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Toward a Resolution of the St.Petersburg Paradox

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Abstract

We study the St.Petersburg paradox from the viewpoint of bounded intelligence. Following Llyod Shapley, we reformulate its coin-tossing gamble introducing a finite budget of the banker, while this is as a resolution in the narrow sense as long as the standard expected reward criterion is adopted. It is still impossible for both banker and people to participate and to generate positive profits. We introduce cognitive bounds to people to modify the expected reward criterion and show that many people are incomparable to between participation and not. This is a rationalistic though people have cognitive bounds, and we take one more step of going to semi-rationalistic behavioral-probability for incomparable alternatives. This shows that some people show positive probabilities of participation in the coin-tossing with a fee producing positive profits for the banker. The last part is formulated as a monopoly market and its activeness is shown by the Mote Carlo simulation method.

Key Words: St.Petersburg Paradox, Shapley's Modification, Expected Utility Theory with Probability Grids, Cognitive Bounds, Bounded Intelligence, Incomparability, behavioral-probability, Monte Carlo Method

1 Introduction

The St.Petersburg (SP for short) paradox has been a long-standing conundrum since the time of Daniel Bernoulli [4]. The present author thinks that the paradox in the narrow sense is resolved by Shapley [24] introducing a budget constraint for the banker. However, this raises the difficulty that the banker would have no incentive to open the gamble market, i.e., having no positive profits from participants or no participants in the gamble market. Adopting Kaneko's [16] expected utility theory with probabilistic grids and cognitive bounds for people, we show that participation or not is incomparable for some people. A new approach to incomparability in terms of a behavioral-probability, along Luce's [18] idea of probabilistic preferences, implies that some people may incline to participate, dependent upon their cognitive bounds, and the banker has positive profits. We evaluate this, by a few criteria, as a resolution of the SP paradox. Here, we discuss what has been resolved for the SP paradox, what remains, and what should be required to obtain a resolution.

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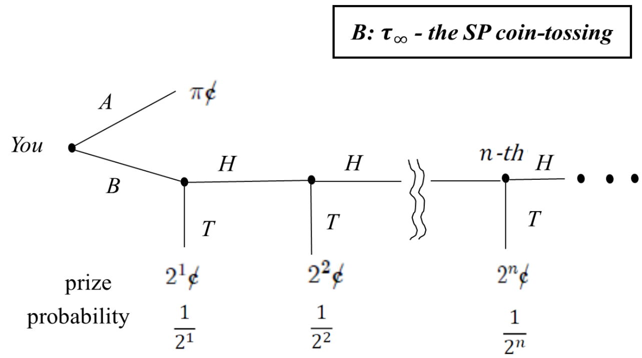

Figure 1: the SP gamble $g_\infty = [\pi, \tau_\infty]$

1.1 The SP gamble and a budget for the banker

The original SP coin-tossing gamble $g_\infty = [\pi, \tau_\infty]$ is as follows: *You* decide to participate or not in the coin-tossing τ_∞ with a fee π¢ ($\pi > 0$): if you participate, you toss a fair coin until it results in the tails, and if it shows the tails at the n-th coin-toss, you get prize 2^n¢ and the gamble is over. The probability of having prize 2^n¢ is $\frac{1}{2^n}$. The profit from participation is $2^n - \pi$, and the profit from non-participation is 0. For simplicity, we focus on the prizes and opportunity reward from non-participation, rather than profits, which is depicted as Fig.1. You choose A or B; π is the (opportunity) reward from A and the rewards from B are the prizes with their probabilities.

The expected reward from B is $\frac{1}{2} \cdot 2^1 + \frac{1}{2^2} \cdot 2^2 + \cdots + \frac{1}{2^n} \cdot 2^n + \cdots = 1 + 1 + \cdots + 1 + \cdots = +\infty$, and this is larger than any finite fee π. Thus, the *expected reward criterion* recommends you to participate in the gamble whatever π is. On the other hand, an ordinary and careful person would choose A unless π is very small. For example, when $\pi = 1000\$$, the probability of having a prize more than $1000\$ = 100,000$¢ is almost negligible, i.e., $1 - (\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{16}}) = \frac{1}{2^{16}} = \frac{1}{65,536}$, since $2^{16} < 100,000 < 2^{17}$; any reasonable person would perhaps choose A. We call this conflict the *St.Petersburg paradox in the narrow sense*, which we write explicitly:

(*): the recommendation by the expected reward criterion disagrees with the answer to be considered by an ordinary person for any large π.

This paradox has been discussed by many people in the literature (cf., Peterson [22] for a survey of discussions). As mentioned in Shapley [24] and [25], (*) is not a logical paradox; instead, it is an "empirical" paradox in a real context ([24], p.440) between the recommendation by the well accepted decision criterion and people's decisions by reflecting on their own minds.[1] Shapley [24] gave a clear-cut resolution of the paradox (*) by giving a budget constraint on the banker. The rules of a gamble in a real context should be concrete, and a (finite) budget is a naturally required component of a gamble.

[1] A salient paradox has societal.effects. "Three crises in mathematics" (cf., Fraenkel *et. al* [10], Chap.I, Section 5) indicates that a social circumstance matters for a paradox. For example, the second crisis happened in the $18 \sim 19$ th centuries about calculus by Newton and Libnitz; no rigorous definition of the concept of a "limit", which is the true basis of calculus, was found then. The cultural and economic success including industrial revolution of western countries in those centuries depended upon calculus, but its basis was shakey. A paradox is highly societal.

Shapley considered an example of a budget: B is bounded by the maximum prize 2^{100}¢, which is an astronomical number. The US federal annual budget for 2018 is about 4.2 trillion dollars, roughly 2^{49}¢. The expected reward from this coin-tossing is $\frac{1}{2} \cdot 2 + \cdots + \frac{1}{2^{47}} \cdot 2^{47} + \frac{1}{2^{48}} \cdot 2^{48} + \frac{1}{2^{49}} \cdot 2^{49} \cdot 2 = 50$¢, where in the last term, the prize is doubly counted since if the $49th$ toss is reached, the prize is 2^{49}¢ independent of the result. Thus, the expected reward is almost negligible to the budget 2^{49}¢. This is a strong example so as to eliminate the paradox (*), but it is not compatible with our ordinary understanding of socioeconomics, i.e., the rules of the gamble do not make sense in a real context.[2]

Shapley concluded the paper [24] with stating reluctancy with the SP paradox for game theory and economics even after the introduction of a budget ([24], p.442). The present author disagrees with his conclusion, and thinks that the SP paradox is a key for considerations of events with large benefits and simultaneously very large damages with very small risks. These may be features shared by a lot of real world events (e.g., "black swan" in Taleb [29]). Careful considerations of the SP paradox may give clues for formal studies of such real-world events as well as reflections upon economics/game theory.

The coin-tossing gamble τ_∞ includes the other type of infinity, i.e., infinitesimal small probabilities. Ignoring very small probabilities has almost the same effect as introducing a budget on prizes (cf., Peterson [22] and Smith [28]). Since, however, the latter is an objective constraint on a gamble and the former is a subjective constraint on the abilities of people, these play very different roles in our development.

We look at the questions we address:

(i) what is wrong *only* with an introduction of a budget while keeping the expected reward criterion unchanged?

(ii) in what sense is the situation a paradox?

For the question (i), we consider a budget compatible with our empirical world. We consider a few examples of budgets for a banker and participation fees for ordinary people; an example is

$$2^{\overline{n}} = 2^{21} = 2,097,152¢ \doteqdot 21,000\$ \quad \text{and} \quad \pi = 500¢ = 5\$. \tag{1}$$

The amount $2^{\overline{n}} \doteqdot 21,000\$$ is slightly smaller than the half of the average annual income, about 47 thousand\$, of the OECD 35 countries in 2018. This may be feasible for a trusted banker living in these countries. If $\overline{n}$ is small such as $\overline{n} = 7$, the budget $2^{\overline{n}}$ is surely feasible but the maximum prize, $2^7 = 128$¢, neither attracts people to participate in the gamble nor gives enough profits to open the market.

We meet a difficulty as long as we keep the expected reward criterion. The SP coin-tossing gamble with a budget is described as Fig.2, where $2^{\overline{n}}$¢ is the budget of the banker and the possible prizes from B is up to $2^{\overline{n}}$¢ and the probability distribution is given as

$$\tau_{\overline{n}}(2^t) = \begin{cases} \frac{1}{2^t} & \text{if } t \leq \overline{n} - 1 \\ \frac{1}{2^{\overline{n}-1}} & \text{if } t = \overline{n}. \end{cases} \tag{2}$$

The prizes are the same as those of the original gamble but at the $\overline{n}$-th toss the prize is $2^{\overline{n}}$¢ either in the heads or tails described in the dotted balloon in Fig.2. The expected reward from

[2] Aumann [2] stated that instead of having a finite budget, (*) could be naturally avoided by assuming boundedness on a utility function. However, the introduction of a budget is unavoidable as long as the rules of a gamble should be specific and meaningful in a real world setting. This does not "*exclude empirically unbelievable prospects from at least hypothetical consideraiton*" (Shapley [25], footnote 11, p.449). The point here is: such prospects should be excluded from the rules of the gamble in a real context.

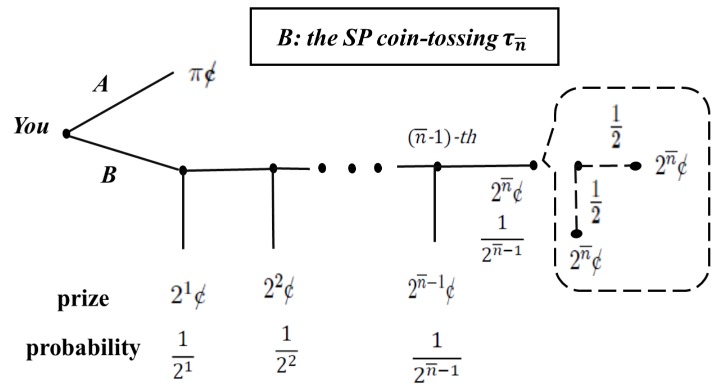

Figure 2: The SP gamble $g_{\overline{n}} = [\pi, \tau_{\overline{n}}]$ with a budget

B is $\frac{1}{2} \cdot 2^1 + \cdots + \frac{1}{2^{\overline{n}-1}} \cdot 2^{\overline{n}-1} + \frac{1}{2^{\overline{n}}} \cdot 2^{\overline{n}} \cdot 2 = \overline{n} + 1$. Also, in the numerical example of (1), the expected reward from B is $\overline{n}+1 = 22$¢ and is much smaller than $\pi = 500$¢, implying that people do not participate in the SP gamble.

For simplicity, we assume $\pi \neq \overline{n} + 1$ throughout the paper. The *expected reward criterion* recommends people to participate in the SP gamble $g_{\overline{n}} = [\pi, \tau_{\overline{n}}]$ if and only if $\pi < \overline{n} + 1$. This is also a condition for the banker not to open the market, because $\overline{n} + 1$ is the average expected expenditure and is greater than the average revenue π. The situation is summarized in Table 1.1; either no people participate in the gamble, or the banker does not open the market. This holds for any budget $2^{\overline{n}}$ and participation fee π with $\pi \neq \overline{n} + 1$. Thus, the SP gamble market is always vacuous.

Table 1.1; the SP market is vacuous

	$\pi < \overline{n} + 1$	$\pi > \overline{n} + 1$
People	participate	not
Banker	not	opens the SP market

A way out from the above vacuousness is to introduce "an intrinsic utility of a gamble" for people. A model is given by Fishburn [9], Schmidt [23], and an extensive argument on the model and related literature was given in Diecidue, *et.al* [6].[3] The literature looked for utility of gambling to explain possible choices of a gamble by modifying the classical expected utility theory. We do not deny the possibility that some people enjoy a gamble itself. Instead, we reflect upon foundations of economics/game theory and look for an explanation by giving a restriction on a free use of any probabilities in $[0, 1]$, which is a cognitive bound on people. Otherwise, we keep the underlying philosophy of expected utility theory.

Consider the above question (*ii*). A paradox includes an inconsistency between two views each of which is accepted in society and/or academia. This prevents us from having the two views at the same time; no theory including the two views is allowed. In formal logic, inconsistency of a theory means that some statement in the theory and its negation are provable; non-existence of

[3] Friedman-Savage [11] gave an argument of the utility of gambling and insurance by assuming "risk lover" for some domain and "risk-averse" for a different domain. A criticism is found in Markowitz [19]. See also Diecidue, *et. al* [6].

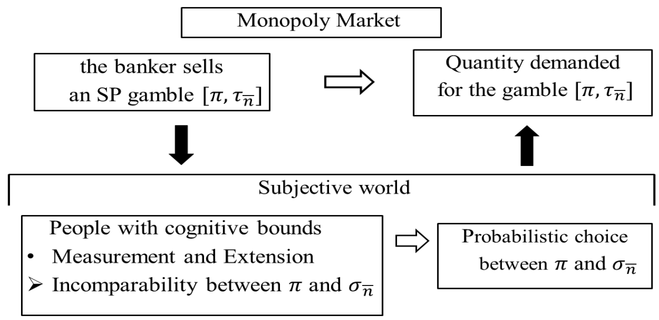

Figure 3: Monopoly market of an SP gamble

such a statement is consistency. The completeness theorem (cf., Mendelson [20], p.34 and p.72) states that consistency is equivalent to the existence of a model (i.e., specification of parameters) of the theory. A resolution of a paradox is given by a theory including the two views, together with specifications of parameters acceptable in society and/or academia.

In the case of the SP paradox, one view is our ordinary feeling about gambles, and the other is the expected reward criterion. Table 1.1 states that a general view obtained by introducing a budget avoiding the paradox (*) is vacuous. We still feel that some or many people would participate in some SP coin-tossing gamble and the banker is interested in selling it to people. After all, what we should do is to construct a theory together with some specifications of parameters compatible with these views.

1.2 A monopoly market and criteria for a resolution

We develop a monopoly market theory toward a resolution of the SP paradox in a broader sense than in that of (*). It consists of a banker and many people. We list three criteria for a theory as a resolution:

C1(non-vacuous): there is a coin-tossing gamble $\tau_{\overline{n}}$ such that it attracts some people and, at the same time, it produces positive profits;

C2(ordinary): elements of the theory are natural relative to our ordinary understanding of socio-economics, though some may deviate from the standard literature;

C3(sensitive): predictions from the theory are compatible with socio-economics for some parameter values but not for other values.

Criterion C1 is to avoid the vacuousness mentioned in Table 1.1. In C2, the term "natural" is understood in an informal manner; the above example with the maximum prize $2^{\overline{n}} = 2^7 = 128$¢ could be rejected by C2 as "uninteresting". This indicates that social and/or individual experiences are included in C2. Also, C2 means that it could be empirically or experimentally tested, but either is far beyond the scope of the paper. However, we cannot avoid some social and/or individual subjective elements, especially in C2. C3 requires the theory have a capability of selecting some parameter values and rejecting some others.

The monopoly market theory is depicted in Fig.3 and will be evaluated from the criteria C1, C2, and C3. The theory consists of the description of people's choice behavior between the gamble $[\tau_{\overline{n}}, \pi]$ and no-participation, and of the rationale of how the banker makes a decision.

People and the banker are asymmetric. First, people are many but the banker is single. Second, when $[\tau_{\overline{n}}, \pi]$ is given, each person evaluates $\tau_{\overline{n}}$ and π, but the banker's concern is the average profits from people's participation. Choice behavior of each person is based on a modification of the expected reward criterion with cognitive bounds, but the banker's rationale for opening the market is based on the average expected profits minus costs, but it has no effective cognitive bounds. Our concern is the existence of a $[\tau_{\overline{n}}, \pi]$ for the banker meeting the rationale. We do not assume the demand (function) for $[\tau_{\overline{n}}, \pi]$ is known to the banker.[4]

The banker's rationale is as follows; the banker borrows the show-money $2^{\overline{n}}$¢ and facility costs from a financial institution, and after the market, the banker should return this amount with some interests. We adopt an index *ROI* (*return of investment*) to express this capacity of paying the additional interests, though the costs may include more. For example, when the banker borrows that amount with the interest rate $r\%$, the index *ROI* should be larger than $r\%$ to judge $[\tau_{\overline{n}}, \pi]$ to be profitable and to open the market. We will see that this index is reasonably positive for some parameter values.

The key is people's participation, which relies upon people's bounded understanding of gamble $[\tau_{\overline{n}}, \pi]$; we develop a partial version of the expected utility theory with cognitive bounds in Kaneko [16] and its probabilistic extension. In this development, we adopt the concept of *bounded intelligence*, meaning a person's finitely bounded ability of logical and numerical calculations. This is a part of the general idea of bounded rationality, due to Simon [27]. To emphasize the nature of our approach, we use the term of bounded intelligence rather than bounded rationality.[5,6]

Our descriptions of decision making and choice behavior for a person take three steps; the first two constitute *rationalistic* decision-making, and the third step is a *semi-rationalistic* choice. The first two steps are: *Step M*: measurement of utilities from relevant pure alternatives; and *Step E*: extension of these measured utilities to the SP lottery. In Step M, person i measures utility by digging his mind, and in Step E, he extend the utilities measured to the SP lottery.

However, person i has a cognitive bound, expressed as $\frac{1}{2^{\rho_i}}$, which is the smallest unit of probability for perceivable probabilities. His understanding of the announced $\tau_{\overline{n}}$ is truncated as the distribution σ_{ρ_i}, depicted in Fig.4. Still, it follows the interpretation of sequential coin-tossing; before the coin-tossing, he thinks about the possible results from the first toss, second toss, up to the ρ_i-th toss. On the other hand, the maximum prize $2^{\overline{n}}$ included in the announced gamble $\tau_{\overline{n}}$ is assumed to remain in his mind; he keeps the entire picture of Fig.4 including the show-money.

In Step M, person i uses a *measurement scale* of utility from a pure alternative, adopting the maximum prize $2^{\overline{n}}$ as the *upper reference point* and 0 as the *lower reference point* 0, and the probability grids within cognitive bounds. The scale has the base unit $\frac{1}{2^{\rho_i}}$ so that the total

[4] In the standard micro-economics, the monopolist (banker in our case) knows the demand function from consumers (cf., Heyek [13], Chap.V, Section 2), which is a possible continuation for the behavior of the banker. However, Simon's [26] argument of "satisficing with an aspiration" may be a more direct continuation.

