution concept, based on the notion of solution mapping, as initially examined by Alparslan Gök et al. ([7]) at the early stages of interval game literature, and solve interval games by applying this solution concept. On the one hand, interval solution concepts are currently playing a central role in the literature and are defined as a (possibly empty or singleton) set of n-dimensional interval vectors. However, such a solution does not directly answer the question of how to allocate realized outcomes between players after uncertainties are removed. Branzei et al. ([13]) tackled this problem as an independent analysis. However, clear instructions to provide an agreeable *ex-post* allocation do not seem to have been derived. On the other hand, our solution concept does not have to consider this issue, since it is defined as a mapping that assigns n-dimensional real-valued vectors, rather than intervals, to each outcome realization (see also Ishihara and Shino ([19]) and Ishihara et al. ([20])).

As previously discussed, our solution concept is based on the solution mapping of Alparslan Gök et al. ([7]). However, we propose a new solution concept, called **Shapley mapping**, and apply it to general n-person interval games. As we will discuss in detail in the subsequent sections, Shapley mapping has some advantages compared with the existing solution concepts of interval games. First, as argued above, it does not have to consider "how to handle interval solution concepts" after uncertainties are removed. Second, it is defined without using interval subtraction. Finally, it is characterized by a standard axiomatization without any restrictions on coverage of the game, that is, it is axiomatized for general n-person interval games.

The rest of this paper is organized as follows. Section 2 reviews coalition form and interval games and solution concepts in interval games. Section 3 examines the existing solution concept and solution mapping by providing some examples. Section 4 defines the **Shapley solution mapping** and characterizes the mapping by a standard axiomatization corresponding to the one of Shapley ([28]). Section 5 concludes the paper.

2 Coalition form games and interval games

2.1 Coalition form games

An n-person coalition form game or a TU game is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that associates a real number $v(S) \in \mathbb{R}$ with each set $S \in 2^N$, so that $v(\emptyset) = 0$. Number $v(S)$ is called the worth of S. We refer to S and N as a coalition and grand coalition, respectively. Worth $v(S)$ is the transferable utility (typically, the amount of money) that coalition S can obtain itself and divide among its members in any possible way. Let CG and CG^N be the set of all coalition form games and all n-person coalition form games, respectively. We hereafter occasionally denote n-person coalition form games simply by v instead of an expression of (N, v). For coalition form game v and coalition $\{1, \ldots, k\} \in 2^N$, we write $v(1, \ldots, k)$ instead of $v(\{1, \ldots, k\})$.

Representative solution concepts for the characteristic function form games include the von Neumann and Morgenstern stable set (von Neumann and Morgenstern [31]), set of imputation, core (Gillies [16]), Shapley value (Shapley [28]), nucleolus (Schmeidler [27]), bargaining sets (Aumann and Maschler [9]), and kernel (Davis and Maschler [14]). In this study, we use the notion of the Shapley ([28]) value.

For an n-person characteristic function form game $v \in CG$, the n-dimensional real-valued vector of $\phi(v) = (\phi_1(v), ..., \phi_i(v), ..., \phi_n(v))$ that satisfies the following (1) is called the **Shapley value** of v:

$$\phi_i(v) = \sum_{S:i \in S} \frac{(s-1)!(n-s)!}{n!} \{v(S) - v(S \setminus \{i\})\}. \tag{1}$$

2.2 Interval games

Similarly to an n-person characteristic function form game (N, v), an n-person **interval game** is a pair (N, w), where $N = \{1, 2, ..., n\}$ is a set of players and w a characteristic function. An interval game differs from a characteristic function form game in that w assigns a closed interval to each coalition instead of a real number. Formally, letting $I(\mathbb{R})$ be the set of all closed and bounded intervals in $\mathbb{R}$, $w : 2^N \to I(\mathbb{R})$ is the characteristic function that assigns to each coalition $S \in 2^N$ a closed interval $w(S) \in I(\mathbb{R})$ that satisfies $w(\emptyset) = [0, 0]$. Interval $w(S)$ is called the **worth set** of S and the minimum and the maximum of $w(S)$ are denoted by $\underline{w}(S)$ and $\overline{w}(S)$, respectively, that is, $w(S) = [\underline{w}(S), \overline{w}(S)]$. An interval game (N, w) considers a situation in which the players face "interval uncertainty" in that they know a coalition S could have $\underline{w}(S)$ as the minimal reward and $\overline{w}(S)$ as the maximal one, but do not know *ex ante* which one between them would be realized. Let IG and IG^N be the set of all interval games and all n-person interval games, respectively. We hereafter denote the n-person characteristic function form games by w instead of a pair (N, w). For an interval game w and a coalition $\{1, ..., k\} \in 2^N$, we write $w(i, ..., k)$ instead of $w(\{1, ..., k\})$. In an interval game w, players i and j are symmetric if $w(S \cup \{i\}) = w(S \cup \{i\})$, for any coalition $S \subset N \setminus \{i, j\}$.

We provide some interval calculus notations for the following analysis. Let $I = [\underline{I}, \overline{I}]$, and $J = [\underline{J}, \overline{J}]$ be two closed intervals. The sum of I and J, denoted by $I + J$, is given as $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$. Next, following Alparslan Gök et al. ([4]), the partial subtraction operator denoted by " $-$ " is defined as:

$$I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}].$$

Note that the partial subtraction operator is only defined for an ordered interval pair, that is, two closed intervals $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ satisfying $\overline{J} - \underline{J} \leq \overline{I} - \underline{I}$. When this condition is violated, $I - J$ cannot be defined as an interval, since $\underline{I} - \underline{J} > \overline{I} - \overline{J}$. Finally, Moore's ([24]) subtraction operator, which we denote as " $-_{(M)}$ "is given by:

$$I -_{(M)} J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]. \tag{2}$$

As opposed to the partial subtraction operator, Moore's operator can be defined for any interval pairs $(I, J) \in I(\mathbb{R}) \times I(\mathbb{R})$.

For two different interval games $w', w'' \in IG$, the sum of the interval games $w' + w'' \in IG$ is also an interval game itself, defined by $(w' + w'')(S) = w'(S) + w''(S)$ for every $S \in 2^N$. For an interval game $w \in IG$, let $v_{\underline{w}} \in CG$ be a characteristic function form game generated from an interval game w, so that $v_{\underline{w}}(S) = \underline{w}(S)$ for every $S \in 2^N$, and $v_{\overline{w}} \in CG$ be a characteristic function game generated from an interval game w, so that $v_{\overline{w}}(S) = \overline{w}(S)$ for every $S \in 2^N$.

