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Does Ambiguity Generate Demand for Options?

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Abstract

This study examines the optimal investment strategies for risk-and-ambiguity-averse investors and characterizes conditions under which ambiguity induces investors to buy or sell options. Under identical constant relative risk aversion utility functions, we show that ambiguity-averse investors should sell portfolio insurance. In particular, when investors' relative risk aversion is less than or equal to two, ambiguity-averse investors should sell options at any realization values of a reference asset. In addition, if the relative risk aversion is greater than two, we demonstrate that ambiguity-averse investors should sell options at smaller and buy them at higher realization values of the reference portfolio.

Keywords: Ambiguity, Multiple prior model, Options demand, Kullback–Leibler divergence

JEL Classification: G11, G22

1 Introduction

Over the past 30 years, notable progress in financial economics has been made in the area of ambiguity and aversion toward it. Ambiguity refers to a situation wherein a decision maker cannot recognize a precise probability distribution to

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describe the uncertainty. These situations can be modeled by using a set of priors that contains more than one distribution. The differences between decision-making models under ambiguity are due to their axiomatized foundations on how to select the optimal probability distribution from a set of plausible distributions.

The subjective expected utility model of Savage (1954) was the first to propose an axiomatized foundation of choice problems under ambiguity. However, his model is a simple extension of the classical von-Neumann–Morgenstern decision-making theory, which is why it has faced severe criticisms from researchers such as Ellsberg (1961). The maxmin expected utility model of Gilboa and Schmeidler (1989) provides a rational alternative model to describe the Ellsberg's paradox. The maxmin model claims that an agent is rational to select a probability distribution that yields the lowest expected utility in the set of priors. Then, the agent considers the optimal behavior that maximizes their felicity under the most pessimistic distribution. The α-maxmin expected utility model of Ghirardato, Maccheroni, and Marinacci (2004) and the smooth ambiguity aversion model of Klibanoff, Marinacci, and Mukerji (2005) (henceforth, KMM) add the flexibility to the maxmin expected utility model by using additional parametric values or functionals to describe the strength of ambiguity aversion. In particular, the smooth ambiguity aversion model of KMM uses an additional function to describe the agent's preference for the variability of his/her felicity on the different probability distributions. This enable us to use the simple analogue of the Arrow–Pratt coefficient of risk aversion to define ambiguity aversion.

Given these theoretical models, portfolio choice problems under ambiguity have become a major topic in financial economics over the past 10 years. One distinct feature of the decision-making models under ambiguity is that the relevant prior under which an agent considers the optimal choice changes according to the agent's behavior. In other words, selecting the prior and optimal behavior cannot be separated under ambiguity. This feature is illustrated in the seminal works by Cao et al. (2005) and Easley and O'Hara (2009), which show the optimality of non-participation in the market. This theoretical result implies that ubiquitous non-participation behavior can be rationalized due to the presence of ambiguity and the maxmin expected utility model, while it is optimal under quite restrictive conditions in the classical expected utility model. Furthermore, as examined in the next section, Gollier (2011) consider the relationship between ambiguity aversion and optimal exposure for an ambiguous asset under the KMM model. He showed that greater ambiguity aversion does not necessarily yield a reduction in optimal exposure for the asset.

In spite of the detailed insights provided in previous studies, these models and most studies conducted under ambiguity focus on the simple portfolio choice problem and adopt the assumption of one safe and one uncertain asset. As a result, previous studies do not sufficiently illustrate how to cope with ambiguity by using the various types of financial derivatives available in the market.[1] As far as we know, therefore, this is the first study to examine an investor's optimal use of derivatives under ambiguity especially in the setting similar to the model of Leland (1980) and Franke, Stapleton, and Subrahmanyam (1998).

Ambiguity has potential to shed light on an unsolved issue in the financial economics theory: what is the fundamental cause of demand for options? In fact, although the trading volume of options is not small in financial markets, this problem has not been sufficiently addressed from theoretical perspectives. This is because the Black–Scholes (1973) arbitrage argument for derivatives pricing assumes that financial derivatives are essentially redundant assets in markets: one can replicate the payoff of derivative securities by combining the underlying asset and risk-free bond. Therefore, investors need not possess the actual derivatives, and these "redundant" assets cannot bring richer investment opportunities for the investors.

To reconcile theory and reality, a large number of studies try to explain the divergence from the Black–Scholes argument. For example, Leland (1985), Boyle and Vorst (1992), and Grannan and Swindle (1996) examine the effect of transaction costs on the replication strategy. Additionally, Shleifer and Vishny (1997) emphasize that intermediaries cannot hedge perfectly because of capital constraints. However, few studies have been conducted to characterize the nature of investors who wish to hold these financial derivatives due to supply and demand attributes. In fact, as stressed in Leland (1980), since financial derivatives are traded among investors, that is, financial derivatives are zero net supply in the market, we can understand the derivatives market only in the context of an equilibrium analysis.

In this study, we adopt the robust control model of Hansen and Sargent (2001) to analyze how ambiguity affects investors' demand for options. A distinct feature of the robust control model compared to the other decision-making models under ambiguity is that instead of using a set of priors that contains more than one plausible distribution, ambiguity together with the strength of ambiguity aversion can be implicitly specified by using some parameterized distance function from the reference probability distribution. Thus, the Hansen–Sargent model can describe the situation wherein an agent who has a reference probability distribution

[1] As associated literature, Wong (2016) and Osaki and Wong (2017) use the KMM model to analyze optimal hedging under ambiguity in the classical corporate investment problem.

is concerned with the possibility of misspecification and attempts to find a robust strategy.

There are two main reasons for using the model: (i) we can derive the optimal shape of the payoff function by using the standard calculus of variations and (ii) we can analyze the effect of ambiguity by specifying a single-parameter that describes the strength of ambiguity and the aversion toward it. It should be noted that while the latter reason is also a notable feature of the maxmin expected utility model, it has been criticized, leading to the development of alternative models such as KMM (2005). In this paper, however, all agents in an economy are assumed to have identical risk-preference and thus this inflexibility is not restrictive, but rather simplifies the analysis of the effects of ambiguity.

The remainder of this paper is organized as follows. In Section 2, we consider the optimal linear exposure to an ambiguous risky asset under the robust control model. In Section 3, investors' optimal payoff functions are derived. In Section 4, we examine the investors' demand for contingent claims on the ambiguous asset under an equilibrium model and characterize the nature of investors who should buy and sell options for hedging purposes. In Section 5, we adopt the similar representative agent model as in Leland (1980) and discuss the result. In Section 6, we summarize the findings of the study and draw conclusions.

2 Portfolio Choice Problem Under Robust Control Model

In this section, we examine the optimal linear strategy under the Hansen–Sargent robust control model. The linear strategy implies that the investors' payoff functions are proportional to the terminal values of a reference portfolio. Therefore, under a real market environment wherein an investor can trade the reference asset dynamically, the investor takes the linear strategy if and only if the investor follows the "buy and hold" policy.

