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Net Borda rules with desirability

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Abstract We apply the concept of *desirability* of alternatives to the clas-sic Borda scoring system. We employ a linear order over the finite set of alternatives and an outside option and define the desirability as follows: each alternative is (un)desirable if and only if it is better (worse) than the out-side option. Additionally, we assume that each voter assigns the Borda scores to all alternatives and the outside option. Thus, there is an extra gap of one point between desirable and undesirable alternatives. We then define two–type *net Borda* rules which output the highest–scored alternatives. The first is the type–I net Borda rule which allows that the social choice includes the outside option. The second is the type–II net Borda rules which does not allow that the social choice includes the outside option. We show an advantage in consid-ering the desirability of alternatives by comparing the outputs of type-II net Borda and classic Borda rules since their co–domains are the same. Further-more, we provide an axiomatic characterisation of the type–II net Borda rule and find that the type–II net Borda rule satisfies more proper conditions than the classic Borda rule when we consider the desirability of alternatives.

Keywords Net Borda rules · Desirability · The outside option · approval–cancellation · cyclic cancellation · aggregated cancellation

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Net Borda rules with desirability

1 Introduction

The classic Borda rule chooses the winners of an election based on aggregated scores of alternatives, which are assigned by every voter as follows: 0, 1, 2, ..., points, for example, are assigned to all alternatives from the bottom to the top. However, each preference ranking includes only the relative evaluation criterion for the alternatives, such as 'a is better than b'. We thus propose new Borda rules, called *net Borda rules*, considering the absolute evaluation criterion for the alternatives, that is, *desirability*.

To define the desirability of alternatives, we use an *outside option* introduced by Roth and Sotomayor (1990). The outside option is equivalent to an empty set and indicates 'choosing nothing'. In this study, we consider a one–shot election, therefore, we do not regard the outside option as the status quo. If we do that, the voters cannot express the absolute evaluation for the existing alternatives.

We then employ a linear order over the set of all alternatives and the outside option, and define the desirability as follows: each alternative is (*un*)*desirable* if and only if it is better (worse) than the outside option. The following example illustrates an advantage in considering the desirability of alternatives: Suppose that there are two alternatives a and b, and 100 voters. If the number of voters who prefer a to b is 51 and that of voters who prefer b to a is 49, the classic Borda rule chooses a. However, under the same setting, if a is desirable for the 51 voters, b is desirable and a is undesirable for the remained 49 voters, and no one thinks that b is undesirable, b might be more appropriate as the winner.

The net Borda rule determines the set of winners along the same way as the classic Borda rule, but with a difference. Each voter assigns Borda scores to all alternatives and the outside option. Then, there exists an extra gap between desirable and undesirable alternatives. The net Borda rules output alternatives (or the outside option) whose aggregated scores are the highest.

This study proposes two–type net Borda rules. The first is called *type–I net Borda rule* allowing that the set of winners includes the outside option. This setting is appropriate to decide the destination of a trip, restaurant for dinner, and so forth. Even if a group plans to go a trip, the group will cancel it when the outside option gets the highest score because of pandemic of a new virus. The second is *type–II net Borda rule* which does not allow that the set of winners includes the outside option.[1] This setting is useful for any situations that a society must choose an existing alternative, such as political election, a program for the school festival, and so forth.

Furthermore, if there is no difference between the Borda scores (alternatively, desirability) of all alternatives and the outside option, the net Borda

[1] If the outside option is in the top, but not unique, the net Borda rule selects all the top alternatives except for the outside option.

rule is equivalent to the approval voting (alternatively, classic Borda) rule.[2] Thus, the net Borda rules have the concepts of approval voting and Borda rules.

Several existing social choice rules have similar concepts to desirability, such as preference approval voting and fallback voting rules, analysed by Brams and Sanver (2009). Brams and Sanver (2009) expressed preference–approval as follows: $ab \mid c$ indicating that a is better than b, b is better than c, a and b are approved, and c is disapproved. The two voting rules consider only the preference rankings of approved alternatives after dividing all alternatives into two groups: approved and disapproved ones.[3] They thus violate *Pareto efficiency*: If there are two alternatives a and b, disapproved by all voters, the two voting rules choose both a to b although every voter prefers a to b, on the other hand, the net Borda rule chooses a.

We have two main results. First, we compare the sets of winners by using the type–II net Borda and classic Borda rules because the co–domains of these two rules are the same. By using our program in Python 3, we calculate a ratio that the type–II net Borda and classic Borda rules have different outcomes. We found that if $|V|$ is an even number and $|X|$ is sufficiently large, the outputs of C^V_{nbII} and C^V_{br} will have large differences.

Second, we characterise the type–II net Borda rule by *neutrality*, *reinforcement*, *faithfulness*, and *aggregated cancellation*.[4] We prepare these conditions by extending axioms which were used to characterise the classic Borda rule in Young (1974). *Neutrality* requires that the set of winners do not depend on the names of alternatives(, but it depends on the outside option since the outside option is not allowed to be a winner). *Reinforcement* requires that if the intersection of sets of winners for any two disjoint voter sets is not empty, then the intersection is equal to the set of winners for the union of those two voter sets. Since we use *reinforcement*, we assume that a voter set is variable. *Faithfulness* requires that if there exists only one voter in the society, the winner will be the best alternative of the voter. *Aggregated cancellation* requires that, for every alternative, if the number of voters who prefer the alternative to other alternatives and the outside option is equal to the number of voters who have the opposite preference rankings, all alternatives should be included in the outcome.

[2] When we employ the approval voting rule, we assume that ever voter has a dichotomous order. The dichotomous order is a special case of complete preorder (also known as weak order). If a voter has a dichotomous order, the voter divides the finite alternative set into two groups, that is, the approval and disapproval groups. Furthermore, all alternatives in each group are assumed to be indifferent, and every alternative in the approval group is assumed to be better than every alternative in the disapproval group.

[3] Strictly speaking, there is a difference between $\mid$ and $\emptyset$. For example, $a \mid b$ implies that a is better than b since every element in the approved alternative set is better than every element in the disapproved alternative set. Nothing more, nothing less. However, $aP\emptyset Pb$ indicates that a is desirable and b is undesirable since a is better than $\emptyset$ and $\emptyset$ is better than b. Thus, Brams and Sanver (2009) used $\mid$ to consider the preference rankings of approved alternatives.

[4] We do not characterise the type–I net Borda rule since it is essentially equivalent to the classic Borda rule when we regard the outside option as an alternative.

The remainder of this paper is structured as follows. Section 2 reports our notations and definitions. Section 3 discusses the advantage of the (type–II) net Borda rule by comparing with the classic Borda rule. Section 4 introduces axioms to characterise the type–II net Borda rule, and Section 5 shows its axiomatisation. Finally, Section 6 provides concluding remarks and future research directions.

2 Preliminaries

Let $V \subset \mathbb{Z}_{>0}$ be a finite and variable voter set. Assume that $|V| \geq 1$, where $\mathbb{Z}_{>0}$ is the set of positive integers and $|V|$ indicates the cardinality of V.[5] Suppose that X is the finite alternative set and $|X| \geq 2$. The alternatives in X are denoted by a, b, c, and so forth.

