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**Abstract**

This paper develops revealed preference analysis of an individual choice model where an agent is a weak preference maximizer, under the assumption that a choice function, rather than a choice correspondence, is observed. In particular, we provide a revealed preference test for such model, and then provide conditions under which we can surely say whether some alternative is indifferent / weakly preferred / strictly preferred to another, solely from the information of the choice function. Furthermore, interpreting a choice correspondence as sets of potential candidates of alternatives that could be chosen from each feasible set, we analyze which alternatives must be, or cannot be a member of the choice correspondence: sharp lower and upper bounds of this underlying choice correspondence are given. As an assumption on observability of data, we assume that the choice function is defined on a non-exhaustive domain, so our results are applicable to data analysis even when only a limited data set is available.

KEYWORDS: Revealed preference; Choice function; Choice correspondence; Weak preference; Bounded rationality

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# 1 Introduction

Let $X$ be a finite set of alternatives, and $A \subseteq X$ be a set of feasible alternatives for an agent. Following standard choice theory, an agent chooses the most preferred alternative according to her preference, which is commonly assumed to be a strict preference. In testing if an agent's behavior can be accounted for by this standard framework, the theory of revealed preference is one of the most prevailing methods for economists. It is typically assumed that a choice function $f : \mathcal{D} \to X$ is observed, where $\mathcal{D}$ is an arbitrary collection of subsets of $X$, and given any feasible set $A \in \mathcal{D}$, $f(A)$ is the chosen alternative from set $A$. It is well known that choice function $f$ is consistent with the standard choice framework, if and only if it obeys the *strong axiom of revealed preference (SARP)*, which requires that the (strict) direct revealed preference relation $\succ^R$ is acyclic, where $\succ^R$ is defined as follows: $x'' \succ^R x'$ if there exists $A \in \mathcal{D}$ such that $x'' = f(A)$, $x' \in A$, and $x' \neq x''$.

However, many experimental studies report that violation of SARP is not rare at all, and various alternative models have been proposed to account for such seemingly irrational behavior. One approach is to relax the full-rationality assumption of the agent, and cast some sort of boundedness on her cognitive capacity: in particular, allow that the agent oversees some alternatives in the feasible set. Some examples are the Limited Attention model and Overwhelming Choice model, which are introduced in Masatlioglu, Nakajima, and Ozbay (2012) and Lleras, Masatlioglu, Nakajima, and Ozbay (2017) respectively. Another approach is to assume multi-step decision making, where the agent has some criterion/criteria other than her preference. Under such models, given a set of feasible alternatives, a shortlist is made according to the criterion/criteria, and then she maximizes her preference within the shortlist. Some examples are Rational Shortlisting in Manzini and Mariotti (2007), Transitive Rational Shortlisting in Au and Kawai (2011), Catogorize Then Choose model in Manzini and Mariotti (2012), and Rationalization model in Cherepanov, Feddersen, and Sandroni (2013). The models stated above are common in that an additional structure is added in the process of decision making, which results in a departure from the classical rational agent model. The model proposed by Barberà and Neme (2016), namely the $r$-rationality model, allows cyclical choices even though the agent is fully-rational with a single strict preference. This is due to her satisficing behavior, where she chooses one of her $r$-th best (or better) alternative from each $A \in \mathcal{D}$.

In this paper, we consider a fully-rational agent, but relax the classical model in two intuitive aspects. Firstly, we relax the common assumption that the agent has a strict preference, and consider an agent who has a weak preference. In the case where an agent has a weak preference, it is typically assumed in the literature that a choice correspondence $F : \mathcal{D} \to 2^X \backslash \varnothing$ is observed.[1] Given feasible set $A \in \mathcal{D}$, $F(A)$ is interpreted as the set of alternatives that *could* have been chosen from $A$. However, it is practically not possible to observe multiple choices simultaneously. Hence as a second departure from standard theory, we relax this "full-observation" assumption, and assume that only a choice function $f$ is observed. Therefore, we consider a fully-rational agent with a weak preference, who chooses *one* of her most preferred alternatives from each feasible set. In particular, in this paper we provide a necessary and sufficient condition under which a choice function $f$ is consistent with an agent maximizing her weak preference: i.e., we can find a weak preference $\succsim$ such that for every feasible set $A \in \mathcal{D}$, $f(A) \succsim x$ for every $x \in A$.

In fact, without any additional constraint on the weak preference, a choice function is trivially consistent with weak preference maximizing behavior: any choice function is a result of maximizing behavior of an agent who is indifferent between all alternatives in $X$. Therefore, we restrict our attention to *non-degenerate* weak preferences, ones where there exist $x', x'' \in X$ with $x'' \succ x'$. Then, given a choice function that is consistent with weak preference maximization, we go one step further and provide conditions for welfare analysis. In particular, we provide conditions under which we can surely say that some alternative is indifferent/weakly preferred/strictly preferred to another, solely from the information given in the choice function. Moreover, we provide sharp lower and upper bounds of the "underlying" choice correspondence, namely, alternatives that could have been chosen from each feasible set. This allows us to make extrapolation over out-of-sample feasible sets.

Taking into account the practicality of our results, we assume that the domain of choice function $f$ is not necessarily exhaustive: we allow $\mathcal{D}$, the domain of $f$, to be a strict subset of $2^X \backslash \varnothing$. There is a growing literature in choice theory that adopts this limited data assumption, which allows us to carry out empirical applications. Some papers that adopt this limited data assumption are Inoue and Shirai (2016) and De Clippel and Rozen (2018).

**Organization of the paper:** In Section 2, we introduce our model and the concept of rationalizability. A necessary and sufficient condition for rationalizability is given in Section

[1] Throughout this paper, we abuse notation and abbreviate the braces, and write $2^X \backslash \varnothing$ instead of $2^X \backslash \{\varnothing\}$. Similar abbreviation of braces will be used whenever there is no fear of confusion.

2.2. Then, Section 3 is devoted to discussions regarding robust inference of agent's preference and underlying choice correspondence. In particular, in Section 3.1, we derive necessary and sufficient conditions under which we can surely say that some alternative is indifferent/weakly preferred/strictly preferred to another; and in Section 3.2 sharp lower and upper bounds of the underlying choice correspondence are given. We conclude the paper by showing in Section 4 how our model relates with some of the models in the literature akin to ours. Proofs are contained in Appendix.

# 2 The model and rationalization condition

## 2.1 Preliminaries

Let $X$ be a finite set of alternatives, and let $\mathcal{D} \subseteq 2^X \backslash \varnothing$ be a collection of feasible sets. A *weak preference*, denoted by $\succsim$, is a complete, reflexive, and transitive binary relation on $X$, and a *strict preference* is a complete, asymmetric, and transitive binary relation on $X$.[2] A *choice function* is a mapping $f : \mathcal{D} \to X$ with $f(A) \in A$ for every $A \in \mathcal{D}$: that is, $f(A)$ is the chosen alternative from feasible set $A$. It is common in the literature that a choice function is associated with an agent maximizing a strict preference, while in models where agents with weak preferences are considered, choice correspondences are assumed to be observed. In this paper, we adopt a natural assumption that an agent has a weak preference, and the observationally practical assumption that an economist can observe only *one* choice made from each feasible set. Put otherwise, given any feasible set, while the agent's most preferred alternatives is a set in general, i.e., the agent has a choice correspondence, only a part of the underlying choice correspondence is observed. Given this assumption, we shall first address the following question: under what condition on $f$ is it possible to interpret $f$ as a result of weak preference maximizing behavior? A formal definition of this issue is given below.

Definition 1. A choice function $f$ is *rationalizable by a weak preference (or weak preference rationalizable)*, if there exists a complete, reflexive, and transitive binary relation $\succsim$ on $X$ such that for every $A \in \mathcal{D}$, $f(A) \succsim x$ for every $x \in A$.