2.1 Related literature

As explained in the previous section, decision making models under ambiguity since Gilboa and Schmeidler (1989), have an important feature that the relevant prior depends on the decision taken by an agent. In particular, a common feature of the maxmin expected utility model of Gilboa and Schmeidler (1989), the α-

maxmin expected utility model of Ghirardato, Maccheroni and Marinacci (2004), and the smooth ambiguity aversion model of KMM (2005) is that an implicit distribution used by an ambiguity-averse investor is dominated by the distribution used by an ambiguity-neutral investor in the sense of the second-degree stochastic dominance (SSD) orders. Gollier (2011) show, however, these pessimistic shift of the implicit distribution does not necessary yield a reduction in the demand for the ambiguous risky asset. This unintuitive result is first documented by the important work by Rothschild and Stieglitz (1971), which show that an increase in risk (that is, SSD-dominated shift of the distribution) is not sufficient to guarantee a reduction in the demand for the risky asset. Under the KMM model, therefore, an increase in ambiguity aversion does not necessary yield a reduction in the optimal risk exposure by all ambiguity-averse investors.

2.2 MLR and other stochastic orders

There are two distinct approaches to characterize the conditions under which investors reduce their demand for the risky asset: limiting the set of utility functions or searching the set of changes in beliefs. The former approach is in line with Rothschild and Stiglitz (1971), Fishburn and Porter (1976) and Hadar and Seo (1990), who characterize the sufficient conditions of the shape of the utility functions to guarantee a reduction in the demand for the risky asset under FSD or SSD shift of the risky asset. Restricted sufficient conditions derived in their studies lead to taking the latter approach to determine the set of changes in beliefs. The notion of Central Dominance (CD) derived by Gollier (1995) is the equivalent condition of the change in beliefs to guarantee a reduction in the demand for the risky asset by all risk-averse investors. Several important stochastic orders that are shown to be a part of CD: Monotone Likelihood Ratio order (MLR) (Milgrom 1981), Strong Increase in Risk (Meyer and Ormiston 1985), Simple Increase in Risk (Dionne and Gollier 1992), and Monotone Probability Ratio order (MPR) (Eeckhoubt and Gollier 1995). In particular, it can be shown that MLR⇒MPR⇒FSD⇒SSD.[2]

The main objective in this section is to show that there is a close relationship between the notion of MLR and the implicit probability distribution used by an ambiguity-averse agent under the robust control criterion. We first present the definition of MLR.

Definition 1. *Consider random variables $\tilde{x}_i$ with the same support in $[x_-, x_+]$. Let f_i be the probability density function of $\tilde{x}_i$. $\tilde{x}_p$ dominates $\tilde{x}_q$ in the sense of*

[2]See Gollier (2001, pp. 71-72).

MLR if $l(\theta) = \ln f_q(\theta)/f_p(\theta)$ *is non-increasing in* θ.

2.3 The model

Consider an agent under static environment with initial wealth W_0. The agent can invest the wealth into two assets. One is risk-free asset with a rate of return denoted by r_f and the other is an ambiguous risky asset with a return $\tilde{r}$. Let $\tilde{x}$ be the realization value of the excess return of the ambiguous risky asset defined as $\tilde{x} = \tilde{r} - r_f$. The investor has a reference probability measure p, but concerns the possibility of the misspecification. When the investor follows the Hansen–Sargent robust control criterion, the maximization problem of a typical investor can be described as follows:

$$\begin{aligned} &\max_{\alpha} \min_{q \in C} \quad E^q[u(w_0 + \alpha\tilde{x})] \\ &\text{s.t.} \qquad C = \{q \in \Delta(\Omega) : R(q||p) \leq \eta\}, \end{aligned} \tag{1}$$

where $w_0 \equiv W_0(1 + r_f)$, E^q denotes the expectation taken under the probability distribution q, $\Delta(\Omega)$ is a set of probability measures, and $R(q||p)$ denotes the Kullbuck–Leibler (KL) information divergence from p to q, which is defined as

$$R(q||p) = E^q\left[\ln\frac{q}{p}\right]. \tag{2}$$

Almost all problems under the robust control criterion adopt the KL information divergence as a "budget constraint" for selecting probability distribution. It should be noted that the KL information divergence from p to q takes zero if and only if $p(x) = q(x)$ for all x and it never takes negative values. These favorable properties enable us to regard this function as representing the distance between one distribution and another while the KL divergence does not satisfy the other two axioms of distance function—symmetry and triangle inequality.

(1) indicates that the robust control model is a special case of the maxmin expected utility model. That is, the robust control model imposes an additional "budget constraint" in selecting the probability distribution by using the KL information divergence. Hansen and Sargent (2001) noted that this maximization problem is equivalent to

$$\max_{\alpha} \min_{q \in C} \quad E^q[u(w_0 + \alpha\tilde{x})] + \rho R(q||p), \tag{3}$$

for some $\rho > 0$. Under this model, ρ can be viewed as proxy of an ambiguity aversion index. To see this, let us consider two extreme cases:

(1) $\rho \to \infty$: the solution for the minimization problem

$$\max_{q \in \Delta(\Omega)} \quad E^q[u(w_0 + \alpha\tilde{x})] + \rho R(q||p)$$

must be the reference probability measure p, since otherwise $E^q[u(w_0 + \alpha\tilde{x})] + \rho R(q||p) \to \infty$. Thus, this case can be viewed as a situation that the agent is ambiguity-neutral, or equivalently, probabilistically sophisticate.

(2) $\rho \to +0$: the reference probability distribution p has no effect on selecting the probability measure in this case, so that the agent's behavior is indistinguishable from the standard maxmin expected utility model of Gilboa and Schmeidler.

For this reason, in this paper we say that an investor is ambiguity-averse if $0 < \rho < \infty$ and ambiguity-neutral if $\rho \to \infty$.

2.4 Optimal investment under the robust control criterion

Consider this *maxminimization* problem for an ambiguity-averse agent (i.e., an agent with some $0 < \rho < \infty$). Let us first derive the probability distribution under which the agent solves the maximization problem. The minimized probability distribution can be easily obtained by differentiating the objective function with respect to q and setting it to zero. Substituting the minimized probability distribution into the objective function, the maximization problem can be written as

$$\max_{\alpha} \quad -\rho \ln E^p\left[\exp\left(-\frac{u(w_0 + \alpha\tilde{x})}{\rho}\right)\right]. \tag{4}$$

Let $V(\alpha)$ denote this objective function. The optimal risk exposure α^* satisfies the following first-order condition:

$$V'(\alpha^*) = 0 \Leftrightarrow E^p\left[\frac{e^{-\frac{u(w_0+\alpha^*\tilde{x})}{\rho}}}{E^p\left[e^{-\frac{u(w_0+\alpha^*\tilde{x})}{\rho}}\right]} x u'(w_0 + \alpha^* x)\right] = 0. \tag{5}$$

In Appendix A, we show that the objective function $V(\alpha)$ is concave in α, which implies that the first-order condition is also sufficient for a maximum. The concavity of the objective function V also implies that the optimal risk exposure, α^*, is non-negative (i.e., $\alpha^* \geq 0$) if and only if $V'(0) \geq 0$. Thus, similar to the classical expected utility model, an agent with robust control criterion invests non-negative amount of their wealth into the ambiguous risky asset if and only if the expected excess return of the risky asset under the reference probability distribution p is non-negative.