Suppose that $P_i \in \mathscr{P}$ is a linear order over $X \cup \{\emptyset\}$ for each $i \in V$, where $\mathscr{P}$ is the set of all linear orders over $X \cup \{\emptyset\}$. Thus, we assume *unrestricted domain.* Additionally, let each alternative $a \in X$ be (*un*)*desirable* for voter i if and only if $aP_i\emptyset$ ($\emptyset P_i a$). We call the empty set an *outside option.*[6] Let $\mathcal{P} = (P_i, ..., P_{|V|}) \in \mathscr{P}^{|V|}$ be the preference profile for each $V \subset \mathbb{Z}_{>0}$.

2.1 Social choice rules

We define the following three Borda rules: the classic Borda rule and two types of new Borda rules considering the desirability of alternatives, called *net Borda rules.*

Classic Borda rule

Let $C^V : \mathscr{P}^{|V|} \rightrightarrows X$ be a *social choice correspondence* for any $V \subset \mathbb{Z}_{>0}$. C^V outputs a non–empty subset of X. Furthermore, let $n_{aa'}(\mathcal{P})$ be the number of voters who prefer a to a' for all $a \neq a'$. For a given $\mathcal{P} \in \mathscr{P}^{|V|}$ and for all $a \in X$, the *Borda score* of a is denoted by

$$B_a(\mathcal{P}) = \Sigma_{a' \in X \setminus \{a\}}(n_{aa'}(\mathcal{P}) - n_{a'a}(\mathcal{P})).$$

From this equation, every voter ignores the outside option even if his/her linear order is defined over $X \cup \{\emptyset\}$.

The above definition of Borda scores is not standard, but useful to understand the characteristics of the classic Borda rule. Suppose that there are

[5] We do not assume that $|V| \geq 2$ since even if $|V| = 1$, a society may include more than two individuals. For example, suppose that a research group, including three members, chooses the leader. If one of them is a voter and the others are candidates, $|V| = 1$ holds. However, this election is conducted for the group, that is, a small society. Thus, the set of winners should be regarded as the social choice even if $|V| = 1$.

[6] You can see further discussions about the outside option in Chapter 5 of Roth and Sotomayor (1990).

three existing alternatives $a, b, c \in X$ and a voter has the following preference ranking: aP_ibP_ic. If we use the standard Borda scores, the voter assigns 2, 1, and 0 points to a, b, and c, respectively. If we use the above equation, the voter assigns 2, 0, and -2 points to a, b, and c, respectively. By using both Borda scoring systems, we always obtain the same set of winners.

We then define the *classic Borda rule*, denoted by C^V_{br}, as follows:

Definition 1 *Classic Borda rule*

$$C^V_{br}(\mathcal{P}) = \{a \in X \mid a \in argmax_{b \in X} B_b(\mathcal{P})\}.$$

Type–I net Borda rule

Let $\hat{C}^V : \mathscr{P}^{|V|} \rightrightarrows X \cup \{\emptyset\}$ be a *social choice correspondence* for any $V \subset \mathbb{Z}_{>0}$. Since $\hat{C}^V$ outputs a non–empty subset of $X \cup \{\emptyset\}$, $\emptyset$ is allowed to be included in the set of winners. For a given $\mathcal{P} \in \mathscr{P}^{|V|}$ and for all $a \in X \cup \{\emptyset\}$, the *net Borda score* over $X \cup \{\emptyset\}$ is denoted by

$$NB_a(\mathcal{P}) = \Sigma_{a' \in (X \cup \{\emptyset\}) \setminus \{a\}} (n_{aa'}(\mathcal{P}) - n_{a'a}(\mathcal{P})).$$

We then define the *type–I net Borda rule*, denoted by $\hat{C}^V_{nbI}$, as follows:

Definition 2 *Type–I net Borda rule*

$$\hat{C}^V_{nbI}(\mathcal{P}) = \{a \in X \cup \{\emptyset\} \mid a \in argmax_{b \in X \cup \{\emptyset\}} NB_b(\mathcal{P})\}.$$

From Definition 2, $\hat{C}^V_{nbI}$ is equivalent to the Borda rule over $X \cup \{\emptyset\}$ if we regard $\emptyset$ as an alternative.

Type–II net Borda rule

We use the same social choice correspondence as that of the classic Borda rule, C^V since the set of winners must include at least one existing alternative in X. We then define the *type–II net Borda rule*, denoted by C^V_{nbII}, as follows:

Definition 3 *Type–II net Borda rule*

$$C^V_{nbII}(\mathcal{P}) = \{a \in X \mid a \in argmax_{b \in X} NB_b(\mathcal{P})\}.$$

From Definitions 2 and 3, the scoring systems of $\hat{C}^V_{nbI}$ and C^V_{nbII} are the same (the net Borda scoring system). Additionally, from Definitions 1 and 3, the co–domains of C^V_{br} and C^V_{nbII} are the same. The type–I net Borda rule fits to situations that a society can take an election back to the drawing board, such as the destination of a trip, restaurant for dinner, and so forth. If the outside option gets the highest score, the members can cancel their plan. The classic Borda or type–II net Borda rule is appropriate any situations that a society must choose an existing alternative, such as political election, a program for the school festival, and so forth.

3 Comparative analysis of classic Borda and net Borda rules

Before axiomatising the type–II net Borda rule, compare the sets of winners by applying the type–II net Borda and classic Borda rules to illustrate an advantage in considering the desirability of alternatives. We use these two voting rules because their co–domains are the same from Definitions 1 and 3.

In Section 1, we found an essential difference between the two voting rules from the following example: Suppose that there are two alternatives a and b, and 100 voters. If $aPbP\emptyset$ for 51 voters and $bP\emptyset Pa$ for the other 49 voters, the classic Borda rule chooses a since $B_a(\mathcal{P}) = 2$ and $B_b(\mathcal{P}) = -2$. However, b might be more appropriate as the winner from the information about the desirability. Then, the type–II net Borda rule chooses b since $NB_a(\mathcal{P}) = 4$ and $NB_b(\mathcal{P}) = 98$.

However, the classic Borda and type–II net Borda scoring systems are proper similar to each other. Thus, the following exercises show difference between the outcomes of two voting rules. In the exercises, we prepare pairs of $|X|$ and $|V|$ as follows: $(|X|, |V|) \in \{(2, 2), ..., (2, 8), (3, 2), ..., (3, 5), (4, 2), (4, 3), (5, 2)\}$ and compare C^V_{br} and C^V_{nbII} for each preference profile and for each pair of $(|X|, |V|)$ by using our program in Python 3. The samples of $(|X|, |V|)$ are determined base on the performance of a 1.99 GHz personal laptop. We do not treat any case that $|V| = 1$ since $C^V_{nbII}(\mathcal{P}) = C^V_{br}(\mathcal{P})$ always holds true.