It is worth noting that rationalizability of a choice function is vacuous if we cast no further

[2] A binary relation $\succsim$ is: *complete*, if for every $x', x'' \in X$, we have $x'' \succsim x'$ or $x' \succsim x''$; *reflexive*, if $x' \succsim x'$ holds for every $x' \in X$; *transitive*, if $x'' \succsim x'$ and $x' \succsim x$ imply $x'' \succsim x$. A binary relation $\succ$ is *asymmetric*, if $x'' \succ x'$ implies not $x' \succ x''$.

restriction on weak preferences that rationalize $f$. That is, any choice function is rationalizable by a weak preference such that all the alternatives in $X$ are indifferent. Therefore, in this paper, we assume in addition that an agent has a *non-degenerate* weak preference, meaning that there exists a pair of alternatives where one alternative is strictly preferred to the other.

Definition 2. A weak preference $\succsim$ is *non-degenerate* if there exist $x', x'' \in X$ with $x'' \succ x'$.

Throughout this paper, when we use the term "weak preference", let us implicitly assume that the weak preference is non-degenerate unless otherwise stated.

## 2.2 Rationalization condition

Here we derive a necessary and sufficient condition for $f$ to be rationalizable by a weak preference. It is worth noting that there are two papers that refer to this issue. Nishimura, Ok, and Quah (2017) provide a general condition which is applicable to this model, and De Clippel and Rozen (2018) state that this issue is solvable by applying the *enumeration procedure* (see Nishimura, Ok, and Quah (2017) and De Clippel and Rozen (2018) for details). However, for completeness of the paper, we explicitly derive a revealed preference condition here as well.

To begin with, let us assume that choice function $f$ is generated by an agent maximizing her weak preference $\succsim$. Then it is natural to define a *revealed preference relation* $R$ on $X$ such that $x'' R x'$ if there exists $A \in \mathcal{D}$ with $x'' = f(A)$ and $x' \in A$. Note that whenever $x'' R x'$ holds, $x'' \succsim x'$ holds as well. Now let $R^T$ be the transitive closure of $R$, and define binary relation $I$ as follows: $x'' I x'$ if (i) $x'' R^T x'$ and $x' R^T x''$; or (ii) $x' = x''$.[3] In this paper, we denote the transitive closure of any binary relation using the superscript "T". Then note that binary relation $I$ is an equivalence relation (reflexive, symmetric, and transitive), and it provides equivalence classes of $X$.[4] Let us denote by $X/I$ the collection of equivalent classes with respect to $I$, and assume that $X$ is partitioned into $K \in \mathbb{N}$ equivalent classes: $X/I = \{E_1, \ldots, E_K\}$. Then, since we have $x'' \sim x'$ for every $x', x'' \in E_k$ and every $k \in \{1, \ldots, K\}$, and since the agent's weak preference is non-degenerate, it must be the case that $K \geqslant 2$. In fact, this simple condition is not only necessary, but also sufficient for the rationalizability of a choice function by a weak preference.

[3] When we have $x'' R^T x'$, this means that there exists a sequence of alternatives $y^0, y^1, \ldots, y^K \in X$ with $y^0 = x''$, $y^K = x'$, and $y^{k-1} R y^k$ for every $k \in \{1, \ldots, K\}$.

[4] A binary relation $I$ is *symmetric*, if $x'' I x'$ implies $x' I x''$.

Proposition 1. *A choice function $f$ is rationalizable by a weak preference, if and only if the equivalence classes of $X$ with respect to binary relation $I$ has more than or equal to 2 elements.*

Below we give an example of a choice function that is weak preference rationalizable, and show how we can test whether $f$ is rationalizable or not. Choice functions in Examples 3 and 4 are ones that are not rationalizable by a weak preference.

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$ and consider choice function $f$ as in Table 1. We show*

| $A$ | $\{x_1, x_2, x_3, x_4\}$ | $\{x_1, x_2, x_3\}$ | $\{x_1, x_2\}$ |
|---|---|---|---|
| $f(A)$ | $x_2$ | $x_1$ | $x_2$ |

Table 1: Choice function of Example 1.

*that $f$ is weak preference rationalizable. Note that $f(x_1, x_2, x_3, x_4) = x_2$ implies $x_2Rx_1$, $x_2Rx_2$, $x_2Rx_3$, $x_2Rx_4$; $f(x_1, x_2, x_3) = x_1$ implies $x_1Rx_1$, $x_1Rx_2$, $x_1Rx_3$; and $f(x_1, x_2) = x_2$ implies $x_2Rx_1$, $x_2Rx_2$. Then, $I = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)\}$, so we have $X/I = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}\}$. Since $|X/I| = 3 \geqslant 2$, $f$ is weak preference rationalizable. One example of a weak preference that rationalizes $f$ is $x_1 \sim x_2 \succ x_3 \succ x_4$.*

# 3 Making robust inference

In this section, we consider how we can make robust inference about the agent's preference and underlying choice correspondence, given a choice function that is weak preference rationalizable. In particular, we derive conditions under which we can surely determine the relative preference ranking between two specific alternatives, using only the information of the choice function. In addition, we derive sharp lower and upper bounds of the underlying choice correspondence, which is the set of alternatives that the agent could have chosen. Since the bounds of choice correspondence are derived not only for sets in $\mathcal{D}$, but also for unobserved feasible sets, this allows extrapolation on what the agent may choose from feasible sets outside of the observed data set. Throughout this section, we assume that the choice function is rationalizable by a weak preference.

## 3.1 Robust inference of preference

Here we analyze how we can infer the agent's preference from a choice function. Even when a choice function is rationalizable by a weak preference, such weak preference is not uniquely determined in general. Meanwhile, it may still be possible to pin down the relative ranking between two alternatives. Below, we introduce the concept of robust inference of preference, and provide necessary and sufficient conditions for such robust inference.

DEFINITION 3. Let choice function $f$ be rationalizble by a weak preference. Then for $x', x'' \in X$,

- say that $x''$ *and* $x'$ *are robustly indifferent*, if $x'' \sim x'$ holds under every weak preference $\succsim$ that rationalizes $f$, and denote this by $x'' \sim^{\mathrm{r}} x'$;
- say that $x''$ *is robustly weakly preferred to* $x'$, if $x'' \succsim x'$ holds under every weak preference $\succsim$ that rationalizes $f$, and denote this by $x'' \succsim^{\mathrm{r}} x'$;
- say that $x''$ *is robustly strictly preferred to* $x'$, if $x'' \succ x'$ holds under every weak preference $\succsim$ that rationalizes $f$, and denote this by $x'' \succ^{\mathrm{r}} x'$.

The proposition below gives necessary and sufficient conditions for robust inference of preference. Intuitively, $x'' \sim^{\mathrm{r}} x'$ if and only if $x', x''$ are in the same equivalence class; $x'' \succsim^{\mathrm{r}} x'$ holds if and only if $x''$ is in a "weakly superior" equivalence class than that of $x'$; and $x'' \succ^{\mathrm{r}} x'$ holds if and only if any $\succsim$ that rationalizes $f$ with $x' \succ x''$ becomes degenerate.

PROPOSITION 2. *Let choice function $f$ be rationalizable by a weak preference. Then:*

1. *$x'$ and $x''$ are robustly indifferent, if and only if $x' R^T x''$ and $x'' R^T x'$;*
2. *$x''$ is robustly weakly preferred to $x'$, if and only if $x'' R^T x'$;*
3. *$x''$ is robustly strictly preferred to $x'$, if and only if $x'' R^T x R^T x'$ holds for every $x \in X$.*

In the following examples we show how robust inference of preference is done.