2.5 MLR and distribution implied by the robust control criterion

The key relation between the implied distribution used by the agent with the robust control criterion and MLR is summarized in the following lemma.

Lemma 1. *The implicit probability distribution used by an ambiguity-averse investor dominates (is dominated by) the reference probability measure in the sense of MLR when the demand for the ambiguous asset is negative (positive).*

Proof. By (5), the implicit distribution used by the ambiguity-averse agent is given by

$$\frac{e^{-\frac{u(w_0+\alpha x)}{\rho}}}{E^p\left[e^{-\frac{u(w_0+\alpha x)}{\rho}}\right]}. \tag{6}$$

Denoting this by $q(x)/p(x)$ and differentiating the logarithm of $q(x)/p(x)$ with respect to x, we obtain

$$\frac{d\ln q(x)/p(x)}{dx} = -\frac{\alpha u'(w_0+\alpha x)}{\rho}\frac{e^{-\frac{u(w_0+\alpha x)}{\rho}}}{E^p\left[e^{-\frac{u(w_0+\alpha x)}{\rho}}\right]}. \tag{7}$$

To complete the proof, note that

$$\text{sign}\left[\frac{d\ln q(x)/p(x)}{dx}\right] = -\text{sign}\left[\alpha\right]. \tag{8}$$

□

This lemma illustrates an attractive feature of the robust control model: since the notion of MLR is a subset of CD, the robust control model ensures the reduction in the optimal risk exposure of an ambiguity-averse agent without imposing further restricting and complex assumptions on the set of beliefs. Furthermore, this lemma also clarifies that when the excess return of the risky asset under the reference probability distribution is negative,[3] an ambiguity-averse agent *increases* the wealth invested in the asset. This is because, the implicit probability distribution *dominates* the reference probability measure in the sense of MLR if $\alpha < 0$. Similar to the other decision-making models under ambiguity, therefore, the implicit distribution in this model also depends on the behavior taken by an agent. Furthermore, noting the fact that an ambiguity-neutral, or equivalently, a probabilistically sophisticate investor uses the reference probability distribution p yields the following proposition.

Proposition 1. *Under the Hansen–Sargent robust control criterion, an ambiguity-averse agent reduces the absolute amount invested in the ambiguous risky asset compared to an ambiguity-neutral agent.*

Since we only use the fact that an ambiguity-averse agent has some $0 < \rho < \infty$ to derive the results, these can be easily extended to the situation wherein both of the agents are ambiguity-averse. The following corollary therefore follows immediately:

Corollary 1. *Suppose that two types of investors with the Hansen–Sargent robust control criterion differ only in ρ. Then, the investor with higher degree of ambiguity aversion (i.e., an agent with lower ρ) asks less in absolute value of the risky asset than the other.*

Proof. Note that the implicit distribution used by the investor with higher ρ dominates the other in the sense of MLR if and only if $\alpha > 0$. □

This corollary indicates that contrary to the other decision-making models under ambiguity, the Hansen–Sargent robust control model ensures that higher ambiguity aversion necessary yields a reduction in the optimal risk exposure.

3 Optimal Strategies Under Ambiguity

3.1 The complete market model

Taking into consideration the huge progress in markets of financial derivatives, the assumption that investors can only achieve returns proportional to the realized

[3]Note that this is equivalent to $\alpha^* < 0$.

value of the reference asset appears restrictive. Thus, it is natural to allow investors to construct non-linear relation between terminal values of their portfolio and the reference asset. For this purpose, we use the complete market assumption to derive the optimal payoff functions of investors. This approach is adopted in Leland (1980) to analyze the investor's demand for portfolio insurance. After that, several research using this method are also conducted in continuous-time settings, such as Wachter (2004), which examines the effect of mean-reversion property of asset prices on the investor's optimal consumption and investment behavior. In this paper, we adopt essentially the same framework as in Leland (1980) except that we use an endogenously derived pricing kernel to price and analyze investors' demand for financial derivatives instead of assuming the existence of a representative agent in the economy.[4]

We use this framework for two reasons. First, it is difficult to derive a utility function of a representative agent under ambiguity. Second and more important, using an endogenous pricing kernel enable us to incorporate the effect of ambiguity and agents' aversion toward it in a consistent way. In fact, if the market admits a representative agent and investors are allowed to transact in Arrow–Debreu (AD) securities, ambiguity has no effect on investors' behavior since investors can hedge all uncertainty even though they cannot recognize the precise probability distribution of the reference asset. In that case, there is no need to distinguish an economy under ambiguity from the classical model as in Leland (1980). The most satisfactory way of dealing with this problem is to use an endogenous derived pricing kernel by imposing a market-clearing condition. In Section 5, however, we adopt the representative agent framework as in Leland (1980) to inspect the effect of the representative agent's ambiguity aversion on the optimal use of derivatives of individual investor.

3.2 Shape of payoff functions and demand for options

There are three basic types of investment strategies: convex, concave, and linear. The differences among them are literally due to the functional forms of the payoff functions. More precisely,if the payoff functions are twice differentiable, the convex (concave) strategies are characterized by the positive (negative) second derivatives of the payoff functions, and the second derivatives are zero when strategies are linear. In cases wherein the market is complete as assumed in our

[4] Leland (1980) used a representative agent model in an economy consist of heterogenous agents with hyperbolic absolute risk aversion (HARA) utility functions.

model, state price securities enable investors to construct any desirable payoff functions. It can be presumed, however, that investors utilize alternative financial derivatives such as options in a real incomplete market environment. Generally, an investor's payoff function becomes convex by holding options, and it becomes concave by selling them. Therefore, if a convex (concave) strategy is optimal for an investor, the investor should only buy (sell) plain vanilla options. On the other hand, it suffices to use forward or futures contracts if the linear strategy maximizes the investor's expected utility. In fact, Leland (1980) shows that any arbitrary twice continuously differentiable payoff functions can be generated by combining the reference asset and vanilla options on the asset. That is, investors can obtain arbitrary (twice continuously differentiable) non-linear (i.e., non-proportional) relation between the terminal value of the reference asset and their payoff function if a complete set of options are available in the market.

3.3 Portfolio insurance

The main objective of the next section is to examine whether an ambiguity-averse investor is more willing to protect their portfolio. For our purpose, let us first characterize an investor who wish to hold portfolio insurance. As Leland (1980) discussed, there are two reasonable definitions for portfolio insurance: full portfolio insurance and general insurance policy. The full portfolio insurance refers to the situation that a payoff function of an investor has a form

$$W(x_T; x_0) = \mathrm{Max}[x_T, x_0], \tag{9}$$

where x_T and x_0 represent the terminal and initial values of the reference portfolio, respectively. The general insurance policy, on the other hand, refers the payoff function

$$W(x_T; x_0) \tag{10}$$

is a strictly convex function of the terminal value of the reference asset x_T. While the full portfolio insurance is the classical definition of portfolio insurance, the term "general insurance policy" can be rationalized as Leland (1980) wrote "concavity implies greater protection from loss at lower values of the reference portfolio." Although the Leland's claim is reasonable, the strict convexity of their payoff functions is somewhat too strong to characterize investors' incentives for loss aversion.