[5] Simon [27] divided the notion of rationality into substrantive and procedural; the former is a property of a realized choice such as a "rational outcome" and the latter is an attribute of a performance of a system (person). Logical inference ability of a person is regarded as included in the latter. However, the term "bounded rationality" is already used in many ways, and we would like to have clear-cut notion. So, we use the new term "bounded intelligence" to express bounded ability of logical inference and numerical calculation.

[6] Behavioral economics explains experimentally or empirically observed anomalies by introducing new variables or parameters, but keeps the substantive "rationality" in the classical micro-economics sense. See Berg-Gigerenzer [3] for discussions on this issue.

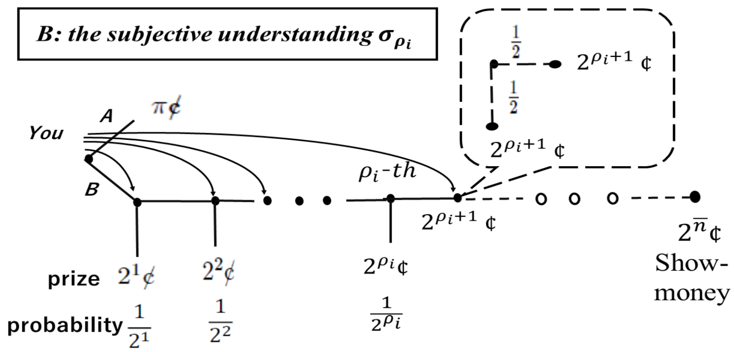

Figure 4: Person i's subjective understanding

length $2^{\overline{n}}$ is divided into into 2^{ρ_i} number of base units. When ρ_i is small, the scale is imprecise, and when ρ_i gets larger, the scale is more precise. In an analogue to the measurement of body temperature, when a thermometer is rough, it tells the body temperature is, say, 36°C, 37°C, or between them, and when it is finer, it measures more accurately though incomparability may remain. To express this incomparability, we adopt a 2-dimensional vector-valued utility function (cf., Fishburn [8]).

It will be shown in Sections 3 and 4 that participation and not are incomparable in many cases, which may be interpreted as "impossibility" to have a resolution of the SP paradox; the above *rationalistic* (or scientific) steps are subject to a difficulty in making decisions. Reflecting upon, however, our own behavior, we find many occasions where rationalistic thinking leads only to incomparability while we are still forced to make a choice by social custom, education etc. In this paper, we take one more step to have semi-rationalistic choice: *Step S*: *semi-rationalistic* probability-choice of participation. We call it *semi-rationalistic*, since it is still based, partially, on results and methods in Steps M and E. This fills incomparability with probabilistic behavior; the basic idea is Luce's [18] theory of probabilistic preferences, while it is entirely reformulated in our context for our incomparable alternatives.[7,8] This shows that people participate in the SP coin-tossing gamble with some positive probability.[9]

In Section 6, we synthesize the external and internal behaviors of the banker and people in a monopoly market theory, depicted in Fig.3, which assumes a distribution of people's cognitive degrees and the return of investment index *ROI* for the banker. We evaluate our entire theory in terms of criteria C1, C2, and C3 with specifications of parameters. A methodological remark is that we adopt the Monte Carlo Simulation method to study the outcomes of the SP coin-tossing for participants, since the stochastic process expressing coin-tossing is not stationary including some bankruptcy possibility for the banker. As a result, however, bankruptcy is observed to be not very important.

This paper is organized as follows: Sections 3 and 4 formulates Steps M and E for measurement of pure alternatives and its extension to a subjective understanding σ_{ρ_i}. Section 5 gives an axiomatic approach to behavioral-probability to incomparability. Section 6 provides a

[7] See Echenique-Saito [7] for a general treatment of Luce's approach and Pleskac [21] for related issues.

[8] Loomes-Sugden [17] discussed the source of probability as an error in probabilistic preferences.

[9] This argument deriving the choice-probability was used in Kaneko [16] for a consideration of Allais's [1] paradox referring to an experimental result by Kahneman-Tversky [15].

formulation of a monopoly market with the banker and people. We analyze the market behavior in terms of Monte Carlo simulation method. We give various concluding remarks in Section 7.

2 Preliminary

Person i's *cognitive degree* ρ_i, a nonnegative integer, defines his cognitive bound $\frac{1}{2^{\rho_i}}$, meaning that he can think about probabilities expressed only as $\frac{\nu}{2^{\rho_i}}$ $(0 \leq \nu \leq 2^{\rho_i})$. When he faces the coin-tossing described by Fig.2, he thinks about the $1st$ toss, $2nd$ toss, ..., and goes to the last toss ρ_i-th in his mind as in Fig.4, which are all within the cognitive bound $\frac{1}{2^{\rho_i}}$. This is a thought experiment conducted in the mind of person i before the actual tossing. His thought is from one degree k to the next $k+1$, where k expresses a cognitive degree within ρ_i.

We define the set of *probability grids*: for each $k = 0, ..., \rho_i$,

$$\Pi_k = \{\tfrac{\nu}{2^k} : 0 \leq \nu \leq 2^k\}. \tag{3}$$

Then, $\Pi_0 = \{0, 1\} \subsetneq \Pi_1 \subsetneq \cdots \subsetneq \Pi_{\rho_i}$. We define the *depth* $\delta(\alpha) = t$ of $\alpha \in \Pi_k$ iff $\alpha \in \Pi_t - \Pi_{t-1}$. It holds that $\delta(\alpha) = t$ if and only if α is expressed as $\frac{\nu}{2^t}$ for some odd ν. Thus, each $\lambda \in \Pi_k$ has at most depth k. The depth $\delta(\cdot)$ will play a crucial role in Section 4.

The cognitive degree ρ_i gives a subjective constraint on i's thought, while the budget degree $\overline{n}$ is an objective constraint on the prizes in the SP gamble. This difference creates a small shift in their effective roles. When $\overline{n} - 1 \geq \rho_i$, the cognitive bound $\frac{1}{2^{\rho_i}}$ permits prizes $2^1, ..., 2^{\rho_i}, 2^{\rho_i+1}$ where the last two have probability $\frac{1}{2^{\rho_i}}$; on the other hand, the budget bound $2^{\overline{n}}$ induces probabilities $\frac{1}{2^1}, ..., \frac{1}{2^{\overline{n}-1}}, \frac{1}{2^{\overline{n}-1}}$ where the last is the probability of the maximum prize $2^{\overline{n}}$ when he reaches the $\overline{n}$-toss. Taking this shift, we let $\widehat{\rho}_i = \min(\rho_i, \overline{n} - 1)$ in order to cover the case of $\overline{n} \leq \rho_i$.

Let $X^*_{\overline{n}} = \{0, 1, ..., 2^{\overline{n}}\}$ be the set of possible monetary payments to person i. When person i faces a choice between A and B in Fig.2, A is represented by π and B by person i's *subjective understanding* σ_{ρ_i}, of the objective probability distribution $\tau_{\overline{n}}$, as of truncated in Fig.4. The set of pure alternatives relevant to the choice between A and B is given as

$$X_{\rho_i} = \{\pi\} \cup \{2^1, 2^2, \cdots, 2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}. \tag{4}$$

Person i's subjective understanding σ_{ρ_i} is defined over the support $\{2^1, 2^2, ..., 2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}$:

$$\sigma_{\rho_i}(2^t) = \begin{cases} \frac{1}{2^t} & \text{if } t \leq \widehat{\rho}_i \\ \\ \frac{1}{2^{\widehat{\rho}_i}} & \text{if } t = \widehat{\rho}_i + 1. \end{cases} \tag{5}$$

When $\rho_i \geq \overline{n}-1$, σ_{ρ_i} is the same as $\tau_{\overline{n}}$, and when $\rho_i \leq \overline{n}-1$, σ_{ρ_i} has the support $\{2^1, 2^2, ..., 2^{\rho_i}, 2^{\rho_i+1}\}$. Person i is to choose one from π and σ_{ρ_i}.

In the step of measurement, each pure alternative is compared with a scale. Here, we adopt the scale $B_k(\overline{x}; \underline{x})$ consisting of the *upper reference point* $\overline{x}$, the *lower reference point* $\underline{x}$, and probability grids $\frac{\nu}{2^k}$ $(0 \leq \nu \leq 2^k)$. More precisely, we assume the maximum prize $2^{\overline{n}}$ for $\overline{x}$ and the zero 0 for $\underline{x}$. The *scale* $B_k(\overline{x}; \underline{x})$ of depth (precision) k is given as the set:

$$B_k(\overline{x}; \underline{x}) = \{[\overline{x}, \alpha; \underline{x}] : \alpha \in \Pi_k\} \tag{6}$$

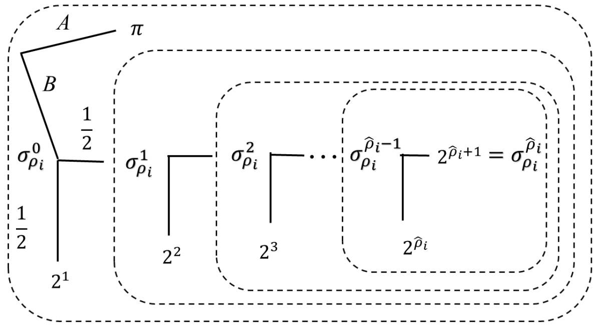

Figure 5: Thought process into the inner lotteries

for $k = 0, 1, ..., \rho_i$. Expression $[\overline{x}, \alpha; \underline{x}]$ means that the upper reference $\overline{x}$ and lower reference $\underline{x}$ happen with probabilities α and $1-\alpha$, which is a *scale lottery*. Then, the scale $B_k(\overline{x}; \underline{x})$ consists of scale lotteries. The scale $B_0(\overline{x}; \underline{x}) = \{[\overline{x}, 1; \underline{x}], [\overline{x}, 0; \underline{x}]\}$ consists of only two lotteries $[\overline{x}, 1; \underline{x}] = \overline{x}$ and $[\overline{x}, 0; \underline{x}] = \underline{x}$, and compares $x \in X^*_{\overline{n}}$ with $[\overline{x}, 1; \underline{x}]$ and $[\overline{x}, 0; \underline{x}]$. The scale $B_1(\overline{x}; \underline{x}) = \{[\overline{x}, 1; \underline{x}], [\overline{x}, \frac{1}{2}; \underline{x}], [\overline{x}, 0; \underline{x}]\}$ contains the middle point $[\overline{x}, \frac{1}{2}; \underline{x}]$. As k increases, the scale $B_k(\overline{x}; \underline{x})$ is getting more precise, up to the most precise scale $B_{\rho_i}(\overline{x}; \underline{x})$ for person i.

The two steps stated in Section 1.1 are more precisely described;

Step M: for each $k = 0, ..., \widehat{\rho}_i$, he measures each $x \in X^*_{\overline{n}}$ by the scale $B_k(\overline{x}; \underline{x})$;

Step E: he extends the measured utilities in Step M to derive his subjective lottery σ_{ρ_i}.

In Step M, person i reflects, using the scale, upon his mind to find a closest lottery in $B_k(\overline{x}; \underline{x})$ to pure alternative $x \in X^*_{\overline{n}}$. In Step E, person i extends these measurements and derives the utility value of σ_{ρ_i}. Thus, he makes a comparison between π and σ_{ρ_i}.

Our description of Step M is richer than Step E in that Step M takes the pure alternatives in $X^*_{\overline{n}}$, while Step E include only the pure alternatives in the description in π and $\tau_{\overline{n}}$. We take this different restrictions so as to have a clear-cut description of Step M and a clear-cut extension. The restriction of Step M to X_{ρ_i} is more faithful to the viewpoint of bounded intelligence.

Fig.5 illustrates a thought process from the outermost $\sigma^0_{\rho_i} = \sigma_{\rho_i}$ into the innermost (rightmost) probability distribution $\sigma^{\widehat{\rho}_i-1}_{\rho_i}$ over $2^{\widehat{\rho}_i}$ and $2^{\widehat{\rho}_i+1}$ with probability $\frac{1}{2}$ each. It is a decomposition process; we describe this process in a faithful manner in that no other elements than those in $\sigma^0_{\rho_i}$ are included. Thus, Step E forms a very partial theory. We can restrict Step M in the same manner, but provide it in a global manner to have a clear-cut conceptual picture. See Remarks 2.1 and 4.1

Steps M and E are connected by Axiom ME in Section 4.2. Fig.6 depicts the relationship between Steps M and E. These two steps meet incomparability. Section 5 analyzes behavioral-probability for the alternatives σ_{ρ_i} and π in the incomparable case.

Kaneko [16] started with preference relations both in Steps M and E, and then derived 2-dimensional vector-utility functions associated with the interval order (cf., Fishburn [8]). In this paper, we adopt vector-utility functions associated with the interval order in Steps M and E.

Let $\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ be the Cartesian product of the set of rationals $\mathbb{Q}$. For two vectors $\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda})$

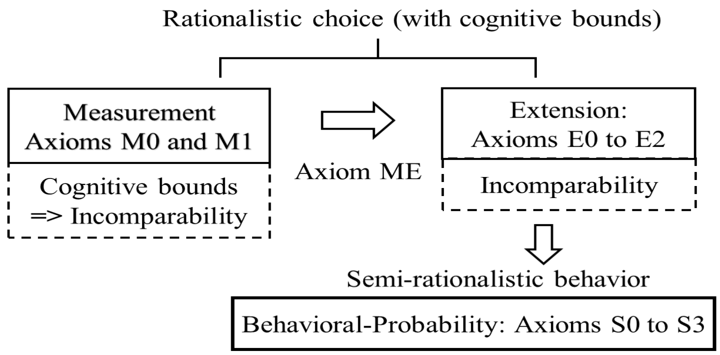

Figure 6: Connections between the axiomatic systems

and $\boldsymbol{\mu} = (\overline{\mu}, \underline{\mu})$ in $\mathbb{Q}^2$ with $\overline{\lambda} \geq \underline{\lambda}$ and $\overline{\mu} \geq \underline{\mu}$, we define the *interval order* $\geq_I$ over $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by

$$\boldsymbol{\lambda} \geq_I \boldsymbol{\mu} \text{ if and only if } \underline{\lambda} \geq \overline{\mu}. \tag{7}$$

The first component $\overline{\lambda}$ of $\boldsymbol{\lambda} = (\overline{\lambda}, \underline{\lambda})$ is the least upper bound of possible utility values, and the second $\underline{\lambda}$ is the greatest lower bound. The comparison $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ means that any upper possible utility value is larger than or equal to any lower possible utility value. This comparison may not be complete. There are three cases:

(a) *strict preference*: $\boldsymbol{\lambda} >_I \boldsymbol{\mu}$ if and only if $\underline{\lambda} > \overline{\mu}$;

(b) *equality*: $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ and $\boldsymbol{\mu} \geq_I \boldsymbol{\lambda}$; in this case, $\overline{\lambda} = \underline{\lambda} = \overline{\mu} = \underline{\mu}$.

(c) *incomparability*: neither $\boldsymbol{\lambda} \geq_I \boldsymbol{\mu}$ nor $\boldsymbol{\mu} \geq_I \boldsymbol{\lambda}$, which is denoted by $\boldsymbol{\lambda} \bowtie_I \boldsymbol{\mu}$.

Incomparability is crucial in our theory.

In Sections 3 and 4, it will be shown that the fee π and subjective understanding σ_{ρ_i} of $\tau_{\overline{n}}$ are incomparable in many cases of cognitive degrees ρ_i, i.e., $\boldsymbol{\lambda} \bowtie_{\rho_i} \boldsymbol{\mu}$ where $\boldsymbol{\lambda} = \boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{\mu} = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$. As stated in Section 1, when people are in incomparability, they are forced to make choices by society. In fact, incomparability is not uniform among people, but some cases show high incomparability and other cases are close to comparability. We extend such non-uniform incomparability in a quantitative manner, which is compatible with our theory since $\boldsymbol{u}_{\rho_i}(\cdot)$ is a 2-dimensional vector function showing often incomparability but the vector representation is still determined up to a positive transformation. This is:

Step S: person i extends incomparability between $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to a probabilistic choice between them.

This step will be given in Section 5. We will calculate the choice probability of σ_{ρ_i} and π in examples. Fig.6 illustrates the three steps from M, E, and S.

Remark 2.1 (Restrictions from the viewpoint of bounded intelligence): Our theory is based on Kaneko's [16] expected utility theory with probability grids, which discusses a derivation of a preference relation over the sets of all lotteries of given degrees. The present paper applies this theory to a comparison between π and σ_{ρ_i}. However, this application needs a lot of specific details and also faces new issues which do not appear in [16]. One issue is a locality of the present theory; although Kaneko's [16] theory was already quite restricted and constructive, it is still a global theory in that a preference relation is considered over the set of all pairs of alternatives. Here, an ultimate comparison is between π and σ_{ρ_i}. This may be regarded as a severe restriction from the standard point of view, but it is reversed from the viewpoint of bounded intelligence. Only small number of comparisons are less restrictive than a global theory.

3 Step M: Measurement of pure alternatives

Let k be a depth with $0 \leq k \leq \rho_i$. A *base utility function* $\boldsymbol{\upsilon}_k$ is a function over $B_k(\overline{x};\underline{x}) \cup X^*_{\frac{*}{n}}$ having a 2-dimensional vector-valued $\boldsymbol{\upsilon}_k(f) = [\overline{\upsilon}_k(f);\underline{\upsilon}_k(f)]$ in $\mathbb{Q}^2$ with $\overline{\upsilon}_k(f) \geq \underline{\upsilon}_k(f)$ for each $f \in B_k(\overline{x};\underline{x}) \cup X^*_{\frac{*}{n}}$. As stated above, the components $\overline{\upsilon}_k(f)$ and $\underline{\upsilon}_k(f)$ are interpreted as the least upper and greatest lower bounds of possible utilities from f. When $\overline{\upsilon}_k(f) = \underline{\upsilon}_k(f)$, person i finds a precise measurement value from f; in this case, we say that $\upsilon_k(f)$ is *singular*. When $\overline{\upsilon}_k(f) > \underline{\upsilon}_k(f)$, he reaches only imprecise measurements; in this case, $\boldsymbol{\upsilon}_k(f)$ is called *non-singular*. Two vector-utility values $\boldsymbol{\upsilon}_k(f)$ and $\boldsymbol{\upsilon}_k(g)$ are compared by the interval order $\geq_I$. We assume two axioms on $\boldsymbol{\upsilon}_k$; the first axiom states that $\boldsymbol{\upsilon}_k$ includes a scale: for $k = 0, ..., \rho_i$.