EXAMPLE 1 (continued). *In this example, since $x_1 R x_2$ and $x_2 R x_1$, it follows that $x_1 \sim^r x_2$. Similarly, for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, we have $x_i R^T x_j$, so $x_i \succsim^r x_j$ follows. Therefore, while there are multiple weak preferences that rationalize $f$, any one of them must obey $x_1 \sim x_2$ and $x_1, x_2 \succsim x_3, x_4$.*

EXAMPLE 2. *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and consider choice function $f$ as in Table 2. Note that we have $X/I = \{\{x_1, x_2, x_3, x_5\}, \{x_4\}\}$, where $x'' R^T x_4$ for every $x'' \in \{x_1, x_2, x_3, x_5\}$. Note*

| $A$ | $\{x_1, x_2, x_3, x_4\}$ | $\{x_1, x_3, x_4\}$ | $\{x_1, x_2, x_5\}$ | $\{x_2, x_5\}$ |
|---|---|---|---|---|
| $f(A)$ | $x_1$ | $x_3$ | $x_2$ | $x_5$ |

Table 2: Choice function of Example 2.

*that for every $x'' \in \{x_1, x_2, x_3, x_5\}$, we have $x'' R^T x R^T x_4$ for every $x \in X$. Therefore, it must be the case that $x'' \succ^r x_4$ for every $x'' \in \{x_1, x_2, x_3, x_5\}$. In fact, the weak preference that rationalizes $f$ is uniquely determined: $x_1 \sim x_2 \sim x_3 \sim x_5 \succ x_4$.*

## 3.2 Inference of underlying choice correspondence

In many papers, a weak preference is commonly associated with a choice correspondence. Given any feasible set $A$, a choice correspondence $F(A)$ is the set of maximum alternatives with respect to the agent's weak preference $\succsim$: $F(A) = \{x' \in A : x' \succsim x \text{ for every } x \in A\}$. In this paper, we make a more practical assumption, and assume that the economist only has access to a choice function $f$. Here, for every feasible set $A$, $f(A)$ can be interpreted as *one* alternative of the "underlying" choice correspondence $F(A)$, where $F(A)$ is the set of alternatives that can potentially be chosen by the agent. Then, a natural question would be: can we make any inference about the underlying choice correspondence $F$, when we observe only a choice function $f$? In this section, this question is addressed, and provide sharp lower and upper bounds of the underlying choice correspondence.

Consider any observed choice function $f$ that is rationalizable by a weak preference. Given a feasible set $A' \in \mathcal{D}$, let us infer what alternatives must/must not be in $F(A')$. First we consider what alternatives must be included in $F(A')$, in other words, we consider the lower bound. Using the results of Proposition 2, it is certain that if $x'$ is robustly weakly preferred to every other alternative in $A'$, then $x'$ must be a member of $F(A')$. This holds because under any $\succsim$ that rationalizes $f$, $x' \succsim x$ holds for every $x \in A'$. In fact, this simple condition characterizes the lower bound for the underlying choice correspondence.

Lemma 1. *Let $f$ be rationalizable by a weak preference. Then, given any $\succsim$ that rationalizes $f$ and any $A' \subseteq X$, $x' \in A'$ must be in $F(A')$ if any only if $x' \succsim^r x$ for every $x \in A'$. Moreover, defining for every $A' \subseteq X$ the set $L(A')$ as below, $L(A')$ is the greatest lower bound of $F(A')$:*

$$L(A') = \{x' \in A' : x' \succsim^r x \text{ for every } x \in A'\}. \quad (1)$$

Deriving upper bound of $F(A')$ is a bit more elaborate. In doing this, we focus on the alternatives that cannot be a member of $F(A')$. First consider an alternative $x' \in A'$ where there exists $x'' \in A'$ such that $x''$ is robustly strictly preferred to $x'$. Then, under any $\succsim$ that rationalizes $f$, we have $x'' \succ x'$, and thus $x'$ can never be in $F(A')$. There is another case where $x'$ cannot be a member of $F(A')$. Note that $x' \in F(A)$ would mean that $x'$ is weakly preferred to $x$ for every $x \in A'$. If setting $x' \succsim x$ for every $x \in A'$ inevitably results in a degenerate $\succsim$, it is not possible for $x'$ to be a member of $F(A')$. The discussion above is summarized in the lemma below.

Lemma 2. *Let $f$ be rationalizable by a weak preference. Then, given any $\succsim$ that rationalizes $f$ and any $A' \subseteq X$, $x' \in A'$ cannot be in $F(A')$ if and only if 1 and/or 2 below holds,*

1. *there exists $x'' \in A'$ with $x'' \succ^r x'$,*
2. (a) *$x'' R^T x'$ for every $x'' \in A'$, and*
   (b) *$X = \bigcup_{x'' \in A'} \{x \in X : x'' R^T x R^T x'\}$.*

*Moreover, defining for every $A' \subseteq X$ the set $U(A')$ as below, $U(A')$ is the least upper bound of $F(A')$:*

$$U(A') = A' \backslash \{x' \in A' : x' \text{ obeys 1 or 2 above }\}. \tag{2}$$

Summarizing the lemmas above, we have the lower and upper bounds of the underlying choice correspondence.

Proposition 3. *Let $f$ be rationalizable by a weak preference. Then for any $\succsim$ that rationalizes $f$, and for every $A \subseteq X$, we have*

$$L(A) \subseteq F(A) \subseteq U(A), \tag{3}$$

*where $L(A)$ and $U(A)$ are defined as (1) and (2) respectively.*

Note that Proposition 3 gives a lower and upper bound of the potentially chosen alternatives for feasible sets $A \subseteq X$ rather than $A \in \mathcal{D}$. Hence this result allows us to make predictions on what the agent may choose when confronting an out-of-sample feasible set. This may be useful in practice, since it is realistically not possible (in many cases) to observe choices from all conceivable feasible sets.

Example 1 (continued). *Here we show how inference of the underlying choice correspondence can be made. Let us focus on feasible set* $\{x_1, x_2, x_3, x_4\}$. *Then we have* $L(x_1, x_2, x_3, x_4) = \{x_1, x_2\}$ *and* $U(x_1, x_2, x_3, x_4) = \{x_1, x_2, x_3, x_4\}$. *Therefore, we can infer that* $\{x_1, x_2\} \subseteq F(x_1, x_2, x_3, x_4) \subseteq \{x_1, x_2, x_3, x_4\}$.

# 4 Relation with existing models

In this section, we relate the observable restrictions of our model with existing choice models. First we see how our model relates with standard rational choice models, and then compare observable restrictions of some closely related non-standard choice models.

## 4.1 Relation with standard rational choice models

Here we take a look at the observable restrictions of our model and standard rational choice models. It is well known that a choice *function* is consistent with maximization of a *strict* preference, if and only if it obeys the *strong axiom of revealed preference (SARP)*, while a choice *correspondence* is consistent with maximization of a *weak* preference, if and only if it obeys the *congruence axiom (CA)*. Formal definitions of these axioms are:

Strong Axiom of Revealed Preference: A choice function $f : \mathcal{D} \to X$ obeys the Strong Axiom of Revealed Preference (SARP), if strict direct revealed preference $\succ^R$ is acyclic, where $x'' \succ^R x' \Leftrightarrow$ there exists $A \in \mathcal{D}$ such that $x'' = f(A)$ and $x' \in A \backslash f(A)$.