For closed intervals $I, J \in I(\mathbb{R})$, $I \leq J$ denotes $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$. Let $\mathbb{I} = (I_1, ..., I_n) \in I(\mathbb{R})^N$ be an n-dimensional closed interval vector, so that $I_i \in I(\mathbb{R})$ for $i \in N$. For $\mathbb{I} \in I(\mathbb{R})^N$, we respectively define $\min \mathbb{I} \in \mathbb{R}^N$ and $\max \mathbb{I} \in \mathbb{R}^N$ as follows:

$$\min \mathbb{I} = (\min I_1, ..., \min I_n), \quad \max \mathbb{I} = (\max I_1, ..., \max I_n). \tag{3}$$

For an interval game $w \in IG$, we say that players i and j are symmetric when $w(S \cup \{i\}) = w(S \cup \{j\})$ holds for any coalition $S \subset N \setminus \{i, j\}$. We say player i is a null player, or simply null, when $w(S) = w(S \cup \{i\})$ holds for any $S \in 2^{N \setminus \{i\}}$.

Note that, for an interval game $w \in IG$, if all worth sets are singletons, that is, $\underline{w}(S) = \overline{w}(S)$ for every $S \in 2^N$, w corresponds to the characteristic function form game $v \in CG$, which is defined as $v(S) = \underline{w}(S) = \overline{w}(S)$. In this case, we say that $v \in CG$ and $w \in IG$ are equivalent or w is equivalent to v. This means that a (classical) characteristic function form game can be regarded as a special case of an interval game with no uncertainty.

3 Solution concepts in interval games

3.1 Existing solution concepts: interval solution concepts and interval Shapley value

In the literature on interval games, the most popular solution concept, which has hitherto played a central part in analyses, is the **interval solution concept**.[4] Interval solution concepts are defined as a (possibly empty or singleton) set of n-dimensional interval vectors. Formally, by letting

[4]Another type of solution concept proposed early in the history of interval game analysis are the **selection-based solution concepts** proposed by Alparslan Gök et al. ([7]). For an interval game $w \in IG$, a function $v : 2^N \to \mathbb{R}$ is called the selection of w if $v(S) \in w(S)$ for each $S \in 2^N$. Note that pair (N, v) constitutes a characteristic function form game. Let $Sel(w)$ be the set of all selections of w. The following two representative selection-based solution

$I_i \in I(\mathbb{R})$ be the interval payoff of player i and $\mathbb{I} = (I_1, ..., I_n) \in I(\mathbb{R})^N$ be an n-dimensional closed interval vector, an interval solution concept in $w \in IG$ assigns a (possibly empty or singleton) set of n-dimensional interval vectors, $\mathbb{I}$. The interval imputation set, interval core, interval stable set (Alparslan Gök et al. [3]), and the interval Shapley value (Alparslan Gök et al. [4]) are classified as part of the interval solution concept. Here, for the following analysis we formally define the interval Shapley value, proposed by Alparslan Gök et al. ([4]) and subsequently axiomatized by Alparslan Gök ([2]) and Alparslan Gök et al. ([6]).[5]

First, let a permutation of N be $\sigma : N \to N$ and the group of all permutations of N be $\pi(N)$. Next, for an interval game $w \in IG^N$ and a permutation σ, we define player i's **marginal contribution** in σ of w as:

$$m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i)). \tag{4}$$

Here, $P_\sigma(i)$ is given by $P_\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, that is, the set of i's predecessors in permutation σ. A marginal contribution vector in σ of w is defined as $m^\sigma(w) = (m_1^\sigma(w), ..., m_n^\sigma(w))$. Based on these setups, **interval Shapley value** $\Phi : IG \to I(\mathbb{R})^N$ is defined by (5):

$$\Phi(w) = (\Phi_1(w), ..., \Phi_n(w)) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w). \tag{5}$$

Then, $\Phi_i(w)$ $(i \in N)$ in (5) can be rearranged as follows:

$$\Phi_i(w) = \sum_{S: i \in S} \frac{(s-1)!(n-s)!}{n!} \{w(S) - w(S \setminus \{i\})\}. \tag{6}$$

It is evident from comparing (6) and (1) that the interval Shapley value is a simple extension of the Shapley value in coalition form games. However, the interval Shapley value is not always definable because the partial subtraction operator is used. To ascertain this, consider examples 3.1 and 3.2. In the former case, the interval Shapley value can be computed, but it cannot in the latter.

concepts of an interval game w, imputation set $I(w)$, and the core set $C(w)$ are given by (Alparslan Gök et al. [7]):

$$\begin{aligned} I(w) &= \cup \{I(v) | v \in sel(w)\}, \\ C(w) &= \cup \{C(v) | v \in sel(w)\}, \end{aligned}$$

where $I(v)$ and $C(v)$ are the imputation and core sets in characteristic function form game $v \in CG$, respectively.

However, analyses using selection-based solution concept have had limited progress and interval solution concepts have become the central and, virtually, the only solution concept applied.

[5]Interval imputation set $II(w)$ and interval core $IC(w)$ of interval game (N, w) are given by:

$$\begin{aligned} II(w) &= \left\{ I = (I_1, ..., I_n) \in I(\mathbb{R})^N \,\middle|\, \sum_{i \in N} I_i = w(N),\ w(i) \preceq I_i,\ \forall\, i \in N \right\} \\ IC(w) &= \left\{ I = (I_1, ..., I_n) \in I(\mathbb{R})^N \,\middle|\, \sum_{i \in N} I_i = w(N),\ w(S) \preceq \sum_{i \in S} I_i,\ \forall\, S \in 2^N \setminus \{\emptyset\} \right\}. \end{aligned}$$

Example 3.1 *This example is based on Alparslan Gök et al. ([4]).* $w(1) = [0,0]$, $w(2) = [0,0]$, $w(3) = [0,0]$, $w(12) = [2,4]$, $w(13) = [2,4]$, $w(23) = [2,4]$, *and* $w(123) = [9,15]$.

Player 1's marginal contributions for each permutation are as follows: $\sigma_1^{(123)}(w) = w(1) - w(\emptyset) = [0,0]$, $\sigma_1^{(132)}(w) = [0,0]$, $\sigma_1^{(213)}(w) = [2,4]$, $\sigma_1^{(231)}(w) = [7,11]$, $\sigma_1^{(312)}(w) = [2,4]$, and $\sigma_1^{(321)}(w) = [7,11]$. Given that the sum of all marginal contributions is $[18,30]$ and $3! = 6$, player 1's interval Shapley value is $\Phi_1(w) = [3,5]$. By similar computations (or symmetry), the interval Shapley value, Φ, in this example is given by:

$$\Phi(w) = ([3,5], [3,5], [3,5]). \tag{7}$$

Example 3.2 *This example is based on Han et al. ([18]).* $w(1) = [0,2]$, $w(2) = [1/2,3/2]$, $w(3) = [1,2]$, $w(12) = [2,3]$, $w(13) = [3,4]$, $w(23) = [4,4]$, *and* $w(123) = [6,7]$.