Axiom M0(Determination of a scale):(1)(Upper and lower references):

$$\overline{\upsilon}_k([\overline{x},1;\underline{x}]) = \underline{\upsilon}_k([\overline{x},1;\underline{x}]) > \overline{\upsilon}_k([\overline{x},0;\underline{x}]) = \underline{\upsilon}_k([\overline{x},0;\underline{x}]). \tag{8}$$

(2)(Expected utility for scale lotteries): for all $[\overline{x},\alpha;\underline{x}] \in B_k(\overline{x};\underline{x})$,

$$\boldsymbol{\upsilon}_k([\overline{x},\alpha;\underline{x}]) = \alpha\boldsymbol{\upsilon}_k([\overline{x},1;\underline{x}]) + (1-\alpha)\boldsymbol{\upsilon}_k([\overline{x},0;\underline{x}]). \tag{9}$$

Axiom M0(1) means that $\boldsymbol{\upsilon}_k$ adopts the two reference points $[\overline{x},1;\underline{x}] = \overline{x} = 2^n$ and $[\overline{x},0;\underline{x}] = \underline{x} = 0$, and states that $\overline{x}$ is strictly better than $\underline{x}$. Also, it requires $\boldsymbol{\upsilon}_k([\overline{x},1;\underline{x}])$ and $\boldsymbol{\upsilon}_k([\overline{x},0;\underline{x}])$ be singular. M0(2) is regarded as the *expected utility property* over the scale lotteries; the value $\boldsymbol{\upsilon}_k([\overline{x},\alpha;\underline{x}])$ is expressed as a convex combination of the values $\boldsymbol{\upsilon}_k([\overline{x},1;\underline{x}])$ and $\boldsymbol{\upsilon}_k([\overline{x},0;\underline{x}])$ of two reference points $\overline{x} = [\overline{x},1;\underline{x}]$ and $\underline{x} = [\overline{x},0;\underline{x}]$ with weights α and $1-\alpha$. The value $\boldsymbol{\upsilon}_k([\overline{x},\alpha;\underline{x}])$ is singular, too. Since the set of values $\{\boldsymbol{\upsilon}_k([\overline{x},\alpha;\underline{x}]) : [\overline{x},\alpha;\underline{x}] \in B_k(\overline{x};\underline{x})\}$ with $\geq_I$ is isomorphic to the set scale Π_k, as well as $B_k(\overline{x};\underline{x})$, with the natural order on Π_k, the function $\boldsymbol{\upsilon}_k$ is a representation of Π_k and $B_k(\overline{x};\underline{x})$.

The next axiom, M1, describes how a pure alternative $x \in X^*_{\frac{*}{n}}$ is measured.

Axiom M1 (Measurement by the smallest unit in a scale): For any $k = 0, 1, ..., \rho_i$,

(1): $\boldsymbol{\upsilon}_k(x) \sqsupseteq \boldsymbol{\upsilon}_{k+1}(x)$, i.e., $\overline{\upsilon}_k(x) \geq \overline{\upsilon}_{k+1}(x)$ and $\underline{\upsilon}_k(x) \leq \underline{\upsilon}_{k+1}(x)$;

(2): $\boldsymbol{\upsilon}_k(x) = [\overline{\upsilon}_k(x);\underline{\upsilon}_k(x)]$ is expressed as either $[\overline{\upsilon}_k([\overline{x},\frac{\nu+1}{2^k};\underline{x}]);\underline{\upsilon}_k([\overline{x},\frac{\nu}{2^k};\underline{x}])]$ or $\boldsymbol{\upsilon}_k([\overline{x},\frac{\nu}{2^k};\underline{x}])$ for some ν $(0 \leq \nu < 2^k)$

Axiom M1(1) states that the measurement of utility is more accurate as k is larger; the strict case (either $\overline{\upsilon}_k(x) > \overline{\upsilon}_{k+1}(x)$ or $\underline{\upsilon}_k(x) < \underline{\upsilon}_{k+1}(x)$) is denoted by $\boldsymbol{\upsilon}_k(x) \sqsupset \boldsymbol{\upsilon}_{k+1}(x)$. Moreover, Axiom M1(2) means that the measurement of x is characterized by the unit interval in the scale $B_k(\overline{x};\underline{x})$ containing x, or the measurement is done exactly. In the former case, it is represented by the two values of scale lotteries $\overline{\upsilon}_k([\overline{x},\frac{\nu+1}{2^k};\underline{x}]) = \underline{\upsilon}_k([\overline{x},\frac{\nu+1}{2^k};\underline{x}])$ and $\overline{\upsilon}_k([\overline{x},\frac{\nu}{2^k};\underline{x}]) = \underline{\upsilon}_k([\overline{x},\frac{\nu}{2^k};\underline{x}])$.

Preciseness of the utility value $\boldsymbol{\upsilon}_k(x)$ is increasing with respect to depth k.

Lemma 3.1 (Accuracy of measurement). Assume Axioms M0 and M1. Then, for all k,

(1): $\boldsymbol{\upsilon}_{k-1}(x) \sqsupset \boldsymbol{\upsilon}_k(x)$ or $\boldsymbol{\upsilon}_{k-1}(x) = \boldsymbol{\upsilon}_k(x)$ is singular;

(2): if $\boldsymbol{\upsilon}_{k-1}(x) \sqsupset \boldsymbol{\upsilon}_k(x)$, then $\boldsymbol{\upsilon}_{k-2}(x) \sqsupset \boldsymbol{\upsilon}_{k-1}(x)$, and if $\boldsymbol{\upsilon}_{k-1}(x) = \boldsymbol{\upsilon}_k(x)$, then $\boldsymbol{\upsilon}_k(x) =$

$\boldsymbol{\upsilon}_{k+1}(x)$.

Proof.(1): Suppose $\boldsymbol{\upsilon}_{k-1}(x) = [\overline{\upsilon}_{k-1}([\overline{x}, \frac{\nu}{2^{k-1}}; \underline{x}]); \underline{\upsilon}_{k-1}([\overline{x}, \frac{\nu}{2^{k-1}}; \underline{x}])]$. By Axiom M1(2), $\boldsymbol{\upsilon}_{k-1}(x) = \boldsymbol{\upsilon}_k(x)$ and they are singular. Next, suppose $\boldsymbol{\upsilon}_{k-1}(x) = [\overline{\upsilon}_{k-1}([\overline{x}, \frac{\nu+1}{2^{k-1}}; \underline{x}]);\ \underline{\upsilon}_{k-1}([\overline{x}, \frac{\nu}{2^{k-1}}; \underline{x}])]$. Then, $\overline{\upsilon}_{k-1}(x) > \underline{\upsilon}_{k-1}(x)$ by Axiom M0. In this case, the interval between $[\overline{x}, \frac{\nu+1}{2^{k-1}}; \underline{x}]$ and $[\overline{x}, \frac{\nu}{2^{k-1}}; \underline{x}]$ is divided into two intervals from $[\overline{x}, \frac{2\nu+2}{2^k}; \underline{x}] = [\overline{x}, \frac{\nu+1}{2^{k-1}}; \underline{x}]$ to $[\overline{x}, \frac{2\nu+1}{2^k}; \underline{x}]$, and from $[\overline{x}, \frac{2\nu+1}{2^k}; \underline{x}]$ to $[\overline{x}, \frac{2\nu}{2^k}; \underline{x}] = [\overline{x}, \frac{\nu}{2^{k-1}}; \underline{x}]$. By Axiom M1, $\boldsymbol{\upsilon}_k(x)$ has the interval of length $\frac{1}{2^k}$ or 0. In either case, $\boldsymbol{\upsilon}_{k-1}(x) \sqsupset \boldsymbol{\upsilon}_k(x)$.

(2): The first claim follows from (1) and Axiom M1(2). Taking the contrapositive of this, we obtain the second claim by (1).■

Axiom M1 describes a process for person i to find the utility values $\boldsymbol{\upsilon}_0(x), \boldsymbol{\upsilon}_1(x), ..., \boldsymbol{\upsilon}_{\rho_i}(x)$ by digging cognitive layers in his mind. Here, the base assumption is the existence of $\boldsymbol{\upsilon}_0(x), ..., \boldsymbol{\upsilon}_{\rho_i}(x)$ for a given x hidden in his mind. Under Axiom M1, this existence implies a utility function $\theta_{\overline{n}} : X^*_{\overline{n}} \to \mathbb{Q}_{[0,1]}$, which we call a *latent utility function*. This theorem is stated from the viewpoint of the outside analyzer, and it gives a great convenience.

Theorem 3.1 (Latent utility function). Let $\{\boldsymbol{\upsilon}_k\}_{k=0}^{\rho_i}$ be a given sequence of base utility functions with Axiom M0. Then, $\{\boldsymbol{\upsilon}_k\}_{k=0}^{\rho_i}$ satisfies Axiom M1 for all $x \in X^*_{\overline{n}}$ if and only if there is a function $\theta_{\overline{n}} : X^*_{\overline{n}} \to \mathbb{Q}_{[0,1]}$ such that for any $k = 0, ..., \rho_i$,

(a): if $\frac{\nu+1}{2^k} > \theta_{\overline{n}}(x) > \frac{\nu}{2^k}$, then $\boldsymbol{\upsilon}_k(x) = [\overline{\upsilon}_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]); \underline{\upsilon}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])]$;

(b): if $\theta_{\overline{n}}(x) = \frac{\nu}{2^k}$, then $\boldsymbol{\upsilon}_k(x) = \boldsymbol{\upsilon}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$.

Proof.(*If*): Axiom M1(2) follows directly from (a) and (b). Noting that $\theta_{\overline{n}}(x)$ is independent of k, M1(1) follows also from (a) and (b).

(*Only-if*): We show the existence of a $\theta_{\overline{n}} : X^*_{\overline{n}} \to \mathbb{Q}_{[0,1]}$ with (a) and (b). Let $x \in X^*_{\overline{n}}$. Let $\theta_{\overline{n}}(x) = (\overline{\upsilon}_{\rho_i}(x) + \underline{\upsilon}_{\rho_i}(x))/2$. When $\overline{\upsilon}_{\rho_i}(x) = \underline{\upsilon}_{\rho_i}(x)$, $\theta_{\overline{n}}(x)$ belongs to $\Pi_{\rho_i} \subseteq \Pi_{\rho_i+1}$. When $\boldsymbol{\upsilon}_{\rho_i}(x)$ is non-singular, by Axiom M1, $\boldsymbol{\upsilon}_{\rho_i}(x) = [\overline{\upsilon}_k([\overline{x}, \frac{\nu+1}{2^{\rho_i}}; \underline{x}]); \underline{\upsilon}_k([\overline{x}, \frac{\nu}{2^{\rho_i}}; \underline{x}])]$ for some ν. Hence, $\theta_{\overline{n}}(x) = (\overline{\upsilon}_{\rho_i}(x) + \underline{\upsilon}_{\rho_i}(x))/2 = \frac{2\nu+1}{2^{\rho_i+1}}$. Since $\overline{\upsilon}_k(x)$ is weakly decreasing and $\underline{\upsilon}_k(x)$ is weakly increasing with $\overline{\upsilon}_k(x) \geq \underline{\upsilon}_k(x)$ for all $k \leq \rho_i$, we have $\overline{\upsilon}_k(x) \geq \theta_{\overline{n}}(x) \geq \underline{\upsilon}_k(x)$ for all $k \leq \rho_i$. If $\frac{\nu+1}{2^k} > \theta_{\overline{n}}(x) > \frac{\nu}{2^k}$, then, by Axiom M1.(1), we have $\boldsymbol{\upsilon}_k(x) = [\overline{\upsilon}_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]); \underline{\upsilon}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])]$, and if $\theta_{\overline{n}}(x) = \frac{\nu}{2^k}$, then $\boldsymbol{\upsilon}_k(x) = \boldsymbol{\upsilon}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])$.■

Notice that a latent utility function $\theta_{\overline{n}}(x)$ is uniquely determined if and only if $\theta_{\overline{n}}(x) = \overline{\upsilon}_k(x) = \underline{\upsilon}_k(x)$ for some $k \leq \rho_i$. Otherwise, it is not uniquely determined.

This theorem connects a bridge between the point of view of person i and that of the outside analyzer. As stated already, $\boldsymbol{\upsilon}_0(x), \boldsymbol{\upsilon}_1(x), ..., \boldsymbol{\upsilon}_{\rho_i}(x)$ mean that they are hidden in the mind of person i and are found step by step by digging his mind from a shallow cognitive degree to deeper. In this sense, $\theta_{\overline{n}}(x)$ is latent for person i. As long as Axiom M1 is assumed, however, the sequence goes to the value of a latent utility function $\theta_{\overline{n}}(x)$ and the values are determined by (a) and (b) of Theorem 3.1. Person i cannot avoid the digging process, but the outside analyzer can use $\theta_{\overline{n}}(x)$, which gives a convenient representation conceptually as well as computationally.

The latent utility function $\theta_{\overline{n}}(\cdot)$ derived in Theorem 3.1 has no constraint with respect to $x \in X^*_{\overline{n}}$. Since $X^*_{\overline{n}}$ is the set of monetary payments, it would be natural to assume that $\theta_{\overline{n}}(x)$ is at least weakly monotone function of x, i.e., $\theta_{\overline{n}}(x+1) \geq \theta_{\overline{n}}(x)$ for all $x = 0, ..., 2^{\overline{n}} - 1$. It could be natural also to assume some structure; for example, concavity over $X^*_{\overline{n}}$, i.e., $\theta_{\overline{n}}(x) - \theta_{\overline{n}}(x-1) \geq \theta_{\overline{n}}(x+1) - \theta_{\overline{n}}(x)$ for all $x = 1, ..., 2^{\overline{n}} - 1$. This is interpreted as weakly risk-averse. One example is the *risk-nuetral* one:

$$\theta_{\overline{n}}^{RN}(x) = \frac{x}{2^{\overline{n}}} \quad \text{for } x \in X_{\overline{n}}^*. \tag{10}$$

We assume this $\theta_{\overline{n}}^{RN}$ for our consideration of a resolution of the SP paradox. The latent function $\theta_{\overline{n}}^{RN}(x)$ is entirely risk neutral, but for $k < \overline{n}$, if $\frac{\nu+1}{2^k} > \theta_{\overline{n}}^{RN}(x) > \frac{\nu}{2^k}$, then $\boldsymbol{\upsilon}_k(x)$ is represented by interval $[\overline{\upsilon}_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]); \underline{\upsilon}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}])]$. This interval is larger, i.e., measurement is coarser for a smaller k. This will cause incomparability, which will be discussed specifically for participation fee π and the subjective understanding σ_{ρ_i} of the SP lottery $\tau_{\overline{n}}$ in Section 4.2.

A risk-averse example for the latent utility function is $\theta_{\overline{n}}^{Rt}(x) = \sqrt{x/2^{\overline{n}}}$ for $x \in X_{\overline{n}}^*$, and a risk-lover example is $\theta_{\overline{n}}^{Sq}(x) = (x/2^{\overline{n}})^2$, for which is convex.

Axiom M0 determines the utility function $\boldsymbol{\upsilon}_k$ over $B_k(\overline{x}; \underline{x})$. Axiom M1 uniquely extends this $\boldsymbol{\upsilon}_k$ over X_{ρ_i}; thus, these axioms determine the function $\boldsymbol{\upsilon}_k$ uniquely over $B_k(\overline{x}; \underline{x}) \cup X_{\rho_i}$. In fact, this determination depends only upon $\boldsymbol{\upsilon}_k([\overline{x}, 1; \underline{x}])$ and $\boldsymbol{\upsilon}_k([\overline{x}, 0; \underline{x}])$.[10] Using this fact, we can normalize $\boldsymbol{\upsilon}_k$ as

$$\boldsymbol{\upsilon}_k([\overline{x}, 1; \underline{x}]) = [\overline{x}; \overline{x}] \text{ and } \boldsymbol{\upsilon}_k([\overline{x}, 0; \underline{x}]) = [\underline{x}; \underline{x}]. \tag{11}$$

Under this normalization, Axiom M1 is expressed as follows:

$$\boldsymbol{\upsilon}_k(x) = \begin{cases} [(\nu+1)\cdot 2^{\overline{n}-k}; \nu \cdot 2^{\overline{n}-k}] & \text{if } (\nu+1)\cdot 2^{\overline{n}-k} > \theta_{\overline{n}}(x)\cdot 2^{\overline{n}} > \nu \cdot 2^{\overline{n}-k} \\ \\ [\nu \cdot 2^{\overline{n}-k}; \nu \cdot 2^{\overline{n}-k}] & \text{if } \theta_{\overline{n}}(x)\cdot 2^{\overline{n}} = \nu \cdot 2^{\overline{n}-k}. \end{cases} \tag{12}$$

For example, the first case of (12) follows from Axiom M1 and (11); $\overline{\upsilon}_k([\overline{x}, \frac{\nu+1}{2^k}; \underline{x}]) = \frac{\nu+1}{2^k} \cdot \overline{x} + 0 = (\nu+1)\cdot 2^{\overline{n}-k}$ and $\underline{\upsilon}_k([\overline{x}, \frac{\nu}{2^k}; \underline{x}] = \frac{\nu}{2^k} \cdot \overline{x} + 0 = \nu \cdot 2^{\overline{n}-k}$. The second case is similar. We use (12) since it is convenient for the calculation purpose.

In the following, we suppose condition (10). Under this condition, we state a few specific facts for calculation purposes. When $\rho_i \geq \overline{n}$, it holds that $2^{\overline{n}-k} \leq 1$ for k with $\rho_i \geq k \geq \overline{n}$. This implies that the lower case of (12) holds; thus,

$$\text{if } \rho_i \geq k \geq \overline{n}, \text{ then } \boldsymbol{\upsilon}_k(x) = [x; x] \text{ for all } x \in X_{\rho_i}. \tag{13}$$

Thus, when the cognitive degree ρ_i is precise enough, all alternatives x are measured precisely. The other extreme is that k ($\leq \rho_i$) is very small. Then, the case $\nu = 0$ in the upper case of (12) is specifically important, and is written explicitly:

$$\text{if } 2^{\overline{n}-k} > x, \text{ then } \upsilon_k(x) = [2^{\overline{n}-k}; 0]. \tag{14}$$

The set of pure alternatives X_{ρ_i} has only two types $\{\pi\}$ and $\{2^1, ..., 2^{\widehat{\rho}_i+1}\}$; only π may not be a binary number. Either (14) or the assertion of (13) holds for a binary number x. This is stated as Lemma 3.2 under $\theta_{\overline{n}}(x) = \theta_{\overline{n}}^{RN}(x)$ for all $x \in X_{\overline{n}}^*$.

