WINPEC Working Paper Series No.E2013
September 2020

The Education Sector and Economic Growth: A First Study of the Uzawa Model

Kazuyuki Sasakura

Waseda INstitute of Political EConomy
Waseda University
Tokyo, Japan

The Education Sector and Economic Growth: A First Study of the Uzawa Model

Kazuyuki Sasakura*

Abstract: The Uzawa-Lucas model is a benchmark model in endogenous growth theory. But Lucas (1988) is so influential that the Uzawa-Lucas model is virtually the Lucas model. This paper distinguishes between the Uzawa (1965) model and the Lucas model, and examines the Uzawa model in detail. It is certain that the two models have much in common. However, there are also important differences. Economically the Uzawa model assumes full employment, whereas the Lucas model admits unemployment. Mathematically the maximum growth rate must be smaller than the rate of time preference in the former, whereas the opposite must hold in the latter.

Key words: Education Sector, Economic Growth, Uzawa-Lucas Model
JEL classification: E13, O41, O43

1. Introduction

The Uzawa-Lucas model is a symbolic name for an endogenous growth theory regarding education (or learning) as the engine of economic growth. Under the influence of Schultz (1961) who emphasized the importance of investment in human beings for explaining the puzzle of large discrepancies between the rates of growth of inputs and outputs, Uzawa (1965) introduced the education sector into Solow's (1956) neoclassical growth model. A crucial part of the Uzawa model is a *linear* labor efficiency function which makes it possible that there exists a persistent growth in a two-sector model of production and education. It was a natural extension of Solow's exogenous growth model and the earliest endogenous model connecting education and economic growth.[1]

Unfortunately, despite its striking originality, the model had been forgotten for a long time partly because of the downward trend in growth theory as a whole in the 1970's and 1980's. Then, it is Lucas (1988) who resuscitated it. Lucas (1988) took as a serious problem the observed diversity across countries in per capita income levels and its consequences for human welfare, saying "Once one starts to think about them,

---

* Faculty of Political Science and Economics, Waseda University, 1-6-1 Nishiwaseda, Shinjuku-ku, Tokyo 169-8050, Japan. E-mail: sasakura@waseda.jp

[1] Nelson and Phelps (1966) is also an incipient theoretical attempt to connect economic growth with education, though their model is not an endogenous one.

it is hard to think about anything else," as is often cited. He needed a theory of economic development to cope with such a problem, and it is the Uzawa model that was chosen as the theoretical basis. In the Lucas model too, an indispensable part is a *linear* learning function which is the counterpart of Uzawa's labor efficiency function.[2]

Unlike in the Uzawa model, however, there is an external effect in the production sector of the Lucas model. The growth rate is depressed due to the effect. Such a situation corresponds, Lucas (1988) argues, to actual economies. Anyway, with or without an external effect, the Uzawa model with a linear labor efficiency function and the Lucas model with a linear learning function have much in common both economically and mathematically. Hence the name the Uzawa-Lucas model. Since Lucas (1988), numerous studies have been made under the name of the Uzawa-Lucas model (or sometimes the Lucas-Uzawa model).

As for examples without an external effect, Caballé and Santos (1993) shows that every positive initial condition converges to some steady state. Faig (1995) introduces government consumption as well as private consumption and analyzes the response to technology and government spending shocks. Ortigueira (1998) examines the implications of tax policies. Boucekkine and Ruiz-Tamarit (2008) and Chilarescu (2011) pursue rigorously transitional dynamics toward a unique steady state by virtue of closed-form solutions. Canton (2002) (in discrete time) and Tsuboi (2018) (in usual continuous time) analyze the stochastic Uzawa-Lucas model with uncertainty in the education sector. And among them is Lucas (1990) too who proposes the best structure of income taxation using the CES production function and the utility function with leisure.

On the other hand, as for examples with an external effect in the production sector as in Lucas (1988), Mulligan and Sala-i-Martin (1993) calculate the steady state and simulate transitional paths toward it. Xie (1994) shows that when the external effect is relatively strong, there exists a continuum of equilibrium paths starting the same initial condition against Lucas's conjecture.[3] Gómez (2003) derives a fiscal policy which leads to the first-best optimum equilibrium. Hiraguchi (2009) obtains a closed-form solution by applying the method of Boucekkine and Ruiz-Tamarit (2008). Finally, as for other examples, Chamley (1993) and Kuwahara (2017) prove the existence of multiple steady states in the Uzawa-Lucas model with an

[2] Romer (1990) also adopts Uzawa's linearity assumption in the specification of his design function.

[3] See also footnote 20 below.

external effect in the education sector.

As is seen from the above, the Uzawa-Lucas model has been widely used as a benchmark model in endogenous growth theory, and for certain will be so. But I wish to point out that what is called the Uzawa-Lucas model is in fact the Lucas (1988) model, not the Uzawa (1965) model. Indeed, they resemble each other considerably as said above, but there are also differences which cannot be ignored economically or mathematically. Economically, the Uzawa model assumes full employment of workers, whereas the Lucas model admits (voluntary) unemployment. That is, in the former model workers are always employed either in the production sector or in the education sector, and without workers in the education sector there is no economic growth. In the latter model a worker must attend school (or rather read books at home) instead of working in order to increase human capital (i.e., embodied knowledge and skills) and as a result enhance his/her wage as well as economic growth. Mathematically, the crucial difference is that the utility function is *linear* in consumption in the Uzawa model, whereas it is the CRRA in the Lucas model. Needless to say, the CRRA utility function is generally accepted in economics. All authors cited above (except Uzawa) adopt it. On the other hand, a linear utility function belongs to a special class in economics. It can be regarded mathematically as a degenerate or limiting case of the CRRA utility function. Thus, judging from such a relation between the two utility functions, one might be tempted to infer that the Uzawa model is a limiting case of the Lucas model. However, as will be shown below, it is not. The two models cannot be applied to the same economy.[4]

The purpose of this paper is to examine the Uzawa model in detail. As far as I know, the model has never been analyzed even briefly. So this is a first study of it. The paper is structured as follows. Section 2 compares the Lucas model and the Uzawa model and pays attention to five conditions for the steady state in the Uzawa model. Section 3 considers economic implications of the five conditions, while Section 4 specifies the labor efficiency function in the Uzawa model and derives some results. Section 5 concludes.