Player 2's marginal contributions for permutation (123) cannot be computed because $m_2^{(123)}(w) = w(12) - w(1) = [2,3] - [0,2]$ but $2 - 0 > 3 - 2$.

To address the so-called "interval subtraction problem," Alparslan Gök et al. ([2], [4], [6]) restricted the coverage of interval games to size monotonic interval games. An interval game $w \in IG$ is called **size monotonic** when the following condition (8) holds for every pair of coalitions (S,T) satisfying $S \subseteq T$:

$$\overline{w}(S) - \underline{w}(S) \leq \overline{w}(T) - \underline{w}(T). \tag{8}$$

When an interval game is size monotonic, $\underline{w}(T) - \underline{w}(S) \leq \overline{w}(T) - \overline{w}(S)$ for every (S,T) with $S \subseteq T$. This implies that a player's marginal contribution defined in (4) can be defined as a closed interval. Example 3.2 presents the case where the marginal contribution cannot be derived because this is not size monotonic and such a game is thus excluded from analyses such as Alparslan Gök et al. ([2], [4], [6]).

As a different approach to address the "interval subtraction problem," Han et al. ([18]) used Moore's subtraction operator (Moore [24]) to define an interval solution concept called the interval Shapley-like value. As shown by the expression of (2), Moore's subtraction operator can be defined for any interval pairs. For example, different from the partial subtraction operator, we can compute player 2's marginal contributions for permutation (123) in Example 3.2 as: $\sigma_2^{(123)}(w) = w(12) -_{(M)} w(1) = [2,3] -_{(M)} [0,2] = [2-2, 3-0] = [0,3]$, which is a closed interval. Letting Φ^M be the interval Shapley-like value, Φ^M is given by:

$$\Phi^M(w) = \left(\left[\frac{11}{12}, \frac{31}{12}\right], \left[\frac{7}{6}, \frac{17}{6}\right], \left[\frac{23}{12}, \frac{43}{12}\right]\right). \tag{9}$$

Now, as an example, assume that a grand coalition forms and $6 \in w(123)$ is realized after uncertainties are removed. In this case, the problem we need to solve is how 6 is allocated among players 1, 2, and 3. However, as evident from (7) and (9), interval solution concepts such as the interval Shapley value and interval Shapley-like value do not provide a clear answer to this problem. Branzei et al.'s ([13]) is the only analysis to address this issue. Based on their proposed procedure, for the realization of $6 \in w(123)$, the interval Shapley-like value leads to a real-valued payoff vector of $(11/12, 7/6, 23/12)$.[6] However, the sum of payoffs is 4, which is strictly smaller than realized outcome 6. This observation clearly implies that interval Shapley-like value does not satisfy efficiency.[7]

Based on these arguments, in the next subsection, we examine the notion of solution mapping as an alternative to the interval solution concept.

3.2 A new solution concept: solution mapping

Considering the underlying situation that interval game analyses essentially assume, an alternative approach in constructing a solution concept may be appropriate. As previously discussed, an interval-game analysis essentially assumes the following as the underlying situation players face. First, players face payoff uncertainties, represented by the worth sets, and negotiate over a "rule" or "protocol" which specifies a way to allocate an outcome among players *ex ante*, before uncertainties are removed. Second, after the uncertainties are removed and one of the outcomes in the worth set is realized, the realized outcome is allocated based on the agreed rule or protocol in the *ex ante* negotiation.[8] Specifically, when the grand coalition forms, the realized outcome in $w(N)$ is allocated. Based on this interpretation, we argue that a solution concept of interval games should be a mapping that assigns n-dimensional real-valued vectors to each realization in the worth set of the grand coalition, rather than directly specifying n-dimensional interval vectors as the interval solution concepts. This is the main idea of the **solution mapping** notion that we employ as a solution concept.

Formally, solution mapping is defined as follows. First, we consider a mapping $\kappa : [a, b] \to \mathbb{R}^n$ that assigns n-dimensional payoff vectors to each realization in closed interval $[a, b]$. Second, let the set of all κ be $K(\mathbb{R}^n)$ and $F : IG \to K(\mathbb{R}^n)$ be a map that assigns to each interval game w a unique mapping $F(w) \in K(\mathbb{R}^n)$. Finally, when the domain of $F(w)$ is the worth set of grand coalition $w(N) = [\underline{w}(N), \overline{w}(N)]$, that is, $F(w) : [\underline{w}(N), \overline{w}(N)] \to \mathbb{R}^n$, we call F a solution mapping in n-person interval games. $F(w)(t)$ assigns n-dimensional payoff vectors to each realization $t \in [\underline{w}(N), \overline{w}(N)]$ for an interval game $w \in IG$.

In fact, such a mapping has already been proposed by Alparslan Gök et al. ([7]) as the ψ^{α}-

[6] Here, we employ a "one-step procedure" to translate an n-dimensional interval vector into an n-dimensional real-valued vector, as proposed in [13].

[7] Specifically, "efficiency" means the "efficiency on realizations of the worth set for the grand coalition" in that the sum of elements (= real numbers) in an n-dimensional real-valued vector has to be equal to a realization of the worth set for the grand coalition. Hereafter, "efficiency" refers to this efficiency on the realizations of the worth set for the grand coalition. The efficiency axiom which we will examine in Section 4 relates to this efficiency type. On the other hand, in the context of the existing interval solution concept, another efficiency type is examined, namely the sum of the elements (= intervals) in an n-dimensional interval vector has to be equal to the worth set (= interval) of the grand coalition. To distinguish it from the former efficiency, we call this the "efficiency on the worth set of the grand coalition."

[8] This situation is similar to that in Habis and Herings's ([17]) TUU game, where uncertainties are introduced into cooperation games from a different approach. Namely, they constructed a two-stage model in which the ex-ante first stage has multiple coalition form games, and one of them is realized and played by the players in the second stage.

value. However, their analysis considered only two-person interval games and the value was defined by using additional exogeneous parameters and characterized by imposing non-necessary standard axioms for its axiomatization. Moreover, no subsequent analysis of the solution mapping has been conducted since. As such, in the next section, we propose a new solution mapping, called Shapley mapping, by not using additional parameters and apply the mapping to general n-person interval games. It is shown that the Shapley mapping can be axiomatized with the standard axioms of efficiency, individual rationality, symmetry, and an interval version of additivity.