Congruence Axiom: A choice correspondence $F : \mathcal{D} \to 2^X \backslash \varnothing$ obeys the Congruence Axiom (CA), if for every $A \in \mathcal{D}$ and $x', x'' \in A$, $x' \in F(A)$ and $x'' R^T x'$ imply $x'' \in F(A)$.[5]

First of all, note that a strict preference is a special case of a weak preference, so whenever a choice function obeys SARP, it is rationalizable by a weak preference, i.e., SARP implies weak preference rationalizability. On the other hand, $f$ can be weak preference rationalizable even when there is a cycle with respect to $\succ^R$, so the other direction does not hold.[6]

Note that CA is a condition on a choice correspondence, so it is not possible to directly compare with weak preference rationalizability. In fact, the only case where a choice correspondence $F$ that obeys CA is weak preference rationalizable is when $F$ turns out to be

[5]See Richter (1966) for details of both the strong axiom of revealed preference and the congruence axiom.
[6]Choice functions in Examples 1 and 2 are weak preference rationalizable, but violate SARP.

"single-valued", i.e., $|F(A)| = 1$ for every $A \in \mathcal{D}$. However, in this case, CA boils down to SARP. Meanwhile, it is possible to see how the underlying choice correspondence of $f$ relates with CA. Suppose that choice function $f$ is rationalizable by a weak preference $\succsim$. Then we have a choice correspondence $F$ such that $F(A)$ is the set of $\succsim$-maximum alternatives for every $A \in \mathcal{D}$. This choice correspondence obviously satisfies CA. On the other hand, suppose that some choice correspondence obeys CA. Then $F(A)$ is the set of $\succsim$-maximum alternatives for every $A \in \mathcal{D}$, for some weak preference $\succsim$. Then, defining a choice function $f$ so that $f(A) \in F(A)$ for every $A \in \mathcal{D}$, this choice function is weak preference rationalizable. Therefore, it seems plausible to regard weak preference rationalizability as a counterpart of CA, under the assumption that there is limitation of observability of choices: we cannot observe multiple simultaneous choices from a given feasible set.

## 4.2 Relation with some non-standard choice models

Here we show how our model relates with some non-standard choice models in the literature. In particular, we show that the Limited Attention/Overwhelming Choice models and weak preference rationalizability are observationally independent, and we give an example of a choice function that is $r$-rationalizable (with $r = 2$) but not weak preference rationalizable.

To begin with, let us go through a brief summary of the models we deal with in this section. Limited Attention/Overwhelming Choice models assume that some feasible alternatives are a priori excluded from agent's consideration, due to limitation recognition capacity. That is, given a feasible set $A \in \mathcal{D}$, an agent maximizes her strict preference on some subset $\Gamma(A) \subseteq A$, which is called a *consideration set* at $A$. Limited Attention and Overwhelming Choice models differ in the assumptions casted on the structure of *consideration mapping* $\Gamma : 2^X \backslash \varnothing \to 2^X \backslash \varnothing$, a mapping that specifies consideration set for each feasible set $A \subseteq X$. Limited Attention model assumes that an alternative must have been considered if removal of that alternative results in a change of choice:

$$f(A) \neq f(A \backslash x) \Longrightarrow x \in \Gamma(A), \tag{4}$$

while Overwhelming Choice assumes that an alternative considered at a large feasible set is considered in a smaller set:

$$x \in A' \subseteq A'' \text{ and } x \in \Gamma(A'') \Longrightarrow x \in \Gamma(A'). \tag{5}$$

Then, we say that *choice function $f$ is rationalizable by Limited Attention/Overwhelming Choice model*, if there exists a strict preference $\succ$ and consideration mapping $\Gamma$ obeying (4)/(5) such that for every $A \in \mathcal{D}$, $f(A) \succ x$ for every $x \in A \backslash x$.

In fact, Rational Shortlisting in Manzini and Mariotti (2007), Categorize Then Choose model in Manzini and Mariotti (2012), and Rationalization model in Cherepanov, Feddersen, and Sandroni (2013) are special cases of Overwhelming Choice, and Transitive Rational Shortlisting in Au and Kawai (2011) is a special case of both Limited Attention and Overwhelming Choice. Therefore, showing observational independence between Limited Attention/Overwhelming Choice and weak preference rationalizability shows that weak preference rationalizability is observationally independent from many leading bounded rationality models.

Another model that is closely related to ours, but slightly different from "limited consideration" type models stated above, is the *$r$-rationality* model in Barberà and Neme (2016). In this model, it is assumed that an agent has a strict preference, and given feasible set $A$, an agent chooses *one* of her $r$-best alternative ($r \in \mathbb{N}$) within $A$. While any choice function $f$ is $|X|$-rationalizable, we show in Example 4 that there is a choice function that is 2-rationalizable but not weak preference rationalizable. This implies that $r$-rationalizability is not observationally nested in weak preference rationalizability.

We first show, in Example 3 that consistency with Limited Attention/Overwhelming Choice does not imply weak preference rationalizability. Examples 1 and 2 respectively show that weak preference rationalizability does not imply consistency with Limited Attention or Overwhelming Choice. Finally, Example 4, which is an example used in Barberà and Neme (2016), gives a choice function that is 2-rationalizable but not weak preference rationalizable.

Example 3. *Let $X = \{x_1, x_2, x_3\}$ and consider choice function $f$ as in Table 3. We first show*

| $A$ | $\{x_1, x_2, x_3\}$ | $\{x_1, x_2\}$ | $\{x_2, x_3\}$ |
|---|---|---|---|
| $f(A)$ | $x_1$ | $x_2$ | $x_3$ |

Table 3: Choice function of Example 3.

*that $f$ is not rationalizable by a weak preference. It holds that $x_i I x_j$ for $i, j \in \{1, 2, 3\}$, and* *thus* $X/I = \{\{x_1, x_2, x_3\}\}$. *Since* $|X/I| = 1$, $f$ *is not weak preference rationalizable. Now see that the preference* $x_2 \succ x_1 \succ x_3$ *and consideration mapping* $\Gamma$ *as in Table 4 are consistent with both Limited Attention and Overwhelming Choice models. Therefore, choice function*

| $A$ | $\{x_1, x_2, x_3\}$ | $\{x_1, x_2\}$ | $\{x_1, x_3\}$ | $\{x_2, x_3\}$ |
|---|---|---|---|---|
| $\Gamma(A)$ | $\{x_1, x_3\}$ | $\{x_1, x_2\}$ | $\{x_1, x_3\}$ | $\{x_3\}$ |

Table 4: Consideration mapping $\Gamma$ for Example 3.

$f$ *is not rationalizable by a weak preference, but is rationalizable by Limited Attention and Overwhelming Choice.*

Example 1 (continued). *We show that* $f$ *is not consistent with Overwhelming Choice (recall that it is weak preference rationalizable). Note that under Overwhelming Choice, it must be the case that* $x_1$ *is strictly preferred to* $x_2$, *and* $x_2$ *is strictly preferred to* $x_1$.[7] *Thus there is no strict preference that can rationalize* $f$, *and* $f$ *is not consistent with Overwhelming Choice.*

Example 2 (continued). *We show that* $f$ *is not consistent with Limited Attention (recall that it is weak preference rationalizable). Note that under Limited Attention, it must be the case that* $x_1$ *is strictly preferred to* $x_2$, *and* $x_2$ *is strictly preferred to* $x_1$.[8] *Thus there is no strict preference that can rationalize* $f$, *and* $f$ *is not consistent with Limited Attention.*

Example 4. *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *and consider choice function* $f$ *defined on* $2^X \backslash \varnothing$ *as below:*

- *for* $A \in 2^X \backslash \varnothing$ *such that* $|A| = 2$,
  - *if* $A = \{x_1, x_5\}$, $f(A) = x_5$,
  - *otherwise,* $f(x_i, x_j) = x_i$, *where* $i < j$,
- *for* $A \in 2^X \backslash \varnothing$ *such that* $|A| = 3$,
  - *if* $x_3 \in A$, *then* $f(A) = x_3$,
  - *if* $x_4 \in A$ *and* $x_3 \notin A$, *then* $f(A) = x_4$,

[7]It follows from (5) that $x_2$ attracts attention at $\{x_1, x_2, x_3\}$, which in turn implies $x_1$ is strictly preferred to $x_2$. Similarly, provided $f$ follows Overwhelming Choice, it must be the case that $x_1$ attracts attention at $\{x_1, x_2, x_3, x_4\}$, which in turn implies that $x_2$ is strictly preferred to $x_1$.