First, we show an example to find out our exercises. Suppose that $X = \{a, b\}$, $|V| = 5$. Then, the numbers of linear orders over X and $X \cup \{\emptyset\}$ are two and six, respectively. If we employ the classic Borda rule, there are 32 preference profiles. Table 2 reports the possible Borda scores of a and b when $C^V_{br}(\mathcal{P}) = \{a\}$. If $C^V_{br}(\mathcal{P}) = \{b\}$, we can obtain the other results by switching a and b in Table 2. Thus, it is enough to consider only Cases I–III. Cases I-III have one, five, and 10 preference profiles, respectively (16 in total).

Note that ab indicates aPb. The same applies to other preference rankings.

Table 1: The possible scores of a and b when $C^V_{br}(\mathcal{P}) = \{a\}$

Case	n_{ab}	n_{ba}	B_a	B_b
I	5	0	5	0
II	4	1	3	-3
III	3	2	1	-1

Next, consider the outputs of type–II net Borda rule in Cases I–III of Table 2. If we employ the type–II net Borda rule, there are 7,776 preference profiles, and Cases I-III have 243, 1,215, and 2,430 preference profiles, respectively.

In Case I, $C^V_{nbII}(\mathcal{P}) = C^V_{br}(\mathcal{P})$ because aPb for all voters.

In Case II, consider the worst situation for a and check whether C^V_{nbII} outputs b. Suppose that $\emptyset PaPb$ for four voters and $bP\emptyset Pa$ for one voter.

Then, the net Borda scores of a and b are -2 and -6, respectively. Thus, C^V_{nbII} outputs a. From this result, we have that $C^V_{nbII}(\mathcal{P}) = C^V_{br}(\mathcal{P})$ in Case II.

In Case III, $C^V_{nbII}(\mathcal{P})$ becomes $\{a, b\}$ or $\{b\}$ with 520 out of the 2,430 preference profiles. Table 3 shows the outputs of C^V_{nbII} in Case III. For instance, assume that $\emptyset PaPb$ for three voters and $bP\emptyset Pa$ for two voters. Then, the net Borda scores of a and b are -4 and -2, respectively, and $C^V_{nbII}(\mathcal{P}) = \{b\}$. This result makes sense because a is undesirable for the all five voters, two voters strictly prefer b to a, and b is desirable for the same two voters. Thus, the outputs of C^V_{nbII} is more intuitive than that of C^V_{br}.

Table 2: The outputs of C^V_{nbII} in Case III

# of voters whose preferences are						scores			
$ab\emptyset$	$a\emptyset b$	$\emptyset ab$	$ba\emptyset$	$b\emptyset a$	$\emptyset ba$	NB_a	NB_b	C^V_{nbII}	$C^V_{nbII} \neq C^V_{br}$
3	0	0	2	0	0	6	4	$\{a\}$	
3	0	0	1	1	0	4	4	$\{a,b\}$	✓
3	0	0	1	0	1	4	2	$\{a\}$	
3	0	0	0	2	0	2	4	$\{b\}$	✓
3	0	0	0	1	1	2	2	$\{a,b\}$	✓
3	0	0	0	0	2	2	0	$\{a\}$	
2	1	0	2	0	0	6	2	$\{a\}$	
2	1	0	1	1	0	4	2	$\{a\}$	
2	1	0	1	0	1	4	0	$\{a\}$	
2	1	0	0	2	0	2	2	$\{a,b\}$	✓
2	1	0	0	1	1	2	0	$\{a\}$	
2	1	0	0	0	2	2	-2	$\{a\}$	
2	0	1	2	0	0	4	2	$\{a\}$	
2	0	1	1	1	0	2	2	$\{a,b\}$	✓
2	0	1	1	0	1	2	0	$\{a\}$	
2	0	1	0	2	0	0	2	$\{b\}$	✓
2	0	1	0	1	1	0	0	$\{a,b\}$	✓
2	0	1	0	0	2	0	-2	$\{a\}$	
1	2	0	2	0	0	6	0	$\{a\}$	
1	2	0	1	1	0	4	0	$\{a\}$	
1	2	0	1	0	1	4	-2	$\{a\}$	
1	2	0	0	2	0	2	0	$\{a\}$	
1	2	0	0	1	1	2	-2	$\{a\}$	
1	2	0	0	0	2	2	-4	$\{a\}$	
1	1	1	2	0	0	4	0	$\{a\}$	
1	1	1	1	1	0	2	0	$\{a\}$	
1	1	1	1	0	1	2	-2	$\{a\}$	
1	1	1	0	2	0	0	0	$\{a,b\}$	✓
1	1	1	0	1	1	0	-2	$\{a\}$	
1	1	1	0	0	2	0	-4	$\{a\}$	
1	0	2	2	0	0	2	0	$\{a\}$	

$ab\emptyset$	$a\emptyset b$	$\emptyset ab$	$ba\emptyset$	$b\emptyset a$	$\emptyset ba$	NB_a	NB_b	C^V_{nbII}	$C^V_{nbII} \neq C^V_{br}$
1	0	2	1	1	0	0	0	$\{a,b\}$	✓
1	0	2	1	0	1	0	-2	$\{a\}$	
1	0	2	0	2	0	-2	0	$\{b\}$	✓
1	0	2	0	1	1	-2	-2	$\{a,b\}$	✓
1	0	2	0	0	2	-2	-4	$\{a\}$	
0	3	0	2	0	0	6	-2	$\{a\}$	
0	3	0	1	1	0	4	-2	$\{a\}$	
0	3	0	1	0	1	4	-4	$\{a\}$	
0	3	0	0	2	0	2	-2	$\{a\}$	
0	3	0	0	1	1	2	-4	$\{a\}$	
0	3	0	0	0	2	2	-6	$\{a\}$	
0	2	1	2	0	0	4	-2	$\{a\}$	
0	2	1	1	1	0	2	-2	$\{a\}$	
0	2	1	1	0	1	2	-4	$\{a\}$	
0	2	1	0	2	0	0	-2	$\{a\}$	
0	2	1	0	1	1	0	-4	$\{a\}$	
0	2	1	0	0	2	0	-6	$\{a\}$	
0	1	2	2	0	0	2	-2	$\{a\}$	
0	1	2	1	1	0	0	-2	$\{a\}$	
0	1	2	1	0	1	0	-4	$\{a\}$	
0	1	2	0	2	0	-2	-2	$\{a,b\}$	✓
0	1	2	0	1	1	-2	-4	$\{a\}$	
0	1	2	0	0	2	-2	-6	$\{a\}$	
0	0	3	2	0	0	0	-2	$\{a\}$	
0	0	3	1	1	0	-2	-2	$\{a,b\}$	✓
0	0	3	1	0	1	-2	-4	$\{a\}$	
0	0	3	0	2	0	-4	-2	$\{b\}$	✓
0	0	3	0	1	1	-4	-4	$\{a,b\}$	✓
0	0	3	0	0	2	-4	-6	$\{a\}$	

Form the above results, if $|X| = 2$ and $|V| = 5$, the ratio that $C^V_{nbII} \neq C^V_{br}$ is equal to 1040/7776 (almost 13%).