2. The Lucas Model vs. the Uzawa Model

The economy under consideration starts from time 0 and has population $L(t)$ at

---

[4] A linear utility function is not denied at least theoretically. For example, Arrow (1962), Romer (1986), and Aghion and Howitt (1992) also make use of it to demonstrate their endogenous growth models. On the other hand, Mankiw et al. (1985) cast a serious doubt on the CRRA utility function from an empirical point of view. Lucas (2003) discusses the merits and demerits of the CRRA utility function.

time t which is equal to total supply of labor and grows at a constant rate n

$$\dot{L}(t) = nL(t), L(0) > 0. \tag{1}$$

First, let us review the basic structure of the Lucas (1988) model (without an external effect) as compactly as possible.[5] The production technology is described by the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha}\big(h(t)u(t)L(t)\big)^{1-\alpha}, 0 \leq u(t) \leq 1, 0 < \alpha < 1, \tag{2}$$

where $Y(t)$, $K(t)$, $h(t)$, and $u(t)$ represent aggregate net output, aggregate physical capital, human capital per worker, and the fraction of a unit time of a worker devoted to producing output, respectively. huL is total effective labor at time t.[6] The equation of physical capital accumulation is written as

$$\dot{K} = K^{\alpha}(huL)^{1-\alpha} - Lc, \tag{3}$$

where c is per capita consumption and so Lc is aggregate consumption at time t.

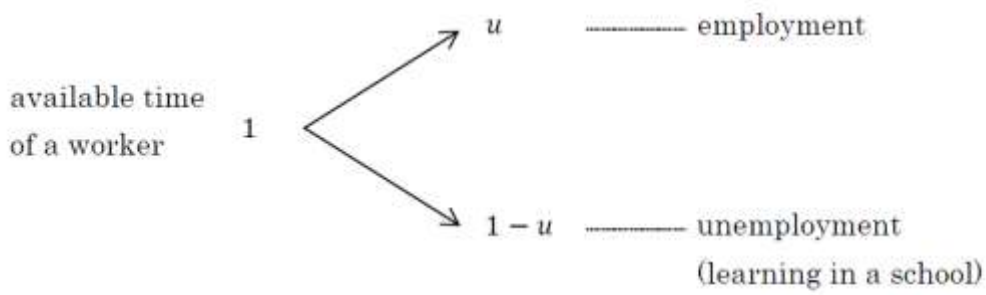

Figure 1. The Lucas Model

The accumulation of human capital is assumed to follow

$$\dot{h} = \delta(1-u)h, \delta > 0. \tag{4}$$

---

[5] Although it is the influence of the external effect of human capital in the production sector that is the selling point of Lucas (1988), this paper focuses on the case of no external effect in order to compare the Lucas model and the Uzawa model.
[6] In what follows, time t of variables is dropped unless confusion is involved.

$1-u$ on the right-hand side is the fraction of available time $(=1)$ of a worker devoted to increasing human capital. How? Lucas (1988) mentions as an example a worker going to school instead of working. In this sense (4) is often referred to as a learning function. Then, $1-u$ can be interpreted as a *rate of voluntary unemployment*. Figure 1 shows such a worker's choice between working and *not working*. When $u=1$, i.e., all workers produce output without going to school at all, human capital does not change. On the other hand, when $u=0$, i.e., all workers go to school without producing output at all, the growth rate of human capital reaches a maximum feasible rate δ.

By controlling c and u under the constraints (3) and (4) consumers (or workers) maximize the discounted sum of utility

$$\int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} L\, e^{-\rho t} dt, \sigma, \rho > 0,$$

where ρ is a rate of time preference and σ is the degree of relative risk aversion (the inverse of which is the intertemporal elasticity of substitution). In order to solve this problem, construct the current-value Hamiltonian with θ_1 and θ_2 as costate variables with respect to (2) and (4) respectively

$$H = \frac{c^{1-\sigma}}{1-\sigma} L + \theta_1[K^\alpha (huL)^{1-\alpha} - Lc] + \theta_2[\delta(1-u)h].$$

Applying Pontryagin's Maximum Principle and arranging the results yields the Euler equation

$$\sigma \frac{\dot{c}}{c} = \alpha K^{\alpha-1}(huL)^{1-\alpha} - \rho, \tag{5}$$

and the steady-state values of the fraction u and the growth rate of human capital $\dot{h}/h$[7]

$$u^* = \frac{\rho - n - \delta(1-\sigma)}{\sigma\delta},$$

[7] A superscript $*$ means a steady-state value or a value on a balanced growth path in what follows.

$$\left(\frac{\dot{h}}{h}\right)^* = \delta(1-u^*) = \frac{\delta-(\rho-n)}{\sigma}. \tag{6}$$

The economically meaningful situation $0 < u^* < 1$ requires that

$$\delta(1-\sigma) < \rho - n < \delta. \tag{7}$$

It follows from (5) that in the steady state

$$\alpha {K^*}^{\alpha-1}(h^* u^* L)^{1-\alpha} = \rho + \sigma\left(\frac{\dot{h}}{h}\right)^*. \tag{8}$$

This is well known as the modified-golden-rule state.

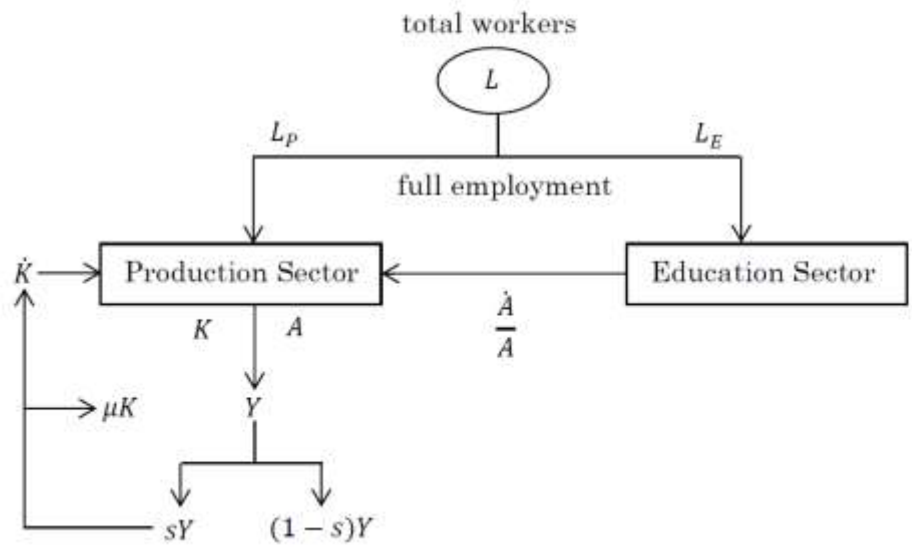

Figure 2. The Uzawa Model

Next turn to the Uzawa (1965) model. Population evolves in the same way as (1). But it is divided into those who work in the production sector, L_P, and those who work in the education sector, L_E, as follows:

$$L = L_P + L_E.$$

Simply speaking, L_P is the number of people producing goods in factories, while L_E is that of people teaching in schools to improve the quality of labor (or labor efficiency). Thus, there is *no unemployment* at each moment of time t, as shown in Figure 2.[8]

Production technology is described by a general form of neoclassical production function with constant returns to scale with respect to physical capital K and labor L_P

$$Y = F(K, AL_P), L_P = uL, 0 \leq u \leq 1, \tag{9}$$

where Y, A, and u represent respectively aggregate gross output, labor efficiency, and the ratio of workers employed in the production sector to total workers. AL_P $(= AuL)$ is total effective labor in the production sector at time t.

