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Optimal Consumption Under Different Resolution Times of Uncertainty

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Abstract

This study compares the optimal consumption amount for a risk-averse agent under different uncertainty resolution times in a time-separable utility setting. We show that an agent with a positive third derivative utility function (i.e., a prudent agent) reduces his/her consumption amount when the uncertainty resolution is postponed. We also demonstrate that an agent does not change consumption behavior under different uncertainty resolution times if and only if the agent has a quadratic utility.

Keywords: Background risk, Time-separable utility, Prudence, Optimal consumption, Timing of uncertainty resolution
JEL Classification: G11, G22, G51

1 Introduction

An agent who reduces his/her current consumption amount in the presence of an uninsurable background risk is said to be prudent. Given that the behavior of an agent is described using the von Neumann–Morgenstern utility function, u, the pioneering works of Leland (1968) and Sandmo (1970) show that agents are prudent if and only if the third derivative of their utility function is positive. In other words, an agent with $u''' > 0$ saves more when faced with an uninsurable background risk that has a non-positive expected value than otherwise.

Kimball's seminal works (1990, 1993) advance the analysis a step further and propose a novel measure of the strength of the precautionary saving motive. This measure is referred to as absolute prudence, which is analogously defined as the Arrow–Pratt measure of risk aversion applied to the marginal utility function multiplied by -1. The optimal saving/consumption problem between agents can be investigated directly through a comparison of their absolute prudence or the precautionary premiums.

In this paper, we focus on the effect of the timing of existing background risks on an agent's optimal saving/consumption behavior. Specifically, we assume that the uncertainty that might

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negatively impact wealth is postponed and explore whether the agent saves more or less compared to the case when the uncertainty is not postponed. This problem is relevant not only in studies of financial economics, but also in the macroeconomic context. For example, if the late resolution of uncertainty has a negative impact on an agent, then he/she has an incentive to shift such a risk to younger generations and prefers to bear the older generations' risks when he/she is young. Thus, our study may be applied to problems regarding the optimal risk shifting scheme, such as annuities across generations.

The remainder of this article proceeds as follows. Section 2 presents the model and demonstrates the equivalence between the positivity of the third derivative of a utility function and the reduction of current consumption under late resolution of uncertainty. Section 3 provides the concluding remarks.

2 The Model

For simplicity, we explore a two-period model. The model, however, can be easily extended to, for instance, an n-period model.

The following assumptions are made throughout this paper:

Assumption 1. *Two-period economy.* $t \in \{0, 1, 2\}$ *denote the point in time, referred to as dates.*

Assumption 2. *An agent has an arbitrarily differentiable risk-averse time-separable von Neumann–Morgenstern utility function; that is, there exist functions* $u_t = u$, $t = \{0, 1, 2\}$ *defined on some interval* $D \subseteq \mathbb{R}$ *such that*[1]

$$U(C_0, C_1, C_2) = u(C_0) + u(C_1) + u(C_2).$$

Assumption 3. *The agent has an initial endowment,* W_0.

Assumption 4. *The interest rate in the economy is zero, and there are no investing opportunities.*

Assumption 5. *The subjective discounting factor of the agent is unity. That is, the agent evaluates tomorrow's unit consumption to be exactly the same as that of today.*

Assumption 6. *There is an uninsurable, independent background risk, denoted by* X.

Assumption 7. X *is continuous random variables so that the support of* $(W_0 + X)/3$ *is in the interior of* D*, which is denoted by* $IntD$.

Note that Assumptions 4 and 5 are necessary to focus on the analysis of the preference for the timing of the uncertainty resolution. In addition, to recognize the necessity of Assumption 7, suppose that the utility function u is defined on D =[a, b]. Let x be a realization value of X. Since the agent consumes no less than a, if $C_0 = C_1 = a$, then C_2 becomes $W_0 + x - 2a$. Conversely, since the agent consumes no more than b, if $C_0 = C_1 = b$, then C_2 becomes $W_0 + x - 2b$. Under Assumption 7, therefore, the agent can choose C_0 and C_1 in Int$D$ while C_2 is also in Int$D$ for any realization values x.