[8]It follows from (4) that $x_2$ attracts attention at $\{x_1, x_2, x_3, x_4\}$, which in turn implies $x_1$ is strictly preferred to $x_2$. Similarly, provided $f$ follows Limited Attention, it must be the case that $x_1$ attracts attention at $\{x_1, x_2, x_5\}$, which in turn implies that $x_2$ is strictly preferred to $x_1$.

- $f(x_1, x_2, x_5) = x_2$,

- *for* $A \in 2^X \backslash \varnothing$ *such that* $|A| = 4$,

$$f(x_1, x_2, x_3, x_4) = x_4, \quad f(x_1, x_2, x_3, x_5) = x_3, \quad f(x_1, x_2, x_4, x_5) = x_4,$$
$$f(x_1, x_3, x_4, x_5) = x_4, \quad f(x_2, x_3, x_4, x_5) = x_3,$$

- $f(X) = x_3$.

*We first show that* $f$ *is* 2*-rationalizable. Consider strict preference* $x_4 \succ x_3 \succ x_2 \succ x_1 \succ x_5$*. Then, at every feasible set* $A \in 2^X \backslash \varnothing$*,* $f(A)$ *is either her favorite, or 2nd favorite alternative, and thus* $f$ *is* 2*-rationalizable by preference* $\succ$*. To show that* $f$ *is not weak preference rationalizable, it suffices to see choices on* $A$ *where* $|A| = 2$*: we have* $x_1 R x_2 R x_3 R x_4 R x_5 R x_1$*, which implies that* $X/I = \{\{x_1, x_2, x_3, x_4, x_5\}\}$*. Since* $|X/I| = 1$*,* $f$ *is not weak preference rationalizable.*

# Appendix

## Proof of Proposition 1

Since necessity is already proved above, here we show that the other direction holds as well. Suppose that $X/I = \{E_1, \ldots, E_K\}$ with $K \geqslant 2$. Let us define a binary relation $\rhd$ on $X/I$ as follows: $E_j \rhd E_k$ if there exist $x'' \in E_j$ and $x' \in E_k$ with $x'' R^T x'$ and not $x' R^T x''$. For future reference, let us present the following lemma.

LEMMA 3. $E_j \rhd E_k, x'' \in E_j$*, and* $x' \in E_k$ *implies* $x'' R^T x'$ *and* **not** $x' R^T x''$**.**

Using this binary relation $\rhd$, let us define binary relations $\mathcal{P}$ and $\mathcal{I}$ as follows:

- $x'' \mathcal{P} x'$ if there exist $j, k$ such that $x'' \in E_j, x' \in E_k$, and $E_j \rhd E_k$,
- $x'' \mathcal{I} x'$ if there exist $k$ such that $x, x'' \in E_k$.

Then, define binary relation $\mathcal{R}$ to be the union of $\mathcal{I}$ and $\mathcal{P}$, i.e., $\mathcal{R} = \mathcal{I} \cup \mathcal{P}$. This means that $\mathcal{I}$ and $\mathcal{P}$ are the symmetric and asymmetric components of $\mathcal{R}$ respectively. Note that by definition of $\mathcal{R}$ and Lemma 3, $x'' \mathcal{R} x'$ implies $x'' R^T x'$.

Lemma 4. *Binary relation $\mathcal{R}$ is consistent, that is, for every $x^1, \ldots, x^L \in X$,*

$$x^1 \mathcal{R} x^2 \mathcal{R} \cdots \mathcal{R} x^L \Longrightarrow \textbf{not } x^L \mathcal{P} x^1. \tag{6}$$

Since binary relation $\mathcal{R}$ is consistent, by Suzumura's Extension Theorem, there exists a complete, reflexive, and transitive extension of $\mathcal{R}$, which we denote by $\succsim$.[9] Under this extension, $x''\mathcal{P}x'$ implies $x'' \succ x'$, so this binary relation is non-degenerate. Now take any $A \in \mathcal{D}$. Note that we have $f(A)\mathcal{R}x$ for every $x \in A$, so $f(A) \succsim x$ holds for every $x \in A$. Thus, $\succsim$ rationalizes choice function $f$.

## Proof of Lemma 3

By definition of $\rhd$, $E_j \rhd E_k$ means that there exist $y^j \in E_j$ and $y^k \in E_k$ such that $y^j R^T y^k$ and not $y^k R^T y^j$. Meanwhile, $x'', y^j \in E_j$ means that $x'' R^T y^j$ and $y^j R^T x''$, and $x', y^k \in E_k$ means that $x' R^T y^k$ and $y^k R^T x'$. Since binary relation $R^T$ is transitive, $x'' R^T x'$ follows. If we assume by way of contradiction that $x' R^T x''$, then transitivity of $R^T$ implies that $y^k R^T y^j$, which is a contradiction.

## Proof of Lemma 4

Note that by Lemma 3, we have $x^1 R^T x^L$. By definition of $\mathcal{P}$, we have $x''\mathcal{P}x'$ if and only if there exist $E', E'' \in X/I$ with $x' \in E'$, $x'' \in E''$, and $E'' \rhd E'$, which in turn implies $x'' R^T x'$ and not $x' R^T x''$. Hence, since we have $x^1 R^T x^L$, it is impossible to have $x^L \mathcal{P} x^1$.

## Proof of Proposition 2

First we prove sufficiency of 1. Take any weak preference $\succsim$ that rationalizes $f$. Then, for every $A \in \mathcal{D}$, it must be the case that $f(A) \succsim x$ for every $x \in A$, therefore whenever $x'' R x'$, then $x'' \succsim x'$ holds. Therefore, $x' R^T x''$ and $x'' R^T x'$ imply $x' \succsim x''$ and $x'' \succsim x'$, which in turn imply $x' \sim x''$. Necessity of 1 is proved by showing the contrapositive. Suppose that "$x' R^T x''$ and $x'' R^T x'$" does not hold. There are two essential cases to consider: (i) $x'' R^T x'$ but not $x' R^T x''$; and (ii) $x', x''$ are not related through $R$. Under case (i), we can follow the proof of Proposition 1, and we have a weak preference $\succsim$ that rationalizes $f$ with $x'' \succ x'$. Under case

[9] For details of Suzumura's Extension Theorem, see Suzumura (1976). A comprehensive summary of extension theorems are given in Andrikopoulos (2009).

(ii), we have $x' \in E'$, $x'' \in E''$ such that $E', E'' \in X/I$ are not related through $\rhd$.[10] Let us define binary relation $\rhd'$ on $X/I$ such that $\rhd' = \rhd \cup \{(E'', E')\}$, and parallel to the proof of Proposition 1, define binary relations $\mathcal{P}', \mathcal{I}'$ on $X$ using $\rhd'$.

LEMMA 5. *Binary relation $\mathcal{R}' = \mathcal{I}' \cup \mathcal{P}'$ is consistent.*

Then, there is a completion of $\mathcal{R}'$, namely $\succsim$, that rationalizes $f$ with $x'' \succ x'$. In both cases (i) and (ii), we have the desired result.