As with the example, we calculate the ratio that $C^V_{nbII} \neq C^V_{br}$ for each pair $(|X|, |V|) \in \{(2,2), ..., (2,8), (3,2), ..., (3,5), (4,2), (4,3), (5,2)\}$. Mathematically, the number of preference profiles is $|\mathscr{P}|^{|V|} = [(|X|+1)!]^{|V|}$. Then, each ratio, denoted by r$(|X|, |V|)$, is calculated as follows:

$$\mathrm{r}(|X|, |V|) = \frac{\#\{\text{preference profiles that } C^V_{nbII} \neq C^V_{br}\}}{[(|X|+1)!]^{|V|}}.$$

Table 4 reports the results. Unfortunately, we could not express the numerator of ratio as a function of $(|X|, |V|)$.[7] However, we can observe the relationship between the ratio and $(|X|, |V|)$ from Table 4.

Table 3: The ratio that $C^V_{nbII} \neq C^V_{br}$

$\lvert X\rvert$	$\lvert V\rvert$	$r(\lvert X\rvert, \lvert V\rvert)$	%
2	2	$8/6^2$	22.222
2	3	$24/6^3$	11.111
2	4	$288/6^4$	22.222
2	5	$1040/6^5$	13.374
2	6	$10160/6^5$	21.776
2	7	$38640/6^5$	13.803
2	8	$358848/6^5$	21.365
3	2	$120/24^2$	20.833
3	3	$2268/24^3$	16.406
3	4	$65448/24^4$	19.727
3	5	$1492800/24^5$	18.748
4	2	$2568/120^2$	17.833
4	3	$296424/120^3$	17.154
5	2	$80400/120^5$	15.509

First, if $|V|$ is an even number, the effect of increasing $|V|$ on the ratio is negative since

- $r(2, |V|) > r(2, |V| + 1)$ for $|V| \in \{2, 4, 6\}$,
- $r(3, |V|) > r(3, |V| + 1)$ for $|V| \in \{2, 4\}$,
- $r(4, |V|) > r(4, |V| + 1)$ for $|V| = 2$,
- $r(2, |V|) > r(2, |V| + 2)$ for $|V| \in \{2, 4, 6\}$, and

However, if $|X| \geq 5$ or 6, we may obtain the opposite results according to the bottom of Table 4.

Second, if $|V|$ is an odd number, the effect of increasing $|V|$ on the ratio is positive since

- $r(2, |V|) < r(2, |V| + 1)$ for $|V| \in \{3, 5, 7\}$,
- $r(3, |V|) < r(3, |V| + 1)$ for $|V| = 3$,
- $r(2, |V|) < r(2, |V| + 2)$ for $|V| \in \{3, 5\}$, and

Similarly, if $|X| \geq 5$ or 6, we may obtain the opposite results according to the bottom of Table 4.

Third, the effect of increasing $|X|$ with a fixed $|V|$ on the ratio is non–negative since

$$r(|X|, |V|) \geq r(|X| + 1, |V|)$$

[7] The sequence $\{8, 24, 288, 1040, ...\}$ and any related sequences have not registered in The On-Line Encyclopedia of Integer Sequences (OEIS), founded in 1964 by N. J. A. Sloane (URL: https://oeis.org/).

for $(|X|, |V|) \in \{(2,2),(2,3),(2,5),(3,2),(3,3),(4,2),(2,4)\}$

In a nutshell, if (1) $|V|$ is a sufficiently small odd number or (2) $|V|$ is an even number and $|X|$ is sufficiently large, there will be little differences between C^V_{nbII} and C^V_{br}. However, if $|V|$ is an even number and $|X|$ is sufficiently large, the outputs of C^V_{nbII} and C^V_{br} will have large differences. Thus, when $|V|$ is an even number and $|X|$ is sufficiently large, we should employ C^V_{nbII}, rather than C^V_{br}. Additionally, if the outcome can include the outside option, we can also use $\hat{C}^V_{nbI}$.

4 Axioms

As said earlier, the type–I net Borda rule is essentially equivalent to the classic Borda rule if we regard $\emptyset$ as an alternative. We thus introduce axioms, which are extensions of axioms in Young (1974) or Hansson and Sahlquist (1976),[8] to characterise C^V_{nbII} in the following manner:

Suppose that λ is a permutation such that $\lambda : X \to X$, and Λ is the set of all permutations on X. Let $\psi_\emptyset$ be a permutation of preference profiles, fixing the positions of $\emptyset$ even if each preference profile is transformed into another one by $\psi_\emptyset$, such that $\psi_\emptyset : \mathscr{P}^{|V|} \to \mathscr{P}^{|V|}$, and $\Psi_\emptyset$ is the set of all $\psi_\emptyset$'s on $\mathscr{P}^{|V|}$. For an arbitrary $V \subset \mathbb{Z}_{>0}$, every $\lambda \in \Lambda_X$ induces $\psi_\emptyset \in \Psi_\emptyset$, in other words, there exists a bijection whose domain and co–domain are Λ and $\Psi_\emptyset$, respectively.

First, C^V satisfies *neutrality* if

$$C^V(\psi_\emptyset(\mathcal{P})) = \{\lambda(a) \in X \mid a \in C^V(\mathcal{P})\}$$

for any $\lambda \in \Lambda$ and for any $\mathcal{P} \in \mathscr{P}^{|V|}$. Thus, *neutrality* requires that no result should depend on the names of alternatives in X, and the result depends on the name of the outside option.

Second, C^V satisfies *reinforcement* if

$$C^{V_1}(\mathcal{P}_1) \cap C^{V_2}(\mathcal{P}_2) \neq \emptyset \Rightarrow C^{V_1}(\mathcal{P}_1) \cap C^{V_2}(\mathcal{P}_2) = C^{V_1 \cup V_2}(\mathcal{P}_1 + \mathcal{P}_2)$$

for any $\mathcal{P}_1 \in \mathscr{P}^{|V_1|}$ and for any $\mathcal{P}_2 \in \mathscr{P}^{|V_2|}$ such that $V_1, V_2 \subset \mathbb{Z}_{>0}$ and V_1 and V_2 are disjoint, where $\mathcal{P}_1 + \mathcal{P}_2 \in \mathscr{P}^{|V_1 \cup V_2|}$. *Reinforcement* requires that if any two disjoint societies have common winners, then the set of winners for the union of those two societies should include only the common winners.

Third, C^V satisfies *faithfulness* if

$$V = \{i\} \Rightarrow C^V(P_i) = \{\hat{a}(P_i)\},$$

for all $P_i \in \mathscr{P}$, where $\hat{a}(P_i)$ is i's best alternative in X. This axiom requires that if there exists only one voter in the society, the winner will be the best

[8] Smith (1973) and Fishburn and Gehrlein (1976) characterised the classic Borda rule which was defined as a social welfare function. Any social welfare function outputs a social preference ranking over the alternative set, rather than the set of winners. For further discussion, see for example, Saari (1990), and Chebotarev and Shamis (1998).

alternative of the voter. Thus, if the outside option is the best for the voter, the winner should be the second–best element of the voter in $X \cup \{\emptyset\}$.

We then introduce four *cancellation* conditions since axiomatisation of C^V_{nbII} is not straightforward since the co–domain of C^V_{nbII} is X.