To see this, Figure 1 illustrates two payoff functions for investors those who are willing to protect their portfolio by using options. While the payoff function in

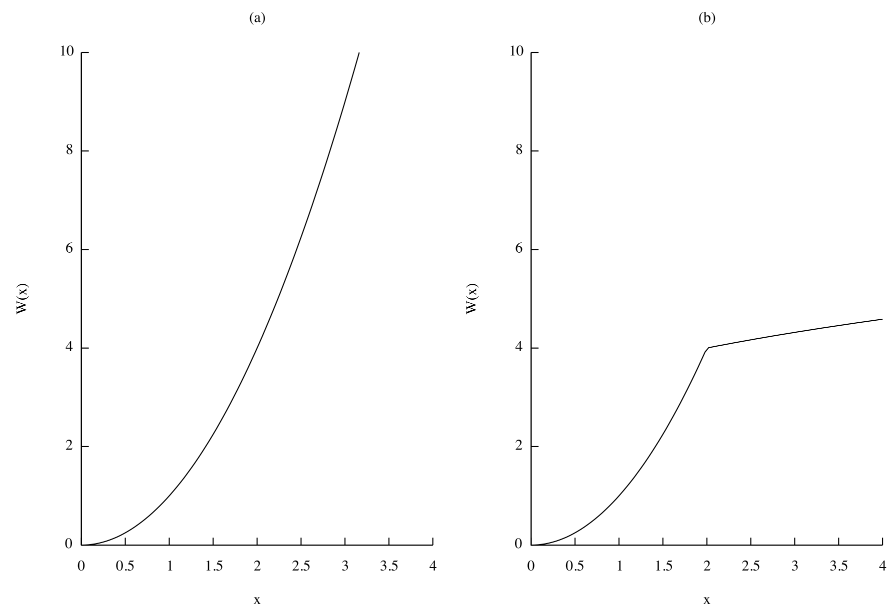

Figure. 1 Examples of payoff functions for loss-averse investors

case (a) exhibits strict convexity, the payoff function in case (b) is neither convex nor concave. A common feature of (a) and (b) is that these payoff functions exhibit convexity at the lower values of the reference portfolio. In this paper, therefore, we say that an investor should buy (sell) portfolio insurance if the payoff function is convex (concave) at lower terminal values of the reference portfolio.

3.4 The model

We proceed the analysis to derive the optimal payoff function. The framework adopted in this section is essentially the same as in the previous section except that investors can utilize endogenously priced AD securities to construct arbitrary payoff functions. Under the assumptions, the maximization problem for an investor can be written as

$$\begin{aligned} \max_{W_i} \min_{q} \quad & E^q[u_i(W_i(x))] \\ \text{s.t.} \quad & \begin{cases} R(q||p) \leq \eta \\ E^p[m(W_i(x) - W_0)] = 0 \end{cases}, \end{aligned} \tag{11}$$

where $u_i(\cdot)$ is the utility function of investor i, E^q and E^p are expectations under probability measures q and p, respectively, and m is the pricing kernel in this economy. Thus, in this model, an investor chooses an optimal probability measure under the constraint of the Kullback–Leibler divergence and then selects the optimal payoff function W_i under the budget constraint.

As in the previous section, we adopt the Lagrange multiplier method to solve this. To derive the optimal payoff function, first note that we can separate this maximization problem into two parts since the budget constraint is not restrictive to minimize the expected utility over the probability distribution. Therefore, we can first focus on the minimization problem: deriving the probability distribution under which the investors construct their optimal payoff functions.

3.5 The optimal payoff function

Let the Lagrange multiplier concerning the distributional constraint be ρ, then (11) is precisely the same as

$$\begin{aligned} \max_{W_i}\min_{q} \quad & E^q[u_i(W_i(x))] + \rho_i R(q||p) \\ \text{s.t.} \quad & E^p[m(W_i(x) - W_0)] = 0. \end{aligned} \tag{12}$$

The following proposition summarizes the result.

Proposition 2. *The optimal solution to (12) is given by*

$$W_i'(x) = \frac{m'(x)}{m(x)} \left(-\frac{u_i'(W_i(x))^2}{\rho_i} + u_i''(W_i(x)) \right)^{-1} u_i'(W_i(x)). \tag{13}$$

Proof. See Appendix B. □

As already stated, ρ's value is the measure of each investor's strength of beliefs as well as aversion toward ambiguity. When the investor is ambiguity-neutral, that is, $\rho_i \to \infty$, (13) is reduced to

$$W_i'(x) = -\frac{m'(x)}{m(x)} \frac{u_i'(W_i(x))}{u_i''(W_i(x))}. \tag{14}$$

This is exactly the same as the result obtained by Leland (1980)[5] and other authors in the literature. In this case of an ambiguity-neutral investor, the first derivative of

[5]We can obtain the same result as in Leland (1980) by replacing the pricing kernel, m, with the Radon–Nikodym derivative for the risk-neutral distribution relative to the physical one.

the optimal payoff function should be the value of risk-tolerance multiplied by the rate of change of the pricing kernel that is determined in the market equilibrium. Comparing (13) and (14), it can be recognized that the effect of ambiguity and the aversion to it appears as a negative term added to the negative second derivative of the utility function. As a result, the ambiguity and the investor's aversion toward it make investors to be less risk-tolerant. This result is consistent with the results obtained in the previous section.

4 Option Demand Under the Market Equilibrium

4.1 The optimal payoff function under the market equilibrium

It is reasonable to suppose that options or other types of financial derivatives are zero net supply in the market. Thus, under the market equilibrium, the sum of payoff functions should coincide with the realization value of the market portfolio itself, that is, $\sum_i W_i(x) = x$. Taking the summation of (13) over i and imposing the restated equilibrium condition, $\sum_i {W_i}'(x) = 1$, (13) can be rewritten as[6]

$$ {W_i}'(x) = \left\{ \sum_i {u_i}'(W_i) \left[-\frac{{u_i}'^2(W_i)}{\rho_i} + {u_i}''(W_i) \right]^{-1} \right\}^{-1} \left[-\frac{{u_i}'^2(W_i)}{\rho_i} + {u_i}''(W_i) \right]^{-1} {u_i}'(W_i). \tag{15} $$

Let us define

$$ T_i(W_i) \equiv - \left[-\frac{{u_i}'^2(W_i)}{\rho_i} + {u_i}''(W_i) \right]^{-1} {u_i}'(W_i), \tag{16} $$

as the risk tolerance of an ambiguity-averse investor i. Then, (15) can be rewritten as

$$ W_i^{'}(x) = \left[\sum_i T_i(W_i) \right]^{-1} T_i(W_i). \tag{17} $$

Differentiating this equation with respect to x and using (17), we have

$$ \text{sign}\left[W_i''(x) \right] = \text{sign} \left[T_i'(W_i) - \sum_i \frac{T_i(W_i)}{\sum_i T_i(W_i)} T_i'(W_i) \right]. \tag{18} $$

[6]We use W_i instead of $W_i(x)$ when it is more convenient.

