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# Calculating a Giffen Good

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**Abstract**

This paper provides a simple example of the utility function with two consumption goods which can be calculated *by hand* to produce a Giffen good. It is based on the theoretical result by Kubler, Selden, and Wei (2013). Using a model of portfolio selection with a risk-free asset and a risky asset, they showed that the risk-free asset becomes a Giffen good if the utility belongs to the HARA family. This paper investigates their result further in a usual microeconomic setting, and derives the conditions for one of the consumption goods to be a Giffen good from a broader perspective.

Key words: HARA family, Decreasing relative risk aversion, Giffen good, Slutsky equation, Ratio effect

JEL classification: D11, D01, G11

## 1 Introduction

Since Marshall (1895) mentioned a possibility of a Giffen good, economists have been trying to find it theoretically and empirically. Although their sincere efforts must be respected, it is also recognized among them that such a good seldom, if ever, shows up as Marhsall already said.[1] But the recent theoretical result by Kubler, Selden, and Wei (2013) seems to be a solid example for a Giffen good. Using a model of portfolio selection with a risk-free asset and a risky asset, they showed that there always exists a parameter set which assures that the risk-free asset becomes a Giffen good if the utility belongs to the HARA (hyperbolic absolute risk aversion) family with decreasing absolute risk aversion (DARA) and decreasing relative risk aversion (DRRA).[2]

As is well known, Arrow (1971) proposed both DARA and increasing relative risk aversion (IRRA) because the former means that a risky asset is a normal good, while the latter means that a risk-free asset is a normal good the wealth elasticity of the demand for which exceeds one as is observed in reality. In this sense the assumption of DRRA adopted by Kubler et al. (2013) is not a usual one. But DRRA is not completely ignored in economics and finance.[3] Then, their result (also endorsed by computer simulations) is still very informative since it gives the conditions for a risk-free asset to be a Giffen good which can be written explicitly.

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[1] For the recent developments, see Doi, Iwasa, and Shimomura (2009) and Heijman and von Mouche (2012). In fact a Giffen good was first analyzed by Simon Gray in 1815. See Masuda and Newman (1981).

[2] For a further consideration of the relationship between the HARA family and a Giffen good, see also Kannai and Selden (2014).

[3] See, for example, Campbell and Cochrane (1995), Gollier (2001), and Meyer and Meyer (2005).

This paper investigates their result further in a usual microeconomic setting where the risk-free asset and the risky asset are changed to the first and second consumption goods, respectively. In such a setting a problem of DRRA does not worry us any longer because the utility function proposed in this paper has standard properties from a microeconomic point of view. The rest of the paper is organized as follows. In Section 2 a utility maximization problem of a consumer is directly solved to obtain the conditions for the first good to be a Giffen good. In Section 3 the same problem is analyzed by means of two kinds of decompositions of the price effect. In Section 4 these analyses are compared and summarized. Section 5 makes a concluding remark.

## 2 Consumer's Utility Maximization

Let $q_1$ and $q_2$ be the quantities demanded of good 1 and good 2, respectively. Then, the utility function investigated in this paper is written as

$$u(q_1, q_2) = \alpha_1 \frac{(\beta_1 q_1 + \beta_{21} q_2 - \beta_3)^{1-\gamma}}{1-\gamma} + \alpha_2 \frac{(\beta_1 q_1 + \beta_{22} q_2 - \beta_3)^{1-\gamma}}{1-\gamma}, \tag{1}$$

and a consumer maximizes (1) with respect to $q_1$ and $q_2$ under a budget constraint

$$p_1 q_1 + p_2 q_2 = I, \tag{2}$$

where $p_1$ and $p_2$ are respectively the prices of good 1 and good 2, and $I$ is nominal income. As for parameters the following are assumed:

Assumption 1: $\alpha_1, \alpha_2, \beta_1, \beta_{21}, \beta_{22}, \beta_3, \gamma > 0$.
Assumption 2: $\beta_{21} p_1 > \beta_1 p_2 > \beta_{22} p_1$.