Define output per capita and capital-labor ratio respectively as $y = Y/L$ and $k = K/L$. Then (9) can be rewritten per capita as

$$y = Auf\left(\frac{k}{Au}\right), f\left(\frac{k}{Au}\right) = F\left(\frac{k}{Au}, 1\right). \tag{10}$$

Let s be denoted by the gross rate of saving. Then, the equation of physical capital accumulation can be written in per capita terms as

$$\dot{k} = sy - (n + \mu)k, 0 \leq s \leq 1, \tag{11}$$

where μ is the rate of depreciation of physical capital. The quality of labor is governed by the labor efficiency function

$$\dot{A} = A\phi\left(\frac{L_E}{L}\right), L_E = (1 - u)L, \phi' \geq 0, \phi'' \leq 0. \tag{12}$$

The distinctive feature of (12) is that it is linear in A, as stressed in the introduction. Lucas's learning function (4) shares the same feature, i.e., it is linear in h. $1 - u$ in (12) stands for the ratio of workers employed in the education sector to total workers. It implies, say, that the higher the ratio, not the number, of teachers becomes, the faster the labor efficiency of factory workers improves, but the extent to which the

---

[8] Phelps (1966) constructs a model similar to Uzawa (1965) in which labor force is so allocated to the production sector and the research sector as to maximize consumption at each point in time. Full employment of labor is assumed in his model too.

labor efficiency improves will be diminishing.[9] In order for u to lie between 0 and 1 it is assumed that

$$\phi(1) < \rho - n < \phi(0) + \phi'(0). \tag{13}$$

It should be emphasized here that the Uzawa model is quite different economically from the Lucas model because it is “teachers” that contribute to the rate of economic growth in the former, whereas it is “students” in the latter.[10]

By controlling s and u under the constraints (11) and (12) consumers (or workers) maximize the discounted sum of aggregate consumption $(1-s)Y$ $(= (1-s)yL = cL)$[11]

$$U = \int_0^\infty (1-s)y\, e^{-(\rho-n)t} dt. \tag{14}$$

In order to solve this problem, construct the Hamiltonian with q and v as costate variables with respect to (11) and (12) respectively

---

[9] Although mathematically equivalent, it may be more desirable to write (12) as $\dot{A}/A = \phi(1-u)$, as Uzawa (1965, p. 19) just writes, in order to avoid misunderstanding. That is, (12) says that the growth rate of labor efficiency $\dot{A}/A$ is a (nonlinear) function of the educational workers' ratio $1-u$ (see again Figure 2). Similarly the learning function (4) says that the growth rate of human capital $\dot{h}/h$ is a (linear) function of the time spent on schooling $1-u$. The term $(1-u)h$ is not the part of human capital which is used in the education sector. It should be added, however, that this interpretation does not apply to the Uzawa-Lucas-type model constructed by King and Rebelo (1990) in which human capital is assumed to be allocated between the two sectors.

[10] Schultz (1961, p. 9) categorizes human investment into five forms: (i) health facilities and services, (ii) on-the-job training, (iii) formally organized education at the elementary, secondary, and higher levels, (iv) study program for adults that are not organized by firms, including extension programs notably in agriculture, and (v) migration of individuals and families to adjust to changing job opportunities. Then, the Uzawa model corresponds to (iii), while the Lucas model to (iv).

[11] The objective to be maximized in Uzawa (1965) seems to be $\int_0^\infty (1-s)ye^{-\rho t}\, dt$ $(= \int_0^\infty c\, e^{-\rho t} dt)$. It corresponds to Millian criterion (average utilitarianism). However, he suggests an alternative (14) in Uzawa (1965, p. 20) too. It corresponds to Benthamite criterion (total utilitarianism) which is adopted in the Lucas model. In order for the Uzawa model to be comparable with the Lucas model, I adopt (14) as the objective. See Marsiglio and La Torre (2012) for the two criteria in the context of the Lucas-Uzawa model. Correctly speaking, the right-hand side of (14) should be $L(0)\int_0^\infty (1-s)y\, e^{-(\rho-n)t} dt$, but dropping the term $L(0)$ does not affect the utility maximization.

$$H = \left[(1-s)Auf\left(\frac{k}{Au}\right) + q\left(sAuf\left(\frac{k}{Au}\right) - (n+\mu)k\right) + v(A\phi(1-u))\right]e^{-(\rho-n)t}.$$

It is convenient here to introduce a new variable $x = k/A = K/AL$, that is, physical capital per unit of effective labor. Then, applying Pontryagin's Maximum Principle and arranging the results leads to the following relations which characterize the steady state in the Uzawa model:[12]

$$\phi(1-u^*) + u^*\phi'(1-u^*) = \rho - n, 0 < u^* < 1, \tag{15}$$

$$f'\left(\frac{x^*}{u^*}\right) - \mu = \rho, \tag{16}$$

$$f\left(\frac{x^*}{u^*}\right) - \frac{x^*}{u^*}f'\left(\frac{x^*}{u^*}\right) = v^*\phi'(1-u^*), \tag{17}$$

$$s^*f\left(\frac{x^*}{u^*}\right) = \frac{x^*}{u^*}[n+\mu+\phi(1-u^*)], 0 < s^* < 1, \tag{18}$$

$$q^* = 1. \tag{19}$$

Proceeding from (15) to (18) it is easy to confirm that four steady-state values, u^*, x^*, v^*, s^*, are uniquely determined in a recursive way. This is a matter of mathematics. Then, what do (15)-(19) mean economically? Uzawa (1965) is very technical and says little about economic implications. Then, it is the task of the next section to make them clear.