4 Shapley mapping and its axiomatization

4.1 Shapley mapping σ^*

For an n-person interval game $w \in IG$, we define the **Shapley mapping** as a solution mapping that we employ in the following analysis.

First, for the realization in the worth set of the grand coalition $t \in w(N)$, α is uniquely determined and satisfies $t = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N)$.[9] Second, we define a characteristic function form game v_w^α generated from w by α as:

$$v_w^\alpha(S) = (1-\alpha)\underline{w}(S) + \alpha\overline{w}(S) \quad for\ every\ S \in 2^N.$$

Note that $v_w^\alpha(N) = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N) = t$. By letting $\phi(v_w^\alpha) = (\phi_1(v_w^\alpha), ..., \phi_i(v_w^\alpha), ..., \phi_n(v_w^\alpha))$ be the Shapley value in coalition form game v_w^α, **Shapley mapping** $\sigma^*(w) : [\underline{w}(N), \overline{w}(N)] \to \mathbb{R}^n$ is defined as:

$$\sigma^*(w)(t) = \phi(v_w^\alpha). \tag{10}$$

Note that $\sigma^*(w)(t)$ can be rearranged as follows:

$$\begin{aligned}
\sigma_i^*(w)(t) &= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{v_w^\alpha(S) - v_w^\alpha(S\setminus\{i\})\} \\
&= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}[\{(1-\alpha)\underline{w}(S) + \alpha\overline{w}(S)\} - \{(1-\alpha)\underline{w}(S\setminus\{i\}) + \alpha\overline{w}(S\setminus\{i\})\}] \\
&= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}[(1-\alpha)\{\underline{w}(S) - \underline{w}(S\setminus\{i\})\} + \alpha\{\overline{w}(S) - \overline{w}(S\setminus\{i\})\}] \\
&= (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}),
\end{aligned} \tag{11}$$

where $\phi_i(\cdot)$ is player i's Shapley value in coalitional form games $v_{\underline{w}}$ and $v_{\overline{w}}$.

For the interval game in Example 3.2, when $6 \in w(123)$ is realized, Shapley mapping σ^* gives

[9] It should be noted that there exists a case for which v_w^α cannot be defined. This happens when $v(N)$ is singleton and $V(S)$ is not for some $S \in 2^N \setminus N$. We exclude such a "degenerate case" since it departs from the underlying situation of interval games, where interval uncertainty exists regarding the realization of the outcome in the grand coalition.

the following real-valued vector:

$$\sigma^*(w)(6) = \left(\frac{5}{4}, 2, \frac{11}{4}\right).$$

Note that the sum of payoffs is 6, which is equal to the realization of the worth set of the grand coalition. Therefore, efficiency is satisfied (see also footnote 7). Now, we consider other numerical examples of three-person interval games.

Example 4.1 *This example is based on Alparslan Gök ([2]) and Palanci et al. ([26]).* $w(1) = [7,7]$, $w(2) = [0,0]$, $w(3) = [0,0]$, $w(12) = [12,17]$, $w(13) = [7,7]$, $w(23) = [0,0]$, $w(123) = [24,29]$.

This interval game is size monotonic. Therefore, the interval Shapley value exists. Letting Φ be the interval Shapley value, the $\Phi(w)$ in this example is given by $\Phi(w) = ([27/2,16],[13/2,9],[4,4])$. Moreover, the interval Banzhaf value proposed by Palanci et al. ([26]) also exists and by letting Φ^B be the value, $\Phi^B = ([25/2,15],[11/2,8],[3,3])$. However, Φ^B does not satisfy the efficiency of the worth set of the grand coalition (again, see footnote 7), that is, $[25/2,15] + [11/2,8] + [3,3] \neq [24,29]$. It should be noted that Palanci et al. ([26]) also examined the validity of the interval egalitarian rule, which gives this example an interval vector, $([8,29/3],[8,29/3],[8,29/3])$. While this rule obviously satisfies the efficiency of the worth set of the grand coalition, it is known that it does not satisfy the null player property, since this rule can give a null player an interval other than $[0,0]$.[10]

Shapley mapping, however, gives the following three-dimensional real-valued vectors for the realizations of the worth set of the grand coalition—24, 26.5, and 29 (the minimum, midpoint, and maximum value of $w(123)$, respectively):

$$\sigma^*(w)(24) = \left(\frac{27}{2}, \frac{13}{2}, 4\right), \quad \sigma^*(w)(26.5) = \left(\frac{59}{4}, \frac{31}{4}, 4\right), \quad \sigma^*(w)(29) = (16, 9, 4).$$

Comparing Shapley mapping σ^* with interval Shapley mapping Φ, $\sigma^*(w)(24)$ and $\sigma^*(w)(29)$ are same as the vectors consisting of the lower and upper bounds of each element in $\Phi(w)$, respectively. Furthermore, each element of $\sigma^*(w)(26.5)$ is identical to the midpoint of the corresponding player's interval in $\Phi(w)$. In general, regarding the relationship between Shapley mapping and the interval Shapley value, the following holds:[11]

Theorem 4.1 *Let Φ and σ^* be the interval Shapley value and Shapley mapping, respectively. For the realization of the worth set of the grand coalition, $t \in w(N)$, in an interval game $w \in IG^n$, let $\alpha \in [0,1]$ be a number satisfying $t = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N)$. When w is size monotonic, the following holds:*

$$\sigma^*(w)(t) = (1-\alpha)\min\Phi(w) + \alpha\max\Phi(w). \tag{12}$$

[10]Regarding the null player property, see Subsection 4.2.

[11]As for operators *min* and *max*, see (3).