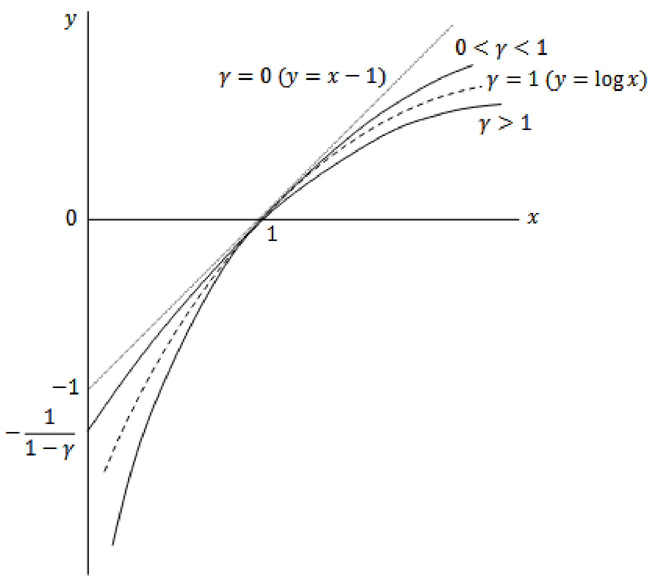


Figure 1. Graphs of $y = \frac{x^{1-\gamma}-1}{1-\gamma}, x > 0$

The utility function (1) is the sum of the two terms on the right-hand side either of which is of the form $x^{1-\gamma}/(1-\gamma), x > 0$. So, it is useful to know what it looks like. To do so, first draw curves of the function $y = (x^{1-\gamma}-1)/(1-\gamma), x > 0$ on the $x$-$y$ plane as in Figure 1. They are all upward sloping and pass through point $(1, 0)$. They can be classified into three kinds depending the value of $\gamma$. When $0 < \gamma < 1$, the curve lies in the region between the (dotted) straight line $y = x - 1$ and the (dashed) curve of the natural logarithm $y = \log x$. When $\gamma > 1$, the curve lies in the region below the curve of the natural logarithm. It is known that the curve in either region approaches that of the natural logarithm. So the function $y = (x^{1-\gamma} - 1)/(1-\gamma)$ for $\gamma = 1$ is often regarded to coincide with $y = \log$.[4] A curve of $y = x^{1-\gamma}/(1-\gamma)$ is obtained by shifting a curve of $y = (x^{1-\gamma} - 1)/(1-\gamma)$ by $1/(1-\gamma)$.[5]

In order to know the sign of bordered Hessian $|U|$ of (1), let us calculate

$$|U| = -u_2^2 u_{11} + 2u_1 u_2 u_{12} - u_1^2 u_{22},$$

where $u_i = \partial u/\partial q_i$ and $u_{ij} = \partial^2 u/\partial q_i \partial q_j$, $i, j = 1, 2$. For convenience, put $A = \beta_1 q_1 + \beta_{21} q_2 - \beta_3 (> 0)$, and $B = \beta_1 q_1 + \beta_{22} q_2 - \beta_3 (> 0)$. Then,

$$\begin{aligned}
u_1 &= \alpha_1 \beta_1 A^{-\gamma} + \alpha_2 \beta_1 B^{-\gamma}, \\
u_2 &= \alpha_1 \beta_{21} A^{-\gamma} + \alpha_2 \beta_{22} B^{-\gamma}, \\
u_{11} &= -\alpha_1 \beta_1^2 \gamma A^{-\gamma-1} - \alpha_2 \beta_1^2 \gamma B^{-\gamma-1}, \\
u_{12} &= -\alpha_1 \beta_1 \beta_{21} \gamma A^{-\gamma-1} - \alpha_2 \beta_1 \beta_{22} \gamma B^{-\gamma-1}, \\
u_{22} &= -\alpha_1 \beta_{21}^2 \gamma A^{-\gamma-1} - \alpha_2 \beta_{22}^2 \gamma B^{-\gamma-1}.
\end{aligned}$$

Substituting the above results into $|U|$ and arranging yields

$$|U| = \alpha_1 \alpha_2 \beta_1^2 \gamma (\beta_{21} - \beta_{22})^2 (\alpha_1 A^{-2\gamma} B^{-\gamma-1} + \alpha_2 A^{-\gamma-1} B^{-2\gamma-1}) > 0. \tag{3}$$

The positivity of $|U|$ means that the utility function (1) is a strictly quasi-concave function and the marginal rate of substitution is decreasing as usual in microeconomics.