To prove sufficiency of 2, first suppose that $x'' R^T x'$ holds: i.e., there exist $y^1, y^2, \ldots, y^L \in X$ such that $x'' = y^1 R y^2 R \cdots R y^L = x'$. This in turn means that for every $\ell \in \{1, \ldots, L\}$, there exists $A^\ell \in \mathcal{D}$ such that $y^\ell = f(A^\ell)$ and $y^{\ell+1} \in A^\ell$. Then, for any $\succsim$ that rationalizes $f$, it must be the case that $y^\ell \succsim y^{\ell+1}$ for every $\ell$. By transitivity of $\succsim$, we have $x'' \succsim x'$. Necessity of 2 is proved by showing the contrapositive. Suppose that $x'' R^T x'$ does not hold. This means that there exist $E', E'' \in X/I$ with $x' \in E'$, $x'' \in E''$, and $E' \neq E''$. There are two cases that we must consider: (i) $E', E''$ are not related through $\rhd$; and (ii) $E' \rhd^T E''$.[11] For case (i), apply the proof of necessity of 1, and for case (ii), apply the proof in Proposition 1. In either case, we have $\succsim$ that rationalizes $f$ with $x' \succ x''$.

To prove sufficiency of 3, suppose that $x'' R^T x R^T x'$ holds for every $x \in X$. By results in 2, under any non-degenerate weak preference $\succsim$ that rationalizes $f$, it must be the case that $x'' \succsim x \succsim x'$ holds for every $x \in X$. Now suppose by way of contradiction that $x'' \sim x'$ holds. Then, by assumption, it follows that $y' \sim y''$ for every $y', y'' \in X$, which contradicts that $\succsim$ is non-degenerate.

Proof of necessity of 3 will be done by showing the contrapositive. As in the previous cases, we construct a non-degenerate weak preference $\succsim$ that rationalizes $f$ with $x' \succsim x''$, when $X \neq \{x : x'' R^T x R^T x'\}$. There are two major cases that we consider.

<u>Case I: $x'' R^T x'$ does not hold.</u> Within this case, if $x' R^T x''$, then by results in 2, we have $x' \succsim^r x''$, and $x' \succsim x''$ holds under any $\succsim$ that rationalizes $f$. If $x', x''$ are unrelated via $R$, then again applying the logic in the proof on necessity of 2, we have $\succ$ that rationalizes $f$ with $x' \succ x''$. This completes the proof for case I.

<u>Case II: $x'' R^T x'$, but there exists $x \in X$ that does not exhibit $x'' R^T x R^T x'$.</u> Prior to present-

[10] Recall that binary relation $\rhd$ is defined in the proof of Proposition 1.
[11] Note that $\rhd^T$ is the transitive closure of $\rhd$.

ing a proof for this case, let us partition $X$ into the following three sets:

$$Y = \{x \in X : x'' R^T x R^T x'\},$$
$$Y_1 = \{x \in X : x R^T x' \text{ does not hold}\},$$
$$Y_2 = \{x \in X : x R^T x' \text{ holds, but } x'' R^T x \text{ does not hold}\}.$$

Note that in this case, $Y_1 \cup Y_2 \neq \varnothing$. There are two subcases that we consider: (II-i) $Y_1 \neq \varnothing$; and (II-ii) $Y_2 \neq \varnothing$.

In case (II-i), take any $\bar{y} \in Y_1$ such that $\bar{y} \in \bar{E}$ for some $\bar{E} \in X/I$, where $\bar{E}$ is minimal with respect to $\rhd$, i.e., $\bar{E} \rhd^T E$ for no $E \in X/I$. Such an $\bar{E}$ exists because $X$, and thus $X/I$, is finite. Now define binary relations $\mathcal{P}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows: $y''\mathcal{P}y'$ if $y'' \in X\backslash\bar{E}$ and $y' \in \bar{E}$; $y''\mathcal{I}_1 y'$ if $y', y'' \in X\backslash\bar{E}$; and $y''\mathcal{I}_2 y'$ if $y', y'' \in \bar{E}$. Then, let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{R} = \mathcal{I} \cup \mathcal{P}$.

Lemma 6. *Binary relations $\mathcal{I}$ and $\mathcal{P}$ are symmetric and asymmetric components of $\mathcal{R}$ respectively, and $\mathcal{R}$ is consistent.*

Then there is a complete, reflexive, and transitive extension $\succsim$ of $\mathcal{R}$, and the following lemma shows that this weak preference rationalizes $f$.

Lemma 7. *The weak preference $\succsim$ is non-degenerate, and $f(A) \succsim x$ holds for every $x \in A$ and every $A \in \mathcal{D}$.*

Finally, since $x', x'' \in X\backslash\bar{E}$, $x''\mathcal{I}x'$ holds by construction of $\mathcal{I}$. This implies that we have $x'' \sim x'$, which completes the proof for case (II-i).

In case (II-ii), take any $\bar{y} \in Y_2$ such that $\bar{y} \in \bar{E}$ for some $\bar{E} \in X/I$, where $\bar{E}$ is maximal with respect to $\rhd$, i.e., $E \rhd^T \bar{E}$ for no $E \in X/I$. Now define binary relations $\mathcal{P}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows: $y''\mathcal{P}y'$ if $y'' \in \bar{E}$ and $y' \in X\backslash\bar{E}$; $y''\mathcal{I}_1 y'$ if $y', y'' \in X\backslash\bar{E}$; and $y''\mathcal{I}_2 y'$ if $y', y'' \in \bar{E}$. Then, let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{R} = \mathcal{I} \cup \mathcal{P}$.

Lemma 8. *Binary relations $\mathcal{I}$ and $\mathcal{P}$ are symmetric and asymmetric components of $\mathcal{R}$ respectively, and $\mathcal{R}$ is consistent.*

Then there is a complete, reflexive, and transitive extension $\succsim$ of $\mathcal{R}$, and the following lemma shows that this weak preference rationalizes $f$.

Lemma 9. *The weak preference $\succsim$ is non-degenerate, and $f(A) \succsim x$ holds for every $x \in A$ and every $A \in \mathcal{D}$.*

Finally, since $x', x'' \in X \setminus \bar{E}$, $x'' \mathcal{I} x'$ holds by construction of $\mathcal{I}$. This implies that we have $x'' \sim x'$, which completes the proof for case (II-ii).

## Proof of Lemma 5

Take any $x^1, x^2, \ldots, x^L \in X$ such that $x^1 \mathcal{R}' x^2 \mathcal{R}' \cdots \mathcal{R}' x^L$. If there is no $\ell$ such that $x^\ell \in E'', x^{\ell+1} \in E'$, then by Lemma 4, $x^L \mathcal{P}' x^1$ cannot hold. Therefore, there must exist $\ell$ such that $x^\ell \in E''$, $x^{\ell+1} \in E'$, and thus $x^\ell \mathcal{P}' x^{\ell+1}$. In fact, such $\ell$ is unique, i.e., there cannot be $m \neq \ell$ such that $x^m, x^\ell \in E''$ and $x^{m+1}, x^{\ell+1} \in E'$.[12] Thus we have $x^1 \mathcal{R}' x^2 \mathcal{R}' \cdots \mathcal{R}' x^\ell \mathcal{P}' x^{\ell+1} \mathcal{R}' \cdots \mathcal{R}' x^L$, which implies $x^1 R^T x^\ell \mathcal{P}' x^{\ell+1} R^T x^L$, where $x^1 \notin E'$ and $x^L \notin E''$. Suppose by way of contradiction that $x^L \mathcal{P}' x^1$. However, this means that $x^L R^T x^1$ must follow, which in turn implies that $x^{\ell+1} R^T x^\ell$, a contradiction.