Proof When $w \in IG$ is size monotonic, $\Phi(w)$ can be defined, and the following holds for every $i \in N$:

$$\begin{aligned}
\Phi_i(w) &= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{w(S) - w(S\setminus\{i\})\} \\
&= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{[\underline{w}(S),\ \overline{w}(S)] - [\underline{w}(S\setminus\{i\}),\ \overline{w}(S\setminus\{i\})]\} \\
&= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{[\underline{w}(S) - \underline{w}(S\setminus\{i\}),\ \overline{w}(S) - \overline{w}(S\setminus\{i\})]\} \\
&= \left[\sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{\underline{w}(S) - \underline{w}(S\setminus\{i\})\},\ \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{\overline{w}(S) - \overline{w}(S\setminus\{i\})\}\right] \\
&= [\phi_i(v_{\underline{w}}),\ \phi_i(v_{\overline{w}})]. \qquad (13)
\end{aligned}$$

Equality (13) holds because $\alpha = 0$ when $t = \underline{w}(N)$, thus $\sigma_i^*(w)(\underline{w}(N)) = \phi_i(v_{\underline{w}})$, and because $\sigma_i^*(w)(\overline{w}(N)) = \phi_i(v_{\overline{w}})$ based on the same argument. Further, (13) implies $\min \Phi_i(w) = \phi_i(v_{\underline{w}})$ and $\max \Phi_i(w) = \phi_i(v_{\overline{w}})$. Therefore, the following holds:

$$(1-\alpha)\min \Phi_i(w) + \alpha \max \Phi_i(w) = (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}) = \sigma_i^*(w)(t). \qquad (14)$$

The last equality in (14) holds from (11). Since (14) holds for every $i \in N$, (12) is true. ■

Theorem 4.1 implies some equivalency between the interval Shapley value and Shapley mapping, in the sense that the allocation for a realization of the worth set of the grand coalition proposed by Shapley mapping is identical to the real-valued vector obtained by dividing internally each interval in the interval Shapley value by the ratio associated with the realization of the worth set. Considering this equivalence and the fact that the interval Shapley value is only defined in a size monotonic interval game, Shapley mapping has high generality and applicability to all interval games. Furthermore, the Shapley mapping can be characterized by a standard axiomatization. The next section provides its axiomatization.

4.2 Axiomatization of the Shapley mapping

This subsection proposes a set of axioms and shows there exists a unique solution mapping satisfying the properties of efficiency, symmetry, null player property, and interval game version of additivity, and this solution mapping is the Shapley mapping.

In particular, for an interval game w and a solution mapping σ, we consider the following axioms.

- Axiom 1-1: Efficiency [**EF**]

$$\Big(\sum_{i\in N} \sigma_i(w)(t) = t\Big)\ \Big(\forall w \in IG\Big)\ \Big(\forall t \in w(N)\Big).$$

- Axiom 2-1: Symmetry [**SYM**]

$$\Big(\sigma_i(w)(t) = \sigma_j(w)(t)\Big)\ \Big(\forall w \in IG\ where\ \ i\ and\ j\ are\ symmetric\Big)\ \Big(\forall t \in w(N)\Big).$$

- Axiom 3-1: Null player property [**NP**]

$$\Big(\sigma_i(w)(t) = 0\Big)\ \Big(if\ w(S) = w(S \cup \{i\})\ \forall S \in 2^N\Big)\ \Big(\forall t \in w(N)\Big).$$

- Axiom 4-1: Additivity-1 [**AD1**]

$$\Big(\sigma_i(w' + w'')(t' + t'') = \sigma_i(w')(t') + \sigma_i(w'')(t'')\Big)$$
$$\Big(\forall w', w'' \in IG\Big)\ \Big(\forall t' \in w'(N)\Big)\ \Big(\forall t'' \in w''(N)\Big)\ \Big(\forall i \in N\Big).$$

- Axiom 4-2: Additivity-2 [**AD2**]
  For an $\alpha \in [0,1]$ and $w', w'' \in IG$, we define $t' \in w'(N)$ and $t'' \in w''(N)$ as :

$$\begin{aligned} t' &= (1-\alpha)\underline{w'}(N) + \alpha\overline{w'}(N) \\ t'' &= (1-\alpha)\underline{w''}(N) + \alpha\overline{w''}(N). \end{aligned}$$

  Then,

$$\Big(\sigma_i(w' + w'')(t' + t'') = \sigma_i(w')(t') + \sigma_i(w'')(t'')\Big)$$
$$\Big(\forall w', w'' \in IG\Big)\ \Big(\forall \alpha \in [0,1]\Big)\ \Big(\forall i \in N\Big).$$

  Note that, when all worth sets are singleton, both in w' and w'', AD2 becomes identical to Shapley's ([28]) axiom of additivity for the characteristic function form games equivalent to w' and w''.

Axiom EF asserts that all $t \in w(N)$ is allocated to either player and no residual exists. Note that this notion corresponds to the "efficiency on realizations of the worth set for the grand coalition" discussed in footnote 7. Axiom SYM argues that only what a player can obtain on its own in the game should matter, not its specific name or label. Axiom NP asserts that a zero payoff should be assigned to a null player. Axioms AD1 and AD2 essentially come from the additivity axiom in Shapley ([28]), which considers interval game $(w' + w'') \in IG$, defined as $(w' + w'')(S) = w'(S) + w''(S)\ \forall S \in 2^N$ for two different games $w', w'' \in IG$. Specifically, AD1 asserts that, when σ gives $\sigma_i(w')(t')$ to player i for realization $t' \in w'(N)$ in $w' \in IG$ and $\sigma_i(w'')(t'')$ to i for realization $t'' \in w''(N)$ in $w'' \in IG$, in the sum game $(w' + w'') \in IG$ σ should give $\sigma_i(w')(t') + \sigma_i(w'')(t'')$ to player i for realization $t' + t'' \in (w' + w'')(N)$. AD2 is similar to AD1, but differs in that it imposes restrictions on t' and t'', so that these are generated by the common "ratio" $\alpha \in [0,1]$ regarding the worth sets of the grand coalition. Therefore, AD1 is stronger than AD2 in terms of the condition that a solution concept has to be satisfied.

We first present below the so-called "impossibility theorem" regarding AD1 and other axioms.

Lemma 4.1 *There exists no solution mapping that simultaneously satisfies EF, SYM, NP, and AD1.*

Proof To derive the contradiction, we assume a solution mapping σ satisfies EF, SYM, NP, and AD1, and consider the following three-person interval game, w, and its partition w_1, w_2 and w_3, that is, $w = w_1 + w_2 + w_3$:

$$\begin{cases} w(1) &= [0,1] \\ w(2) &= [0,1] \\ w(3) &= [0,1] \\ w(12) &= [0,2] \\ w(13) &= [0,2] \\ w(23) &= [0,2] \\ w(123) &= [0,3] \end{cases} \quad \begin{cases} w_1(1) &= [0,1] \\ w_1(2) &= [0,0] \\ w_1(3) &= [0,0] \\ w_1(12) &= [0,1] \\ w_1(13) &= [0,1] \\ w_1(23) &= [0,0] \\ w_1(123) &= [0,1] \end{cases} \quad \begin{cases} w_2(1) &= [0,0] \\ w_2(2) &= [0,1] \\ w_2(3) &= [0,0] \\ w_2(12) &= [0,1] \\ w_2(13) &= [0,0] \\ w_2(23) &= [0,1] \\ w_2(123) &= [0,1] \end{cases}$$

$$\begin{cases} w_3(1) &= [0,0] \\ w_3(2) &= [0,0] \\ w_3(3) &= [0,1] \\ w_3(12) &= [0,0] \\ w_3(13) &= [0,1] \\ w_3(23) &= [0,1] \\ w_3(123) &= [0,1] \end{cases}$$