The first-order condition for utility maximization is written as

$$\frac{u_1}{u_2} = \frac{\alpha_1 \beta_1 A^{-\gamma} + \alpha_2 \beta_1 B^{-\gamma}}{\alpha_1 \beta_{21} A^{-\gamma} + \alpha_2 \beta_{22} B^{-\gamma}} = \frac{p_1}{p_2}.$$

Arranging it gives

$$\frac{B}{A} = \frac{\beta_1 q_1 + \beta_{22} q_2 - \beta_3}{\beta_1 q_1 + \beta_{21} q_2 - \beta_3} = \kappa, \tag{4}$$

where

$$\kappa = \left( \frac{\alpha_2}{\alpha_1} \frac{\beta_1 p_2 - \beta_{22} p_1}{\beta_{21} p_1 - \beta_1 p_2} \right)^{\frac{1}{\gamma}}. \tag{5}$$

By Assumptions 1 and 2 $\kappa$ is positive. It is also obvious that $\kappa$ is a decreasing function of $p_1$, which can be written as $\kappa' = d\kappa/dp_1 < 0$. So it is assumed that

[4] As far as I know, there is no such figure as Figure 1 in which the three curves for $0 < \gamma < 1$, $\gamma = 1$, and $\gamma > 1$ are drawn *at the same time.*

[5] When $\gamma = 1$, therefore, the utility function (1) may be written as $\alpha_1 \log(\beta_1 q_1 + \beta_{21} q_2 - \beta_3) + \alpha_2 \log(\beta_1 q_1 + \beta_{22} q_2 - \beta_3)$, which is of the type referred to as the Klein-Rubin or Stone-Geary utility function.

Assumption 3: $\beta_{22} - \beta_{21}\kappa > 0$.

Assumption 3 also implies that $\kappa < 1$ because $\kappa < \beta_{22}/\beta_{21} < 1$ by Assumption 2. As will be seen, Assumption 3 is crucial for the existence of a Giffen good.[6]

The demands for good 1 and good 2 are solutions to simultaneous equations (4) and (2). Solving them leads to

$$q_1^* = \frac{(1-\kappa)\beta_3 p_2 - (\beta_{22} - \beta_{21}\kappa)I}{D}, \tag{6}$$

$$q_2^* = \frac{(1-\kappa)(\beta_1 I - \beta_3 p_1)}{D}, \tag{7}$$

where

$$\begin{aligned} D &= (1-\kappa)\beta_1 p_2 - (\beta_{22} - \beta_{21}\kappa)p_1 \\ &= \beta_1 p_2 - \beta_{22} p_1 + \kappa(\beta_{21} p_1 - \beta_1 p_2). \end{aligned}$$

Note that $D > 0$ by Assumptions 1 and 2. Then, the demand $q_1^*$ for good 1 is a decreasing function of $I$ by Assumption 3, while the demand $q_2^*$ for good 2 is an increasing function of $I$. In other words, good 1 is always an inferior good, while good 2 is always a normal good, as far as $q_1^* > 0$ and $q_1^* > 0$. So it is assumed that

Assumption 4: $I_{\text{min}} < I < I_{\text{max}}$, where

$$I_{\text{min}} = \frac{\beta_3 p_1}{\beta_1}, \quad I_{\text{max}} = \frac{(1-\kappa)\beta_3 p_2}{\beta_{22} - \beta_{21}\kappa}.$$