3. Economic Implications

Suppose that the economy under examination starts on a balanced growth path at time 0 and continues to stay on it.[13] Then, five conditions for the steady state (15)-(19) always hold. First of all, let us look at (18). Using $x = k/A$ and $\dot{A}/A = \phi(1-u)$, the capital accumulation equation (11) can be rewritten as

$$\dot{x} = suf\left(\frac{x}{u}\right) - [n+\mu+\phi(1-u)]x. \tag{20}$$

Thus, (18) means $\dot{x} = 0$, that is, physical capital per unit of effective labor remains

---

[12] Equations (15)-(18) are the same respectively as Equations (36)-(39) in Uzawa (1965, p. 25).

[13] For the reason see the next paragraph.

constant.[14] In other words, the growth rate of aggregate capital is the sum of the growth rates of population and labor efficiency.

In order to understand what (16) and (15) mean, remember that in neoclassical theory the real rate r of interest and the real rate w of wage coincide respectively with the marginal productivity of capital (MPK) net of the depreciation rate (μ) and the marginal productivity of labor (MPL). That is,

$$r = MPK - \mu = \frac{\partial F(K, AL_P)}{\partial K} - \mu = f'\left(\frac{x}{u}\right) - \mu, \tag{21}$$

$$w = MPL = \frac{\partial F(K, AL_P)}{\partial L_P} = A\left[f\left(\frac{x}{u}\right) - \frac{x}{u} f'\left(\frac{x}{u}\right)\right]. \tag{22}$$

(16) is obtained in Uzawa (1965) as the condition on which costate variable q remains constant ($\dot{q} = 0$) and takes a value of one at the same time. According to (21), (16) means that the steady-state value r^* of the real rate of interest is equal to the rate of time preference ρ. This seems to be related to the modified-golden-rule state (8). In the Lucas model where σ is assumed to be positive, the first term on the right-hand side of the Euler equation (5) represents the MPK for the Cobb-Douglas (2). And it is also the real rate r of interest.[15] Then, the Euler equation implies that if $r > \rho$ ($r < \rho$), more (less) saving to increase (decrease) future per capita consumption is desirable for utility maximization. In particular, (8) says that in the steady state r^* exceeds ρ by $\sigma(\dot{h}/h)^*$. So it may induce one to think that (16) is a limiting case of (8) as σ tends to 0. But it is not. In fact, as Uzawa (1965) shows, if the economy starts from an initial value $x(0)$ other than the steady state x^* in his model, $x(t)$ tends to x^* with $s = 1$ and $q > 1$ in the case of $x(0) < x^*$, but with $s = 0$ and $q < 1$ in the case of $x(0) > x^*$.[16] It means that (5) with $\sigma = 0$ does not

---

[14] In fact (18) is obtained from (20) by setting $\dot{x} = 0$ in Uzawa (1965).

[15] Recall that Y in (2) is net output. So the rate of capital depreciation does not appear in (5).

[16] See also Intriligator and Smith (1966) who consider the optimal allocation of new scientists between teaching and research careers using Pontryagin's Maximum Principle and obtain the result similar to that of Uzawa (1965). Optimal solutions of such a type are known as bang-bang control. Probably because of the seemingly unrealistic character of transitional paths, the Uzawa model has not been taken so seriously as the Lucas model. I wish to make comments on this point in three ways. First, transitional dynamics of the Uzawa model may be regarded as an *extreme* simplification of a complicated real economy. In the Lucas model, on the other hand, the transitional path is difficult to calculate in spite of the specifications of functions therein, and then the *extreme* assumption of $\sigma = \alpha$ (α is capital share in (2)) is often used in the literature for the analysis of transitional dynamics. Second, in the Uzawa model the economy arrives at the steady state within a *finite* time, whereas in the Lucas model, as is usual with optimal growth models, the economy does not reach the

describe the Uzawa model. It is worth stressing here that (8) is just a special case of (5). Therefore, (16) is not a limiting case of (5) despite their close resemblance.

(15) is obtained in Uzawa (1965) as the condition on which costate variable v remains constant ($\dot{v}=0$) under (17) and (19). At first glance it is hard to understand what it means. Then, I would like to rely on Lucas (1990, p. 298) who provides the condition the ratio u must satisfy at each moment of time t on an optimum growth path. It can be written with some modifications as follow:

$$w(t) = \phi'(1-u)\int_t^{\infty} \exp\left[-\int_t^{s}(r(\tau)-n)\,d\tau\right] u(s)w(s)ds. \tag{23}$$

The left-hand side is the real wage (22) at time t if people of the number uL work in the production sector. The right-hand side is the product of the marginal increase in the growth rate of labor efficiency ($\phi'(1-u)$) when "some" of the people move to the education sector and the discounted sum of the resulting increased real wages from time t on. Both sides must be equal on an optimum path and the utility maximizing value of u is determined by solving (23). In the steady state of the Uzawa model (23) becomes

$$w^*(0) = \phi'(1-u^*)\int_0^{\infty} \exp\left[-\int_0^{s}(r^*-n)\,d\tau\right] u^*w^*(0)e^{\phi(1-u^*)s}ds, \tag{24}$$

because of (12) and (22). Using the fact that $r^*=\rho$ due to (21) and arranging (24) leads to (15).

In general, a costate variable in optimal control theory measures the marginal contribution of the corresponding state variable to the objective.[17] In the case of the Uzawa model, it means $q=\partial U/\partial k$ and $v=\partial U/\partial A$. In the steady state the maximized U can be calculated as

$$\begin{aligned} U^* &= \int_0^{\infty}(1-s^*)y^*\,e^{-(\rho-n)t}dt \\ &= \int_0^{\infty}(1-s^*)A^*u^*f\left(\frac{x^*}{u^*}\right)e^{-(\rho-n)t}dt \\ &= A(0)\int_0^{\infty}(1-s^*)u^*f\left(\frac{x^*}{u^*}\right)e^{-[\rho-n-\phi(1-u^*)]t}dt \end{aligned}$$

---

steady state *forever*. Third, the usefulness of endogenous growth models may not depend on their transitional paths. For example, there is *no* transitional dynamics in the *AK* model, but it gives wide applications as Barro (1990) shows for instance.