First, C^V satisfies *cancellation* if

$$[n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P}) \ \ \forall a \neq b, a, b \in X \cup \{\emptyset\}] \Rightarrow C^V(\mathcal{P}) = X.$$

This axiom requires that if $n_{ab}(\mathcal{P})$ is cancelled out by $n_{ba}(\mathcal{P})$ for all $a \neq b$, $a, b \in X \cup \{\emptyset\}$, X should be the outcome.

Second, C^V satisfies *approval–cancellation* if

$$[n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P}) \wedge n_{a\emptyset}(\mathcal{P}) = n_{b\emptyset}(\mathcal{P}) \ \ \forall a \neq b, a, b \in X] \Rightarrow C^V(\mathcal{P}) = X.$$

This requires that if $n_{ab}(\mathcal{P})$ is cancelled out by $n_{ba}(\mathcal{P})$ for all $a \neq b$, $a, b \in X$ and $n_{a\emptyset}(\mathcal{P})$ is the same for every $a \in X$, X should be the outcome. Trivially, *approval–cancellation* implies *cancellation–II. Approval–cancellation* shows the feature of the net Borda rules. The part '$n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P}) \wedge ...$ for all $a, b \in X$ implies that $C^V(\mathcal{P}) = X$' is related to the concept of the Borda rule, and the other part '$... \wedge n_{a\emptyset}(\mathcal{P}) = n_{b\emptyset}(\mathcal{P})$ for all $a, b \in X$ implies that $C^V(\mathcal{P}) = X$' is related to the concept of the approval voting rule.

Third, C^V satisfies *cyclic cancellation* if

$$[n_{ar}(\mathcal{P}) = n_{br}(\mathcal{P}) \ \ \forall a, b \in X, \ \forall r \in \{1, ..., |X| + 1\}] \Rightarrow C^V(\mathcal{P}) = X,$$

where $n_{ar}(\mathcal{P})$ is the number of voters whose rth best element is a. This axiom requires that if every $a \in X$ is appeared at each rank the same times, X should be the outcome.

Finally, C^V satisfies *aggregated cancellation* if

$$[NB_a(\mathcal{P}) = NB_b(\mathcal{P}) \ \ \forall a, b \in X] \Rightarrow C^V(\mathcal{P}) = X.$$

This axiom requires that if the net Borda scores of all alternatives in X are the same, X should be the outcome. *Aggregated cancellation* implies *approval–cancellation* and *cyclic cancellation*.

5 Characterisation

First, we consider to characterise C^V_{nbII} by *neutrality*, *reinforcement*, *faithfulness*, and *cancellation*. However, the classic Borda rule, C^V_{br}, satisfies the four axioms (and *approval–cancellation*). C^V_{br} violates *cyclic cancellation*. For instance, let $|V| = 3$, $X = \{a, b\}$, $aP_1bP_1\emptyset$, $bP_2\emptyset P_2a$, and $\emptyset P_3aP_3b$. If C^V satisfies *cyclic cancellation*, the outcome should be X. However, $C^V_{br}(\mathcal{P}) = \{a\}$.

Second, we consider to characterise C^V_{nbII} by *neutrality*, *reinforcement*, *faithfulness*, *cancellation*, and *cyclic cancellation*. Even if we add *cyclic cancellation*, there still exists another social choice rule satisfying the above five axioms. Suppose that the social choice rule is denoted by C^V_{bi} and called the *bisection* rule. Let s_r^{bi} be a score of the rth best element in $X \cup \{\emptyset\}$ for every voter. Let $\boldsymbol{s}^{bi} = (s_1^{bi}, ..., s_{|X|+1}^{bi})$ be the bisection scoring vector such that

$$\boldsymbol{s}^{bi} = \begin{cases} (|X|/2, ..., 2, 1, 0, -1, -2, ..., -|X|/2) & (|X|+1 \text{ is odd}) \\ ((|X|+1)/2, ..., 2, 1, -1, -2, ..., -(|X|+1)/2) & (|X|+1 \text{ is even}). \end{cases}$$

Then, C_{bi}^V includes elements whose aggregated scores are the highest according to the above scoring system $\boldsymbol{s}^{bi}$. C_{bi}^V violates *approval–cancellation*. For example, suppose that $V = \{1, 2\}$, $X = \{a, b, c\}$, $aP_1bP_1cP_1\emptyset$, and $cP_2bP_2aP_2\emptyset$. Then, $C_{bi}^V(\mathcal{P}) = \{b\}$.

Third, we consider to characterise C_{nbII}^V by *neutrality*, *reinforcement*, *faithfulness*, *approval–cancellation*, and *cyclic cancellation*. Unfortunately, there exists another social choice rule satisfying the above five axioms. Suppose that the social choice rule is denoted by C_{tb}^V and called the *twisted bisection* rule. Let s_r^{tb} be a score of the rth best element in $X \cup \{\emptyset\}$ for every voter when a rank of $\emptyset$ is not the same for every voter, and let $\bar{s}_r^{tb}$ be a score of the rth best element in X for every voter when a rank of $\emptyset$ is the same for every voter. Then, we denote the two–type twisted bisection scoring vectors by $\boldsymbol{s}^{tb}$ and $\bar{\boldsymbol{s}}^{tb}$, respectively. Assume that $\boldsymbol{s}^{tb} = \boldsymbol{s}^{bi}$. Additionally, suppose that

$$\bar{\boldsymbol{s}}^{tb} = \begin{cases} ((|X|-1)/2, ..., 2, 1, 0, -1, -2..., -(|X|-1)/2) & (|X| \text{ is odd}) \\ (|X|/2, ..., 2, 1, -1, -2, ..., -|X|/2) & (|X| \text{ is even}). \end{cases}$$

C_{tb}^V then includes elements whose scores are the highest according to the above scoring system $\boldsymbol{s}^{tb}$ and $\bar{\boldsymbol{s}}^{tb}$. C_{tb}^V violates *aggregated cancellation*. Suppose that $|V| = 8$, $X = \{a, b, c\}$, $cP_i\emptyset P_iaP_ib$, $i = 1, 2, 3$, $\emptyset P_4aP_4bP_4c$, and $bP_jaP_jcP_j\emptyset$, $j = 5, 6, 7, 8$. Then, $C_{tb}^V(\mathcal{P}) = \{a\}$. However, $C_{nbII}^V(\mathcal{P}) = X$.

Finally, Theorem 1 shows the necessary and sufficient conditions to derive C_{nbII}^V.

Theorem 1 *$C^V = C_{nbII}^V$ if and only if C^V satisfies neutrality, reinforcement, faithfulness, and aggregated cancellation.*

Proof It is trivial that C_{nbII}^V satisfies *neutrality*, *reinforcement*, *faithfulness*, and *aggregated cancellation*. We then assume that C^V satisfies the four axioms and prove the following two lemmas.