Since the risk tolerance, T_i, is positive, we can interpret $T_i / \sum_i T_i$ as a weight function specified in terms of each investor's risk tolerance in the market. Thus, we can restate (18) by using an expectation symbol, E^*, as

$$\operatorname{sign}\left[W_i''(x)\right] = \operatorname{sign}\left[T_i{}'(W_i) - E^*\left[T'\right]\right]. \tag{19}$$

Using this fact, we can obtain the following lemma.

Lemma 2. *In the above economy, buying (selling) options are optimal if and only if the first derivative of the investor's risk tolerance function is higher (lower) than the market average.*

Proof. As discussed in the previous section, an investor should buy (sell) options if W_i'' is positive (negative). Then, this follows immediately from (19). □

4.2 Ambiguity and demand for options under the equilibrium model

For further investigation of the model, suppose that there are only two types of investors: an ambiguity-averse investor with some $\rho > 0$ and an investor with $\rho \to \infty$. The latter is said to be ambiguity-neutral or probabilistically sophisticate based on the reasons explained in Section 2.3. The investors are also assumed to have the same risk-preference.

Lemma 2 implies that if the first derivative of risk tolerance of the ambiguity-neutral investor is less than that of the ambiguity-averse investor, then ambiguity induces demand for options. The risk tolerance function of the ambiguity-averse agent is given by

$$T_a(W_i) = \frac{u'(W_i)}{-u''(W_i) + \frac{u'(W_i)^2}{\rho}}, \tag{20}$$

where the subscript a denotes that it is for an ambiguity-averse investor. Then, a direct calculation yields

$$T_a'(W_i) = -\frac{1 - \frac{u'(W_i)}{u''(W_i)}\left[\frac{u'''(W_i)}{u''(W_i)} - \frac{u'(W_i)}{\rho}\right]}{\left[1 - \frac{u'(W_i)^2}{\rho u''(W_i)}\right]^2}. \tag{21}$$

Thus, if we can compare (21) under different values of ρ, whether ambiguity induces demand for options can be examined. Unfortunately, however, (21) is difficult to manipulate without imposing further restrictions on the shape of the utility function since those derivatives also depend on W_i.

To proceed the analysis, therefore, we further assume the utility function is constant relative risk aversion (CRRA), i.e., $u(W_i) = {W_i}^{1-\gamma}/1-\gamma$. Since the first derivative of the risk tolerance function for an ambiguity-neutral investor, T', is constant, and is equal to $1/\gamma$ under CRRA preference, it allows us to obtain sharper results. We can then prove the following proposition:

Proposition 3. *Suppose that there are two types of investor in the market: an ambiguity-averse investor with some ρ and an ambiguity-neutral (i.e., $\rho \to \infty$) investor. The investors are also assumed to have the same CRRA utility function, $u(W_i) = {W_i}^{1-\gamma}/1-\gamma$. Then, the optimal payoff function for the ambiguity-averse investor exhibits strict concavity when $\gamma \le 2$.*

Proof. See Appendix C. □

4.3 Ambiguity and demand for portfolio insurance

The restricting condition, $\gamma \le 2$, induces us to impose some further assumptions to derive more concrete results on the effect of ambiguity on the optimal use of derivatives. For this purpose, note that $x \to +0$ implies $W_i \to +0$ from the market-clearing condition, $\sum_i W_i = x$ and from the fact that $W_i > 0$. Furthermore, if W_i is unbounded from above, then $x \to \infty$ implies $W_i \to \infty$. Thus, we can investigate the limiting cases where $x \to +0$ and $x \to \infty$ by taking the limits of W_i. The result is summarized as the following proposition:

Proposition 4. *Suppose that investors have the same CRRA preference, then portfolio insurance is sold from the ambiguity-averse investor to the ambiguity-neutral investor. Furthermore, when $\gamma > 2$, the ambiguity-averse investor should sell options at lower realization values of the reference asset and buy options at higher values.*

Proof. See Appendix D. □

Therefore, contrary to our intuition, this proposition indicates that ambiguity aversion reduces investors' incentives for risk management by using options. In other words, the ambiguity-neutral agent is more willing to avoid losses under the equilibrium model.

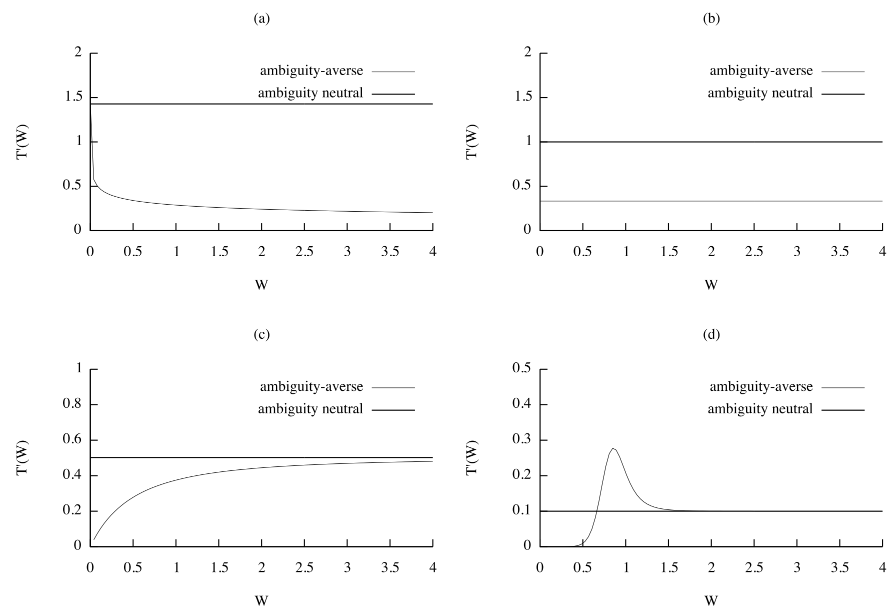

Figure. 2 This figure illustrates T_a' and T' under four cases: (a) $0 < \gamma < 1$; (b) $\gamma = 1$; (c) $1 < \gamma < 2$; and (d) $2 < \gamma$. Under all cases, we can see that T_a' is less than T' at $W_i \to +0$, which indicates that the ambiguity-averse agent should sell the portfolio insurance to the ambiguity-neutral agent. It is also noteworthy that when $\gamma \leq 2$ (i.e., (a), (b) and (c)), $T_a' \leq T'$ at all realization value of W_i, and when $\gamma > 2$ (i.e., (d)), $T_a' \leq T'$ at lower values of W_i, while $T_a' \geq T'$ at higher values of W_i, which illustrate results obtained in Proposition 3 and 4.