Note that players 2 and 3 are null players in w_1, 1 and 3 are null players in w_2, and 1 and 2 are null players in w_3. Therefore, letting t_1,t_2,t_3 be realizations in w_1, w_2 and w_3, respectively ($t_1 \in w_1(123)$, $t_2 \in w_2(123)$ and $t_3 \in w_3(123)$), the following holds.

$$\sigma(w_1)(t_1) = (t_1,0,0), \quad \sigma(w_2)(t_2) = (0,t_2,0), \quad \sigma(w_3)(t_3) = (0,0,t_3)$$

From AD1, it follows that:

$$\sigma(w)(t_1 + t_2 + t_3) = \sigma(w_1)(t_1) + \sigma(w_2)(t_2) + \sigma(w_3)(t_3) = (t_1,t_2,t_3),$$

which contradicts SYM unless $t_1 = t_2 = t_3$, which does not always hold. ■

For the following analysis, we select AD2 instead of AD1 as the interval game version of additivity. The main result of this paper is shown by Theorem 4.2.

Theorem 4.2 *Shapley mapping σ^*, as defined in (10), is the unique solution mapping that satisfies EF, SYM, NP, and AD2.*

To prove Theorem 4.2, we first posit the following lemma.

Lemma 4.2 *Shapley mapping σ^* satisfies EF, SYM, NP, and AD2.*

Proof EF and SYM are obvious from the definition of the Shapley mapping. For NP, letting i be a null player in interval game w, yields:

$$\begin{aligned}\sigma_i^*(w)(t) &= \sum_{S:i\in S}\frac{(s-1)!(n-s)!}{n!}\{v_w^\alpha(S)-v_w^\alpha(S\setminus\{i\})\}\\ &= \sum_{S:i\in S}\frac{(s-1)!(n-s)!}{n!}\cdot 0\\ &= 0.\end{aligned}$$

For AD2, from (11), it follows that:

$$\begin{aligned}\sigma_i^*(w')(t')+\sigma_i^*(w'')(t'') &= (1-\alpha)\phi_i(v_{\underline{w'}})+\alpha\phi_i(v_{\overline{w'}})+(1-\alpha)\phi_i(v_{\underline{w''}})+\alpha\phi_i(v_{\overline{w''}})\\ &= (1-\alpha)\{\phi_i(v_{\underline{w'}})+\phi_i(v_{\underline{w''}})\}+\alpha\{\phi_i(v_{\overline{w'}})+\phi_i(v_{\overline{w''}})\}\\ &= (1-\alpha)\phi_i(v_{\underline{w'}}+v_{\underline{w''}})+\alpha\phi_i(v_{\overline{w'}}+v_{\overline{w''}})\\ &= (1-\alpha)\phi_i(v_{\underline{w}})+\alpha\phi_i(v_{\overline{w}})\\ &= \sigma_i^*(w)(t).\ \blacksquare\end{aligned}$$

Then, we show Lemmas 4.3–4.6.

Lemma 4.3 *For a coalition $R\in 2^N\setminus\{\emptyset\}$, we define a characteristic function form game $v_R\in CG$ as:*

$$v_R(S)=\begin{cases}1 & \text{if} \quad R\subset S\\ 0 & \text{otherwise.}\end{cases}$$

Then, for an interval game $w\in IG$, there uniquely exist $2(2^n-1)$ real numbers, denoted by $\underline{c_R}$ and $\overline{c_R}$ for $R\in 2^N\setminus\{\emptyset\}$ (note: the number of R is 2^n-1), which satisfy

$$v_{\underline{w}}=\sum_{R\subset N}\underline{c_R}v_R,\quad v_{\overline{w}}=\sum_{R\subset N}\overline{c_R}v_R,$$

and $\underline{c_R}$ and $\overline{c_R}$ are determined by:

$$\underline{c_R}=\sum_{T\subset R}(-1)^{r-t}v_{\underline{w}}(T),\quad \overline{c_R}=\sum_{T\subset R}(-1)^{r-t}v_{\overline{w}}(T),$$

where r and t denote the number of players in the coalition of R and T, respectively.

Proof The proof is essentially identical to Shapley's [28].

Lemma 4.4 *For a coalition $R\in 2^N\setminus\{\emptyset\}$ in interval game $w\in IG$, (15) holds:*

$$w+\sum_{R:\underline{c_R}>\overline{c_R}}[-\underline{c_R},\ -\overline{c_R}]v_R=\sum_{R:\underline{c_R}\le\overline{c_R}}[\underline{c_R},\ \overline{c_R}]v_R. \tag{15}$$

Proof For a coalition $R \in 2^N \setminus \{\emptyset\}$ satisfying $\underline{c_R} \leq \overline{c_R}$, $[\underline{c_R},\ \overline{c_R}]$ can be defined as a closed interval, and For a coalition $R \in 2^N \setminus \{\emptyset\}$ satisfying $\underline{c_R} > \overline{c_R}$, $[-\underline{c_R},\ -\overline{c_R}]$ can be defined as a closed interval. From Lemma 4.3, it holds that:

$$\begin{aligned} v_{\underline{w}} &= \sum_{R \subset N} \underline{c_R} v_R = \sum_{R:\underline{c_R} \leq \overline{c_R}} \underline{c_R} v_R + \sum_{R:\underline{c_R} > \overline{c_R}} \underline{c_R} v_R \\ v_{\overline{w}} &= \sum_{R \subset N} \overline{c_R} v_R = \sum_{R:\underline{c_R} \leq \overline{c_R}} \overline{c_R} v_R + \sum_{R:\underline{c_R} > \overline{c_R}} \overline{c_R} v_R. \end{aligned}$$

Therefore,

$$[v_{\underline{w}},\ v_{\overline{w}}] + \sum_{R:\underline{c_R} > \overline{c_R}} [-\underline{c_R},\ -\overline{c_R}] v_R = \sum_{R:\underline{c_R} \leq \overline{c_R}} [\underline{c_R},\ \overline{c_R}] v_R. \quad \blacksquare$$

Lemma 4.5 *Let σ be a solution mapping in $w \in IG$ satisfying EF, SYM, NP, and AD2. Then, for an interval game of $[\underline{c},\ \overline{c}]v_R \in IG$ ($\underline{c} \leq \overline{c}$) and a realization $t = (1-\alpha)\underline{c} + \alpha\overline{c}$:*

$$\sigma_i([\underline{c},\ \overline{c}]v_R)(t) = \begin{cases} \dfrac{t}{r} & \textit{if} \quad i \in R \\ 0 & \textit{otherwise} \end{cases}, \tag{16}$$

where r is the number of players in $R \in 2^N \setminus \{\emptyset\}$.