Lemma 3.2 (Shape of $\boldsymbol{\upsilon}_k(x)$ for a binary x). Let $x = 2^{\overline{n}-k^*}$ for some k^* ($0 \leq k^* \leq \overline{n}$). Then,

$$\boldsymbol{\upsilon}_k(x) = \begin{cases} [2^{\overline{n}-k}; 0] & \text{if } k < k^* \\ \\ [x; x] & \text{if } k \geq k^*. \end{cases}$$

[10] This further implies that the utility representation $\boldsymbol{\upsilon}_k$ of $\succsim_k$ is uniquely determined up to a positive linear transformation (cf., Section 3 in Kaneko [16]).

Proof. Let $k < k^*$. Then $x = 2^{\overline{n}-k^*} < 2^{\overline{n}-k}$. By (14), $\boldsymbol{\upsilon}_k(x) = [2^{\overline{n}-k}; 0]$. Let $k \geq k^*$. Since $2^{\overline{n}-k} \leq 2^{\overline{n}-k^*}$, we have $x = 2^{\overline{n}-k^*} = 2^{k-k^*} \cdot 2^{\overline{n}-k}$. Let $\nu = 2^{k-k^*}$. Then, $x = \nu \cdot 2^{\overline{n}-k} = \frac{\nu}{2^k} \cdot \overline{x}$. By (12), we have $\boldsymbol{\upsilon}_k(x) = [\nu \cdot 2^{\overline{n}-k}; \nu \cdot 2^{\overline{n}-k}] = [x; x]$.■

4 Step E: Extension to have a choice from π and σ_{ρ_i}

We define a process of deriving the utility function $\boldsymbol{u}_{\rho_i}$ to make a comparison between fee π and subjective understanding σ_{ρ_i} of a coin-tossing $\tau_{\overline{n}}$ given by (2). we use new utility functions $\boldsymbol{u}_{\rho_i}(\cdot), ..., \boldsymbol{u}_{\rho_i - \widehat{\rho}_i}(\cdot)$ for this process, and then we add Axiom ME to make a connection between $\boldsymbol{u}_k(\cdot)$ and the base utility function $\boldsymbol{\upsilon}_k(\cdot)$ given in Section 3.

4.1 Process to evaluate σ_{ρ_i}

The ultimate goal of the extension process is to evaluate $\boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i}) = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ to make a comparison with $\boldsymbol{u}_{\rho_i}(\pi) = \boldsymbol{\upsilon}_{\rho_i}(\pi)$. It starts separating the smallest prize 2^1 from the other part $\sigma^1_{\rho_i}$; this is depicted in Fig.5, where the the other part is expressed as the subtree in the second outermost broken-line rectangular. It is expressed as the compound lottery $\frac{1}{2}2^1 * \frac{1}{2}\sigma^1_{\rho_i}$, meaning that each of 2^1 and $\sigma^1_{\rho_i}$ happens with probability $\frac{1}{2}$. These components are is evaluated by $\boldsymbol{u}_{\rho_{i-1}}$, i.e., $\frac{1}{2} \cdot \boldsymbol{u}_{\rho_i - 1}(2^1) + \frac{1}{2} \cdot \boldsymbol{u}_{\rho_i - 1}(\sigma^1_{\rho_i})$. The term $\boldsymbol{u}_{\rho_i - 1}(\sigma^1_{\rho_i})$ needs to be further decomposed such as in the third rectangular in Fig.5; the second prize 2^2 is separated from $\sigma^2_{\rho_i}$, and these are evaluated by $\boldsymbol{u}_{\rho_i - 2}$. Repeating this decomposition and evaluation, person i goes to the innermost rectangular to finish this process. We will show that this process leads to the formula (15) by connecting the process to the result in Section 3;

$$\boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i}) = \sum_{k=1}^{\widehat{\rho}_i} \frac{1}{2^k} \cdot \boldsymbol{\upsilon}_{\rho_i - k}(2^k) + \frac{1}{2^{\widehat{\rho}_i}} \cdot \boldsymbol{\upsilon}_{\rho_i - \widehat{\rho}_i}(2^{\widehat{\rho}_i + 1}). \tag{15}$$

The very last term has the same probability coefficient as the last term of the summation, because of the truncation rule mentioned after (2). The formula (15) is the final goal of Step E.

We prepare a few concepts. Recall $X_{\rho_i} = \{\pi\} \cup \{2^1, ..., 2^{\widehat{\rho}_i + 1}\}$ given in (4). First, let

$$L_l(X_{\rho_i}) = \{f : X_{\rho_i} \to \Pi_l : \sum_{x \in X_{\rho_i}} f(x) = 1\}, \tag{16}$$

where l is a nonnegative integer. We say that $(\boldsymbol{u}, \Delta)$ is a *legitimate pair* iff $\boldsymbol{u} = (\overline{u}, \underline{u})$ is defined over some nonempty subset Δ of $L_l(X_{\rho_i})$ for some l and its region is $\mathbb{Q}^2$ satisfying $\overline{u}(f) \geq \underline{u}(f)$ for all $f \in \Delta$. We say that $\langle(\boldsymbol{u}_{\rho_i}, \Delta^0), (\boldsymbol{u}_{\rho_i - 1}, \Delta^1), ..., (\boldsymbol{u}_{\rho_i - \widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ is a *trajectory of the decomposition process* iff $(\boldsymbol{u}_{\rho_i - k}, \Delta^k)$ is a legitimate pair for each $k = 0, 1, ..., \widehat{\rho}_i$. The length of a trajectory can be shown to be $\widehat{\rho}_i$ (without counting $(\boldsymbol{u}_{\rho_i}, \Delta^0)$) as a result from our axioms for the process, but to avoid unnecessary complications, we use the length $\widehat{\rho}_i$ as given.

In Kaneko [16], a lottery is simply a probability distribution, but here, it has the order structure of coin-tossing. A trajectory $\langle(\boldsymbol{u}_{\rho_i}, \Delta^0), (\boldsymbol{u}_{\rho_i - 1}, \Delta^1), ..., (\boldsymbol{u}_{\rho_i - \widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ goes along this order structure, but its evaluation is made before the coin-tossing. The depths involved in Δ^0 must be the most precise among those domains $\Delta^0, \Delta^1, ..., \Delta^{\widehat{\rho}_i}$. The following axioms are described along this order structure. Note that the depth structure used in [16] is opposite in that it is ordered from the roughest probability grids to more precise ones.

The first axiom states that the process aims to evaluate utility values from participation fee π and his subjective understanding σ_{ρ_i} of $\tau_{\overline{n}}$.

Axiom E0 (Target): The domain of $\boldsymbol{u}_{\rho_i}$ is $\Delta^0 = \{\pi, \sigma^0_{\rho_i}\}$, where $\sigma^0_{\rho_i} = \sigma_{\rho_i}$.

We give two axioms on connections between adjacent pairs $(\boldsymbol{u}_{\rho_i-k}, \Delta^k)$ and $(\boldsymbol{u}_{\rho_i-(k+1)}, \Delta^{k+1})$ in a trajectory. The following definition plays a key role for this connection. Let f, f^1, f^2 be three lotteries in $L_l(X_{\rho_i})$ with some $l \geq 1$. We say that $\{f^1, f^2\}$ is a *decomposition* of f iff

$$f(x) = \frac{1}{2} \cdot f^1(x) + \frac{1}{2} \cdot f^2(x) \ \text{ for all } x \in X_{\rho_i}; \tag{17}$$

$$\text{for } t = 1, 2,\ \delta(f^t(x)) < \delta(f(x)) \ \text{ for all } x \in X_{\rho_i} \text{ with } \delta(f(x)) > 0. \tag{18}$$

In Fig.5, subjective lottery $\sigma^k_{\rho_i}$ is decomposed to 2^{k+1} and $\sigma^{k+1}_{\rho_i}$. These are candidates for decomposed lotteries f^1 and f^2. In general, a lottery may have multiple decompositions.

We say that $f \in L_l(X_{\rho_i})$ is *mixed* iff $0 < \delta(f(x))$ for some $x \in X_{\rho_i}$.

Axiom E1 (Decomposition): Let $k = 0, ..., \widehat{\rho}_i - 1$ and let f be mixed. Then, $f \in \Delta^k$ if and only if f has a decomposition $\{f^1, f^2\}$ such that $f^1 \in \Delta^{k+1}$ and $f^2 \in \Delta^{k+1}$.

When $f \in \Delta^k$ is mixed, f is decomposed into two lotteries in Δ^{k+1}, and conversely, if f is obtained by combining two lotteries f^1, f^2 in Δ^{k+1}, then f belongs to Δ^k. By (18), the depths of lotteries in Δ^k are decreasing, and the process stops when Δ^k has no mixed lottery. This happens for $k = \widehat{\rho}_i - 1$; thus the length of a trajectory is uniquely determined to be $\widehat{\rho}_i$.

The next axiom connects the utility values of $(\boldsymbol{u}_{\rho_i-k}, \Delta^k)$ to those of $(\boldsymbol{u}_{\rho_i-(k+1)}, \Delta^{k+1})$ via decompositions.

Axiom E2 (Reduction of utility value): For $k = 0, ..., \rho_i - 1$, if $f \in \Delta^k$ has a decomposition $\{f^1, f^2\}$, then

$$\boldsymbol{u}_{\rho_i-k}(f) = \tfrac{1}{2} \cdot \boldsymbol{u}_{\rho_i-(k+1)}(f^1) + \tfrac{1}{2} \cdot \boldsymbol{u}_{\rho_i-(k+1)}(f^2). \tag{19}$$

The subscripts of the utility functions are numbered along the decomposition process, and contain the information of permissible depths. For example, person i evaluates the utility value $\boldsymbol{u}_{\rho_i}(f)$ from the viewpoint of cognitive degree ρ_i. Then, he enters the scope of probability weight $\frac{1}{2}$ covering a decomposition $\{f^1, f^2\}$, and evaluates each of $\{f^1, f^2\}$ is from the viewpoint of cognitive degree $\rho_i - 1$. Thus, $\boldsymbol{u}_{\rho_i-1}$ is used with the outer weights $\frac{1}{2}$ in (19).

Let us define the probability distributions $\sigma^0_{\rho_i}, \sigma^1_{\rho_i}, ..., \sigma^{\widehat{\rho}_i}_{\rho_i}$ in Fig.5: for each $l = 0, ..., \widehat{\rho}_i$, each $\sigma^l_{\rho_i}$ with the support $\{2^{l+1}, ..., 2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}$ is given as

$$\sigma^l_{\rho_i}(2^t) = \begin{cases} \frac{1}{2^{t-l}} & \text{if } l+1 \leq t \leq \widehat{\rho}_i \\ \\ \frac{1}{2^{t-1-l}} & \text{if } t = \widehat{\rho}_i + 1. \end{cases} \tag{20}$$

The first $\sigma^0_{\rho_i}$ is σ_{ρ_i} itself, and $\sigma^1_{\rho_i}$ is defined over the support $\{2^2, ..., 2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}$. In general, $\sigma^l_{\rho_i}$ is defined over the support $\{2^{\ell+1}, ..., 2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}$. The last $\sigma^{\widehat{\rho}_i}_{\rho_i}$ has the support $\Delta^{\widehat{\rho}_i} = \{2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}$, and the decomposition is not applied to either $2^{\widehat{\rho}_i}$ or $2^{\widehat{\rho}_i+1}$, and the process does not go any further. We stipulate to write $\sigma^{\widehat{\rho}_i}_{\rho_i} = 2^{\widehat{\rho}_i+1}$; then $\Delta^{\widehat{\rho}_i} = \{2^{\widehat{\rho}_i}, \sigma^{\widehat{\rho}_i}_{\rho_i}\}$. We have the following theorem from Axioms E0 to E2: the domains Δ^0, $\Delta^1, ..., \Delta^{\widehat{\rho}_i}$ are uniquely determined, but the utility functions $\boldsymbol{u}_{\rho_i}, \boldsymbol{u}_{\rho_i-1}, ..., \boldsymbol{u}_{\rho_i-\widehat{\rho}_i}$ are determined within some freedom, which will be explained below. A proof will be given below.

Theorem 4.1 (Decomposition trajectory): Suppose $\Delta^0 = \{\pi, \sigma^0_{\rho_i}\}$. A trajectory $\langle(\boldsymbol{u}_{\rho_i}, \Delta^0)$, $(\boldsymbol{u}_{\rho_i-1}, \Delta^1)$, ..., $(\boldsymbol{u}_{\rho_i-\widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ satisfies Axioms E1 and E2 if and only if for each $k = 1, ..., \widehat{\rho}_i$,

$$\Delta^k = \{2^k, \sigma^k_{\rho_i}\}; \tag{21}$$

$$\boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i}) = \sum_{t=1}^{k} \frac{1}{2^t} \cdot \boldsymbol{u}_{\rho_i-t}(2^t) + \frac{1}{2^k} \cdot \boldsymbol{u}_{\rho_i-k}(\sigma^k_{\rho_i}). \tag{22}$$

Under Axioms E0, Axioms E1 and E2 form necessary and sufficient conditions for a trajectory to be the same as what is described in Fig.5. As remarked stated, the domain Δ^k in (21) is uniquely determined and $\Delta^{\rho_i} = \{2^{\widehat{\rho}_i}, \sigma^{\widehat{\rho}_i}_{\rho_i}\} = \{2^{\widehat{\rho}_i}, 2^{\widehat{\rho}_i+1}\}$. (22) is an intermediate expression obtained in the decomposition process up to k. Plugging $k = \widehat{\rho}_i$ to (22) and $\sigma^{\widehat{\rho}_i}_{\rho_i} = 2^{\widehat{\rho}_i+1}$, we obtain

$$\boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i}) = \sum_{k=1}^{\widehat{\rho}_i} \frac{1}{2^k} \cdot \boldsymbol{u}_{\rho_i-k}(2^k) + \frac{1}{2^{\widehat{\rho}_i}} \cdot \boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(2^{\widehat{\rho}_i+1}). \tag{23}$$

This formula differs from our goal (15) in that the utility functions in the right-hand side are $\boldsymbol{u}_{\rho_i-k}$'s in (23) and $\boldsymbol{v}_{\rho_i-k}$'s in (15). Also, these utility values $\boldsymbol{u}_{\rho_i-1}(2^1), ..., \boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(2^{\widehat{\rho}_i})$, $\boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(2^{\widehat{\rho}_i+1})$ are still arbitrary. This arbitrariness is fixed when they are connected to the base utility functions $\boldsymbol{v}_{\rho_i-k}$'s given in Step M.

Axiom ME (Bridge): For each $k = 0, ..., \widehat{\rho}_i$, if $x \in \Delta^k \cap X_{\rho_i}$, then $\boldsymbol{u}_{\rho_i-k}(x) = \boldsymbol{v}_{\rho_i-k}(x)$.

Suppose Axioms M0, M1 for $\boldsymbol{v}_{\rho_i}, ..., \boldsymbol{v}_{\rho_i-\widehat{\rho}_i}$ and E0 to E2 for $\langle(\boldsymbol{u}_{\rho_i}, \Delta^0), ..., (\boldsymbol{u}_{\rho_i-\widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$. Then, Axiom ME connects these axiomatic systems; we have the target result (15). Since $\boldsymbol{u}_{\rho_i}(\pi) = \boldsymbol{v}_{\rho_i}(\pi)$ is also fixed by Axiom ME, $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ are compared by the interval order $\geq_I$.

We give one remark on a possible restriction of the domain of $\boldsymbol{v}_{\rho_i-k}$.

Remark 4.1 (Implications with respect to bounded intelligence for formula (15)). In Section 3, the base utility function $\boldsymbol{v}_{\rho_i-k}$ is defined first on the scale lotteries $B_{\rho_i}(\overline{x}; \underline{x})$ by Axiom M0, and then it is defined over $X^*_{\overline{n}} = \{0, ..., 2^{\overline{n}}\}$ by Axiom M1. However, since $\boldsymbol{u}_{\rho_i-k}$ is defined over $\Delta^k \cap X_{\rho_i}$ and since $\boldsymbol{u}_{\rho_i-k} = \boldsymbol{v}_{\rho_i-k}$ is required by Axiom ME for $\Delta^k \cap X_{\rho_i}$, the target set $X^*_{\overline{n}} = \{0, ..., 2^{\overline{n}}\}$ in Axiom M1 for $\boldsymbol{v}_{\rho_i-k}$ can be restricted to X_{ρ_i}. Hence, the calculation of formula (15) can be done by concentrating on X_{ρ_i}. This is an implication with respect to bounded intelligence. Nevertheless, Axiom M0 required $\boldsymbol{v}_{\rho_i-k}$ be defined over the scale lotteries $B_{\rho_i}(\overline{x}; \underline{x})$. Some restriction on the scale lotteries is also possible, but it needs some systematic consideration.

Now, we go to the proof of Theorem 4.1. In general, a lottery $f \in L_k(X_{\rho_i})$ may have multiple decompositions. This multiplicity makes an application of Kaneko's [16] theory more complicated. In the case of SP lottery $\sigma^k_{\rho_i}$, however, we can prove uniqueness, which simplifies the application of Axiom E2..

Lemma 4.1 (Unique decomposition). For each $k \leq \widehat{\rho}_i - 1$, lottery $\sigma^k_{\rho_i}$ has the unique decomposition consisting of 2^{k+1} and $\sigma^{k+1}_{\rho_i}$.

Proof. It is easy to see that $\{2^{k+1}, \sigma^{k+1}_{\rho_i}\}$ is a decomposition of $\sigma^k_{\rho_i}$. We show its uniqueness.