Lemma 1 *$NB_a(\mathcal{P}_1) = NB_a(\mathcal{P}_2)$ for all $a \in X$ implies that $C^{V_1}(\mathcal{P}_1) = C^{V_2}(\mathcal{P}_2)$, where $V_1, V_2 \subset \mathbb{Z}_{>0}$, $\mathcal{P}_1 \in \mathscr{P}^{|V_1|}$, and $\mathcal{P}_2 \in \mathscr{P}^{|V_2|}$.*

Proof Let $\mathcal{P}_1 \in \mathscr{P}^{|V_1|}$ and $\mathcal{P}_2 \in \mathscr{P}^{|V_2|}$ such that $V_1, V_2 \subset \mathbb{Z}_{>0}$, $V_1 \cap V_2 = \emptyset$, and $NB_a(\mathcal{P}_1) = NB_a(\mathcal{P}_2)$ for all $a \in X$. Let $\mathcal{P}_3 \in \mathscr{P}^{|V_3|}$ such that V_3 and $V_1 \cup V_2$ are disjoint and $NB_a(\mathcal{P}_1 + \mathcal{P}_3) = NB_b(\mathcal{P}_1 + \mathcal{P}_3)$ for all $a, b \in X$. By *reinforcement* and *aggregated cancellation*, $C^{V_1 \cup V_3}(\mathcal{P}_1 + \mathcal{P}_3) = C^{V_2 \cup V_3}(\mathcal{P}_2 + \mathcal{P}_3) = X$ and

$$C^{V_1}(\mathcal{P}_1) = C^{V_1}(\mathcal{P}_1) \cap X = C^{V_1 \cup V_2 \cup V_3}(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) = X \cap C^{V_2}(\mathcal{P}_2) = C^{V_2}(\mathcal{P}_2).$$

Furthermore, let $\mathcal{P}_4 \in \mathscr{P}^{|V_4|}$ such that V_4 is a clone of V_1, including different voters who have the same preference profile as voters in V_1. Thus, V_4 is assumed to be disjoint from $V_1 \cup V_2$. Then, if V_1 and V_2 are not disjoint, $C^{V_4}(\mathcal{P}_4) = C^{V_2}(\mathcal{P}_2)$ from the above result. Since $C^{V_1}(\mathcal{P}_1) = C^{V_4}(\mathcal{P}_4)$, $C^{V_1}(\mathcal{P}_1) = C^{V_2}(\mathcal{P}_2)$. □

We denote $t(\in \mathbb{Z}_{\geq 0})$ clone sets of V which have the same preference profile by $tV = V_1 \cup \cdots \cup V_t$. Furthermore, we denote a preference profile of tV by $t\mathcal{P} = \mathcal{P}_1 + \cdots + \mathcal{P}_t$ such that $\mathcal{P}_1 = \cdots = \mathcal{P}_t$, where $\mathbb{Z}_{\geq 0}$ is the set of non–negative integers. By *reinforcement*, $C^V(\mathcal{P}) = C^{tV}(t\mathcal{P})$ for any $t \in \mathbb{Z}_{>0}$. Additionally, for any $V \subset \mathbb{Z}_{>0}$, $C^{tV}(t\mathcal{P}) = X$ if $t = 0$. Take $\mathcal{P}, \mathcal{P}' \in \mathscr{P}^{|V|}$ such that $C^V(\mathcal{P}) = X$. By *reinforcement*,

$$C^{V \cup 0V}(\mathcal{P} + 0\mathcal{P}') = C^V(\mathcal{P}) = X;$$

$$C^{V \cup 0V}(\mathcal{P} + 0\mathcal{P}') = C^{0V}(0\mathcal{P}') \cap C^V(\mathcal{P}).$$

If $C^{0V}(0\mathcal{P}') \neq X$, the first equation does not hold. We thus obtain that $C^{tV}(t\mathcal{P}) = X$ if $t = 0$ for any $V \subset \mathbb{Z}_{>0}$.

We then extend the population of t to the set of non–negative rational numbers $\mathbb{Q}_{\geq 0}$ and suppose that $q\mathcal{P}$ is a preference profile of $q(\in \mathbb{Q}_{\geq 0})$ clone sets of V, and $C^{qV}(q\mathcal{P}) = C^V(\mathcal{P})$ for all $q \in \mathbb{Q}_{>0}$, where $\mathbb{Q}_{\geq 0}$ is the set of positive rational numbers. This extension is consistent with the original setting since $C^{tV}(t\mathcal{P}) = C^{t'V}(t'\mathcal{P})$ for all $t, t' \in \mathbb{Z}_{>0}$ if and only if $C^{V''}(\mathcal{P}'') = C^{qV''}(q\mathcal{P}'')$ for all $q \in \mathbb{Q}_{\geq 0}$, where $V'' = tV$, $\mathcal{P}'' = t\mathcal{P}$ and $q = t'/t$.

Next, let $a^m_{\mathcal{P}}$ be an alternative which has the mth highest net Borda score $NB_{a^m_{\mathcal{P}}}(\mathcal{P})$. Thus, $NB_{a^1_{\mathcal{P}}}(\mathcal{P}) \geq \cdots \geq NB_{a^{|X|+1}_{\mathcal{P}}}(\mathcal{P})$. $\mathbf{NB}(\mathcal{P}) = (NB_{a^1_{\mathcal{P}}}(\mathcal{P}), \ldots, NB_{a^{|X|+1}_{\mathcal{P}}}(\mathcal{P}))$ indicates the vector of net Borda scores.

Suppose that $\mathcal{P}_{2,a^m_{\mathcal{P}}} = (P_1, P_2)$ such that $a^m_{\mathcal{P}}$ is the best alternative for both voters, and their preference rankings of remaining alternatives are opposite. The voter set corresponding to $\mathcal{P}_{2,a^m_{\mathcal{P}}}$ is denoted by $V_{a^m_{\mathcal{P}}}$. For any $m \in \{1, ..., |X| + 1\}$, $|V_{a^m_{\mathcal{P}}}| = 2$. Thus, $C^{V_{a^m_{\mathcal{P}}}}(\mathcal{P}_{2,a^m_{\mathcal{P}}}) = \{a^m_{\mathcal{P}}\}$ if $a^m_{\mathcal{P}} \in X$ and $C^{V_{a^m_{\mathcal{P}}}}(\mathcal{P}_{2,a^m_{\mathcal{P}}}) = X$ if $a^m_{\mathcal{P}} = \emptyset$ from Lemma 1, *faithfulness*, *reinforcement*, and *aggregated cancellation*. Furthermore, for all $m \in \{1, ..., |X| + 1\}$, suppose that

$$\mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^m_{\mathcal{P}}}) = (NB_{a^1_{\mathcal{P}}}(\mathcal{P}_{2,a^m_{\mathcal{P}}}), \ldots, NB_{a^{|X|+1}_{\mathcal{P}}}(\mathcal{P}_{2,a^m_{\mathcal{P}}})).$$

We then have that

$$\begin{aligned}
\mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^1_{\mathcal{P}}}) &= (2|X|, -2, \ldots, -2),\\
\mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^2_{\mathcal{P}}}) &= (-2, 2|X|, \ldots, -2),\\
&\vdots\\
\mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^{|X|+1}_{\mathcal{P}}}) &= (-2, \ldots, -2, 2|X|).
\end{aligned}$$

From the above setting, we can state $\mathbf{NB}(\mathcal{P})$ as a linear combination of $\mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^m_{\mathcal{P}}})$'s as follows:

$$\mathbf{NB}(\mathcal{P}) = \Sigma_{m=1}^{|X|+1} q_m \mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^m_{\mathcal{P}}}),$$

where $q_m \in \mathbb{Q}$ for all $m \in \{1, ..., |X|+1\}$ and $\mathbb{Q}$ is the set of rational numbers. Additionally, $NB_{a^m_{\mathcal{P}}}(\mathcal{P}) = 2q_m|X| - 2\Sigma_{l \neq m} q_l$. Thus,

$$NB_{a^m_{\mathcal{P}}}(\mathcal{P}) - NB_{a^n_{\mathcal{P}}}(\mathcal{P}) = 2(|X|+1)(q_m - q_n),$$

and since $m < n$ implies that $NB_{a^m_{\mathcal{P}}}(\mathcal{P}) - NB_{a^n_{\mathcal{P}}}(\mathcal{P}) \geq 0$, $m < n$ also implies that $q_m - q_n \geq 0$. Furthermore, $NB_{a^m_{\mathcal{P}}}(\mathcal{P}) = NB_{a^n_{\mathcal{P}}}(\mathcal{P})$ if and only if $q_m = q_n$.