Needless to say, $q_2^* > 0$ for $I > I_{\text{min}}$, and $q_1^* > 0$ for $I < I_{\text{max}}$. Now the following proposition has been proved:

*Proposition 1: Under Assumptions 1-4, good 1 is an inferior good.*

[6] Using the natural logarithm, Assumption 3 can be written as

$$(0 <)\ \gamma < \log\left(\frac{\alpha_2}{\alpha_1}\frac{\beta_1 p_2 - \beta_{22} p_1}{\beta_{21} p_1 - \beta_1 p_2}\right)\left(\log\frac{\beta_{22}}{\beta_{21}}\right)^{-1}.$$

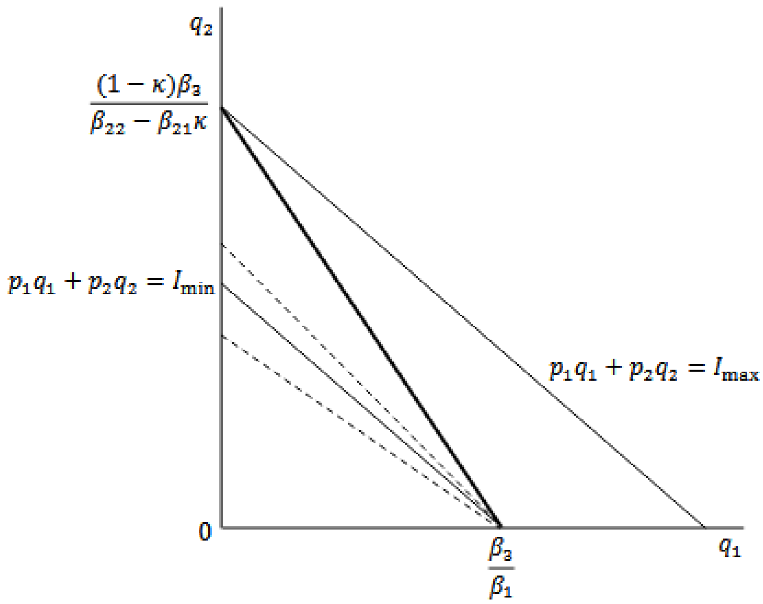


Figure 2. Expansion Path

The above arguments are summarized using the $q_1$-$q_2$ plane as in Figure 2. There are two downward sloping dashed lines in the positive orthant. The lower one corresponds to $A = \beta_1 q_1 + \beta_{21} q_2 - \beta_3 = 0$, and the upper one corresponds to $B = \beta_1 q_1 + \beta_{22} q_2 - \beta_3 = 0$. It is only in the region above the upper one that both $A > 0$ and $B > 0$. There are also two parallel solid lines. The lower one is a budget line for $I = I_{\text{min}}$, and the upper one is a budget line for $I = I_{\text{max}}$. It is only in the region between the two budget lines that both $q_1^* > 0$ and $q_2^* > 0$. Note that by Assumption 2 the slope of the budget lines lies between the slope of the $A = 0$ line and that of the $B = 0$ line, i.e., $\beta_1/\beta_{21} < p_1/p_2 < \beta_1/\beta_{22}$. Optimal point $(q_1^*, q_2^*)$ is a point of contact of a budget line and an indifference curve (which is not shown). A locus of such points is called an expansion path. Solving (6) for $I$ and substituting the result into (7) yields the equation of the expansion path

$$q_2 = -\frac{(1-\kappa)\beta_1}{\beta_{22} - \beta_{21}\kappa} q_1 + \frac{(1-\kappa)\beta_3}{\beta_{22} - \beta_{21}\kappa}.$$

This expansion path is drawn as a bold straight line. As income increases from $I_{\text{min}}$ to $I_{\text{max}}$, the corresponding optimal point goes up along the expansion path from point $(\beta_3/\beta_1, 0)$ on the horizontal axis to point $(0, (1-\kappa)\beta_3/(\beta_{22} - \beta_{21}\kappa))$ on the vertical axis. As is seen from Figure 2 too, good 1 is always an inferior good.