[1] u being differentiable is equivalent to u being differentiable in the interior of D.

We first consider an agent with X at date 2. To derive a condition for an agent's optimal consumption at date 0, we first define the agent's value function, which can be expressed as

$$v(z) = \max_{C_1 \in \text{Int}D} u(C_1) + E_1[u(z - C_1 + X)], \tag{1}$$

where the subscript indicates that the expectation is taken conditionally on date 1 information. Under our independence assumption (Assumption 6), date 1 information does not improve the precision of the conditional expectation value for X, and (1) is restated as

$$v(z) = \max_{C_1 \in \text{Int}D} u(C_1) + E_0[u(z - C_1 + X)]. \tag{2}$$

Since the function is maximized with respect to C_1, the resultant function depends solely on z, which is stochastic at date 0. With this value function at hand, the maximization problem for the optimal consumption at date 0 can be written as

$$\max_{C_0 \in \text{Int}D} u(C_0) + E_0[v(W_0 - C_0)]. \tag{3}$$

Since the objective function is concave in C_0, the optimal consumption, C_0^*, is characterized by the first-order condition:

$$u'(C_0^*) - E_0[v'(W_0 - C_0^*)] = 0. \tag{4}$$

By using the envelope theorem, $v'(z) = E_0[u'(z - C_1 + X)]$, (4) can be rewritten as

$$u'(C_0^*) - E_0[u'(W_0 - C_0^* - C_1 + X)] = 0. \tag{5}$$

Since there is no uncertainty at date 1, $C_0^* = C_1$ is necessary for the optimality; the agent can otherwise improve his/her felicity by transferring the amount consumed from the larger to the smaller. Therefore, substituting $C_0^* = C_1$ into (5) yields

$$u'(C_0^*) - E_0[u'(W_0 - 2C_0^* + X)] = 0. \tag{6}$$

Next, consider an agent with X at date 1. In this case, there is no uncertainty at date 2, $C_1 = C_2$ is necessary for optimality. By using the envelope theorem, this implies that the value function takes a form:

$$v'(z) = u'\left(\frac{z}{2}\right). \tag{7}$$

The agent's maximization problem can be described as

$$\max_{C_0 \in \text{Int}D} u(C_0) + E_0[v(W_0 - C_0 + X)]. \tag{8}$$

Using (7), optimal consumption can be characterized by[2]

[2]Note that this maximization is equivalent to

$$\max_{C_0 \in \text{Int}D} u(C_0) + 2E_0\left[u\left(\frac{W_0 - C_0 + X}{2}\right)\right]. \tag{9}$$

$$u'(C_0^*) - E_0\left[u'\left(\frac{W_0 - C_0^* + X}{2}\right)\right] = 0. \tag{10}$$

Let $C_0(\mathrm{LR})$ and $C_0(\mathrm{ER})$ be optimal consumption satisfying (6) and (10), respectively.[3] To compare $C_0(\mathrm{LR})$ and $C_0(\mathrm{ER})$, we use a useful instrument called "diffidence theorem," introduced by Gollier (2001). We first summarize the diffidence theorem as the next lemma.

Lemma 1. *(Diffidence theorem)*

$$E[f(X)] = 0 \Rightarrow E[g(X)] \leq 0 \Leftrightarrow \exists m \quad g(x) \leq mf(x) \quad \forall x.$$

Proof. See Gollier (2001). □

This useful result by Gollier enables us to convert a seemingly complex problem into a tractable one. As an application of the diffidence theorem, we can prove a simple relation between the shape of the utility function and the sign of $C_0(\mathrm{ER}) - C_0(\mathrm{LR})$. The following proposition summarizes the result.