Suppose that $\{\tau_1, \tau_2\}$ is a decomposition of $\sigma^k_{\rho_i}$. Since the support of $\sigma^k_{\rho_i}$ is $\{2^{k+1}, ..., 2^{\widehat{\rho}_i+1}\}$ by (20), it holds by (17) that

$$\frac{1}{2} \cdot \tau_1(2^t) + \frac{1}{2} \cdot \tau_2(2^t) = \sigma^k_{\rho_i}(2^t) \text{ for } t = k+1, ..., \widehat{\rho}_i + 1. \tag{24}$$

Since $\sigma^k_{\rho_i}(2^{k+1}) = \frac{1}{2}$, we have $\frac{1}{2}\tau_1(2^{k+1}) + \frac{1}{2}\tau_2(2^{k+1}) = \frac{1}{2}$. By (18), $\delta(\tau_1(2^{k+1})) < \delta(\sigma^k_{\rho_i}(2^{k+1})) = 1$ and $\delta(\tau_2(2^{k+1})) < \delta(\sigma^k_{\rho_i}(2^{k+1})) = 1$, which implies that each of $\tau_1(2^{k+1})$ and $\tau_2(2^{k+1})$ is 0 or 1. Since $\frac{1}{2}\tau_1(2^{k+1}) + \frac{1}{2}\tau_2(2^{k+1}) = \frac{1}{2}$, at least one of $\tau_1(2^{k+1})$ and $\tau_2(2^{k+1})$ is 0. Thus,

$$\tau_1(2^{k+1}) = 0 \text{ or } \tau_2(2^{k+1}) = 0. \tag{25}$$

Let $\tau_1(2^{k+1}) = 0$; the other case $\tau_2(2^{k+1}) = 0$ is parallel. Then, $\tau_2(2^{k+1}) = 1$; thus, $\tau_2(2^t) = 0$ for all $t = k+2, ..., \widehat{\rho}_i + 1$. So, τ_2 is pure outcome 2^{k+1}. It remains to show $\tau_1 = \sigma^{k+1}_{\rho_i}$. By (24), τ_1 is obtained from $\sigma^k_{\rho_i}$ by restricting the support to $\{2^{k+2}, ..., 2^{\widehat{\rho}_i+1}\}$ with normalization by multiplying by 2. The resulting lottery is $\tau_1 = \sigma^{k+1}_{\rho_i}$. We have proved $\{\tau_1, \tau_2\} = \{\sigma^{k+1}_{\rho_i}, 2^{k+1}\}$.$\blacksquare$

Proof of Theorem 4.1 (Only-if): Suppose that $\langle(\boldsymbol{u}_{\rho_i}, \Delta^0), (\boldsymbol{u}_{\rho_i-1}, \Delta^1), ..., (\boldsymbol{u}_{\rho_i-\widehat{\rho}_i}, \Delta^{\widehat{\rho}_i})\rangle$ satisfies Axioms E1 and E2. Repeating Axiom E1 and Lemma 4.1, we have $\Delta^k = \{2^k, \sigma^k_{\rho_i}\}$ for $k = 1, ..., \widehat{\rho}_i$. Applying Axioms E2 and E1 to each $\sigma^k_{\rho_i}$, we have (22) for $k = 1, ..., \widehat{\rho}_i$.

(If): Axiom E1 follows from Lemma 4.1. Axiom E2 is proved as follows: by (15), $\boldsymbol{u}_{\rho_i-k}(\sigma^k_{\rho_i}) = 2^k \cdot \boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i}) - \sum_{t=1}^k 2^{k-t} \cdot \boldsymbol{u}_{\rho_i-t}(2^t) = \frac{1}{2} \cdot [2^{k+1} \cdot \boldsymbol{u}_{\rho_i}(\sigma^0_{\rho_i}) - \sum_{t=1}^{k+1} 2^{k+1-t} \cdot \boldsymbol{u}_{\rho_i-t}(2^t)] + \frac{1}{2} \cdot 2^0 \cdot \boldsymbol{u}_{\rho_i-(k+1)}(2^{k+1}) = \frac{1}{2} \cdot \boldsymbol{u}_{\rho_i-(k+1)}(\sigma^{k+1}_{\rho_i}) + \frac{1}{2} \cdot \boldsymbol{u}_{\rho_i-(k+1)}(2^{k+1})$.$\blacksquare$

4.2 Value $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ under Axioms M0-M1, E0-E2, and ME

Our concern is a comparison between the utility values $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ for person i with cognitive degree ρ_i. Under the risk-nuetrality condition (10), $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i-k}(2^k)$ are determined in terms of the maximum prize $\overline{n}$, cognitive degree ρ_i, and k. This is given by (12). It follows from this and and the result (15) that $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ must be expressed in terms of $\overline{n}$ and ρ_i. However, the result (15) is indirectly expressed in terms of $\overline{n}$ and ρ_i in that a variable k occurs in the formula. Here, we express $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ directly by $\overline{n}$ and ρ_i. In the following, we assume Axioms M0-M1, E0-E2, and ME as well as (10).

Theorem 4.2 (Value of $\boldsymbol{u}_{\rho_i}(\boldsymbol{\sigma}_{\rho_i})$).
(1): Let $\rho_i \geq \overline{n} - 1$. Then $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\overline{n}+1; \overline{n}+1]$.
(2): Let $\rho_i \leq \overline{n} - 2$. Then $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [(\rho_i + 1) \cdot 2^{\overline{n}-\rho_i}; 0]$.

Proof.(1): Recall $\widehat{\rho}_i = \min(\rho_i, \overline{n} - 1)$. Let $\rho_i \geq \overline{n} - 1$. First, for $t \leq \widehat{\rho}_i$, we calculate $\boldsymbol{u}_{\rho_i-t}(2^t)$ using (12); since $0 < 2^t = 2^{t+\rho_i-\overline{n}} \cdot 2^{\overline{n}-\rho_i}$, we have the second case of (12) and thus $\boldsymbol{u}_{\rho_i-t}(2^t) = [2^{t+\rho_i-\overline{n}} \cdot 2^{\overline{n}-\rho_i}; 2^{t+\rho_i-\overline{n}} \cdot 2^{\overline{n}-\rho_i}] = [2^t; 2^t]$. Consider $\boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(2^{\overline{n}})$. Since $2^{\overline{n}} = 2^{\overline{n}} \cdot 2^0$, we have $\boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(2^{\overline{n}}) = [2^{\overline{n}}; 2^{\overline{n}}]$ by the second case of (12). Plugging these to the formula (15), we have $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = \sum_{t=1}^{\widehat{\rho}_i} \frac{1}{2^t} \cdot \boldsymbol{u}_{\rho_i-t}(2^t) + \frac{1}{2^{\widehat{\rho}_i}} \cdot \boldsymbol{u}_{\rho_i-\widehat{\rho}_i}(2^{\widehat{\rho}_i+1}) = [\widehat{\rho}_i + 2; \widehat{\rho}_i + 2] = [\overline{n}+1, \overline{n}+1]$.

(2): Suppose that $\rho_i \leq \overline{n} - 2$. Let $t \leq \rho_i$. First, we calculate $\boldsymbol{u}_{\rho_i-t}(2^t)$ using (12); since $0 < 2^t < 2^t \cdot 2^{(\overline{n}-\rho_i)} = 2^{\overline{n}-(\rho_i-t)}$, we have the first case of (12); from this, we have

$$\boldsymbol{u}_{\rho_i-t}(2^t) = [2^{\overline{n}-(\rho_i-t)}; 0] \text{ for } t \leq \rho_i. \tag{26}$$

Since $\rho_i \leq \overline{n} - 2$, i.e., $\rho_i + 2 \leq \overline{n}$, we have $0 < 2^{\rho_i+1} < 2^{\rho_i+1} \cdot 2^1 \leq 2^{\overline{n}} = 2^{\overline{n}-0}$. By the first case of (12), $\boldsymbol{u}_0(2^{\rho_i+1}) = [2^{\overline{n}}; 0]$. Thus,

$$\text{if } \rho_i \leq \overline{n} - 2, \text{ then } \boldsymbol{u}_0(2^{\rho_i+1}) = [2^{\overline{n}}; 0]. \tag{27}$$

Now, we calculate $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\overline{u}_{\rho_i}(\sigma_{\rho_i}); \underline{u}_{\rho_i}(\sigma_{\rho_i})]$. Using (26) and (27), we have, for $\rho_i \leq \overline{n} - 2$,

$$\begin{aligned}
\overline{u}_{\rho_i}(\sigma_{\rho_i}) &= \sum_{t=1}^{\rho_i} \frac{1}{2^t} \cdot \overline{u}_{\rho_i - t}(2^t) + \frac{1}{2^{\rho_i}} \cdot \overline{u}_0(2^{\rho_i+1}) \\
&= \sum_{t=1}^{\rho_i} \frac{1}{2^t} \cdot 2^{\overline{n}-\rho_i} \cdot 2^t + \frac{1}{2^{\rho_i}} \cdot 2^{\overline{n}} = \rho_i \cdot 2^{\overline{n}-\rho_i} + 2^{\overline{n}-\rho_i} = (\rho_i + 1) \cdot 2^{\overline{n}-\rho_i}.
\end{aligned}$$

The lower bound $\underline{u}_{\rho_i}(\sigma_{\rho_i})$ is given $\sum_{t=1}^{\rho_i} \frac{1}{2^t} \cdot 0 + \frac{1}{2^{\rho_i}} \cdot 0 = 0.$■

To have comparisons between $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$, we use the interval order $\geq_I$. We state a simple observation from Theorem 4.2 and (12).

Lemma 4.2 (Precise cognitive bound). Let $\rho_i \geq \overline{n} - 1$ and π an even number. Then,

(1): both $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{u}_{\rho_i}(\pi)$ are singular;

(2): $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) \geq_I \boldsymbol{u}_{\rho_i}(\pi)$ if and only if $\overline{n} - 1 \geq \pi$.

Proof. (2) follows from (1). Consider (1). By Theorem 4.2.(2), $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\overline{n}+1; \overline{n}+1]$. We change $\rho_i \geq \overline{n} - 1$ into $\overline{n} - \rho_i \leq 1$. We show that $\pi = \nu \cdot 2^{\overline{n}-\rho_i}$ for some ν. Indeed, If $\overline{n} - \rho_i = 1$, then $\pi = \nu \cdot 2^{\overline{n}-\rho_i} = \nu \cdot 2$ for some ν, since π is an even number. If $\overline{n} - \rho_i \leq 0$, then, $\pi = \nu \cdot 2^{\overline{n}-\rho_i}$ for some ν. By (12), $\boldsymbol{v}_{\rho_i}(\pi) = [\pi; \pi]$.■

The formula (12) for $\boldsymbol{u}_{\rho_i}(\pi)$ as well as Theorem 4.2 for $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ are still abstract. Let us see what would happen with comparisons between them in an numerical example. The example has the salient feature that $\boldsymbol{u}_{\rho_i}(\pi)$ and $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ are incomparable in many cases, which leads us to Section 5.

Example 4.1 (Small): Let $\overline{n} = 17$ and $\pi = 500$, that is, the maximum prize is $2^{17} = 131,072$¢ $\doteqdot 1,300$\$ and the participation fee is 500¢ $= 5$\$. It is simpler to calculate $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\overline{u}_{\rho_i}(\sigma_{\rho_i}); \underline{u}_{\rho_i}(\sigma_{\rho_i})]$ than $\boldsymbol{u}_{\rho_i}(\pi) = [\overline{u}_{\rho_i}(\pi); \underline{u}_{\rho_i}(\pi)]$. We can directly apply Theorem 4.2 to have $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\overline{u}_{\rho_i}(\sigma_{\rho_i}); \underline{u}_{\rho_i}(\sigma_{\rho_i})]$. However, we need to return to the formula (12) to have $\boldsymbol{u}_{\rho_i}(\pi) = [\overline{u}_{\rho_i}(\pi); \underline{u}_{\rho_i}(\pi)]$, Lemma 3.2 is not applied to $x = 500$.

Let us calculate $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ by applying Theorem 4.2. By Theorem 4.2.(1), we have $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [(\rho_i+1) \cdot 2^{\overline{n}-\rho_i}; 0]$ for $\rho_i \leq \overline{n} - 2 = 15$, and by Theorem 4.2.(2), we have $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [\overline{n}+1; \overline{n}+1]$ for $\rho_i = 16, 17$. These are written in the second and third rows in Tables 4.1 and 4.2. For example, when $\rho_i = 3$, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [(\rho_i + 1) \cdot 2^{\overline{n}-\rho_i}; 0] = [4 \cdot 2^{14}; 0]$. Observe that $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ is non-singular up to $\rho_i = 15$, and that the width of $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ is almost exponentially decreasing with ρ_i, and when $\rho_i = 16$, it is singular, i.e., $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [18; 18]$ is the same as the expected reward from $\tau_{\overline{n}}$.

On the other hand, $\boldsymbol{u}_{\rho_i}(\pi)$ is obtained by (12). The cognitive degrees from $\rho_i = 0$ to 17 are divided into three cases. In Table 4.1, $\boldsymbol{u}_{\rho_i}(\pi) = \boldsymbol{v}_{\rho_i}(\pi) = [2^{n-\rho_i}; 0]$ is to $\rho_i = 0$ to 8, then, the first case of (12) is applied to $\rho_i = 9$ to 12, and finally, the second case is to $\rho_i \geq 13$. The values are written in the fourth and fifth rows. The width of $\boldsymbol{u}_{\rho_i}(\pi) = \boldsymbol{v}_{\rho_i}(\pi)$ is almost exponentially decreasing, and when $\rho_i \geq 13$, this utility value gets singular.

Table 4.1; example with $\overline{n} = 17$

ρ_i	0	1	2	3	4	5	6	7	8
$\overline{u}_{\rho_i}(\sigma_{\rho_i})$	2^{17}	$2 \cdot 2^{16}$	$3 \cdot 2^{15}$	$4 \cdot 2^{14}$	$5 \cdot 2^{13}$	$6 \cdot 2^{12}$	$7 \cdot 2^{11}$	$8 \cdot 2^{10}$	$9 \cdot 2^{9}$
$\underline{u}_{\rho_i}(\sigma_{\rho_i})$	0	0	0	0	0	0	0	0	0
$\overline{u}_{\rho_i}(\pi)$	2^{17}	2^{16}	2^{15}	2^{14}	2^{13}	2^{12}	2^{11}	2^{10}	2^{9}
$\underline{u}_{\rho_i}(\pi)$	0	0	0	0	0	0	0	0	0
Comparisons	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$
$\eta_{\overline{n},\pi}[\rho_i]$	1/2	3/4	5/6	7/8	9/10	11/12	13/14	15/16	17/18

Table 4.2; example with $\overline{n} = 17$

ρ_i	9	10	11	12	13	14	15	16	17
$\overline{u}_i(\sigma_{\rho_i})$	$10 \cdot 2^{8}$	$11 \cdot 2^{7}$	$12 \cdot 2^{6}$	$13 \cdot 2^{5}$	$14 \cdot 2^{4}$	$15 \cdot 2^{3}$	$16 \cdot 2^{2}$	18	18
$\underline{u}_i(\sigma_{\rho_i})$	0	0	0	0	0	0	0	18	18
$\overline{u}_i(\pi)$	$2 \cdot 2^{8}$	$4 \cdot 2^{7}$	$8 \cdot 2^{6}$	$16 \cdot 2^{5}$	500	500	500	500	500
$\underline{u}_i(\pi)$	2^{8}	$3 \cdot 2^{7}$	$7 \cdot 2^{6}$	$15 \cdot 2^{5}$	500	500	500	500	500
Comparisons	$\bowtie$	$\bowtie$	$\bowtie$	π	π	π	π	π	π
$\eta_{\overline{n},\pi}[\rho_i]$	17/20	15/22	9/24	0	0	0	0	0	0

In the last rows of these tables, $\bowtie$ from $\rho_i = 0$ to 11 means that $\boldsymbol{u}_i(\sigma_{\rho_i})$ and $\boldsymbol{u}_i(\pi)$ are incomparable, and for $\rho_i \geq 12$, π (not participate) is strictly preferred to σ_{ρ_i} (participate). It is important to see the structure of incomparability differs significantly between the cases $\rho_i \leq 8$ and $9 \leq \rho_i \leq 11$. Even for these ρ_i, the inequalities determining incomparability differs significantly, for example, for $\rho_i = 0$, $\boldsymbol{u}_{\rho_i}(\pi) = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [2^{17}; 0]$, but for $\rho_i = 8$, $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) = [9 \cdot 2^9; 0]$, $\boldsymbol{u}_{\rho_i}(\pi) = [2^9; 0]$. Incomparability $\bowtie$ does not reflect these detailed differences. To capture them, we will introduce semi-rationalistic choice probability in Section 5. The last row gives choice-probabilities for σ_{ρ_i}, which will be explained in there.

5 Semi-rationalistic Choice for Incomparable Alternatives

We described preferences of person i with cognitive degree ρ_i over his subjective understanding σ_{ρ_i} of the SP coin-tossing $\tau_{\overline{n}}$ and participation fee π. As shown in Example 4.1, in many cases, σ_{ρ_i} and π are incomparable. This may appear to mean the impossibility of a resolution of the SP paradox. However, behavioral consideration of incomparability suggests that some people still participate in the SP coin-tossing. In this section, we give a probabilistic approach to this incomparability, and show that it is quite possible to have participations of some people in the coin-tossing.

5.1 Forced semi-rationalistic choice; reduction to rationalistic choices

People are often unable to make choices by rational thinking for various reasons, for example, cognitive bounds, lack of precise information etc. Nevertheless, they are often required to make choices by norm, social pressure. One famous exception is *Buridan's donkey* (cf., Zupko [30]); a donkey faces two carrots in the same distances (up to his cognitive ability) and he cannot choose the right carrot or the left, and eventually dies of starvation. Our incomparability is applied to this example.[11] The other exception is that some people consciously refuse to make choices because the description of a situation is not enough.

The incomparability results obtained in Sections 3 and 4 dictate the existence of a limit to have a conscious decision by his free will; the theory stops at preferences or incomparabilities. When person i finds a preference between σ_{ρ_i} and π, he chooses the preferred. If he reaches incomparability, what is the next step? So far, the theory suggests nothing for him. In the real world, however, majority of people make choices because they have been forced to make choices by social norm, authorities, etc.[12]

Suppose that person i is forced to make a choice. He has already used his capacity for his rationalistic choice and reached incomparability between σ_{ρ_i} and π. Now, he need to use different sources to make a choice. The theory in Sections 3 and 4 is restricted in the explicitly formulated process of measurement and extension, except for the assumption that the utility is hidden in the mind of person i. People have many more experiences and practices, even including similar processes. Person i digs memories in past choice practices to have a choice between incomparable σ_{ρ_i} and π, while trying deviations from the rationalistic method as small as possible.