Since a Giffen good is always an inferior good, let us examine whether good 1 can be a Giffen good. Differentiating $q_1^*$ in (6) with respect to $p_1$ leads to

$$\begin{aligned}
\frac{\partial q_1^*}{\partial p_1} &= \frac{1}{D^2}\{(-\beta_3 p_2 + \beta_{21} I)\kappa' D \\
&\quad + [(1-\kappa)\beta_3 p_2 - (\beta_{22} - \beta_{21}\kappa) I][-\kappa'(\beta_{21} p_1 - \beta_1 p_2) + (\beta_{22} - \beta_{21}\kappa)]\} \\
&= \frac{1}{D^2}\Big(\beta_3 p_2\{-\kappa' D + (1-\kappa)[-\kappa'(\beta_{21} p_1 - \beta_1 p_2) + (\beta_{22} - \beta_{21}\kappa)]\} \\
&\quad - \{-\beta_{21}\kappa' D + (\beta_{22} - \beta_{21}\kappa)[-\kappa'(\beta_{21} p_1 - \beta_1 p_2) + (\beta_{22} - \beta_{21}\kappa)]\} I\Big) \\
&= \frac{1}{D^2}\Big(\beta_3 p_2\{-\kappa'(\beta_{21} - \beta_{22}) p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)\} \\
&\quad - \{-\kappa'(\beta_{21} - \beta_{22})\beta_1 p_2 + (\beta_{22} - \beta_{21}\kappa)^2\} I\Big),
\end{aligned}$$

where $\kappa' < 0$ as has been seen above. Define $I_1^G$ such that $\partial q_1^*/\partial p_1 = 0$ for $I_1^G$. Then,

$$I_1^G = \frac{\beta_3 p_2[-\kappa'(\beta_{21} - \beta_{22})p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)]}{-\kappa'(\beta_{21} - \beta_{22})\beta_1 p_2 + (\beta_{22} - \beta_{21}\kappa)^2}. \tag{8}$$

Note that $I_1^G > 0$ by Assumptions 1-3. And it can be said that $\partial q_1^*/\partial p_1 \gtreqless 0$ for $I \lesseqgtr I_1^G$. So good 1 becomes a Giffen good if $I < I_1^G$ *and* $I_{\min} < I_1^G < I_{\max}$.

Before examining the magnitude of $I_1^G$ it is helpful to introduce another income level. For that purpose, differentiate $q_2^*$ in (7) with respect to $p_1$. Then,

$$\begin{aligned}
\frac{\partial q_2^*}{\partial p_1} &= \frac{1}{D^2}\{[-\kappa'(\beta_1 I - \beta_3 p_1) - (1-\kappa)\beta_3]D \\
&\quad +(1-\kappa)(\beta_1 I - \beta_3 p_1)[-\kappa'(\beta_{21}p_1 - \beta_1 p_2) + (\beta_{22} - \beta_{21}\kappa)]\} \\
&= \frac{1}{D^2}\Big(-\{-\kappa'\beta_3 p_1 D + (1-\kappa)\beta_3 D + \beta_3 p_1(1-\kappa)[-\kappa'(\beta_{21}p_1 - \beta_1 p_2) + (\beta_{22} - \beta_{21}\kappa)]\} \\
&\quad +\{-\beta_1\kappa' D + \beta_1(1-\kappa)[-\kappa'(\beta_{21}p_1 - \beta_1 p_2) + (\beta_{22} - \beta_{21}\kappa)]\}I\Big) \\
&= \frac{1}{D^2}\Big(-\beta_3\{-\kappa'(\beta_{21} - \beta_{22})p_1^2 + (1-\kappa)^2\beta_1 p_2\} \\
&\quad +\beta_1\{-\kappa'(\beta_{21} - \beta_{22})p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)\}I\Big).
\end{aligned}$$

Define $I_1^Z$ such that $\partial q_2^*/\partial p_1 = 0$ for $I = I_1^Z$. Then,

$$I_1^Z = \frac{\beta_3}{\beta_1}\frac{-\kappa'(\beta_{21} - \beta_{22})p_1^2 + (1-\kappa)^2\beta_1 p_2}{-\kappa'(\beta_{21} - \beta_{22})p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)}. \tag{9}$$

$I_1^Z > 0$ by Assumptions 1-3. And $\partial q_2^*/\partial p_1 \gtreqless 0$ for $I \gtreqless I_1^Z$.