Since $\Sigma_{m=1}^{|X|+1} \mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^m_{\mathcal{P}}}) = \mathbf{0}$, we can redefine $\mathbf{NB}(\mathcal{P})$ in another way.

$$\begin{aligned} \mathbf{NB}(\mathcal{P}) &= \Sigma_{m=1}^{|X|}[(q_m - q_{m+1})\Sigma_{l \leq m} \mathbf{NB}_{\mathcal{P}}(\mathcal{P}_{2,a^l_{\mathcal{P}}})] \\ &= \mathbf{NB}[\Sigma_{m=1}^{|X|}((q_m - q_{m+1})\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}})]. \end{aligned}$$

This holds true because $q_m - q_{m+1} \geq 0$ for all $m \in \{1, ..., |X|\}$.

By using these results, we prove Lemma 2, which shows that

$$C^V(\mathcal{P}) = \begin{cases} \{a^m_{\mathcal{P}} \in X \mid NB_{a^m_{\mathcal{P}}}(\mathcal{P}) = NB_{a^1_{\mathcal{P}}}(\mathcal{P})\} & (a^1_{\mathcal{P}} \neq \emptyset) \\ \{a^m_{\mathcal{P}} \in X \mid NB_{a^m_{\mathcal{P}}}(\mathcal{P}) = NB_{a^2_{\mathcal{P}}}(\mathcal{P})\} & (a^1_{\mathcal{P}} = \emptyset) \end{cases}$$

if C^V depends only on the net Borda scores.

Lemma 2 C^V *depends only on the net Borda scores if and only if* $C^V = C^V_{nbII}$.

Proof The 'if' part is trivial. Thus, we prove the 'only if' part in the following manner.

First, we should prove that

$$C^{\cup_{l \leq m} V_{a^l_{\mathcal{P}}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}}) = \begin{cases} X & ([a^1_{\mathcal{P}} = \emptyset, m = 1] \vee m = |X|+1) \\ \{a^1_{\mathcal{P}}, ..., a^m_{\mathcal{P}}\} \setminus \{\emptyset\} & (\text{otherwise}). \end{cases}$$

If (1) $a^1_{\mathcal{P}} = \emptyset$ and $m = 1$ or (2) $m = |X|+1$, $C^{\cup_{l \leq m} V_{a^l_{\mathcal{P}}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}}) = X$ by *aggregated cancellation.*

We then consider the other cases. By way of contradiction, assume that there exists $n > m$ such that $a^n_{\mathcal{P}} \in C^{\cup_{l \leq m} V_{a^l_{\mathcal{P}}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}})$. By *neutrality*,

$$NB_{a^1_{\mathcal{P}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}}) = \cdots = NB_{a^m_{\mathcal{P}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}});$$

$$NB_{a^m_{\mathcal{P}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}}) > NB_{a^{m+1}_{\mathcal{P}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}});$$

$$NB_{a^{m+1}_{\mathcal{P}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}}) = \cdots = NB_{a^{|X|+1}_{\mathcal{P}}}(\Sigma_{l \leq m} \mathcal{P}_{2,a^l_{\mathcal{P}}})$$

imply that $(\{a_{\mathcal{P}}^{m+1}, ..., a_{\mathcal{P}}^{|X|+1}\} \setminus \{\emptyset\}) \subset C^{\cup_{l \leq m} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m} \mathcal{P}_{2,a_{\mathcal{P}}^l})$. If $a_{\mathcal{P}}^{m+1} \neq \emptyset$, by *reinforcement* and *faithfulness*,

$$C^{\cup_{l \leq m+1} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m+1} \mathcal{P}_{2,a_{\mathcal{P}}^l}) = C^{\cup_{l \leq m} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m} \mathcal{P}_{2,a_{\mathcal{P}}^l}) \cap C^{V_{a_{\mathcal{P}}^{m+1}}}(\mathcal{P}_{2,a_{\mathcal{P}}^{m+1}}) = \{a_{\mathcal{P}}^{m+1}\}.$$

However, by *neutrality*, $a_{\mathcal{P}}^{m+1} \in C^{\cup_{l \leq m+1} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m+1} \mathcal{P}_{2,a_{\mathcal{P}}^l})$ must imply that $\{a_{\mathcal{P}}^1, ..., a_{\mathcal{P}}^m\} \subset C^{\cup_{l \leq m+1} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m+1} \mathcal{P}_{2,a_{\mathcal{P}}^l})$ since

$$NB_{a_{\mathcal{P}}^1}(\Sigma_{l \leq m+1} \mathcal{P}_{2,a_{\mathcal{P}}^l}) = \cdots = NB_{a_{\mathcal{P}}^{m+1}}(\Sigma_{l \leq m+1} \mathcal{P}_{2,a_{\mathcal{P}}^l}).$$

That is a contradiction. Similarly, if $a_{\mathcal{P}}^{m+1} = \emptyset$, we can obtain the same contradiction by considering $a_{\mathcal{P}}^{m+2}$ and $C^{\cup_{l \leq m+2} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m+2} \mathcal{P}_{2,a_{\mathcal{P}}^l})$. Thus, $C^{\cup_{l \leq m} V_{a_{\mathcal{P}}^l}}(\Sigma_{l \leq m} \mathcal{P}_{2,a_{\mathcal{P}}^l}) = \{a_{\mathcal{P}}^1, ..., a_{\mathcal{P}}^m\} \setminus \{\emptyset\}$.