4.4 Strength of ambiguity aversion and demand for options: Logarithmic utility case

We now examine the effect of the strength of ambiguity aversion on the optimal use of derivatives. To derive an interesting result, we assume $\gamma = 1$, that is, investors have a logarithmic utility function and differ only in the values of ρ_i. In this case, the first derivative of the risk tolerance function for agent i becomes constant, which enables us to investigate the effect of the strength of the ambiguity aversion ρ_i without detecting W_i. In fact, substituting $\gamma = 1$ into (C.1), we obtain

$$T_a'(W_i) = \left(1 + \frac{1}{\rho_i}\right)^{-1}, \tag{22}$$

which implies that T_a' depends only on ρ_i. A direct calculation yields

$$\frac{\partial T_a'}{\partial \rho_i} = \left(1 + \frac{1}{\rho_i}\right)^{-2} \frac{1}{{\rho_i}^2} > 0, \tag{23}$$

which indicates that T_a' for an investor with a logarithmic utility function is increasing in ρ_i. Then, recalling that options are sold from investors with lower T' to those who have higher values of it, we have the following corollary.

Corollary 2. *Suppose that two types of investors have a logarithmic utility function and differ only in ρ. Then, the investor with higher ρ should buy options from another type.*

Proof. It is immediate from Lemma 1 and (23). □

The result in Corollary 3 can be easily extended to the case wherein there exists more than two types of investors. In that case, investors who have lower ρ than the subjectively weighted market average (see (18)) should be option writers. Therefore, investors under strong ambiguity and strong ambiguity aversion should sell options in this case of logarithmic utility.

5 Option Demand Under the Representative Agent Model

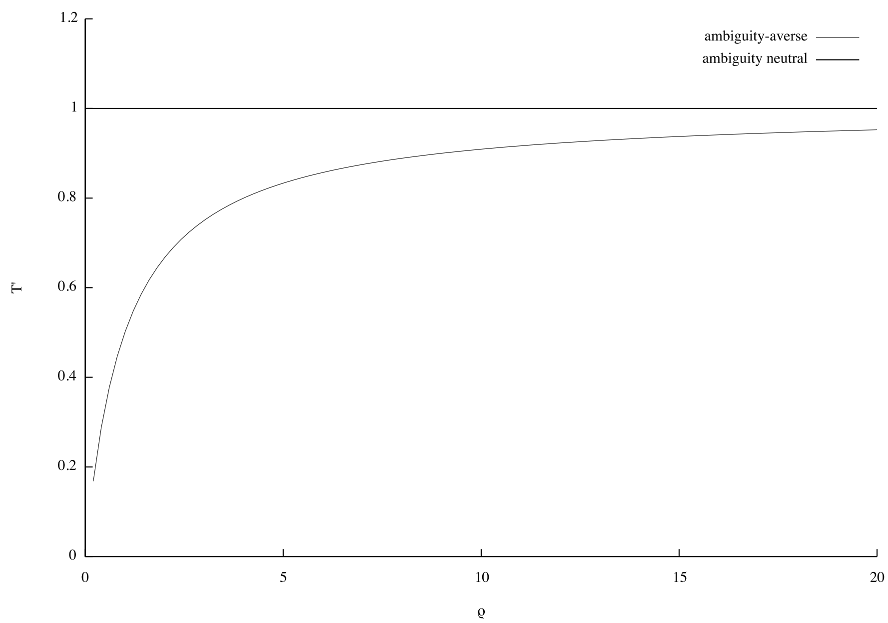

Figure. 3 This figure illustrates the relation between the strength of ambiguity aversion, ρ, and the first derivative of risk tolerance function, T' under the logarithmic utility. We can see that T'_a is an increasing concave function of ρ and T'_a is less than T'.

5.1 *The optimal payoff function under the representative agent model*

Even though the shapes of the utility functions are determined, it is difficult to obtain the closed-form expressions for the optimal payoff functions. This is because under our equilibrium model, analytical shape of the endogenously derived pricing kernel is hard to derive. This motivates us to use the representative agent model as in Leland (1980) though the existence of a representative agent under ambiguity is uncertain. Under the representative agent model, the pricing kernel m is replaced by the risk tolerance function of the representative agent, which implies that (13) becomes

$$W_i{}'(x) = \frac{-\frac{v'(x)^2}{\rho_v} + v''(x)}{v'(x)} \frac{u'(W_i)}{-\frac{u'(W_i)^2}{\rho_u} + u''(W_i(x))}, \tag{24}$$

where v is the utility function of the representative agent, ρ_v and ρ_u are the strength of ambiguity aversion of the representative agent and the individual investor, respectively.[7] Note that the optimal payoff function of the representative agent is, by definition, x, the realization value of the reference portfolio by definition.

5.2 *Ambiguity and demand for options under the representative agent model*

Under this representative agent model, we have the following lemma, which is first derived in Leland (1980).

Lemma 3. *The optimal payoff function exhibits strict convexity if and only if the first derivative of risk tolerance function of the individual is uniformly greater than that of the representative investor.*

Recall that the risk tolerance function of the representative agent, T_r, is given by

$$T_r = \frac{v'(x)}{\frac{v'(x)^2}{\rho_v} - v''(x)}, \tag{25}$$

which is a function of x, while the risk tolerance function of the individual, T_i, is given by

$$T_i = \frac{u'(W_i)}{\frac{u'(W_i)^2}{\rho_u} - u''(W_i)}, \tag{26}$$

[7] The important question whether the ambiguity is priced under equilibrium, that is, whether the representative agent is ambiguity-averse or not is also outside the scope of this study.

which is a function of W_i. Then, Lemma 3 implies that optimal payoff function of the individual i exhibits strict convexity if and only if $T_i' > T_r'$. When the representative investor and the individual have the same CRRA preference, we have the following result.

Corollary 3. *Suppose that the representative agent and the individual have the same CRRA preference. Then, if the representative agent is ambiguity-neutral, the optimal payoff function of the individual exhibits strict concavity on $x > 0$ if and only if RRA is less than or equal to two. In addition, when RRA is greater than two, the payoff function exhibits strict concavity until some point of x and then exhibits strict convexity.*

Proof. Substituting $v(x) = x^{1-\gamma}/1-\gamma$, $u(W_i) = {W_i}^{1-\gamma}/1-\gamma$, and $\rho_v \to \infty$ into T_r and T_i, we have

$$T_r(x) = \frac{x}{\gamma}, \quad and \quad T_i(W_i) = \frac{W_i}{\frac{{W_i}^{1-\gamma}}{\rho_v} + \gamma}. \tag{27}$$

Comparing the first derivative of $T_r(x)$ and $T_i(W_i)$ is exactly the same to what we have done in Proposition 3 and 4. Thus, the rest of the proof is similar to that of Proposition 3 and 4. □

Although this corollary largely depends on the informal assumption of the existence of the representative agent, the result obtained in this representative agent model has an obvious similarity to those in the equilibrium approach. This is because Proposition 3 and 4 assume that there are only two types of investors and thus the equilibrium model in the previous section is essentially the same as the representative agent model in this section.