Proposition 1. *Consider an agent who has an increasing, concave utility function defined on D with a non-zero third derivative, and suppose the derivatives do not alternate in wealth. Then, $C_0(\mathrm{LR}) < C_0(\mathrm{ER})$ if and only if the third derivative of the utility function is positive.*

Proof. To apply the diffidence theorem to our context, note that $C_0(\mathrm{LR}) \leq C_0(\mathrm{ER})$ is equivalent to

$$u'(C_0) - E_0\left[u'\left(\frac{W_0 - C_0}{2} + \frac{X}{2}\right)\right] = 0 \Rightarrow u'(C_0) - E_0[u'(W_0 - 2C_0 + X)] \leq 0. \tag{11}$$

Based on the diffidence theorem, a necessary and sufficient condition for (11) is

$$\exists m \quad u'(C_0) - u'(W_0 - 2C_0 + x) - m\left[u'(C_0) - u'\left(\frac{W_0 - C_0}{2} + \frac{x}{2}\right)\right] \leq 0 \quad \forall x. \tag{12}$$

Denoting the left-hand side of (12) as $H(x, m)$, we should look for the value m, such that $H(x, m) \leq 0$ for any realization values of X. Since $H(3C_0 - W_0, m) = 0$,[4]

$$\left.\frac{\partial H}{\partial x}\right|_{x=3C_0-W_0} = 0, \quad \left.\frac{\partial^2 H}{\partial x^2}\right|_{x=3C_0-W_0} \leq 0 \tag{13}$$

are necessary. Thus,

$$-u''(C_0) + \frac{m}{2}u''(C_0) = 0 \Leftrightarrow m = 2, \tag{14}$$

and substituting this back to H and differentiating twice at $x = 3C_0 - W_0$ results in

$$\frac{1}{2}u'''(C_0) \leq u'''(C_0). \tag{15}$$

[3]"LR" and "ER" denote "late resolution of uncertainty" and "early resolution of uncertainty.

[4]Let $\overline{x}$ and $\underline{x}$ be the lower and upper bound of X, respectively. By using the concavity of u, it can be verified that $(\underline{x} + W_0)/3 \leq C_0 \leq (\overline{x} + W_0)/3$. Then, it follows that $\underline{x} \leq 3C_0 - W_0 \leq \overline{x}$. Therefore, $u'' < 0$ is sufficient to ensure that $3C_0 - W_0$ is in the support of X.

This condition is satisfied if and only if $u''' \geq 0$. Moreover, in the same way, the necessary condition for $C_0(\text{LR}) \geq C_0(\text{ER})$ collapses to $u''' \leq 0$.

To prove sufficiency, substituting $m = 2$ into (12), then a necessary and sufficient condition for (11) becomes

$$u'(C_0) - u'(W_0 - 2C_0 + x) - 2\left[u'(C_0) - u'\left(\frac{W_0 - C_0}{2} + \frac{x}{2}\right)\right] \leq 0 \quad \forall x. \tag{16}$$

Thus, we must show that $u''' \geq 0$ is sufficient for (16). Rearranging (16) leads to

$$u'\left(\frac{W_0 - C_0 + x}{2}\right) - u'(C_0) \leq u'(W_0 - 2C_0 + x) - u'\left(\frac{W_0 - C_0 + x}{2}\right) \quad \forall x. \tag{17}$$

When $x = 3C_0 - W_0$, (17) follows immediately. When $x > 3C_0 - W_0$, note that

$$C_0 < \frac{W_0 - C_0 + x}{2} < W_0 - 2C_0 + x \tag{18}$$

and

$$\frac{W_0 + x - 3C_0}{2} = \frac{W_0 - C_0 + x}{2} - C_0 = W_0 - 2C_0 + x - \frac{W_0 - C_0 + x}{2}. \tag{19}$$

Then, dividing both sides of (17) by $(W_0 + x - 3C_0)/2$ yields

$$\frac{u'(\frac{W_0 - C_0 + x}{2}) - u'(C_0)}{\frac{W_0 + x - 3C_0}{2}} \leq \frac{u'(W_0 - 2C_0 + x) - u'(\frac{W_0 - C_0 + x}{2})}{\frac{W_0 + x - 3C_0}{2}} \quad \forall x > 3C_0 - W_0. \tag{20}$$