Proof First, assume $\underline{c} \geq 0$. If i is not in R, i is a null player in the interval game of $[\underline{c},\ \overline{c}]v_R \in IG$. Therefore, $\sigma_i([\underline{c},\ \overline{c}]v_R)(t) = 0$ from NP. Moreover, since i and j in R are symmetric in $[\underline{c},\ \overline{c}]v_R \in IG$, $\sigma_i([\underline{c},\ \overline{c}]v_R)(t) = \sigma_j([\underline{c},\ \overline{c}]v_R)(t)$ from SYM. Meanwhile, EF follows:

$$\sum_{i \in R} \sigma_i(cv_R)(t) = t.$$

Consequently, $\sigma_i([\underline{c},\ \overline{c}]v_R) = \frac{t}{r}$ holds for $i \in R$.

Next, assume $\underline{c} < 0$. Then, the following holds:

$$\begin{aligned} t - \underline{c} &= (1-\alpha) \cdot 0 + \alpha(\overline{c} - \underline{c}) \\ t &= (1-\alpha) \cdot \underline{c} + \alpha \cdot \overline{c} \\ -\underline{c} &= (1-\alpha) \cdot (-\underline{c}) + \alpha \cdot (-\underline{c}). \end{aligned}$$

Therefore, from AD2:

$$\begin{aligned} \sigma_i([0,\ \overline{c} - \underline{c}]v_R)(t - \underline{c}) &= \sigma_i([\underline{c},\ \overline{c}]v_R + [-\underline{c},\ -\underline{c}]v_R)(t + (-\underline{c})) \\ &= \sigma_i([\underline{c},\ \overline{c}]v_R)(t) + \sigma_i([-\underline{c},\ -\underline{c}]v_R)(-\underline{c}). \end{aligned}$$

Since:

$$\sigma_i([0,\ \overline{c}-\underline{c}]v_R)(t-\underline{c}) = \begin{cases} \dfrac{t-\underline{c}}{r} & \text{if} \quad i \in R \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma_i([-\underline{c},\ -\underline{c}]v_R)(-\underline{c}) = \begin{cases} \dfrac{-\underline{c}}{r} & \text{if} \quad i \in R \\ 0 & \text{otherwise,} \end{cases}$$

(16) also holds when $\underline{c} < 0$. ■

Lemma 4.6 *Assume $\phi_i(v_{\underline{w}})$ and $\phi_i(v_{\overline{w}})$ are i's Shapley values in characteristic function form games $v_{\underline{w}}$ and $v_{\overline{w}}$, respectively. Then, the following holds:*

$$\phi_i(v_{\underline{w}}) = \sum_{R:i\in R} \frac{\underline{c_R}}{r}, \quad \phi_i(v_{\overline{w}}) = \sum_{R:i\in R} \frac{\overline{c_R}}{r}$$

Proof The proof is essentially identical to Shapley's ([28]).

Finally, we prove Theorem 4.2 using Lemmas 4.3–4.6.

Proof of Theorem 4.2 From Lemma 4.2, it suffices to prove the uniqueness of σ^*, that is, letting σ be a solution mapping satisfying EF, SYM, NP, and AD2, $\sigma = \sigma^*$ must be proved.

For a realization of grand coalition $t = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N) \in w(N)$ in $w \in IG$, let $t_R = (1-\alpha)\underline{c_R} + \alpha\overline{c_R}$ be a realization of the grand coalition in $[\underline{c},\ \overline{c}]v_R \in IG$. Then, from Lemma 4.3, $v_{\underline{w}}(N) = \sum_{R\subset N} \underline{c_R}$ and $v_{\overline{w}}(N) = \sum_{R\subset N} \overline{c_R}$ hold. Therefore, $t = \sum_{R\subset N} t_R$ is true. From Lemma 4.4 and AD2, it follows that:

$$\sigma_i(w)(t) + \sum_{R:\underline{c_R}>\overline{c_R}} \sigma_i([-\underline{c_R},\ -\overline{c_R}]v_R)(-t_R) = \sum_{R:\underline{c_R}\leq\overline{c_R}} \sigma_i([\underline{c_R},\ \overline{c_R}]v_R)(t_R).$$

From Lemma 4.5:

$$\sigma_i(w)(t) + \sum_{R\ni i:\underline{c_R}>\overline{c_R}} \frac{-t_R}{r} = \sum_{R\ni i:\underline{c_R}\leq\overline{c_R}} \frac{t_R}{r},$$

$$\sigma_i(w)(t) = \sum_{R:i\in R} \frac{t_R}{r} = \sum_{R:i\in R} \left\{(1-\alpha)\cdot\frac{\underline{c_R}}{r} + \alpha\cdot\frac{\overline{c_R}}{r}\right\} = (1-\alpha)\sum_{R:i\in R}\frac{\underline{c_R}}{r} + \alpha\sum_{R:i\in R}\frac{\overline{c_R}}{r}$$

hold. From Lemma 4.6:

$$\sigma_i(w)(t) = (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}) = \sigma^*(w)(t). \ \blacksquare$$

5 Conclusions

This paper considers the cooperative interval games model. Top this end, we employ the notion of solution mappings as the solution concept to be applied to n-person interval games and show that there exists a unique solution mapping that satisfies the axioms of efficiency, symmetry, null player property, and an interval game version of additivity, and this solution mapping is the Shapley mapping.