We describe an extension of our theory called the semi-rationalistic *behavioral-probability* $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i});\, \boldsymbol{u}_{\rho_i}(\pi))$, which is the probability of person i choosing σ_{ρ_i} when he faces a choice between σ_{ρ_i} and π. It has some similarity to Luce's [18] approach in that both approaches extend the standard theory of preferences in terms of probability. However, the essential part of our approach is between two incomparable alternatives, while Luce's theory is simply a generalization of preferences in terms of probability.[13] We reduce the behavioral-probability $\eta(\boldsymbol{u}_{\rho_i}(\pi); \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}))$ to comparable cases; in other words, we construct a theory of probabilistic choice for incomparable σ_{ρ_i} and π based on the rationalistic-choice theory developed in Sections 3 and 4. We call this the *reduction method to rationalistic choices.*

In order to illustrate this method, consider the value $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}); \boldsymbol{u}_{\rho_i}(\pi)) = \eta([10 \cdot 2^8; 0]; [2 \cdot 2^8; 2^8])$ in Table 4.2 for $\rho_i = 9$. The first interval $[10 \cdot 2^8; 0]$ contains the second $[2 \cdot 2^8; 2^8]$. Focussing on $[10 \cdot 2^8; 0]$, we divide $[10 \cdot 2^8; 0]$ at the point $2 \cdot 2^8$ of the second interval to the right term of (28), and then $[2 \cdot 2^8; 0]$ is divided again at 2^8. Thus, we have the second line of (28);

$$\begin{aligned} [10 \cdot 2^8; 0] \;\; &\rightarrow \;\; [10 \cdot 2^8; 2 \cdot 2^8],\; [2 \cdot 2^8; 0] \\ &\rightarrow \;\; [10 \cdot 2^8; 2 \cdot 2^8],\; [2 \cdot 2^8; 2^8],\; [2^8; 0]. \end{aligned} \tag{28}$$

Each of these intervals can be compared with $\boldsymbol{u}_{\rho_i}(\pi) = [2 \cdot 2^8; 2^8]$ with the interval order $\geq_I$; that is, $[10 \cdot 2^8; 2 \cdot 2^8] \geq_I [2 \cdot 2^8; 2^8]$, $[2 \cdot 2^8; 2^8] = [2 \cdot 2^8; 2^8]$, and $[2^8; 0] \leq_I [2 \cdot 2^8; 2^8]$. We assign probability 1 to the first since the left interval is chosen, the probability $\frac{1}{2}$ to the second since they are identical with respect to utility representations, and the probability 0 to the left $[2^8; 0]$. Thus, $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}); \boldsymbol{u}_{\rho_i}(\pi)) = \eta([10 \cdot 2^8; 0]; [2 \cdot 2^8; 2^8])$ is calculated by reducing it into these

[11] Interpreting this situation by indifference means that he can choose either rather than he cannot choose. This goes to a probablistic choice of each with $\frac{1}{2}$.

[12] In literature, there are many instances treating such problems, e.g., "Sophie's Choice" by William Styron and "Terror" by Ferdinand von Schirachon. In the literature of economics, however, people (experimental subjects) are supposed to answer to questionaires. In the game theory literature, Davis-Maschler [5] asked game theorists about about a questionaire related to the theory in [5], only Martin Shubik made an explicit refusal to.answer to it because of insufficiency of descriptions the theory to answer.

[13] Our theory may look similar to "propensity probability" (cf., Gillies [12], Hájek [14], Section 3.5). It is is interpreted as a (probablisitc) generalization of causality In our theory, causality is hidden in Step M.

subintervals, while we need some weights for summing up these values. Thus, the probability $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}); \boldsymbol{u}_{\rho_i}(\pi))$ of choosing σ_{ρ_i} is obtained in a rationalistic manner.

The reduction method to rationalistic choices is interpreted as follows. Recall that in the rationalistic method given in Sections 3 and 4, person i considers his choice in terms of the elements in the SP gamble of Fig.2 up to his cognitive bound, digging his latent satisfactions/dissatisfactions in his mind, which is the measurement step, and he extends those utility values in a logical manner. In the reduction method, he recalls his past practices of this rationalistic choice, and reduces the present problem of incomparability to a few pieces of past memories. In this sense, this method is not totally rationalistic, in which sense it is semi-rationalistic. The practical meaning will get clearer as an axiomatic representation is developed.

5.2 An axiomatic system determining the semi-rationalistic choice-function

Let us formulate the above reduction method. We work on representing utility vectors, rather than basic objects to be represented. Let Λ be a finite subset of $\mathbb{Q}^2(\geq) = \{\boldsymbol{\alpha} = \langle \overline{\alpha}; \underline{\alpha} \rangle \in \mathbb{Q}^2 : \overline{\alpha} \geq \underline{\alpha}\}$, associated with the interval order $\geq_I$. We start with a given pair $(\Lambda, \geq_I)$. Let us define the following condition (*decomposition closedness*:): for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda$ and $\xi = \overline{\beta}, \underline{\beta}$ with $\overline{\alpha} \geq \xi \geq \underline{\alpha}$,

$$[\overline{\alpha}; \xi] \text{ and } [\xi; \underline{\alpha}] \text{ are in } \Lambda. \tag{29}$$

Here, $\boldsymbol{\alpha}$ is decomposed into $[\overline{\alpha}; \xi]$ and $[\xi; \underline{\alpha}]$ by $\xi = \overline{\beta}$ or $\underline{\beta}$. When $\boldsymbol{\beta} = \boldsymbol{\alpha}$, (29) implies $[\overline{\alpha}; \overline{\alpha}] \in \Lambda$ and $[\underline{\alpha}; \underline{\alpha}] \in \Lambda$. However, the essential case of (29) is $\overline{\alpha} > \xi > \underline{\alpha}$; in this case, a new non-degenerated interval is $[\overline{\alpha}; \xi]$ and $[\xi; \underline{\alpha}]$. This decomposition generates a finite number of subintervals.

We say that Λ is the set *generated* by $\boldsymbol{\lambda}, \boldsymbol{\mu}$, denoted by $\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$, iff Λ satisfies (29) and $\Lambda \subseteq \Lambda'$ for any Λ' that includes $\boldsymbol{\lambda}, \boldsymbol{\mu}$ and satisfies (29). This $\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$ is uniquely determined, nonempty, and consists of a finite number intervals. For example, when $\overline{\lambda} > \overline{\mu} > \underline{\lambda} > \underline{\mu}$, we have

$$\Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu}) = \{[\overline{\lambda}; \overline{\mu}], [\overline{\mu}; \underline{\lambda}], [\underline{\lambda}; \underline{\mu}]\} \cup \{[\overline{\lambda}; \overline{\lambda}], [\overline{\mu}; \overline{\mu}], [\underline{\lambda}; \underline{\lambda}], [\underline{\mu}; \underline{\mu}]\} \cup \{\boldsymbol{\lambda}; \boldsymbol{\mu}\}. \tag{30}$$

In (30), $[\overline{\lambda}; \underline{\mu}]$ is missing.

Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{Q}^2(\geq)$. We define a behavioral-probability function η.

Axiom S0(Choice probability): A behavioral-probability function is given as $\eta : \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})^2 \to Q_{[0,1]}$.

For $\boldsymbol{\alpha}; \boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})^2$, the value $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$ means that person i chooses $\boldsymbol{\alpha}$ with probability $\eta(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Here, we focus on a person i for $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$, but we do not put the subscript i. Our final target is to calculate $\eta(\boldsymbol{\lambda}; \boldsymbol{\mu})$ rather than $\{\eta(\boldsymbol{\alpha}; \boldsymbol{\beta}) : (\boldsymbol{\alpha}; \boldsymbol{\beta}) \in \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})^2\}$. The following axiom is associated with the above interpretation of $\eta(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Axiom S1(Probability): $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta}) + \eta(\boldsymbol{\beta}; \boldsymbol{\alpha}) = 1$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$.

Person i chooses $\boldsymbol{\alpha}$ with probability $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$ and $\boldsymbol{\beta}$ with probability $1 - \eta(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Lemma 5.1 is an implication of Axiom S1, which is the hint discussed in the beginning of Section 5, and plays a key role in the determination of the probability function η.

Lemma 5.1(Equal probability for an equal utility representation): $\eta(\boldsymbol{\alpha}; \boldsymbol{\alpha}) = \frac{1}{2}$ for $\boldsymbol{\alpha} \in \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})$.

Proof. By Axiom S1, we have $\eta(\boldsymbol{\alpha};\boldsymbol{\alpha})+\eta(\boldsymbol{\alpha};\boldsymbol{\alpha})=1$, which implies $\eta(\boldsymbol{\alpha};\boldsymbol{\alpha})=\frac{1}{2}$.■

Notice that since $\boldsymbol{\alpha}=[\overline{\alpha};\underline{\alpha}]$ is a utility representation of some underlying alternatives, this lemma has a non-trivial content.

The next axiom is the connection to the previous theory in Sections 3 and 4: if $\boldsymbol{\alpha},\boldsymbol{\beta}$ are comparable with $\geq_I$, the behavioral-probability $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ coincides with his rationalistic decision expressed by $\geq_I$.

Axiom S2 (Preservation of the interval order): Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$. If $\boldsymbol{\alpha}\geq_I\boldsymbol{\beta}$ and $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})=1$.

It follows from Axioms S1 and S2 that if $\boldsymbol{\alpha}\geq_I\boldsymbol{\beta}$ but $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$, then $\eta(\boldsymbol{\beta};\boldsymbol{\alpha})=0$.

The third axiom states that $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ is decomposed along (29) with some weights. This axiom is well-defined since the domain $\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$ satisfies the condition (29).

Axiom S3 (Proportional reduction to irreducibles): Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$ and $\xi=\overline{\beta}$ or $\underline{\beta}$ with $\overline{\alpha}\geq\xi\geq\underline{\alpha}$, and let $\gamma\in\mathbb{Q}_{[0,1]}$ satisfying $\xi=(1-\gamma)\cdot\overline{\alpha}+\gamma\cdot\underline{\alpha}$. Then,

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta})=\gamma\cdot\eta([\overline{\alpha};\xi];\boldsymbol{\beta})+(1-\gamma)\cdot\eta([\xi;\underline{\alpha}];\boldsymbol{\beta}). \tag{31}$$

The interval $\boldsymbol{\alpha}=[\overline{\alpha};\underline{\alpha}]$ is divided into $[\overline{\alpha};\xi]$ and $[\xi;\underline{\alpha}]$ for $\xi=\overline{\beta}$ or $\underline{\beta}$ with $\overline{\alpha}\geq\xi\geq\underline{\alpha}$. Then, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ is decomposed to the weighted sum of $\eta([\overline{\alpha};\xi];\boldsymbol{\beta})$ and $\eta([\xi;\underline{\alpha}];\boldsymbol{\beta})$ with their weights γ and $1-\gamma$ given by $\xi=(1-\gamma)\cdot\overline{\alpha}+\gamma\cdot\underline{\alpha}$. The weight γ is given as

$$\gamma=\tfrac{\overline{\alpha}-\xi}{\overline{\alpha}-\underline{\alpha}} \text{ and } 1-\gamma=\tfrac{\xi-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}}, \text{ under } \overline{\alpha}-\underline{\alpha}>0. \tag{32}$$

Thus, the weight γ is the proportional of the length $[\overline{\alpha};\xi]$ over $[\overline{\alpha};\underline{\alpha}]$. When $\overline{\alpha}=\underline{\alpha}$, γ is arbitrary but the assertion (31) holds in a trivial sense.

Let us summarize the axioms. Axiom S0 states what are measured, and Axiom S1 requires the way be represented by a probability of a choice from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Axiom S2 requires the behavioral-probability function η to preserve the interval order $\geq_I$. These axioms are basic requirements. On the other hand, Axiom S3 decomposes $\eta(\cdot;\cdot)$ into two subintervals in a proportional manner. By this, these are reduced to the cases satisfying Lemma 5.1 and Axiom S1. By this, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ is proved to be uniquely determined.

Axiom S3 plays an central role for the reduction process leading the unique determination of $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$. We look at how Axiom S3 is used in the reduction process. For an interval $\boldsymbol{\alpha}=\ell[\overline{\alpha};\underline{\alpha}]$, we denote $\ell[\boldsymbol{\alpha}]=\ell[\overline{\alpha};\underline{\alpha}]=\overline{\alpha}-\underline{\alpha}$; it is simply the length of the interval $\boldsymbol{\alpha}$. By (32), Then, $\gamma=\frac{\ell[\overline{\alpha};\xi]}{\ell[\boldsymbol{\alpha}]}$ and $1-\gamma=\frac{\ell[\xi;\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}$. Thus, (31) is expressed as

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta})=\tfrac{\ell[\overline{\alpha};\xi]}{\ell[\boldsymbol{\alpha}]}\cdot\eta([\overline{\alpha};\xi];\boldsymbol{\beta})+\tfrac{\ell[\xi;\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}\cdot\eta([\xi;\underline{\alpha}];\boldsymbol{\beta}). \tag{33}$$

This means that $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ is reduced to a weighted sum of the choice-probabilities of subintervals. In fact, the behavioral-probability of at least one of the subinterval is determined by some other axiom. For example, when $\xi=\overline{\beta}$, the value $\eta([\overline{\alpha};\xi];\boldsymbol{\beta})$ is determined to be 1 by Axiom S2. Then, the same argument is applied to $\eta([\xi;\underline{\alpha}];\boldsymbol{\beta})$ of the second terme. In this manner, the value $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ is reduced to a sum of terms expressed by $\ell[\cdot]$, 1, 0 and $\frac{1}{2}$.

Theorem 5.1 expresses $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ and $\eta(\boldsymbol{\beta};\boldsymbol{\alpha})$ explicitly by $\ell[\cdot]$ with $0,\frac{1}{2},1$ in three cases. After stating the theorem, it will be explained that the three cases are exhaustive. A proof will be given in Section 5.3.

Theorem 5.1 (Semi-rationalistic behavioral-probability). Assume Axiom S0 for $\eta : \Lambda^2 \to \mathbb{Q}_{[0,1]}$, i.e., $\Lambda = \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$. A function $\eta : \Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})^2 \to \mathbb{Q}_{[0,1]}$ satisfies Axioms S1 to S3 if and only if for each $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \Lambda^2$, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta})$ is given by

(0): if $\underline{\alpha} \geq \overline{\beta}$, then

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta} \\ \frac{1}{2} & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta}; \end{cases} \quad \text{and} \quad \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) = \begin{cases} 0 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta} \\ \frac{1}{2} & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta}; \end{cases} \tag{34}$$

(1): if $\overline{\alpha} \geq \overline{\beta} \geq \underline{\alpha} \geq \underline{\beta}$ and both $\boldsymbol{\alpha};\boldsymbol{\beta}$ are non-singular, then

$$\begin{aligned} \eta(\boldsymbol{\alpha};\boldsymbol{\beta}) &= \tfrac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \tfrac{1}{2}\cdot\tfrac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}\cdot\tfrac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\beta}]} + \tfrac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}\cdot\tfrac{\ell[\underline{\alpha};\underline{\beta}]}{\ell[\boldsymbol{\beta}]} \\ \eta(\boldsymbol{\beta};\boldsymbol{\alpha}) &= \tfrac{1}{2}\cdot\tfrac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}\cdot\tfrac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\beta}]}; \end{aligned} \tag{35}$$

(2): if $\overline{\alpha} \geq \overline{\beta} \geq \underline{\beta} \geq \underline{\alpha}$ and $\boldsymbol{\alpha}$ is non-singular, then

$$r(\boldsymbol{\alpha};\boldsymbol{\beta}) = \tfrac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \tfrac{1}{2}\cdot\tfrac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} \quad \text{and} \quad r(\boldsymbol{\beta};\boldsymbol{\alpha}) = \tfrac{1}{2}\cdot\tfrac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} + \tfrac{\ell[\underline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]}. \tag{36}$$

First, let us see that the above three cases of Theorem 4.1 are exhaustive. Since both $r(\boldsymbol{\alpha};\boldsymbol{\beta})$ and $r(\boldsymbol{\beta};\boldsymbol{\alpha})$ are given in the three cases, it suffices to consider the case $\overline{\alpha} \geq \overline{\beta}$. Thus, we consider the following three subcases:

(a): $\underline{\alpha} \geq \overline{\beta}$; (b): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\alpha} \geq \underline{\beta}$; and (c): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\beta} \geq \underline{\alpha}$.

In (a), (0) covers all the cases where $\boldsymbol{\alpha}$ and/or $\boldsymbol{\beta}$ are singular or not. In (b), if $\boldsymbol{\alpha}$ and/or $\boldsymbol{\beta}$ is singular, then this is included in (0). In (c), if $\boldsymbol{\alpha}$ is singular, then it is included in (0). Thus, (0) to (2) of Theorem 5.1 cover all the cases.

Theorem 5.1 states that Axioms S0 to S3 determine the semi-rationalistic behavioral-probability function $\eta(\cdot;\cdot)$ uniquely. Then, this axiomatic system is connected to the axiomatic system M0 and M1 with the risk-neutral latent utility function $\theta_{\overline{n}}^{RN}$ and the system E0 to E2 by $\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$ in Axiom S0 with

$$\boldsymbol{\lambda} = \boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) \text{ and } \boldsymbol{\mu} = \boldsymbol{u}_{\rho_i}(\pi). \tag{37}$$

In the following, we consider this set $\Lambda(\boldsymbol{\lambda};\boldsymbol{\mu})$. Let us look at the example $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i});\boldsymbol{u}_{\rho_i}(\pi))$ $= \eta([10\cdot 2^8;0];[2\cdot 2^8;2^8])$ in Table 4.2 for $\rho_i = 9$. Case (2) is applied to this example;

$$\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i});\boldsymbol{u}_{\rho_i}(\pi)) = \tfrac{10\cdot 2^8 - 2\cdot 2^8}{10\cdot 2^8 - 0} + \tfrac{1}{2}\cdot\tfrac{2\cdot 2^8 - 2^8}{10\cdot 2^8 - 0} = \tfrac{8}{10} + \tfrac{1}{2}\cdot\tfrac{1}{10} = \tfrac{17}{20},$$

which is given in Table 4.2.

Example 4.1 illustrated calculations of involved utility values and their comparisons. As a purpose of considering a resolution of the SP paradox, this example is too small. It would be needed to consider a larger example for it. The following example is based on the parameter values discussed in Section 1.1.