When $u''' = 0$ (i.e., u' is linear), (20) holds with equality. When $u''' > 0$, since

$$\frac{u'(b) - u'(a)}{b - a} < \frac{u'(c) - u'(b)}{c - b} \quad \forall a < b < c, \tag{21}$$

for any strictly convex function u', combining (21) with (18) and (19) implies (20). This proves that $u''' \geq 0$ is sufficient for (16) to hold when $x > 3C_0 - W_0$. When $x < 3C_0 - W_0$, the proof is similar to the case where $x > 3C_0 - W_0$; thus, $u''' \geq 0$ implies $C_0(\text{LR}) \leq C_0(\text{ER})$. Following the same steps, it can also be verified that $u''' \leq 0$ implies $C_0(\text{LR}) \geq C_0(\text{ER})$. Note also that $C_0(\text{LR}) = C_0(\text{ER})$ only if $u''' = 0$ since $C_0(\text{LR}) = C_0(\text{ER})$ is sufficient for $u''' \geq 0$ and $u''' \leq 0$. Furthermore, using the fact that the precautionary premium for an agent with $u''' = 0$ is zero, it can be proved that $u''' = 0$ is sufficient for $C_0(\text{LR}) = C_0(\text{ER})$. This establishes the equivalence between $C_0(\text{LR}) = C_0(\text{ER})$ and $u''' = 0$, which completes the proof. □

An immediate corollary of this proposition is that

Corollary 1. $C_0(\text{LR}) = C_0(\text{ER})$ *if and only if the agent has a quadratic utility function.*

Figure 1-3 illustrate the results of Proposition 1 and Corollary 1. By (17), $C_0(\text{LR}) \leq C_0(\text{ER})$ if and only if $B \leq A$ and $B* \leq A*$ for any realization values x, while $C_0(\text{LR}) \geq C_0(\text{ER})$ if and only if $A \leq B$ and $A* \leq B*$ for any x since the inequality in (17) is reversed. Figure 1 shows that $u''' > 0$ implies $B \leq A$ and $B* \leq A*$, while Figure 2 shows that $u''' < 0$ implies $A \leq B$ and