[17] See Arrow (1968).

$$= A(0)\left[f\left(\frac{x^*}{u^*}\right) - s^* f\left(\frac{x^*}{u^*}\right)\right] u^* \int_0^\infty e^{-u^*\phi'(1-u^*)t}\, dt$$

$$= A(0)\left[f\left(\frac{x^*}{u^*}\right) - \frac{x^*}{u^*} f'\left(\frac{x^*}{u^*}\right) + x^*\phi'(1-u^*)\right] \frac{1}{\phi'(1-u^*)}$$

$$= \frac{f\left(\frac{x^*}{u^*}\right) - \frac{x^*}{u^*} f'\left(\frac{x^*}{u^*}\right)}{\phi'(1-u^*)} A(0) + k(0),$$

because of (10), (12), (15), and (18). Then, it is seen at once that

$$v^* = \frac{\partial U^*}{\partial A(0)} = \frac{f\left(\frac{x^*}{u^*}\right) - \frac{x^*}{u^*} f'\left(\frac{x^*}{u^*}\right)}{\phi'(1-u^*)},$$

$$q^* = \frac{\partial U^*}{\partial k(0)} = 1,$$

which are none other than (17) and (19), respectively.[18]

Now the economic implications of five conditions (15)-(19) have been made clear. There remains, however, one simple question: How are teachers paid their wages? So far, it looks as if the total amount of wages, $w^* L_P^*$, go to workers in the production sector. But it makes no sense of course. Even teachers would never do their job without pay. In other words, they could work in factories for wages. Then, it is natural to assume that they receive wages by $(1-u^*)w^* L_P^*$ from the production sector, while $u^* w^* L_P^*$ is for workers in the production sector.[19]

4. Specifications and Some Results

---

[18] Thus, in the steady state (17) and (24) coincide.

[19] This assumption is similar to that of the Romer (1990) model. In the model there are the production sector and the research sector, and workers (or engineers) in the research sector receive wages from the production sector in reward for the designs they produce in the research sector. See also footnote 2 above.

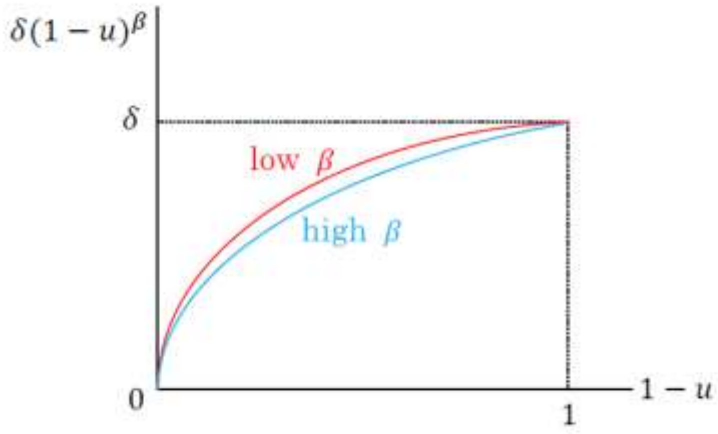

Figure 3. Labor Efficiency Function

This section studies the Uzawa model further by specifying related functions. As is seen from (15), an overwhelmingly important function in the Uzawa model is the labor efficiency function (12). Then, let it be specified as

$$\frac{\dot{A}}{A} = \phi(1-u) = \delta(1-u)^\beta,\ \delta > 0, 0 < \beta < 1, \tag{25}$$

where δ and β represent respectively the potentially maximum growth rate of labor efficiency and the elasticity of the growth rate of labor efficiency with respect to the ratio of workers in the education sector to total workers. The shape of the graph of $\delta(1-u)^\beta$ can been seen from Figure 3 for a low value and a high value of β. Note that the value of $\delta(1-u)^\beta$ is greater in the former case than that in the latter case because of $0 < 1-u < 1$. That is, given δ and $1-u$, the growth rate of labor efficiency decreases as β rises.

Next write $g(u) = \phi(1-u) + u\phi'(1-u)$ which corresponds to the left-hand side of (15). Then, in the case of (25) for $0 < u < 1$,

$$\begin{aligned} g(u) &= \delta(1-u)^{\beta-1}[1-(1-\beta)u] > 0, \\ g'(u) &= \delta\beta(1-\beta)u(1-u)^{\beta-2} > 0, \\ g''(u) &= \delta\beta(1-\beta)(1-u)^{\beta-3}[(1-u)+(2-\beta)u] > 0, \end{aligned} \tag{26}$$

$g(0) = \delta$, $g(1) = \infty$, $g'(0) = 0$, and $g'(1) = \infty$. Remember that $\phi(1-u)$ must satisfy Condition (13). In the case of (25), it becomes

$$\delta = \phi(1) = g(0) < \rho - n < g(1) = \phi'(0) = \infty. \tag{27}$$

It should be noted that the left inequality $\delta < \rho - n$ is diametrically opposite to the right inequality $\rho - n < \delta$ in (7). It can be said, therefore, that the Uzawa model and the Lucas model are not applicable to the same economy.[20]

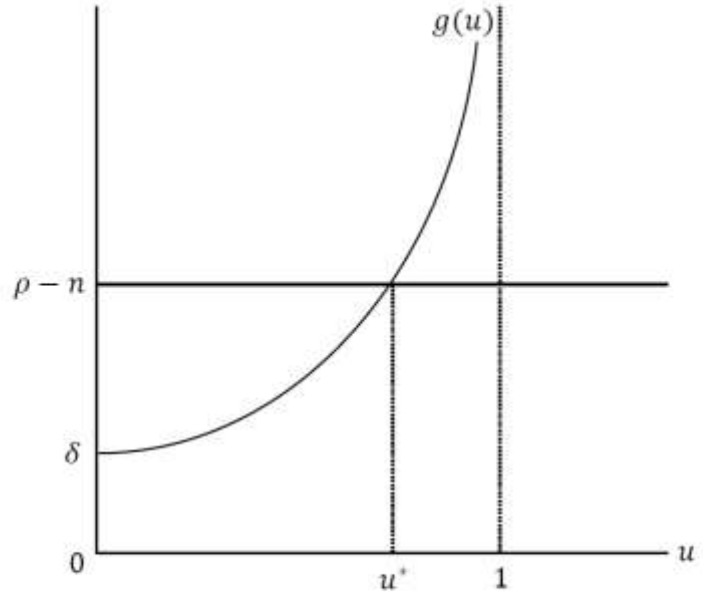

Figure 4. Optimum Ratio u^* of L_P to L

Figure 4 shows how the steady-state value u^* is determined as an intersection of the graph of $g(u)$ and the horizontal line $\rho - n$ in the interval between 0 and 1. As has been seen above, the graph of $g(u)$ is an upward sloping curve, taking a value of $\delta\ (> 0)$ for $u = 0$ and approaching ∞ as u tends to 1. Since $\delta < \rho - n$ by (27), it is apparent that the intersection always exists for $0 < u < 1$ and it is unique. Since the growth rate $\dot{A}/A$ of labor efficiency is a decreasing function of u, it is straightforward to derive Results 1-3 below:

Result 1. The growth rate of labor efficiency is higher the more patient are consumers (i.e., the smaller is ρ).