Among four income levels introduced so far the following lemma holds.

*Lemma 1: Under Assumptions 1-4,* $I_{\min} < I_1^G < I_1^Z < I_{\max}$.
Proof: See Appendix A.

Lemma 1 assures that $I_{\min} < I_1^G < I_{\max}$. Hence the following proposition.[7]

*Proposition 2: Under Assumptions 1-4, good 1 is a Giffen good for* $I_{\min} < I < I_1^G$.

[7] The model of this section and that of Kubler et al. (2013) are much the same mathematically. The utility function (1) corresponds to one of their utility functions $E[W(\tilde{c}_2)] = \pi_{21}\frac{(c_{21}-a)^{-\delta}}{-\delta} + \pi_{22}\frac{(c_{22}-a)^{-\delta}}{-\delta}$, $c_{21} > c_{22} > a > 0$, $\delta > -1$, where $c_{21}$ and $c_{22}$ represent contingent claims with probability $\pi_{21}$ and $\pi_{22}(= 1 - \pi_{21})$. Propositions 1 and 2 in this paper correspond to their Theorem 1 (ii) and Proposition 1, respectively. Two differences are to be mentioned right away. First, income level $I_{\max}$ does not appear in their paper because they assume that a risk-free asset may be held short. Second, income level $I_1^Z$ is not used in their analysis because their proof of the existence of Giffen behavior is confined to the neighborhood of income level $I_1^G$.

## 3 Decompositions of the Price Effect

This section examines the above utility maximization problem by means of two kinds of decompositions of the price effect. First, consider the Slutsky equation:

$$\underbrace{\left.\frac{\partial q_1}{\partial p_1}\right|_{p_2,I=\text{const}}}_{\text{price effect}} = \underbrace{\left.\frac{\partial q_1}{\partial p_1}\right|_{p_2,u=\text{const}}}_{\text{substitution effect}} + \underbrace{\left(-q_1\left.\frac{\partial q_1}{\partial I}\right|_{p_1,p_2=\text{const}}\right)}_{\text{income effect}},$$

where

$$\left.\frac{\partial q_1}{\partial p_1}\right|_{p_2,u=\text{const}} = -\frac{u_1^*(u_2^*)^2}{p_1|U^*|} < 0, \quad -q_1\left.\frac{\partial q_1}{\partial I}\right|_{p_1,p_2=\text{const}} = -q_1^*\frac{u_1^*(u_2^*u_{12}^* - u_1^*u_{22}^*)}{p_1|U^*|}.$$

Superscript $*$ means a value evaluated at optimal point $(q_1^*, q_2^*)$. As is well known, the substitution effect is always negative. So in order for good 1 to be a Giffen good, the income effect must be positive and so large that it dominates the substitution effect with the result the price effect becomes positive.[8]

Remember that $A$ and $B$ appearing the partial derivatives of $u(q_1, q_2)$ are defined respectively as $\beta_1 q_1 + \beta_{21} q_2 - \beta_3$ and $\beta_1 q_1 + \beta_{22} q_2 - \beta_3$. Then, they are evaluated at an optimal point as

$$A^* = \frac{X}{D}, \quad B^* = \kappa\frac{X}{D},$$

where $X = (\beta_{21} - \beta_{22})(\beta_1 I - \beta_3 p_1)$. By Assumptions 2 and 4, $X > 0$. Therefore $A^* > 0$ and $B^* > 0$. For convenience, moreover, put $\chi = X/D$. Then $A^* = \chi$ and $B^* = \kappa\chi$.[9] Using these results as well as (3), bordered Hessian $|U|$ is evaluated at an optimal point as