## Proof of Lemma 6

By construction, it is obvious that $\mathcal{I}$ is symmetric and that $\mathcal{P}$ is asymmetric. Now we show that it is not possible to have $x^1 \mathcal{R} x^2 \mathcal{R} \cdots \mathcal{R} x^L$ and $x^L \mathcal{P} x^1$ simultaneously. Note that $x^L \mathcal{P} x^1$ means that $x^L \in X \setminus \bar{E}$, and $x^1 \in \bar{E}$. Then it follows by definition of $\mathcal{I}$ that $x^1, \ldots, x^L \in \bar{E}$, which is a contradiction.

## Proof of Lemma 7

Since $y'' \succ y'$ holds for $y'' \in X \setminus \bar{E}$ and $y' \in \bar{E}$, $\succsim$ is non-degenerate. Now take any $A \in \mathcal{D}$. To show that $f(A) \succsim x$ for every $x \in A$, it suffices to show that $f(A) \mathcal{R} x$ for every $x \in A$ Note that this holds whenever there does not exist $x \in A$ with $x \mathcal{P} f(A)$. Suppose by way of contradiction that $x \mathcal{P} f(A)$ holds for some $x \in A$. This means that $x \in X \setminus \bar{E}$ and $f(A) \in \bar{E}$. Meanwhile, we have $f(A) R x$, so it follows that $\bar{E} \rhd E(x)$, where $E(x)$ is the equivalence class of $x$. This contradicts that $\bar{E}$ was chosen to be minimal with respect to $\rhd$.

## Proof of Lemma 8

By construction, it is obvious that $\mathcal{I}$ is symmetric and that $\mathcal{P}$ is asymmetric. Now we show that it is not possible to have $x^1 \mathcal{R} x^2 \mathcal{R} \cdots \mathcal{R} x^L$ and $x^L \mathcal{P} x^1$ simultaneously. Note that $x^L \mathcal{P} x^1$

[12] Suppose to the contrary that such $m$ exists: $x^1 \mathcal{R}' x^2 \mathcal{R}' \cdots \mathcal{R}' x^\ell \mathcal{P}' x^{\ell+1} \mathcal{R}' \cdots \mathcal{R}' x^m \mathcal{P}' x^{m+1} \mathcal{R}' \cdots \mathcal{R}' x^L$. Then $x^{\ell+1} R^T x^m$ must follow, which contradicts that $E', E''$ are not related via $\rhd$.

means that $x^L \in \bar{E}$ and $x^1 \in X \backslash \bar{E}$, and then it follows by definition of $\mathcal{I}$ that $x^1, \ldots, x^L \in X \backslash \bar{E}$, which is a contradiction.

## Proof of Lemma 9

Since $y'' \succ y'$ holds for $y'' \in \bar{E}$ and $y' \in X \backslash \bar{E}$, $\succsim$ is non-degenerate. Now take any $A \in \mathcal{D}$. To show that $f(A) \succsim x$ for every $x \in A$, it suffices to show that $f(A)\mathcal{R}x$ for every $x \in A$. Note that this holds whenever there does not exist $x \in A$ with $x\mathcal{P}f(A)$. Suppose by way of contradiction that $x\mathcal{P}f(A)$ holds for some $x \in A$. This means that $x \in \bar{E}$ and $f(A) \in X \backslash \bar{E}$. Meanwhile, we have $f(A)Rx$, so it follows that $E(f(A)) \rhd \bar{E}$, where $E(f(A))$ is the equivalence class of $f(A)$. This contradicts that $\bar{E}$ was chosen to be maximal with respect to $\rhd$.

## Proof of Lemma 1

Take any $A' \subseteq X$, and suppose that for $x' \in A'$, we have $x' \succsim^{\mathrm{r}} x$ for every $x \in A'$. This means that $x' \succsim x$ for every $x \in A'$, under any $\succsim$ that rationalizes $f$. Therefore, $x' \in F(A')$ must hold for every $\succsim$ that rationalizes $f$. Now we prove necessity by showing the contrapositive. Suppose that there exists $x'' \in A'$ such that $x' \succsim^{\mathrm{r}} x''$ does not hold. Applying Proposition 2, this means that there exists $\succsim$ that rationalizes $f$ with $x'' \succ x'$. Under this $\succsim$, we do not have $x' \in F(A')$.

## Proof of Lemma 2

Take any $A' \subseteq X$, and suppose first that 1 holds: there exists $x'' \in A'$ with $x'' \succ^{\mathrm{r}} x'$. Then for every $\succsim$ that rationalizes $f$, we have $x'' \succ x'$. This in turn means $x' \notin F(A')$ for every $\succsim$ that rationalizes $f$. Suppose that 2 holds: (a) $x''R^T x'$ for every $x'' \in A'$; and (b) $X = \bigcup_{x'' \in A'} \{x \in X : x''R^T xR^T x'\}$. Take any $\succsim$ that rationalizes $f$, and suppose by way of contradiction that $x' \in F(A')$. This means that $x' \succsim x''$ for every $x'' \in A'$. Meanwhile, $x''R^T x'$ for every $x'' \in A'$ means that $x'' \succsim x'$ for every $x'' \in A'$. Moreover, 2-(b) means that for every $x \in X$, there exists $x'' \in A'$ such that $x'' \succsim x \succsim x'$. Summarizing, we have $y'' \sim y'$ for every $y', y'' \in X$, which contradicts that $\succsim$ is non-degenerate.

Necessity is proved through showing the contrapositive. Suppose that both 1 and 2 fail to hold. Then there are two cases to consider: case I is when 1 and 2-(a) fail; and case II is when 1 and 2-(b) fail. Here, for any $x \in X$, let us denote by $E(x)$ the equivalence class that $x$ belongs to.

Case I: 1 and 2-(a) fail. In this case, there exists $x'' \in A'$ such that $x'' R^T x'$ does not hold. Then, we have $E(x') \rhd^T E(x'')$, or $E(x'), E(x'')$ are not related via $\rhd$. Let $\bar{E} = \{x \in X : E(x'') = E(x)$ or $E(x'') \rhd^T E(x)\}$. Note that $f(A') \notin \bar{E}$.[13] Now define binary relations $\mathcal{P}, \mathcal{I}_1, \mathcal{I}_2$ as follows: $y'' \mathcal{P} y'$ if $y'' \in X \backslash \bar{E}$ and $y' \in \bar{E}$; $y'' \mathcal{I}_1 y'$ if $y', y'' \in X \backslash \bar{E}$; and $y'' \mathcal{I}_2 y'$ if $y', y'' \in \bar{E}$. Then let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{R} = \mathcal{I} \cup \mathcal{P}$. By construction of $\mathcal{I}$ and $\mathcal{P}$, it is not possible to have $x^1 \mathcal{R} x^2 \mathcal{R} \cdots \mathcal{R} x^L$ and $x^L \mathcal{P} x^1$ simultaneously: $x^L \mathcal{P} x^1$ means that $x^L \in X \backslash \bar{E}$ and $x^1 \in \bar{E}$, and then $x^1, \ldots, x^L \in \bar{E}$ must hold, which in turn implies that $x^L \mathcal{P} x^1$ is not possible. Thus $\mathcal{R}$ is consistent, and there is a complete, reflexive, and transitive extension $\gtrsim$ of $\mathcal{R}$. Since $y'' \succ y'$ holds for $y'' \in X \backslash \bar{E}$ and $y' \in \bar{E}$, $\gtrsim$ is non-degenerate. Now take any $A \in \mathcal{D}$. Since $f(A) R x$ for every $x \in A$, we have $E(f(A)) = E(x)$ or $E(f(A)) \rhd^T E(x)$. This implies that it is not possible to have $x \in A \cap (X \backslash \bar{E})$ and $f(A) \in \bar{E}$, and thus $x \mathcal{P} f(A)$ never holds for any $x \in A$. Therefore, we have $f(A) \gtrsim x$ for every $x \in A$. By construction of $\mathcal{R}$, we have $x' \mathcal{R} x$ for every $x \in A'$, so $x' \gtrsim x$ holds for every $x \in A'$.[14] This results in $x' \in F(A')$.