Result 2. The growth rate of labor efficiency is higher the faster grows population (i.e., the larger is n).

[20] Nevertheless, as Xie (1994) shows, under certain circumstances $\delta < \rho - n$ must hold in the *original Lucas* (1988) model in order for u^* to lie between 0 and 1 and also in order for any initial condition to converge to u^*. Everything depends!

Result 3. The growth rate of labor efficiency is higher the greater is the potentially maximum growth rate of labor efficiency (i.e., the greater is δ).

δ	β	u^*	$\delta(1-u^*)^\beta$	s^*
0.01	0.5	0.99	0.001	0.18
0.03	0.5	0.84	0.01	0.21
0.04	0.5	0.64	0.03	0.27

Table 1. Numerical Examples

Notes: $\rho - n = 0.05$, $\rho + \mu = 0.11$, and $f\left(\frac{x}{u}\right) = \left(\frac{x}{u}\right)^\alpha$, $\alpha = \frac{1}{3}$.

Interestingly, as is known at once from (6), Results 1-3 apply to the Lucas model too. In my opinion, Result 3 is particularly noteworthy because the endogenized growth rate still depends on the other growth rate, δ, which is assumed to be exogenous. How is δ determined? This crucial question is not answered in either model. Anyway, Table 1 shows numerical examples of the effect of δ on the values u^*, $\delta(1-u^*)^\beta$, and s^*, when production function (9) is the Cobb-Douglas $Y = K^\alpha(AuL)^{1-\alpha}, 0 < \alpha < 1$, or $f\left(\frac{x}{u}\right) = \left(\frac{x}{u}\right)^\alpha$. As is expected from Result 3, a greater value of δ leads to a smaller u^* which in turn implies a higher growth rate of labor efficiency. In addition, it is seen from the table that the saving rate becomes larger. It is already inferred from (18), though.

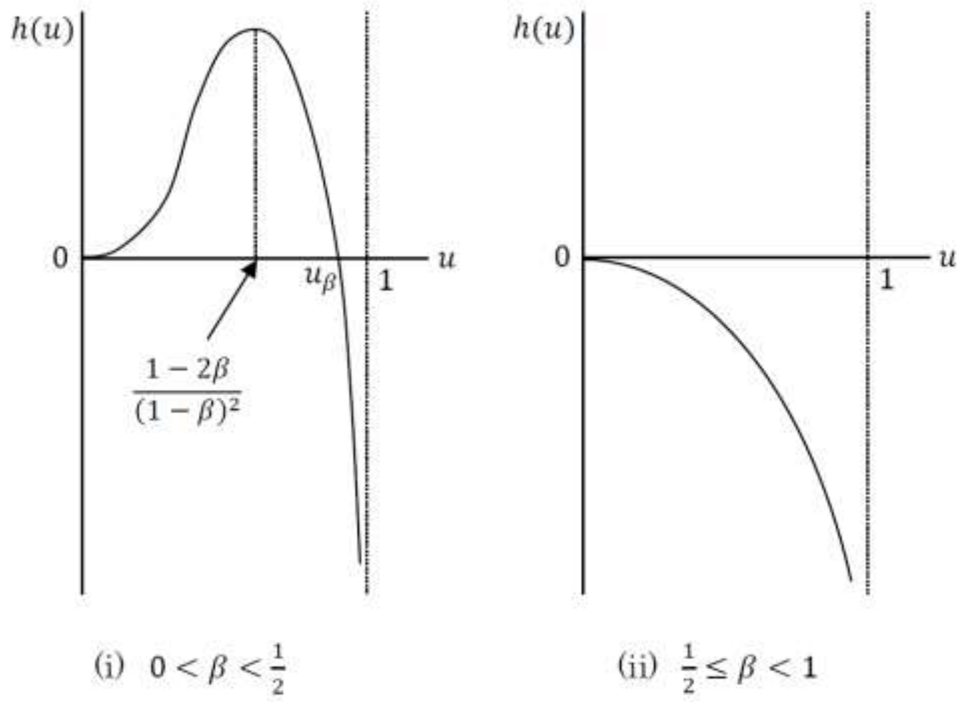

Figure 5. Graphs of $h(u)$

The effect of the elasticity β on the growth rate of labor efficiency is peculiar to the Uzawa model. Then, let us examine it. First, taking the logs of both sides of (26) gives

$$\log g(u) = \log \delta - (1-\beta)\log(1-u) + \log\left(1-(1-\beta)u\right).$$

Next, write the derivative of $\log g(u)$ with respect to β as $h(u)$. Then,

$$h(u) = \frac{d\log g(u)}{d\beta} = \log(1-u) + \frac{u}{1-(1-\beta)u},$$

$$h'(u) = \frac{dh(u)}{du} = \frac{u[(1-2\beta)-(1-\beta)^2 u]}{(1-u)[1-(1-\beta)u]^2}.$$

Moreover, $h(0) = 0$ and $h(1) = -\infty$. It depends on the sign of $h(u)$ whether $g(u)$ is an increasing or a decreasing function of β. And the sign of $h(u)$ depends on the value of β. Two cases of the graph of $h(u)$ are depicted in Figure 5. The left panel (i) represents the case of $0 < \beta < 1/2$. Starting at the origin, the graph of $h(u)$ is upward sloping until $u = \frac{1-2\beta}{(1-\beta)^2}$. After that it begins to decrease and approaches $-\infty$ as u tends to 1. Therefore, there exists $0 < u_\beta < 1$ such that $h(u_\beta) = 0$. Thus, $h(u) > 0$ for $0 < u < u_\beta$, while $h(u) < 0$ for $u_\beta < u < 1$. It turns out that $g(u)$ is an increasing (a decreasing) function of β for $0 < u < u_\beta$ ($u_\beta < u < 1$). The right panel represents of the case of $1/2 \leq \beta < 1$. In this case $h'(u) < 0$ and so $h(u) < 0$ for $0 < u < 1$. Therefore, $g(u)$ is always a decreasing function of β.