From the above result, *reinforcement*, and Lemma 1, we obtain that

$$\begin{aligned} C^V(\mathcal{P}) &= C^V[\Sigma_{m=1}^{|X|}((q_m - q_{m+1})\Sigma_{l \leq m} \mathcal{P}_{2,a_{\mathcal{P}}^l})] \\ &= \begin{cases} \cap_{m: q_m > q_{m+1}}(\{a_{\mathcal{P}}^1, ..., a_{\mathcal{P}}^m\} \setminus \{\emptyset\}) & (a_{\mathcal{P}}^1 \neq \emptyset) \\ X \cap_{m>1: q_m > q_{m+1}} \{a_{\mathcal{P}}^2, ..., a_{\mathcal{P}}^m\} & (a_{\mathcal{P}}^1 = \emptyset) \end{cases} \\ &= \begin{cases} \{a_{\mathcal{P}}^m \in X \mid q_m = q_1\} & (a_{\mathcal{P}}^1 \neq \emptyset) \\ \{a_{\mathcal{P}}^m \in X \mid q_m = q_2\} & (a_{\mathcal{P}}^1 = \emptyset). \end{cases} \end{aligned}$$

Thus,

$$C^V(\mathcal{P}) = \begin{cases} \{a_{\mathcal{P}}^m \in X \mid NB_{a_{\mathcal{P}}^m}(\mathcal{P}) = NB_{a_{\mathcal{P}}^1}(\mathcal{P})\} & (a_{\mathcal{P}}^1 \neq \emptyset) \\ \{a_{\mathcal{P}}^m \in X \mid NB_{a_{\mathcal{P}}^m}(\mathcal{P}) = NB_{a_{\mathcal{P}}^2}(\mathcal{P})\} & (a_{\mathcal{P}}^1 = \emptyset) \end{cases}$$

if all voter sets, which have the following preference profiles: $(q_m - q_{m+1})\Sigma_{l \leq m} \mathcal{P}_{2,a_{\mathcal{P}}^l}$, $m = 1, ..., |X|$, are disjoint. Furthermore, even if some of them are not disjoint, consider their disjoint clone voter sets. We then obtain the same result by the same method of Lemma 1. □

From Lemmas 1 and 2, $C^V = C_{nbII}^V$ if C^V satisfies *neutrality*, *reinforcement*, *faithfulness*, and *aggregated cancellation*. □

We can find the necessity of *aggregated cancellation* for characterising C_{nbII}^V from the characterisation of C_{br}^V. When we characterise C_{br}^V, we use the following *classic cancellation* rather than *aggregated cancellation*.

C^V satisfies *classic cancellation* if

$$[n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P}) \;\; \forall a \neq b, a, b \in X] \Rightarrow C^V(\mathcal{P}) = X.$$

This axiom requires that if $n_{ab}(\mathcal{P})$ is cancelled out by $n_{ba}(\mathcal{P})$ for all $a \neq b$, $a, b \in X$, X should be the outcome. We can rewrite this condition as follows:

If $n_{ab}(\mathcal{P}) = n_{ba}(\mathcal{P})$ for all $a \neq b$, $a, b \in X$, $n_{ab}(\mathcal{P}) - n_{ba}(\mathcal{P}) = 0$ for all $a \neq b$, $a, b \in X$. These equations imply that

$$[B_a(\mathcal{P}) = B_b(\mathcal{P}) = 0 \;\; \forall a \neq b, a, b \in X] \Rightarrow C^V(\mathcal{P}) = X.$$

Thus, *classic cancellation* based on the Borda scoring system corresponds to *aggregated cancellation* based on the net Borda scoring system. Furthermore, C^V_{br} violates *aggregated cancellation*, and C^V_{nbII} violates *classic cancellation*. When we consider the desirability of alternatives, *aggregated cancellation* is more proper condition than *classic cancellation*.

Table 1 shows a summary of the necessary conditions to derive C^V_{br}, C^V_{bi}, C^V_{tb}, and C^V_{nbII}.

Table 4 Summary of the necessary conditions

Cancellation conditions	C^V_{br}	C^V_{bi}	C^V_{tb}	C^V_{nbII}
Cancellation	✓	✓	✓	✓
Approval–cancellation	✓		✓	✓
Cyclic cancellation		✓	✓	✓
Aggregated cancellation				✓
Classic cancellation	✓			

Approval–cancellation⇒*cancellation*.
Aggregated cancellation⇒*approval–cancellation* and *cyclic cancellation*.
Classic cancellation⇒*cancellation* and *approval-cancellation*.
Each rule satisfies *neutrality*, *reinforcement*, and *faithfulness*.

From the above results, the type–II net Borda rule is better than the classic Borda rule from the viewpoint of cancellation conditions: the classic Borda rule violates *cyclic cancellation* and satisfies *classic cancellation* which ignores $\emptyset$. If we ignore voters' preference rankings of $\emptyset$, that is, the information about the desirability of existing alternatives, we might obtain a non–intuitive outcome as with the examples in Section 3, and as a matter of fact, we often do that in the real world.

We confirm the independence of *neutrality*, *reinforcement*, *faithfulness*, and *aggregated cancellation*.

First, the *fixed–order net Borda* rule C^{II}_{fnb} satisfies *reinforcement*, *faithfulness*, and *aggregated cancellation*, but violates *neutrality*. This rule outputs the set of winners as follows: after we obtain the set of winners based on the net Borda rule, the social choice is re–decided based on the fixed and exogenous preference ranking of an 'arbitrator' in the outside of V. Then, the winner becomes the best alternative of the arbitrator within $C^V_{nbII}(\mathcal{P})$ if $C^V_{nbII}(\mathcal{P}) \neq X$, and $C^V_{fnb}(\mathcal{P}) = X$ if $C^V_{nbII}(\mathcal{P}) = X$.

Second, the *quasi–feasible* rule C^V_{qf} satisfies *neutrality*, *faithfulness*, and *aggregated cancellation*, but violates *reinforcement*. $C^V_{qf}(\mathcal{P}) = X$ for any preference profile if $|V| \geq 2$, and the winner becomes the best alternative of a voter if $|V| = 1$.

Third, the *feasible* rule C_f^V satisfies *neutrality*, *reinforcement*, and *aggregated cancellation*, but violates *faithfulness*. This rule is assumed to output X for any preference profile.

Finally, C_{bn}^V, C_{bi}^V, or C_{tb}^V satisfies *neutrality*, *reinforcement*, and *faithfulness*, but violates *aggregated cancellation* from Table 1.

6 Conclusions

This study introduces and characterises two types of net Borda rules based on a preference profile over $X \cup \{\emptyset\}$. The first is the type–I net Borda rule whose co–domain is $X \cup \{\emptyset\}$. The second is the type–II net Borda rule whose co–domain is X. We have two main results.

First, we find that the type–II net Borda rule outputs more intuitive winners than the classic Borda rule by considering a simple example (two alternatives and five voters). Furthermore, by simulation analyses using a program in Python 3, we show that if the number of voters is an even number and that of alternatives is sufficiently large, the outputs of the type–II net Borda and classic Borda rules will have large differences. In these situations, we should employ C_{nbII}^V, rather than C_{br}^V.

Second, we axiomatise the type–II net Borda rule by *neutrality*, *faithfulness*, *reinforcement*, and *aggregated cancellation* and find that the classic Borda rule violates *aggregated cancellation*.

Finally, we provide one possible application of net Borda rules, that is, the Borda rules with multiple outside options. By using more than two outside options, the Borda score which is assigned to each existing alternative might be close to the cardinal utilities of voters when each alternative becomes the winner. However, multiple outside options do not mean 'choosing nothing' such as $\emptyset$ in this study, therefore, the multiple outside options should be regarded as reference points which are outside of the alternative set. Thus, we should add $\emptyset$ besides the multiple reference points if we try to consider the absolute evaluation criterion as well.

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