5.3 A closed-form solution: Logarithmic utility case

To obtain a closed-form expression for the payoff function, we further assume a logarithmic utility function for the both u and v. Substituting $u(W_i) = \ln W_i$ and $u(x) = \ln x$ into (24), we have

$${W_i}'(x) = \frac{1 + \frac{1}{\rho_v}}{x} \frac{W_i(x)}{1 + \frac{1}{\rho_u}}. \tag{28}$$

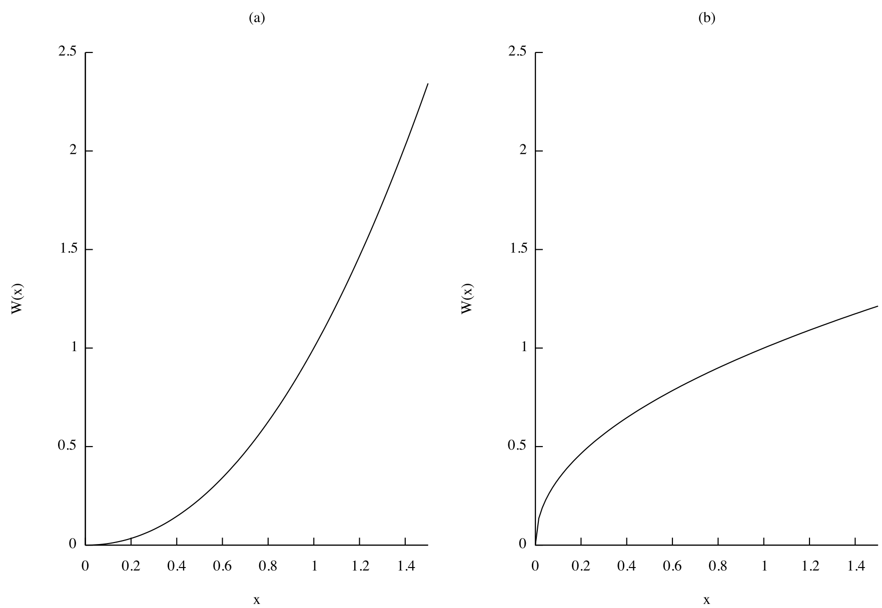

Figure. 4 The optimal payoff functions under the representative agent model

In this case, we can obtain the closed-form expression for the optimal payoff function. A direct calculation shows that the optimal payoff function has the form:

$$W_i(x) = cx^{\frac{1+\rho_v^{-1}}{1+\rho_u^{-1}}}, \tag{29}$$

where $c > 0$ is the constant determined by the budget constraint of the investor. Figure 4 illustrates the optimal payoff functions under two cases: (a) the individual investor i is less ambiguity-averse than the representative agent (i.e., $\rho_v^{-1} - \rho_u^{-1} > 0$); (b) the individual investor i is more ambiguity-averse than the representative agent (i.e., $\rho_v^{-1} - \rho_u^{-1} < 0$). The optimal payoff function exhibits convexity in case (a), which indicates the investor should buy options, while the optimal payoff function exhibits concavity in case (b), which indicates the investor should sell options. Thus, an increase in ambiguity aversion generates supply of options in this logarithmic utility representative agent model, which is also similar to the result obtained in the equilibrium model (see Corollary 3).

6 Conclusion

In this study, we investigate the effect of ambiguity on investors' demand for options. Under the assumption that investors have identical CRRA utility and follow the Hansen–Sargent robust control criterion, this study characterizes the nature of investors who will benefit from holding options, particularly from buying portfolio insurance. The main conclusions of this paper are: (i) ambiguity-neutral investors will buy portfolio insurance from ambiguity-averse investors; (ii) Ambiguity-averse investors will demand options only when the identical RRA is greater than two; (iii) Under the logarithmic utility, investors with relatively stable ambiguity aversion will be option writers.

In summary, except when the identical RRA is greater than two, the answer to the question of whether ambiguity generates demand for options is negative. In particular, we demonstrate how ambiguity induces the supply of portfolio insurance under the CRRA preference. The theoretical reason behind the result is that the risk tolerance of an ambiguity-neutral investor increases more rapidly than that of an ambiguity-averse investor at lower realized values of the reference asset.

Although we derive several strong conclusions about the relationship between ambiguity and the use of options, this study adopts a simple static economy to focus on the role of ambiguity. Thus, one natural extension of our study would be conducting theoretical research in a dynamic and continuous time setting and assuming more general decision-making criteria such as KMM (2005). Furthermore, it could be informative to incorporate preference heterogeneity into the model and analyze the combined effects of risk aversion and ambiguity on investors' demand for options. These problems are left for future research.

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Appendix A: Concavity of the Objective Function in the Hansen–Sargent Robust Control Model

In this appendix, we show the concavity of the objective function under the Hansen–Sargent robust control model. Differentiating $V'(\alpha)$ with respect to α, we obtain

$$-\frac{1}{\rho}E^p\left[e^{-\frac{u}{\rho}}\right]^{-1}\left\{E^p\left[x^2u'^2e^{-\frac{u}{\rho}}\right]+\frac{1}{\rho}E^p\left[xu'e^{-\frac{u}{\rho}}\right]^2E^p\left[e^{-\frac{u}{\rho}}\right]^{-1}+E^p\left[x^2u''e^{-\frac{u}{\rho}}\right]\right\}, \tag{A.1}$$

where we use u instead of $u(w_0+\alpha x)$ to reduce notation. Let E^* denote the expectation under the implicit probability distribution, defined as

$$E^*[Z]=E^p\left[\frac{e^{-\frac{u(w_0+\alpha x)}{\rho}}}{E^p\left[e^{-\frac{u(w_0+\alpha x)}{\rho}}\right]}Z\right], \tag{A.2}$$

then (A.1) can be rewritten as

$$-\frac{1}{\rho}\left\{E^*\left[x^2u'(w_0+\alpha x)^2\right]-E^*\left[xu'(w_0+\alpha x)\right]^2\right\}+E^*\left[x^2u''(w_0+\alpha x)\right]. \tag{A.3}$$

Since $E^*\left[x^2u'(w_0+\alpha x)^2\right]-E^*\left[xu'(w_0+\alpha x)\right]^2$ is the variance of $xu'(w_0+\alpha x)$ under the expectation E^*, it takes a positive value. Combining this observation with the fact that $E^*\left[x^2u''(w_0+\alpha x)\right]<0$ implies that (A.1) takes negative values.