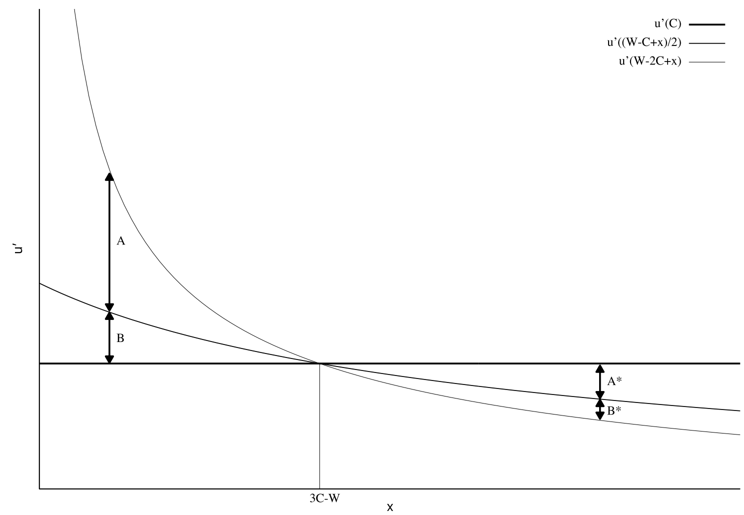

Figure 1: $u''' > 0$

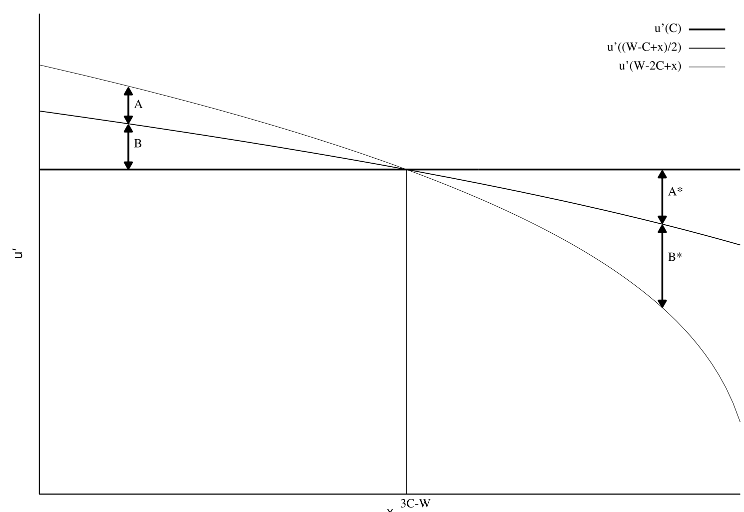

Figure 2: $u''' < 0$

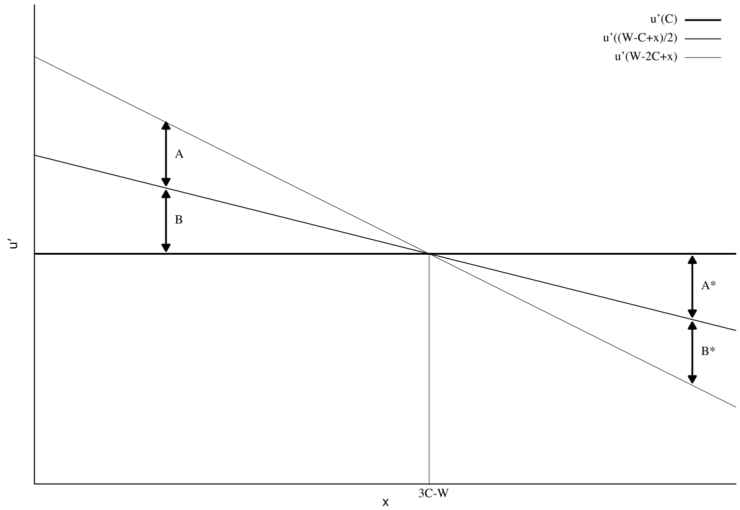

Figure 3: $u''' = 0$

$A* \leq B*$. Furthermore, since $C_0(\mathrm{LR}) = C_0(\mathrm{ER})$ is equivalent to $A = B$ and $A* = B*$ for any realization values x, $C_0(\mathrm{LR}) = C_0(\mathrm{ER})$ if and only if u' is linear (see Figure 3).

Scott and Horvath (1980) and Menegatti (2001) show that the first sentence of the previous proposition also implies that the third derivative of a utility function is positive under more restrictive conditions on D. The key feature of their study is that D is unbounded from above. In that case, it can be verified that the concavity of a utility function implies that the third derivative of utility is positive by using the mean value theorem. Thus, we have the following corollary.

Corollary 2. *Consider an agent who has an increasing, concave utility function defined on D that is unbounded from above, and suppose the derivatives do not alternate in wealth. Then, the agent increases the current consumption during the early resolution of uncertainty.*

Without any restrictions imposed on D, however, concavity is not sufficient to guarantee the positivity of the third derivative. For example,

$$u(z) = -(A - Bz)^{\gamma},$$

where $A - Bz > 0$, $B > 0$, $2 > \gamma > 1$ can be viewed as an example of a utility function that is increasing, concave, and imprudent. It is also clear that the domain of this utility function is bounded from above since $A - Bz > 0$ if and only if $z < A/B$.

3 Conclusion

In this paper, the optimal saving/consumption problem for a risk-averse agent is considered with respect to the timing of uncertainty resolution. We demonstrate that when the occurrence of background risk is postponed, a positive third derivative of the utility function is proved to be a necessary and sufficient condition for increasing optimal savings for an agent under the time-separable utility setting. Compared with previous research, therefore, this study clarifies that the time-separable utility setting implicitly contains a preference for the early resolution of uncertainty in its widely used assumption of $u''' > 0$.

This study presents a strong relation between the sign of the third derivative of the utility function and the preference for the timing of the uncertainty resolution. However, there are still several unanswered questions. In particular, it is natural to consider the relation between the timing of uncertainty resolution and optimal current consumption under the Kreps–Porteus (1978) recursive utility because we expect that the recursive utility will enable us to express the preference for the timing of the uncertainty resolution in a more flexible way. These problems are left for further research.

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