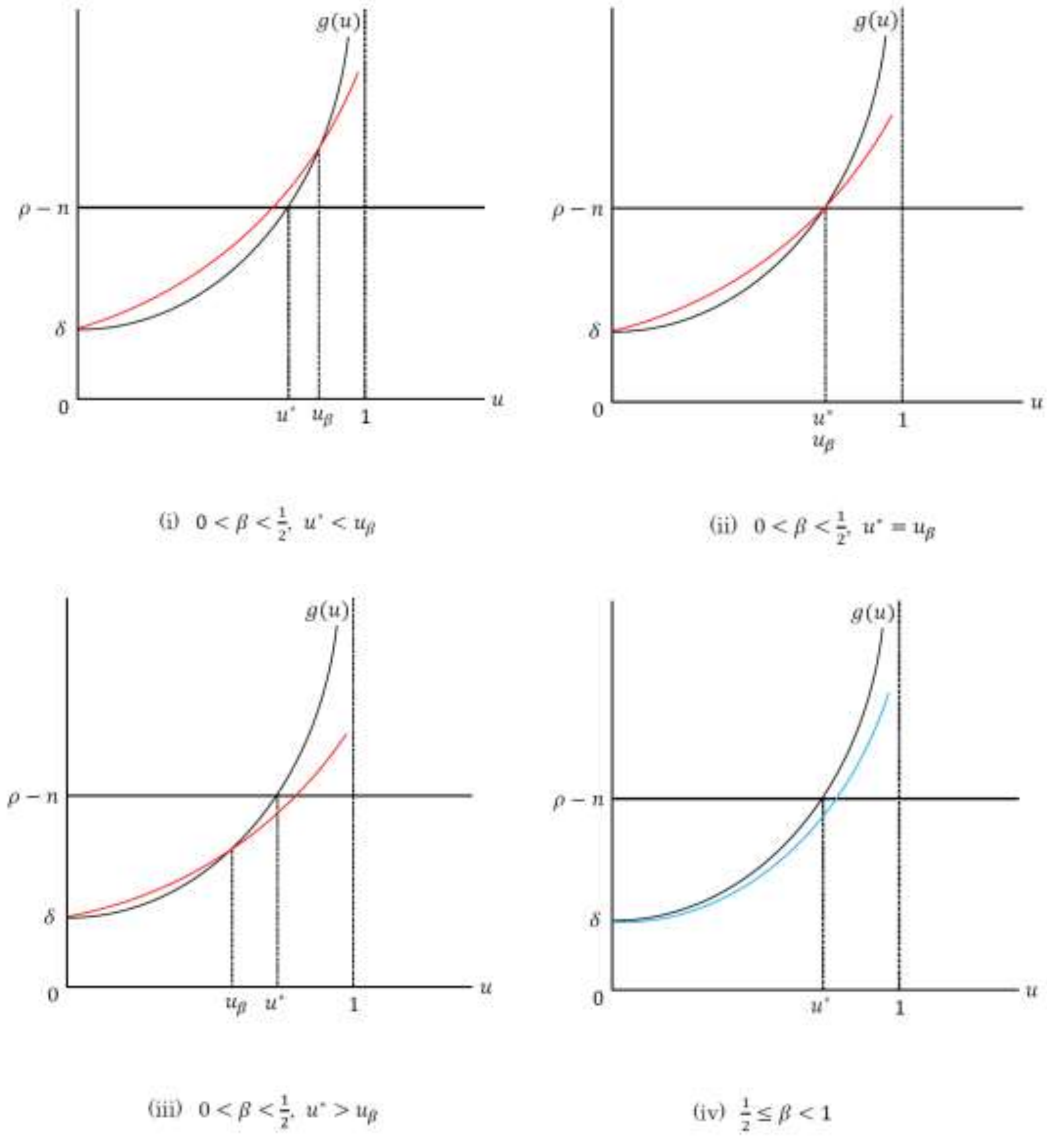

Figure 6. Response of the Steady-State Value of u to a Rise in the Elasticity β

According to the above considerations, the response of the steady-state value of u to a rise in β needs to be investigated for four cases. Such cases are pictured in Figure 6. The first panel (i) represents the case of $0 < \beta < 1/2$ and $u^* < u_\beta$, where u^* is the initial steady-state value of u. When β rises, the graph of $g(u)$ shifts upward (downward) for $0 < u < u_\beta$ ($u_\beta < u < 1$) because $g(u)$ is an increasing (a decreasing) function of β. It means that a rise in β leads to a fall in the steady-state value of u. The second panel (ii) represents the case of $0 < \beta < 1/2$ and $u^* = u_\beta$. In this special case a rise in β does not change the steady-state value of u from $u^*(= u_\beta)$. The third panel (iii) represents the case of $0 < \beta < 1/2$ and $u^* > u_\beta$. In this case, a rise in β leads to an increase in the steady-state value of u. Finally, the fourth panel (iv) represents the case of $1/2 \leq \beta < 1$. In this case, when β rises, the graph of $g(u)$ shift downward for $0 < u < 1$. So there is no need to take account of the initial position of u^*. As is apparent from the panel, a rise in β leads to an increase in the steady-state value of u.

It should be remembered here that the growth rate of labor efficiency is described by (25) and it decreases in response to a rise in β if the value of u remains constant as Figure 3 shows. It is now known, however, that the steady-state value of u changes depending on the value of β and the relation between u^* and u_β as in Figure 6. Taking all these analyses into account, the effect of the elasticity β of the labor efficiency function on the steady-state growth rate of labor efficiency can be stated as follows:

Result 4. For $0 < \beta < 1/2$ the effect of the elasticity β on the growth rate of labor efficiency is indeterminate if the initial steady-state value u^* is smaller than the critical value u_β.
Result 5. For $0 < \beta < 1/2$ a rise in the elasticity β leads to a decrease in the growth rate of labor efficiency if the initial steady-state value u^* is greater than or equal to the critical value u_β.
Result 6. For $1/2 \leq \beta < 1$ a rise in the elasticity β leads to a decrease in the growth rate of labor efficiency irrespective of the initial steady-state value u^*.

In numerical examples in Table 1 β takes a value of $1/2$. Then, according to Result 6, if β rises marginally from $1/2$, each value of u^* increases and the corresponding value of the growth rate of labor efficiency decreases.[21]

---

[21] The same function as (25) is used in Rosen (1976) and Lucas (1990), where the elasticity (β in this paper) is estimated at 0.5 and 0.8, respectively.