Example 5.1 (Large): Consider the example with the budget degree $\overline{n} = 21$ for the banker and participation fee $\pi = 5\$$ (and later $\pi = 50\$, 100\$$). If people's cognitive degrees ρ_i are high and close to $\overline{n}$, the expected revenue is again $\overline{n}+1 = 22$¢, and they do not participate in the coin-tossing. However, since participation could give the large (maximum) prize $2^{21} = 2,097,152$¢ $\doteqdot$

20, 000\$ with the relatively small fee $\pi = 5\$$, some people may think of participation in the coin-tossing. If this induces participations of some people, the banker would receive positive profits unless some person reaches a coin-toss with large prize such as the $21th$ toss. Let us calculate $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i})$ and $\boldsymbol{u}_{\rho_i}(\pi)$, which are done in the same way as in Example 4.1.

Table 5.1; $\rho_i = 0, ..., 10$

ρ_i	0	1	2	3	4	5	6	7	8	9	10
$\overline{u}_{\rho_i}(\sigma_{\rho_i})$	2^{21}	$2 \cdot 2^{20}$	$3 \cdot 2^{19}$	$4 \cdot 2^{18}$	$5 \cdot 2^{17}$	$6 \cdot 2^{16}$	$7 \cdot 2^{15}$	$8 \cdot 2^{14}$	$9 \cdot 2^{13}$	$10 \cdot 2^{12}$	$11 \cdot 2^{11}$
$\underline{u}_{\rho_i}(\sigma_{\rho_i})$	0	0	0	0	0	0	0	0	0	0	0
$\overline{u}_{\rho_i}(\pi)$	2^{21}	2^{20}	2^{19}	2^{18}	2^{17}	2^{17}	2^{15}	2^{14}	2^{13}	2^{12}	2^{11}
$\underline{u}_{\rho_i}(\pi)$	0	0	0	0	0	0	0	0	0	0	0
$\eta_{[\overline{n},\pi]}(\rho_i)$	1/2	3/4	5/6	7/8	9/10	11/12	13/14	15/16	17/18	19/20	21/22

We abbreviate $\eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}); \boldsymbol{u}_{\rho_i}(\pi))$ as $\eta_{[\overline{n},\pi]}(\rho_i)$, and the row "comparisons", since the last row includes this information, i.e., $\eta_{[\overline{n},\pi]}(\rho_i) > 0$ if and only $\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}) \bowtie_{\rho_i} \boldsymbol{u}_{\rho_i}(\pi)$.

Table 5.2; $\rho_i = 11, ..., 20$

ρ_i	11	12	$\Rrightarrow$ 13	14	15	16	17	18	19	20
$\overline{u}_{\rho_i}(\sigma_{\rho_i})$	$12 \cdot 2^{10}$	$13 \cdot 2^9$	$14 \cdot 2^8$	$15 \cdot 2^7$	$16 \cdot 2^6$	$17 \cdot 2^5$	$18 \cdot 2^4$	$19 \cdot 2^3$	$20 \cdot 2^2$	22
$\underline{u}_{\rho_i}(\sigma_{\rho_i})$	0	0	0	0	0	0	0	0	0	22
$\overline{u}_{\rho_i}(\pi)$	2^{10}	2^9	$2 \cdot 2^8$	$4 \cdot 2^7$	$8 \cdot 2^6$	$16 \cdot 2^5$	$32 \cdot 2^4$	$63 \cdot 2^3$	500	500
$\underline{u}_{\rho_i}(\pi)$	0	0	$1 \cdot 2^8$	$3 \cdot 2^7$	$7 \cdot 2^6$	$15 \cdot 2^5$	$31 \cdot 2^4$	$62 \cdot 2^3$	500	500
$\eta_{[\overline{n},\pi]}(\rho_i)$	23/24	25/26	25/28	23/30	17/32	3/34	0	0	0	0

From these tables, we observe that the participation probability is quite high in many cases, e.g., up to $\rho_i = 12$, the participation probability is increasing to 25/26 from 1/2 at $\rho_i = 0$, and after it, it is decreasing (e.g. $\frac{8 \times 2^6}{16 \times 2^6} + \frac{1}{2} \cdot \frac{2^6}{16 \times 2^6} = \frac{8}{16} + \frac{1}{2} \cdot \frac{1}{16} = \frac{17}{32}$ at $\rho_i = 15$) to 0 at $\rho_i = 17$.

The above calculation indicates that our study is moving in a direction of a resolution of the SP paradox. So far, however, our theory includes no consideration of the banker's choice or financial moves. Since we introduced a budget, it would be a problem for the banker to meet a possible bankruptcy. Therefore, we need to consider a theoretical extension including the banker's behavior. This will be given in Section 6.

5.3 Proof of Theorem 5.1

If part: Axiom S1: $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta}) + \eta(\boldsymbol{\beta}; \boldsymbol{\alpha}) = 1$ for all $\boldsymbol{\alpha}; \boldsymbol{\beta} \in \Lambda(\boldsymbol{\lambda}; \boldsymbol{\mu})^2$. This is verified for the cases (0), (1), and (2). This is straightforward for (0). For (1), $\eta(\boldsymbol{\alpha}; \boldsymbol{\beta}) + \eta(\boldsymbol{\beta}; \boldsymbol{\alpha}) = \frac{\ell[\overline{\alpha}; \overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\beta}]} + \frac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \frac{\ell[\underline{\alpha}; \underline{\beta}]}{\ell[\boldsymbol{\beta}]} = \frac{\ell[\overline{\alpha}; \overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot (\frac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\beta}]} + \frac{\ell[\underline{\alpha}; \underline{\beta}]}{\ell[\boldsymbol{\beta}]}) = \frac{\ell[\overline{\alpha}; \overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} = 1$. (2) is similar.

Axiom S2 is proved by (1).

Consider Axiom S3. It holds by (0) that $\eta([\overline{\alpha}; \overline{\beta}]; \boldsymbol{\beta}) = 1$, $\eta(\boldsymbol{\beta}; \boldsymbol{\beta}) = \frac{1}{2}$, and $\eta([\underline{\beta}; \underline{\alpha}]; \boldsymbol{\beta}) = 0$. Consider case (2). Then, by (36),

$$\begin{aligned} \eta([\overline{\alpha}; \overline{\beta}]; \boldsymbol{\beta}) &= \tfrac{\ell[\overline{\alpha}; \overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot 1 + \tfrac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \tfrac{1}{2} + \tfrac{\ell[\underline{\beta}; \underline{\boldsymbol{\alpha}}]}{\ell[\boldsymbol{\alpha}]} \cdot 0 \\ &= \tfrac{\ell[\overline{\alpha}; \overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\alpha}; \overline{\beta}]; \boldsymbol{\beta}) + \tfrac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta(\boldsymbol{\beta}; \boldsymbol{\beta}) + \tfrac{\ell[\underline{\beta}; \underline{\boldsymbol{\alpha}}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\underline{\beta}; \underline{\alpha}]; \boldsymbol{\beta}) \\ &= \tfrac{\ell[\overline{\alpha}; \overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\alpha}; \overline{\beta}]; \boldsymbol{\beta}) + \tfrac{\ell[\overline{\beta}; \underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\beta}; \underline{\alpha}]; \boldsymbol{\beta}). \end{aligned}$$

This is Axiom S3. Case (1) is similarly proved.

Only-If part: It suffices to derive (34), (35), and (36) in the cases (0), (1), and (2) of Theorem 5.1.

(0): Let us see the left statement of (34); the right follows from Axiom S1. Suppose $\underline{\alpha} \geq \overline{\beta}$. If $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = 1$ by Axiom S2, and if $\boldsymbol{\alpha} = \boldsymbol{\beta}$, then $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{1}{2}$ by Lemma 5.1.

(1) and (2): To prove (1) and (2) of Theorem 5.1, first we prove the following lemma.

Lemma 5.3. Let $\boldsymbol{\zeta} = [\overline{\zeta};\underline{\zeta}]$ and $\boldsymbol{\zeta}' = [\overline{\zeta}';\underline{\zeta}']$ with $\overline{\zeta} = \overline{\zeta}'$. Then

(1): if $\overline{\zeta} > \underline{\zeta}' \geq \underline{\zeta}$, then $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{1}{2} \cdot \frac{\ell[\overline{\zeta};\underline{\zeta}']}{\ell[\boldsymbol{\zeta}]}$;

(2): if $\overline{\zeta} > \underline{\zeta} \geq \underline{\zeta}'$, then $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{1}{2} \cdot \frac{\ell[\boldsymbol{\zeta}]}{\ell[\boldsymbol{\zeta}']} + \frac{\ell[\underline{\zeta};\underline{\zeta}']}{\ell[\boldsymbol{\zeta}]}$.

Proof. (1): By Axiom S3 and (33), we have $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{\ell[\overline{\zeta};\underline{\zeta}']}{\ell[\boldsymbol{\zeta}]} \cdot \eta([\overline{\zeta};\underline{\zeta}'];\boldsymbol{\zeta}') + \frac{\ell[\underline{\zeta}';\underline{\zeta}]}{\ell[\boldsymbol{\zeta}]} \cdot \eta([\underline{\zeta}';\underline{\zeta}];\boldsymbol{\zeta})$. Since $[\overline{\zeta};\underline{\zeta}'] = \boldsymbol{\zeta}'$, we have $\eta([\overline{\zeta};\underline{\zeta}'];\boldsymbol{\zeta}) = \frac{1}{2}$ by Lemma 5.1. Since $\boldsymbol{\zeta}' \geq_I [\underline{\zeta}';\underline{\zeta}]$, we have $\eta([\underline{\zeta}';\underline{\alpha}];\boldsymbol{\zeta}) = 0$ by Axioms S1 and S2. Hence, $\eta(\boldsymbol{\zeta};\boldsymbol{\zeta}') = \frac{1}{2} \cdot \frac{\ell(\overline{\zeta};\underline{\zeta}')}{\ell(\boldsymbol{\zeta})}$.

(2): Switching $\boldsymbol{\zeta}$ with $\boldsymbol{\zeta}'$, (2) is reduced to (1).■

Now, we return to the proof of the only-if part. To prove (1) and (2) of Theorem 5.1. We write the assumptions of (1) and (2) explicitly:

(A): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\alpha} \geq \underline{\beta}$, and both $\boldsymbol{\alpha};\boldsymbol{\beta}$ are non-singular;

(B): $\overline{\alpha} \geq \overline{\beta} \geq \underline{\beta} \geq \underline{\alpha}$,and $\boldsymbol{\alpha}$ is non-singular.

Suppose (A). Then,

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell(\overline{\alpha};\overline{\beta})}{\ell(\boldsymbol{\alpha})} \cdot \eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) + \frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \eta([\overline{\beta},\underline{\alpha}];\boldsymbol{\beta}). \tag{38}$$

In the first term of the right-hand side, we have $\eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) = 1$ by Axiom S2. The second term is reduced by Lemma 5.3.(1), and we have $\frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \eta([\overline{\beta},\underline{\alpha}];\boldsymbol{\beta}) = \frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \frac{1}{2} \cdot \frac{\ell([\underline{\alpha};\underline{\beta}])}{\ell(\boldsymbol{\beta})}$. Thus, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell(\overline{\alpha};\overline{\beta})}{\ell(\boldsymbol{\alpha})} + \frac{1}{2} \cdot \frac{\ell(\overline{\beta};\underline{\alpha})}{\ell(\boldsymbol{\alpha})} \cdot \frac{\ell([\underline{\alpha};\underline{\beta}])}{\ell(\boldsymbol{\beta})}$, which is (1) of the theorem.

Consider (B). By (33), we have

$$\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) + \frac{\ell[\overline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}). \tag{39}$$

By Axiom S2, $\eta([\overline{\alpha},\overline{\beta}];\boldsymbol{\beta}) = 1$. When $\boldsymbol{\beta}$ is singular, we $\eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = 0$ by Axiom S2; thus, $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]}$. Suppose that $\boldsymbol{\beta}$ is non-singular. The second term is decomposed into

$$\eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = \frac{\ell[\overline{\beta};\underline{\beta}]}{\ell[\boldsymbol{\beta}]} \cdot \eta([\overline{\beta};\underline{\beta}];\boldsymbol{\beta}) + \frac{\ell[\underline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} \cdot \eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta}).$$

Lemma 5.1, we have $\eta([\overline{\beta};\underline{\beta}];\boldsymbol{\beta}) = \frac{1}{2}$. If $\underline{\beta} = \underline{\alpha}$, then $\frac{\ell[\underline{\beta};\underline{\alpha}]}{\ell[\boldsymbol{\alpha}]} = 0$, and if $\underline{\beta} > \underline{\alpha}$, then by Axiom S2, $\eta([\underline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = 0$. Summing up these, we have $\eta([\overline{\beta};\underline{\alpha}];\boldsymbol{\beta}) = \frac{1}{2}$**.** Plugging this to (39), we have $\eta(\boldsymbol{\alpha};\boldsymbol{\beta}) = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} = \frac{\ell[\overline{\alpha};\overline{\beta}]}{\ell[\boldsymbol{\alpha}]} + \frac{\ell[\boldsymbol{\beta}]}{\ell[\boldsymbol{\alpha}]}$.■

6 Monopoly Market of an SP Coin-tossing Gamble

We formulate a monopoly market of an SP coin-tossing gamble with a banker and people. People's inclinations to participate in the market follow the theory given in Sections 3, 4, and

5. The banker has a criterion, expressed by the index of profit/investment, to open or not the market. The behavior of the market is analyzed by the Monte Carlo simulation method. We show that with some specifications of parameters, some people incline to buy an SP coin-tossing gamble and the banker produces the profits satisfying the criterion.

6.1 Monopoly market with a banker and people

We express the basic parameters of the *SP gamble market* with a banker and people as $(\{0\} \cup M, \overline{n}, \pi, C \mid \boldsymbol{\rho})$:

$S0$: 0 is the *banker facing people* $M = \{1, ..., |M|\}$;

$S1$: $\overline{n} > 0$ is a *budget degree* (natural number) for banker 0, and $2^{\overline{n}}$ is the maximum prize and show-money;

$S2$: π is a *participation fee* which is an even number with $2 \leq \pi$ and $\pi \neq \overline{n} + 1$;

$S3$: $C(\ell)$ is a facility cost when ℓ people participate in the gamble;

$S4$: ρ_i is a *cognitive degree* (natural number) of person $i \in M$ and $\frac{1}{2^{\rho_i}}$ is his *cognitive bound*.

The banker and people are given in $S0$. We consider only $|M| = 1,000$ in the examples in this section. In $S1$, the maximum prize (show-money) $2^{\overline{n}}$ for banker 0 is given. In $S2$, a participation fee π is given, and is assumed to be even for simplicity. Condition $\pi \neq \overline{n} + 1$ is already assumed in Section 1.1. Since the SP gamble market is organized by the banker, he knows $S1$ to $S2$ but does only some people are gathering. The last parameters $\boldsymbol{\rho} = \{\rho_i\}_{i \in M}$ in $S3$ are subjective and unobservable. When person $i \in M$ considers the SP coin-tossing, he may find his own cognitive bound ρ_i. Our concern is to consider whether or not there are some π and $\boldsymbol{\rho} = \{\rho_i\}_{i \in M}$ so that the SP gamble market is actively played and bankruptcy would happen with a small probability.

In the following examples, the facility cost $C(\ell)$ is assumed to be a step function having a step size for 100 people, and the unit size requires cost 200¢ × 100. Formally, it is given as

$$C(\ell) = 200 \cdot 100 \cdot \lceil \tfrac{\ell}{100} \rceil, \tag{40}$$

where $\lceil \frac{\ell}{100} \rceil$ is the smallest natural number not less than $\frac{\ell}{100}$. For example, when $\ell = 120$, two steps are applied, i.e., $C(120) = 200 \cdot 100 \cdot \lceil \frac{120}{100} \rceil = 4,000$¢. In the following, we use \$ rather than ¢ for simplicity.

The banker borrows the show-money $2^{\overline{n}}$ + the facility cost $C(\ell)$ from some financial institute and the banker returns $2^{\overline{n}} + C(\ell)$ with some interests to the institute after the SP market. We do not consider a specific interest rate; but instead, we adopt an index of profits/investments in Section 6.2.

The SP gamble market are formulated in Fig.6. First, the banker 0 decides to open or not the gamble market with a participation fee π to be announced to the people. We study whether or not this would be profitable for the banker, and if it is profitable, it is a candidate for an actual choice of the banker. Our question is whether such an SP gamble exists. After its announcement, each person decides (or inclines) to participate in the gamble. These people go to the coin-tossing in a given order, until all participants finish coin-tossing or the banker goes bankrupt, i.e., banker 0 cannot pay the the maximum prize $2^{\overline{n}}$ to the next person. Since coin-tossing is independent for people and fee π is paid at the time of coin-tossing, the results and ordering of coin-tossing do not affect people's choices.

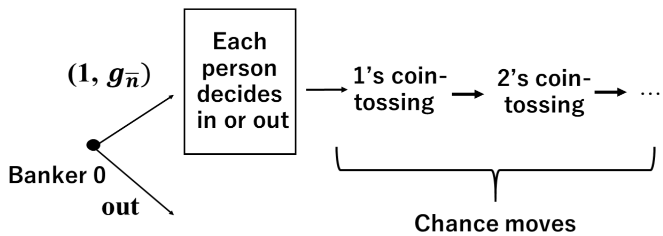

Figure 7: The SP gamble market

Each person's concern is to participate in the coin-tossing with fee π or not; this depends upon the show-money $2^{\overline{n}}$, fee π, and his subjective cognitive bound ρ_i. He knows the rules of the gamble but his effective understanding is restricted by ρ_i. If $\rho_i \geq \overline{n} - 1$, he goes to the last $2^{\overline{n}}$, but if $\rho_i \leq \overline{n} - 1$, then he would stop at 2^{ρ_i+1}. The latter case is our central interest, described in Fig.3. Recall the stipulation $\widehat{\rho}_i = \min(\rho_i, \overline{n} - 1)$.

We consider the following three examples of cognitive degrees $\rho = \{\rho_i\}_{i \in M}$; it is distributed over the people M described in Table 6.1.