Case II: 1 and 2-(b) fail; 2-(a) holds. In this case, there exists $\bar{x} \notin \bigcup_{x'' \in A'} \{x \in X : x'' R^T x R^T x'\}$. Note that we have $\bar{x} \notin \bigcup_{x'' \in A'} E(x'')$, which in turn implies that $E(\bar{x}) \cap [\bigcup_{x'' \in A'} E(x'')] = \varnothing$.[15]

**Fact 1.** *One of the following holds:*

*(i). there exists* $\bar{x} \notin \bigcup_{x'' \in A'} \{x \in X : x'' R^T x R^T x'\}$ *such that* $E \rhd^T E(\bar{x})$ *for no* $E \in X/I$*;*

*(ii). there exists* $\bar{x} \notin \bigcup_{x'' \in A'} \{x \in X : x'' R^T x R^T x'\}$ *such that* $E(\bar{x}) \rhd^T E$ *for no* $E \in X/I$*.*

*Proof.* Take any $\bar{E} \in X/I$ such that $E \rhd^T \bar{E}$ holds for no $E \in X/I$. Finiteness of $X$ and rationalizability of $f$ assures the existence of such $\bar{E}$. If such $\bar{E}$ exhibits $\bar{E} \cap [\bigcup_{x'' \in A'} E(x'')] = \varnothing$, then (i) holds. Otherwise, $E \rhd^T \bar{E}$ for no $E \in X/I$ would imply $\bar{E} \subseteq \bigcup_{x'' \in A'} E(x'')$, meaning that for any $\bar{E} \in X/I$ that is $\rhd$-maximal, there exists $x'' \in A'$ such that $\bar{E} = E(x'')$. To show that (ii) must hold in this case, suppose to the contrary that (ii) fails: for every $\bar{E}$ such that $\bar{E} \rhd^T E$ for no $E \in X/I$, we have $\bar{E} \subseteq \bigcup_{x'' \in A'} E(x'')$.[16] This means that for every $\bar{E} \in X/I$ that is $\rhd$-minimal, there exists $x'' \in A'$ such that $\bar{E} = E(x'')$. Then, for every $E \in X/I$, it

[13]Otherwise, since $f(A') R x'$, we have $x'' R^T x'$, which is a contradiction.

[14]Note that $x' \notin \bar{E}$, so for every $x \in A'$, we have $x' \mathcal{P} x$ or $x' \mathcal{I}_1 x$.

[15]Otherwise, we have some $x'' \in A'$ with $x'' R^T \bar{x}$ and $\bar{x} R^T x''$, which in turn implies that $x'' R^T \bar{x} R^T x'' R^T x'$. Thus $x'' R^T \bar{x} R^T x'$, which contradicts $\bar{x} \notin \bigcup_{x'' \in A'} \{x \in X : x'' R^T x R^T x'\}$.

[16]Note that this is an equivalent statement to the failure of (ii).

follows that "there exists $\hat{x} \in A'$ such that $E(\hat{x}) = E$ or $E(\hat{x}) \rhd^T E$" and "there exists $\tilde{x} \in A'$ such that $E(\tilde{x}) = E$ or $E \rhd^T E(\tilde{x})$." This in turn implies that $\bar{x} \in \bigcup_{x'' \in A'} \{x \in X : x'' R^T x R^T x'\}$ for every $\bar{x} \in X$, contradicting the assumption that 2-(b) fails. $\square$

Now suppose that (i) in Fact 1 holds, and take any $\bar{x}$ as stated there. Then define binary relations $\mathcal{P}, \mathcal{I}_1$, and $\mathcal{I}_2$ as follows: $y''\mathcal{P}y'$ if $y'' \in E(\bar{x})$ and $y' \in X \backslash E(\bar{x})$; $y''\mathcal{I}_1 y'$ if $y', y'' \in X \backslash E(\bar{x})$; and $y''\mathcal{I}_2 y'$ if $y', y'' \in E(\bar{x})$. Now let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{R} = \mathcal{I} \cup \mathcal{P}$. By construction of $\mathcal{I}$ and $\mathcal{P}$, it is not possible to have $x^1 \mathcal{R} x^2 \mathcal{R} \cdots \mathcal{R} x^L$ and $x^L \mathcal{P} x^1$ simultaneously: $x^L \mathcal{P} x^1$ means that $x^L \in E(\bar{x})$ and $x^1 \in X \backslash E(\bar{x})$, and then $x^1, \ldots, x^L \in X \backslash E(\bar{x})$ must hold, which in turn implies that $x^L \mathcal{P} x^1$ is not possible. Thus $\mathcal{R}$ is consistent, and there is a complete, reflexive, and transitive extension $\succsim$ of $\mathcal{R}$. Since $y'' \succ y'$ holds for $y'' \in E(\bar{x})$ and $y' \in X \backslash E(\bar{x})$, $\succsim$ is non-degenerate. Now take any $A \in \mathcal{D}$. Since it is not possible to have $f(A) \notin E(\bar{x})$ and $x \in A \cap E(\bar{x})$, $x \mathcal{P} f(A)$ never holds for any $x \in A$. Therefore, we have $f(A) \succsim x$ for every $x \in A$. By construction of $\mathcal{R}$, we have $x' \mathcal{R} x$ for every $x \in A'$, so $x' \succsim x$ holds for every $x \in A'$. This results in $x' \in F(A')$.

Now suppose that (ii) in Fact 1 holds, and take any $\bar{x}$ as stated there. Then define binary relations $\mathcal{P}, \mathcal{I}_1$, and $\mathcal{I}_2$ as follows: $y''\mathcal{P}y'$ if $y'' \in X \backslash E(\bar{x})$ and $y' \in E(\bar{x})$; $y''\mathcal{I}_1 y'$ if $y', y'' \in X \backslash E(\bar{x})$; and $y''\mathcal{I}_2 y'$ if $y', y'' \in E(\bar{x})$. Now let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{R} = \mathcal{I} \cup \mathcal{P}$. By construction of $\mathcal{I}$ and $\mathcal{P}$, it is not possible to have $x^1 \mathcal{R} x^2 \mathcal{R} \cdots \mathcal{R} x^L$ and $x^L \mathcal{P} x^1$ simultaneously: $x^L \mathcal{P} x^1$ means that $x^L \in X \backslash E(\bar{x})$ and $x^1 \in E(\bar{x})$, and then $x^1, \ldots, x^L \in E(\bar{x})$ must hold, which in turn implies that $x^L \mathcal{P} x^1$ is not possible. Thus $\mathcal{R}$ is consistent, and there is a complete, reflexive, and transitive extension $\succsim$ of $\mathcal{R}$. Since $y'' \succ y'$ holds for $y'' \in X \backslash E(\bar{x})$ and $y' \in E(\bar{x})$, $\succsim$ is non-degenerate. Now take any $A \in \mathcal{D}$. Since it is not possible to have $f(A) \in E(\bar{x})$ and $x \in A \cap [X \backslash E(\bar{x})]$, $x \mathcal{P} f(A)$ never holds for any $x \in A$. Therefore, we have $f(A) \succsim x$ for every $x \in A$. By construction of $\mathcal{R}$, we have $x' \mathcal{R} x$ for every $x \in A'$, so $x' \succsim x$ holds for every $x \in A'$. This results in $x' \in F(A')$.

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