Appendix B: Proof of Proposition 2

As in Section 3.5, the maximization problem is given by

$$\begin{aligned}\max_{W_i}\min_{q}\quad & E^q[u_i(W_i(X))]+\rho R(q||p)\\ \text{s.t.}\quad & E^p[m(W_i(X)-W_0)]=0,\end{aligned} \tag{B.1}$$

where $\rho>0$. To solve this problem, forming the Lagrangian

$$\mathcal{L}=E^q[u_i(W_i(X))]+\rho R(q||p)-\lambda E^p[m(W_i(X)-W_0)] \tag{B.2}$$

and taking the derivative with respect to q yields

$$\mathcal{L}_q=\int\left[u_i(W_i(x))+\rho\ln\left(\frac{q(x)}{p(x)}\right)\right]dx. \tag{B.3}$$

Therefore, we can obtain the optimal probability measure by setting $[\cdot]$ in the above equation to zero:

$$q(x) = p(x)e^{-\frac{u_i(W_i(x))}{\rho}}. \tag{B.4}$$

To make this a probability measure, (B.4) should be normalized as

$$q(x) = \frac{p(x)e^{-\frac{u_i(W_i(x))}{\rho}}}{\int p(x)e^{-\frac{u_i(W_i(x))}{\rho}}dx}. \tag{B.5}$$

Substituting (B.5) into (B.2) yields

$$\mathcal{L} = -\rho\ln E^p\left[e^{-\frac{u_i(W_i(X))}{\rho}}\right] - \lambda E^p[m(W_i(X) - W_0)]. \tag{B.6}$$

Taking the derivative with respect to W_i and setting it equal to zero yields

$$E^p\left[u_i'(W_i^*(X))\frac{e^{-\frac{u_i(W_i^*(X))}{\rho}}}{E^p\left[e^{-\frac{u_i(W_i^*(X))}{\rho}}\right]}\right] - \lambda E^p[m] = 0, \tag{B.7}$$

where we use $*$ to denote optimal strategy. It should be noted that this equation has some similarity to (5). Using the martingale method, we know that the optimal payoff function $W_i^*(x)$ satisfies

$$u_i'(W_i^*(x))\frac{e^{-\frac{u_i(W_i^*(x))}{\rho}}}{E^p\left[e^{-\frac{u_i(W_i^*(x))}{\rho}}\right]} = \lambda m(x) \quad \forall x. \tag{B.8}$$

Differentiating (B.8) with respect to x, solving for λ, and substituting λ back into lead to

$$W_i'(x) = \frac{m'(x)}{m(x)}\left(-\frac{u_i'^2(W_i)}{\rho} + u_i''\right)^{-1} u_i'(W_i). \tag{B.9}$$

Appendix C: Proof of Proposition 3

Let T' be the first derivative of a risk tolerance function for the ambiguity-neutral investor, which is equal to $1/\gamma$. For the ambiguity-averse investor, substituting

$u'(W_i) = W_i^{-\gamma}$ into (21), we have

$$T'_a(W_i) = \frac{\gamma + \frac{\gamma}{\rho} {W_i}^{1-\gamma}}{\left(\gamma + \frac{{W_i}^{1-\gamma}}{\rho}\right)^2}, \tag{C.1}$$

To prove the proposition, we have to show that $T'_a < T'$ holds when $\gamma \leq 2$.

If $0 < \gamma < 1$, taking the limit $W_i \to +0$, we have

$$\lim_{W_i \to +0} T'_a = \frac{1}{\gamma}, \tag{C.2}$$

which shows that $T'_a = T'$ at $W_i = 0$. In addition, if $1 < \gamma$, we have

$$\lim_{W_i \to +0} T'_a = \lim_{W_i \to +0} \frac{\gamma + \frac{\gamma}{\rho} W_i^{1-\gamma}}{\left(\gamma + \frac{W_i^{1-\gamma}}{\rho}\right)^2} = \lim_{W_i \to +0} \frac{\gamma}{2\left(\gamma + \frac{W_i^{1-\gamma}}{\rho}\right)} = 0, \tag{C.3}$$

where the second equality uses the l'Hôpital's rule. Moreover, if $\gamma = 1$, T'_a is also constant and is given by

$$T'_a = \frac{1}{1 + \frac{1}{\rho}}. \tag{C.4}$$

By (C.2) and (C.3), $T''_a \leq 0$ is sufficient to establish that $T'_a \leq T'$. Differentiating (C.1) with respect to W_i, we obtain

$$T''_a(W_i) = \frac{\frac{\gamma(1-\gamma)}{\rho} W_i^{-\gamma} \left(\gamma + \frac{W_i^{1-\gamma}}{\rho}\right)\left(\gamma - 2 - \frac{W_i^{1-\gamma}}{\rho}\right)}{\left(\gamma + \frac{W_i^{1-\gamma}}{\rho}\right)^4}, \tag{C.5}$$

which implies that

$$\text{sign}\left[T''_a(W_i)\right] = \text{sign}\left[(1-\gamma)\left(\gamma - 2 - \frac{W_i^{1-\gamma}}{\rho}\right)\right]. \tag{C.6}$$

Since $W_i > 0$, (C.6) implies that $T''_a(W_i) > 0$ if $1 < \gamma \leq 2$ and $T''_a(W_i) \leq 0$ if $\gamma \leq 1$. Thus, the concave strategy (i.e., $W''_i < 0$) is optimal for the ambiguity-averse agent when $\gamma \leq 1$. For $1 < \gamma$, taking the limit $W_i \to \infty$, we have

$$\lim_{W_i \to \infty} T'_a(W_i) = \frac{1}{\gamma}. \tag{C.7}$$

Combining this with the fact that $T''_a(W_i) \geq 0$ for all $1 < \gamma \leq 2$ yields $T'_a(W_i) \leq T'(W_i)$. This implies that the payoff function of the ambiguity-averse investor exhibits concavity when $1 < \gamma \leq 2$ and completes the proof.

Appendix D: Proof of Proposition 4

Recall that an investor i should buy (sell) portfolio insurance if $W_i'' > 0$ ($W_i'' < 0$) at $x \to +0$. As discussed in Section 4.3, therefore, the market-clearing condition $\sum_i W_i = x$ implies that the investor i should buy (sell) portfolio insurance if $W_i'' > 0$ ($W_i'' < 0$) when $W_i \to +0$. Then, the first part of the proposition follows from Proposition 3 (for the case of $\gamma \leq 2$) and (C.3) (for the case of $\gamma > 2$).

To prove the second part of the proposition, note that for all $\gamma > 2$, $T_a''(W_i) > 0$ holds when $W_i < [\rho(\gamma-2)]^{\frac{1}{1-\gamma}}$ and $T_a''(W_i) < 0$ when $W_i > [\rho(\gamma-2)]^{\frac{1}{1-\gamma}}$. This implies that T_a' is increasing in W_i when $W_i < [\rho(\gamma-2)]^{\frac{1}{1-\gamma}}$ and is decreasing when $W_i > [\rho(\gamma-2)]^{\frac{1}{1-\gamma}}$. Thus, T_a' is maximized at $W_i = [\rho(\gamma-2)]^{\frac{1}{1-\gamma}}$. Substituting $W_i = [\rho(\gamma-2)]^{\frac{1}{1-\gamma}}$ into (C.1), we have

$$T_a'\left((\rho(\gamma-2))^{\frac{1}{1-\gamma}}\right) = \frac{\gamma+\gamma(\gamma-2)}{[\gamma+(\gamma-2)]^2} = \frac{\gamma}{4(\gamma-1)}. \tag{D.1}$$

To complete the proof, we need to verify that $T_a'\left((\rho(\gamma-2))^{\frac{1}{1-\gamma}}\right) > T'(W)$ for all $2 < \gamma$. Thus, we have to show that

$$\frac{\gamma}{4(\gamma-1)} > \frac{1}{\gamma}, \tag{D.2}$$

which is immediate since (D.2) is equivalent to

$$(\gamma-2)^2 > 0. \tag{D.3}$$

Finally, combining this with (C.7) completes the rest of the proof.