Table 6.1; distributions of cognitive degrees (%)

ρ_i	$\cdots$ 7	8	9	10	11	12	13	14	15	16	17	18	19$\cdots$
I: $\varphi_I(\rho_i)$	$\cdots$ 0	5	10	20	30	20	10	5	0	0	0	0	0$\cdots$
II: $\varphi_{II}(\rho_i)$	$\cdots$ 0	0	0	5	10	20	30	20	10	5	0	0	0$\cdots$
III: $\varphi_{III}(\rho_i)$	$\cdots$ 0	0	0	0	0	5	10	20	30	20	10	5	0$\cdots$

In Case I, ρ_i is distributed from $\rho_i = 8$ to 14 with density function φ_I. The cognitive bound of the least precise person is $\frac{1}{2^8} = \frac{1}{256}$, meaning that he thinks only up to the event that the SP coin tossing goes to the 8-*th* toss. The number of people of this type is $1000 \times \frac{5}{100} = 50$. The most precise person has cognitive degree $\rho_i = 14$, i.e., his cognitive bound is $\frac{1}{2^{14}} = \frac{1}{16384}$, meaning that he thinks about up to the 14-*th* coin-toss. In Cases II and III, the distribution of I is shifted to the upper direction by 2 and 4 degrees. Effects of these shifts will be considered after the calculations of the total % of participants.

We consider how many people are expected to buy the SP coin-tossing gamble in Examples 4.1 and 5.1.

Example 6.2 (Small): Let $\overline{n} = 17$ and $\pi = 500¢ = 5\$$. Using the values of $\eta_{\overline{n},\pi}[\sigma_{\rho_i}] = \eta(\boldsymbol{u}_{\rho_i}(\sigma_{\rho_i}); \boldsymbol{u}_{\rho_i}(\pi))$ given in Tables 4.1 and 4.2, we calculate, in Case I,

$$\eta_{[\overline{n},\pi]} = \sum_{\rho_i=0}^{17} \varphi_I(\rho_i) \cdot \eta_{\overline{n},\pi}[\sigma_{\rho_i}] \doteqdot 38.0\%, \tag{41}$$

and in Case II, $\eta_{[\overline{n},\pi]} = \sum_{\rho_i=50}^{17} \varphi_{II}(\rho_i) \cdot \eta_{\overline{n},\pi}[\sigma_{\rho_i}] \doteqdot 7.2\%$. In Case I, 38.0% of people $M = \{1, ..., 1000\}$ incline to participate in the SP coin-tossing gamble, and in Case II, the percentage is smaller, 7.2%. In Case III, no body participate in the gamble. The total %'s are given also for $\pi = 3\$, 10\$, 20\$$, and $50\$$; for these, $\boldsymbol{u}_{\rho_i}(\pi)$ in Tables 4.1 and 4.2 should be recalculated. For $\pi = 1\$$, the banker's profits are negative because of (40); we start this table with $\pi = 3\$$.

Table 6.2; Total % of participants

$\overline{n} = 17 \ \backslash\ \pi$	3\$	5\$	10\$	20\$	50\$
Case I	59.0%	38.0%	15.2%	5%	0%
Case II	21.7%	7.2%	1.1%	0%	0%
Case II	1.3%	0%	0%	0%	0%

This table states that the total percentage decreases as the participation fee π becomes higher. Yet, the detailed numbers differ between Cases I and II; more people participate in Case I than in Case II. This difference is because the cognitive degrees in Case II are distributed in a higher region than in Case I and people with higher cognitive degrees are more cautious.

This example tells that we avoid the absurd consequence in Table 1.1 that when $\overline{n} = 17$ and people have precise cognitive degrees, i.e., $\rho_i \geq 16$, no people are attracted in the gamble if $\pi \geq 19¢$.

Now, we calculate the percentages in the large example with $2^{\overline{n}} = 2^{21}$.

Example 6.2 (Large): Let $\overline{n} = 21$. The maximum prize is $2^{21}¢ \doteqdot 20,972\$$. In the same manner as in (41), we calculate $\eta_{[\overline{n},\pi]}$ in Cases I, II, III for π from 3\$,to 500\$:

Table 6.3; Total % of participants

$\backslash\ \pi$	3\$	5\$	20\$	50\$	100\$	200\$	300\$	500\$
Case I	94.4%	94.1%	78.8%	52.8%	29.3%	11.8%	5.6%	1.4%
Case II	84.3%	81.5%	51.2%	15.3%	5.0%	0.7%	0%	0%
Case III	60.8%	46.4%	8.3%	1.3%	0%	0%	0%	0%

This table shows similar qualitative shapes to that of Table 6.2, but are quite different quantitatively; the numbers in the corresponding columns are much larger here. For example, for $\pi = 5\$$, the total %'s in Cases I, II, and III are 94.1, 81.5, and 46.4 for $\overline{n} = 21$ and 38.0, 7.2, and 0 for $\overline{n} = 17$. This difference is caused by the large show-money $2^{21}¢ \doteqdot 20,972\$$. [14] A question of which case I, II, or III is more suitable is a question on an empirical fact on cognitive bounds. An experimental study of such facts is an open question.

6.2 Dynamics of budgetary changes with an SP coin-tossing

An SP coin-tossing gamble includes large prizes, and it possibly induces the banker to have large payments. Therefore, a market structure to manage coin-tossing for participants should be specified; one important constraint is a temporal budget changing by having participation fees and paying prizes. There are many possible market structures, but here, we consider a simple structure, which still needs to take the banker's bankruptcy into account.

Suppose that people in $M(\overline{n}, \pi) \subseteq M$ have chosen to participate in the coin-tossing, where the cardinality of $M(\overline{n}, \pi)$ is the nearest whole number, ℓ, rounded off from $|M| \cdot \eta_{[\overline{n},\pi]}$. We assume that $M(\overline{n}, \pi)$ is enumerated as $\{1, 2, ..., \ell\}$ for simplicity, and each person $i \in M(\overline{n}, \pi)$

[14] We note that our theory does not include budget constraints for people; Case I may need to have budgets for people.

does coin-tossing one-by-one following the order $1, 2, ..., \ell$, which is illustrated as the latter part in Fig.9. The initial budget of banker 0 is $2^{\overline{n}} + c$. For person i's SP coin-tossing, the budget increases by π but decreases by the prize 2^{k_i} if he reaches the k_i-*th* toss. Coin-tossing follows the rule of the SP coin-tossing gamble, and the objective distribution $\tau_{\overline{n}}$ is relevant - - subjective understanding σ_{ρ_i} is no longer relevant to actual coin-tossing.

Since the SP coin-tossing is repeated, the following rule is effective:

(Guarantee rule): any prize in the coin-tossing $\sigma_{\overline{n}}$ should be paid by the banker.

In the process, the banker may become not able to pay a next possible prize, which is *bankruptcy*. The process may go to the last person ℓ without bankruptcy, but the banker may goes bankrupt before ℓ. Let $(n_1, ..., n_{\ell_B})$ be a finite sequence with $n_i \in \{1, ..., \overline{n}\}$ for $i = 1, ..., \ell_B \leq \ell$; each n_i means that coin-tossing results in the tails in the n_i-th toss. In this case, we denote the budget $B(i)$ in the beginning of person i's coin tossing; person i pays participation fee π and starts coin-tossing. After ℓ_B, the banker goes bankrupt.

We define the sequence of budgets $(B(0), B(1), ..., B(\ell_B))$ by

b0: $B(0) = 2^{\overline{n}}$;

b1: $B(i) = B(i-1) + \pi - 2^{n_i}$ for $i = 1, ..., \ell_B$;

b2: $B(i) + \pi \geq 2^{\overline{n}}$ for $i = 1, ..., \ell_B - 1$,

b3: if $\ell_B < \ell$, then $B(\ell_B) < 2^{\overline{n}} - \pi$.

Before the start, the banker prepares the initial budget $B(0)$. Person 1 pays participation fee π and then makes coin-tossing. If coin tossing goes to n_1, the banker pays 2^{n_1} cents to the banker. After this payments, the resulting money amount becomes $B(1) = B(0) + \pi - 2^{n_1}$ and is brought to the second coin-tossing if $B(1) + \pi \geq 2^{\overline{n}}$. If $B(1) + \pi < 2^{\overline{n}}$, then the banker cannot guarantee the maximum prize $2^{\overline{n}}$ for person 2 and goes bankrupt. These are described in b0 to b3. In general, this process goes up to person ℓ_B with bankruptcy or to the last person ℓ. For simplicity, we include the case $B(\ell) < 2^{\overline{n}} + \pi$ in bankruptcy.

As stated, the banker borrows the show-money $2^{\overline{n}}$ and the facility cost $C(\ell)$ from some financial institution, and the banker should return $2^{\overline{n}} + C(\ell)$ with an interest rate r after the SP market. The banker's income may have a lower bound and his $B(\ell_B) - (1 + r)(2^{\overline{n}} + C(\ell))$ should be larger than the bound. However, we consider only the *return of investment, ROI*, index given as

$$ROI = \frac{B(\ell_B) - (2^{\overline{n}} + C(\ell))}{2^{\overline{n}} + C(\ell)}. \tag{42}$$

When this index ROI is positive, the banker generates a positive profit after returning $(1 + r)(2^{\overline{n}} + C(\ell))$ with $r < ROI$, and if ROI is non-positive, the banker cannot make a profit. We take this index ROI as an economic index to judge whether the banker possibly opens the SP market. The lower bound for his income will be considered in interpretations of this index in addition to the absolute number of $B(\ell_B) - (2^{\overline{n}} + C(\ell))$ in the examples.

The above process $(B(0), B(1), ..., B(\ell_B))$ is a stochastic process. If includes no bankruptcy, the process can be reduced to a stationary process, but since it includes a possibility of bankruptcy, it can not be reduced to a stationary one. It is analytically difficult to evaluate the possibility of bankruptcy and the behavior stochastic process. Instead, we adopt the *Mont Carlo method* to evaluate the distribution of $(B(0), B(1), ..., B(\ell_B))$. Each coin-tossing is implemented by a random generator and a run follows the rules b0-b3. We take $10,000$ iterations of this run.

Separating the facility cost $C(\ell)$, we calculate the average AV of $B(\ell_B) - B(0)$ over such $10,000$ runs where $B(0) = 2^{\overline{n}}$. Using this, the index ROI is expressed as $[AV - C(\ell)]/[B(0) + C(\ell)]$.

We have the calculation results.

Example 6.1 (Small-continuing): In each case, we make $10,000$ iterations of each coin-tossing and take the averages of $B(\ell_B) - 2^{17}$. In the case $\pi = 3\$$ of Table 6.4, $\ell = 590$ people participate in the coin-tossing, and 3.1% of $10,000$ iterations meet bankruptcy. The (average/$10,000$) revenue for the banker is 1623\$. The ROI index is 0.17; the banker makes a positive profits even after paying the facility cost $C(590) = 1200\$$. In Case I, the index ROI is increases up to 10\$ with π even when the number ℓ decreases. In Case II, the index ROI is positive only for $\pi = 5\$$, and in Case III, it is negative.

The bankruptcy frequency is 3.1% in Case I with $\pi = 3\$$. It is decreasing with π, since the accumulation of participation fees prevents a possible bankruptcy; typically, the bankruptcy possibility itself is decreasing for later people. These bankruptcy percentages can be regard as small.

Table 6.4; Case I

$\backslash\pi$	3\$	5\$	10\$	20\$
ℓ	590	380	152	50
$AV\$$	1623	1,799	1482	988
ROI	0.17	0.47	0.63	0.52
b.ruptcy	3.1%	2.0%	0.9%	0.4%

Table 6.5; Case II

$\backslash\pi$	3\$	5\$	10\$	20\$
ℓ	217	72	11	0
$AV\$$	597	343	108	–
ROI	-0.00	0.09	-0.06	–
b.ruptcy	2.9%	1.5%	0.6%	–

Table 6.6; Case III

$\backslash\pi$	3\$	5\$ ...
ℓ	13	0
$AV\$$	36	–
ROI	-0.11	–
b.ruptcy	1.4%	–

What are observed from Tables 6.4 to 6.6? Apparently in Table 6.6, the number of participants in the SP gamble is too small for the banker to have a positive profits, while in Table 6.4, some people participate in it and the banker possibly gets reasonable positive profits. Table 6.5 is closer to Table 6.6. These differences are caused by the differences in distributions of cognitive bounds in Table 6.1. That is, when people have low cognitive bounds, i.e., Case I, more people participate and when people have high cognitive bounds, i.e., Case III, people avoid SP gamble.

Let us go to the main example in this paper.

Example 6.2 (Large-continuing): In this example, the show-money $2^{\overline{n}}$ is 16 times larger than that in Example 6.1. When $2^{\overline{n}}$ becomes larger, more people are incentivized to buy the gamble; for example, in Case I, when $\overline{n} = 17$, the number of participants is 380 for $\pi = 5\$$, but when $\overline{n} = 21$, the number becomes 941. This increases the gross profits from $1,623\$$ to $4,403\$$, but the banker should pay back some interests to the financial institute in addition to the borrowed amounts $2^n + C(\ell)$. The index is $ROI = 0.47$ for $\overline{n} = 17$, but it is $ROI = 0.09$ for $\overline{n} = 21$. When the interest rate is 0.20 between the banker and institute, the banker's profits are negative.

In Table 6.7, the profit/investment index ROI takes the maximum around $\pi = 100\$$; the ROI index is much larger than 0.2 and the absolute profits are also much than the corresponding value in Table 6.4.

Table 6.7; $\overline{n} = 21$ & Case I

$\backslash\ \pi$	3\$	5\$	20\$	50\$	100\$	200\$	300\$	500\$
$\#_{\text{participants}}$	944	941	788	528	293	118	56	14
$AV\$$	2,555	4,403	15,491	26,270	29,219	23,566	16,783	6,997
ROI	0.02	0.09	0.62	1.13	1.33	1.08	0.78	0.32
$_{\text{bankruptcy}}\%$	3.2	2.4	0.7	0.1	0.07	0.04	0.03	0.01

Table 6.8; $\overline{n} = 21$ & Case II

$\backslash\ \pi$	3\$	5\$	20\$	50\$	100\$	200\$	300\$
$\#_{\text{participants}}$	843	815	512	153	50	7	0
$Av\$$	2274	3841	10091	7612	4988	3496	–
ROI	-0.04	0.09	0.40	0.34	-0.01	0.16	–
$_{\text{b.ruptcy}}\%$	3.3	2.1	0.5	0.2	0.1	0.0	–

In Table 6.8, the index ROI and absolute value of profits are smaller than in Table 6.7. In Table 6.9, the index ROI does not exceed 0.10. Thus, a possible trade between the banker and people depends upon the distribution of people's cognitive bounds.

Table 6.9; $\overline{n} = 21$ & Case III

$\backslash\ \pi$	3\$	5\$	20\$	50\$	100\$
$\#_{\text{participants}}$	608	464	83	13	0
$Av\$$	1643	2184	1640	647	–
ROI	0.01	0.05	0.07	0.002	–
$_{\text{bankruptcy}}\%$	3.3	2.1	0.3	0.1	–

Table 6.7 to 9 with $\overline{n} = 21$ considerably differ from Table 6.4 to 6 with $\overline{n} = 17$. In Table 6.7, the numbers of participants may be too large; for example, for $\pi = 100\$$, 293 people would participate out of 1000. Table 6.8 has a less tendency; for $\pi = 20\$$, 512 people participate. Table 6.9 has a much less tendency; for $\pi = 5\$$, 464 people participate. These differences are caused by the differences in the distribution of cognitive bounds in Table 6.1. At least for the present author, the result in Table 6.7 that 293 people would participate with $\pi = 100\$$ in Case I is incompatible with his observations in Japan. The above other two results are less but they sound too much. One possibility is to discount participations in the coin-tossing, since some people's refusal to choose from π and σ_{ρ_i} is interpreted as behavioral equivalent to non-participation.

To have better answers, we need to test more examples with other parameter values, such as $\overline{n}$, distributions of ρ_i, and even refusal percentages. Such calculations may give a better understanding of the structure of the theory, and a hint even on possible experiments.

As long as a resolution of the SP paradox is concerned, some references to ordinary observations/experiences are perhaps unavoidable. The above examples are interpreted as progress toward a resolution of the SP paradox.

7 Conclusions

Let us evaluate our theory relative to the three criteria C1, C2, and C3 for a resolution of the SP paradox in Section 1.2. Criterion C1, to have an SP coin-tossing gamble to attract people and produces reasonably positive profits, is apparently satisfied. Sections 3, 4, and 5 are relevant for C2 and they are more delicate. Section 6 is relevant for C3 and it is less delicate than C2.

Consider Sections 3, 4, and 5 with respect to C2. The methods discussed in Sections 3 and 4 are regarded as natural, since they follow with a general scientific method of measurement, i.e., the measurement step with a well defined scale and the logical extension step. Also, the theory is partial to the targeted alternatives and subjective distribution following the general idea of bounded intelligence, which is a part of Simon's [27] bounded rationality.

Semi-rationalistic choice for incomparability in Section 5 is a generalization of the rationalistic choice in Sections 3 and 4, though people are bounded by cognitive degrees. One element should be discussed; Axiom S3 (proportional decomposition) may be too stringent. The present author is not very happy with this axiom. It may be relaxed, keeping the result (Theorem 5.1) in a qualitative manner. Numerical examples remain with the same tendencies. This relaxation remains open.

Criterion C3 is applied to Section 6; the theory in Sections 3 and 4 include only parameters, budget degree $\overline{n}$ and cognitive degree ρ_i, and theory treats them as arbitrary. Only in Section 6, these parameters are tested with specific values, that is, $\overline{n} = 17$ and 21, and Cases I, II, and III for $\rho = \{\rho_i\}_{i \in M}$. Choice $\overline{n} = 21$ was discussed in Section 1.1, and $\overline{n} = 17$ was considered for the reference purpose. Case II is our main concern, and Cases I and II are given to see sensitivity of the parameter changes. These examples show reasonably positive profits with some parameter values and negative results with other parameter values. Our theory passes Criterion C3. Nevertheless, it would be fruitful to have more numerical tests to have more precise evaluations of our theory for C3.

As long as a resolution of the SP paradox is concerned, some references to ordinary observations/experiences are perhaps unavoidable, which is the nature of criterion C2. To have better answers, we test more examples with other parameter values, such as $\overline{n}$, distributions of ρ_i, and even refusal percentages. Such calculations may give a better understanding of the SP paradox and a hint on further analysis.

The theory may be modified to apply it to similar but different problems where some events take place with very small probabilities; one example, problems of investments. The other class of important problems are; events have huge negative effects with very small probabilities ("black swan" in Teleb [29]). When probabilities are very small, typically, people ignores the probabilities for such event, while they logically aware the events. This is quite close to our theory. It would be a challenge to extend our theory to such seemingly related problems.

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