5. Conclusion

Lucas (1988) is so influential that the Uzawa-Lucas model is virtually the Lucas model. This paper distinguished between the Uzawa model and the Lucas model, and studied the Uzawa model in detail for the first time. Although the two models have much in common both economically and mathematically, this paper focused on the differences between the two. From the economic point of view, the Uzawa model assumes full employment, whereas the Lucas model admits unemployment. To put it another way, it is teachers who contribute to the growth rate in the Uzawa model, whereas it is students in the Lucas model. From the mathematical point of view, the two models are based on the opposite conditions. That is, the potentially maximum growth rate must be smaller than the rate of time preference (less population growth) in the Uzawa model, whereas the opposite must hold in the Lucas model. This implies that the two models cannot be applied to the same economy. One may infer that the Uzawa model with a linear utility function ($\sigma = 0$) is a limiting case of the Lucas model with the CRRA utility function ($\sigma > 0$) because a linear utility function can be regarded as a limiting case of the CRRA utility function. This paper showed that such an inference is not true. In conclusion, it is true that the two models resemble each other considerably, but they are not the same.[22]

References

Aghion, Philippe, and Peter Howitt, 1992, "A Model of Growth through Creative Destruction," *Econometrica,* 60, 323-351.

Arrow, Kenneth J., 1962, "The Economic Implications of Learning by Doing," *Review of Economic Studies*, 29, 155-173.

Arrow, Kenneth J., 1968, "Applications of Control Theory to Economic Growth," in Goerge B. Dantzig and Arther F. Veinott, Jr., eds., *Mathematics of the Decision Sciences, Part 2,* Providence, Rhode Island: American Mathematical Society, 85-119.

Barro, Robert J., 1990, "Government Spending in a Simple Model of Endogenous Growth," *Journal of Political Economy,* 98, S103-S125.

Boucekkine, R., and J. R. Ruiz-Tamarit, 2008, "Special Functions for the Study of

[22] It would be interesting to ask what happens if the linear utility function in the Uzawa model is replaced by the CRRA utility function. This is exactly the case Caballé and Santos (1993) thoroughly examines. Using their results, (13) and (15) in the Uzawa model change respectively to $(1-\sigma)\phi(1) < \rho - n < (1-\sigma)\phi(0) + \phi'(0)$ and $(1-\sigma)\phi(1-u^*) + u^*\phi'(1-u^*) = \rho - n, 0 < u^* < 1$ in the case of the CRRA utility function.

Economic Dynamics: The Case of the Lucas-Uzawa Model," *Journal of Mathematical Economics*, 44, 33-54.

Caballé, Jordi, and Manuel S. Santos, 1993, "On Endogenous Growth with Physical and Human Capital," *Journal of Political Economy*, 101, 1042-1067.

Canton, Erik, 2002, "Business Cycle in a Two-Sector Model of Endogenous Growth," *Economic Theory*, 19, 477-492.

Chamley, Christophe, 1993, "Externalities and Dynamics in Models of "Learning or Doing"," *International Economic Review*, 34, 583-609.

Chilarescu, Constantin, 2011, "On the Existence and Uniqueness of Solution to the Lucas-Uzawa Model," *Economic Modelling*, 28, 109-117.

Faig, Miquel, 1995, "A Simple Economy with Human Capital: Transitional Dynamics, Technoloty Shocks, and Fiscal Policies," *Journal of Macroeconomics*, 17, 421-446.

Gómez, Manuel A., 2003, "Optimal Fiscal Policy in the Uzawa-Lucas Growth with Externalities," *Economic Theory*, 22, 917-925.

Hiraguchi, Ryoji, 2009, "A Note on the Closed-Form Solution to the Lucas-Uzawa Model with Externality," *Journal of Economic Dynamics and Control*, 33, 1757-1760.

Intriligator, Michael D., and Bruce L. R. Smith, 1966, "Some Aspects of the Allocation of Scientific Effort between Teaching and Research," *American Economic Review*, 56, 494-507.

King, Robert G., and Sergio Rebelo, 1990, "Public Policy and Economic Growth: Developing Neoclassical Implications," *Journal of Political Economy,* 98, S126-S150.

Kuwahara, Shiro, 2017, "Multiple Steady States and Indeterminacy in the Uzawa-Lucas Model with Educational Externalities," *Journal of Economics*, 122, 173-190.

Lucas, Robert E., Jr., 1988, "On the Mechanics of Economic Development," *Journal of Monetary Economics,* 22, 3-42.

Lucas, Robert E., Jr., 1990, "Supply-Side Economics: An Analytical Review," *Oxford Economic Papers,* 42, 293-316.

Lucas, Robert E., Jr., 2003, "Macroeconomic Priorities," *American Economic Review,* 93, 1-14.

Mankiw, N. Gregory, Julio Rotemberg, and Lawrence H. Summers, 1985, "Intertemporal Substitution in Macroeconomics," *Quarterly Journal of Economics,* 100, 225-251.

Marsiglio, Simone, and Davide La Torre, 2012, "Population Dynamics and Utilitarian Criteria in the Lucas-Uzawa Model," *Economic Modelling*, 29, 1197-

1204.

Mulligan, Casey B., and Xavier Sala-i-Martin, 1993, "Transitional Dynamics in Two-Sector Models of Endogenous Growth," *Quarterly Journal of Economics,* 108, 739-773.

Nelson, Richard R., and Edmund S. Phelps, 1966, "Investment in Humans, Technological Diffusion, and Economic Growth," *American Economic Review*, 56, 69-75.

Ortigueira, Salvador, 1998, "Fiscal Policy in an Endogenous Growth Model with Human Capital Accumulation," *Journal of Monetary Economics*, 42, 323-355.

Phelps, Edmund S., 1966, "Models of Technical Progress and the Golden Rule of Research," *Review of Economic Studies,* 33, 133-145.

Romer, Paul M., 1986, "Increasing Returns and Long-Run Growth," *Journal of Political Economy,* 94, 1002-1037.

Romer, Paul M., 1990, "Endogenous Technological Change," *Journal of Political Economy,* 98, S71-S102.

Rosen, Sherwin, 1976, "A Theory of Life Earnings," *Journal of Political Economy,* 84, S45-S67.

Schultz, Theodore W., 1961, "Investment in Human Capital," *American Economic Review*, 51, 1-17.

Solow, Robert M., 1956, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 70, 65-94.

Tsuboi, Mizuki, 2018, "Stochastic Accumulation of Human Capital and Welfare in the Uzawa-Lucas Model: An Analytical Characterization," *Journal of Economics*, 125, 239-261.

Uzawa, Hirofumi, 1965, "Optimum Technical Change in an Aggregative Model of Economic Growth," *International Economic Review,* 6, 18-31.

Xie, Danyang, 1994, "Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria," *Journal of Economic Theory*, 63, 97-112.

September 25, 2020