We conclude the analysis by pointing out some topics for further research. First, it is worth providing other axiomatizations of the Shapley mapping. In particular, in the literature on coalition form games, the axiom of additivity is known as controversial because sum game $w' + w''$ may induce a behavior that can differ from the behavior induced by the two games, w' and w'' separetely. Therefore, other axiomatization using axioms such as Young's ([32]) strong monotonicity instead of the additivity would enhance the validity of the Shapley mapping. Second, we can use the solution concepts of the Banzhaf index and discounted Shapley value in the context of coalition form games, and redefine them as solution mappings rather than existing interval solution concepts. Those could be characterized by an axiomatic approach, as shown in this paper. Furthermore, non-singleton set solutions such as the core and stable sets may also be redefined as solution mappings in interval games. Third, once those solutions are defined as solution mappings, it is possible to examine the relationships between them. For example, whether a real valued payoff vector assigned by the Shapley mapping is included in the core mapping *for every realization of the worth set of the grand coalition* or which class of the interval games satisfies this property may be worth examining. Finally, solution mappings can be applied to actual cooperative game situations with uncertainty. For example, a classical game-theoretic analysis of the bankruptcy problem essentially entails the uncertainties that creditors face regarding debtor's future solvency. Here, new insights may be derived by reconstructing the bankruptcy problem as an interval game and applying solution mapping to those games. [12]

References

[1] S. Z. Alparslan Gök. 2010. Cooperative Interval Games: Theory and Applications. *Lambert Academic Publishing*.

[2] S. Z. Alparslan Gök. 2014. On the Interval Shapley Value. *Optimization* 63: 747–755.

[3] S. Z. Alparslan Gök, R. Branzei and S. Tijs. 2008. Cores and Stable Sets for Interval Games. *CentER Discussion Paper*: No. 2008–17.

[4] S. Z. Alparslan Gök, R. Branzei and S. Tijs. 2009. Convex Interval Games. *Journal of Applied Mathematics and Decision Sciences* 2009: 342089.

[12]Branzei et al. ([12]) applied interval games to the bankruptcy problem and focused on credit amount uncertainties. We consider debtor's fiscal condition as a source of uncertainty for debt contracts.

[5] S. Z. Alparslan Gök, R. Branzei and S. Tijs. 2009. Airport Interval Games and their Shapley Value. *Operational Research Decisions* 19: 9–18.

[6] S. Z. Alparslan Gök, R. Branzei and S. Tijs. 2010. The Interval Shapley Value: An Axiomatization. *Central European Journal of Operations Research* 18: 131–140.

[7] S. Z. Alparslan Gök, S. Miquel and S. Tijs. 2009. Cooperation under Interval Uncertainty. *Mathematical Methods of Operational Research* 69: 99–109.

[8] J. P. Aubin. 1974. Coeur et Valeur des Jeux Flous a Paiments Lateraux. *Comptes Rendus de l.Academie des Sciences Paris* 279(A): 891–894.

[9] R. J. Aumann and M. Maschler. 1964. The Bargaining Set for Cooperative Games. *Annals of Mathematics Studies* 52: 443–476.

[10] R. J. Aumann and B. Peleg. 1960. Von Neumann-Morgenstern Solutions to Cooperative Games without Side Payments. *Bulletin of the American Mathematical Society* 66: 173–179.

[11] R. Branzei, O. Branzei, S. Zeynep Alparslan Gök and S. Tijs. 2010. Cooperative Interval Games: A Survey. *Central European Journal of Operations Research* 18: 397–411.

[12] R. Branzei, D. Dimitrov and S. Tijs. 2003. Shapley-Like Values for Interval Bankruptcy Games. *Economic Bulletin* 3: 1–8.

[13] R. Branzei, S. Tijs and S. Z. Alparslan Gök. 2010. How to Handle Interval Solutions for Cooperative Interval Games. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 18 (2): 123–132.

[14] M. Davis and M. Maschler. 1965. The Kernel of a Cooperative Game. *Naval Research Logistics Quarterly* 12: 223–259.

[15] W. Fei, D.-F. Li and Y.-F. Ye. 2018. An Approach to Computing Interval-Valued Discounted Shapley Values for a Class of Cooperative Games under Interval Data *International Journal of General Systems* 47: 794–808.

[16] D. B. Gillies. 1959. Solutions to General Non-Zero-Sum Games, in *Contributions to the Theory of Games*, Volume IV: 47–85.

[17] H. Habis and P. J.-J. Herings. 2011. Transferable Utility Games with Uncertainty. *Journal of Economic Theory* 146: 2126–2139.

[18] W. Han, H. Sun and G. Xu. 2012. A New Approach of Cooperative Interval Games: The Interval Core and Shapley Value Revisited. *Operational Research Letters* 40: 462–468.

[19] S. Ishihara and J. Shino. 2020. A Solution Mapping and its Axiomatization in Two-Person Interval Games. *WIAS Discussion Paper Series* No. 2019-005.

[20] S. Ishihara, J. Shino and S. Yamauchi. 2020. Interval Games and its Solution Concepts: Some Observations and Projections. *Waseda Global Forum* No. 16.

[21] D.-F. Li, Y.-F. Ye and W. Fei. 2020. Extension of Generalized Solidarity Values to Interval-Valued Cooperative Games. *Journal of Industrial and Management Optimization* 16: 919–931.

[22] K. Liang and D. Li. 2019. A Direct Method of Interval Banzhaf Values of Interval Cooperative Games. *Journal of Systems Science and Systems Engineering* 28: 382–391.

[23] F. Meng, X. Chen and C. Tan. 2016. Cooperative Fuzzy Games with Interval Characteristic Functions. *International Journal of Operational Research* 16: 1–24.

[24] R. Moore. 1979. Methods and Applications of Interval Analysis. *Philadelphia: SIAM Studies in Applied Mathematics*.

[25] O. Palanci, S. Z. Alparslan Gök, M. O. Olgun and G.-W. Weber. 2016. Transportation Interval Situations and Related Games. *OR Spectrum* 38: 119–136.

[26] O. Palanci, S. Z. Alparslan Gök and G.-W. Weber. 2015. An Axiomatization of the Interval Shapley Value and on Some interval Solution Concepts. *Contributions to Game Theory and Management* VIII: 243–251.

[27] D. Schmeidler. 1969. The Nucleolus of a Characteristic Function Game. *SIAM Journal of Applied Mathematics* 17: 1163–1170.

[28] L. S. Shapley. 1953. A Value for n-Person Games. *Annals of Mathematics Studies* 28: 307–318.

[29] J. Suijs and P. Borm. 1999. Stochastic Cooperative Games: Superadditivity, Convexity, and Certainty Equivalents. *Games and Economic Behavior* 27: 331–345.

[30] J. Suijs, P. Borm, A. De Waegenaere and S. Tijs. 1999. Cooperative Games with Stochastic Payoffs. *European Journal of Operations Research* 113: 193–205.

[31] J. von Neumann and O. Morgenstern. 1953. *Theory of Games and Economic Behavior, 3rd ed.* Princeton University Press.

[32] H. P. Young. 1985. Monotonic Solutions of Cooperative Games. *International Journal of Game Theory* 14: 65–72.