$$\begin{aligned} |U^*| &= \alpha_1\alpha_2\beta_1^2\gamma(\beta_{21} - \beta_{22})^2\left[\alpha_1(A^*)^{-2\gamma}(B^*)^{-\gamma-1} + \alpha_2(A^*)^{-\gamma-1}(B^*)^{-2\gamma-1}\right] \\ &= (\chi)^{-3\gamma-1}\alpha_1\alpha_2\beta_1^2\gamma(\beta_{21} - \beta_{22})^2(\alpha_1\kappa^{-\gamma-1} + \alpha_2\kappa^{-2\gamma}). \end{aligned}$$

Similar calculations give the evaluated values of the substitution effect and the income effect as follows:

$$\begin{aligned} &-\frac{u_1^*(u_2^*)^2}{p_1|U^*|} \\ &= -\frac{1}{p_1|U^*|}\beta_1\chi^{-\gamma}(\alpha_1 + \alpha_2\kappa^{-\gamma})\chi^{-2\gamma}(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2 \\ &= -\frac{\beta_1\chi^{-3\gamma-1}}{p_1|U^*|}\chi(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2 \end{aligned}$$

[8] Kubler et al. (2013, pp. 1034, 1038, 1047-1048) repeatedly mention the Slutsky decomposition to explain the behavior of a risk-free asset as a Giffen good, but they does not use it.

[9] When $q_1 = q_1^*$ and $q_2 = q_2^*$, the value of the utility function (1) is written as

$$\begin{aligned} u(q_1^*, q_2^*) &= \alpha_1\frac{(A^*)^{1-\gamma}}{1-\gamma} + \alpha_2\frac{(B^*)^{1-\gamma}}{1-\gamma} = \frac{\alpha_1 + \alpha_2\kappa^{1-\gamma}}{1-\gamma}\chi^{1-\gamma} \text{ for } \gamma \neq 1, \\ &= \alpha_1\log(A^*) + \alpha_2\log(B^*) = (\alpha_1 + \alpha_2)\log\chi + \alpha_2\log\kappa \text{ for } \gamma = 1. \end{aligned}$$

Note that $\chi$ is an increasing function of $I$. It follows that the maximized value of utility is an increasing function of $I$, though $q_1^*$ and $q_2^*$ move in the opposite direction to changes in $I$.

$$
\begin{aligned}
&= -\frac{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2}{p_1\alpha_1\alpha_2\beta_1\gamma(\beta_{21}-\beta_{22})(\alpha_1\kappa^{-\gamma-1}+\alpha_2\kappa^{-2\gamma})D}(\beta_1 I-\beta_3 p_1),\\
&-q_1^*\frac{u_1^*(u_2^*u_{12}^*-u_1^*u_{22}^*)}{p_1|U^*|}\\
&= \frac{q_1^*}{p_1|U^*|}\beta_1\chi^{-\gamma}(\alpha_1+\alpha_2\kappa^{-\gamma})\alpha_1\alpha_2\beta_1\gamma(\beta_{21}-\beta_{22})\chi^{-2\gamma-1}(\beta_{22}-\beta_{21}\kappa)\kappa^{-\gamma-1}\\
&= \frac{\beta_1\chi^{-3\gamma-1}}{p_1|U^*|}(\alpha_1+\alpha_2\kappa^{-\gamma})\alpha_1\alpha_2\beta_1\gamma(\beta_{21}-\beta_{22})q_1^*(\beta_{22}-\beta_{21}\kappa)\kappa^{-\gamma-1}\\
&= \frac{\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)(\alpha_1+\alpha_2\kappa^{-\gamma})}{p_1(\beta_{21}-\beta_{22})(\alpha_1\kappa^{-\gamma-1}+\alpha_2\kappa^{-2\gamma})D}[(1-\kappa)\beta_3 p_2-(\beta_{22}-\beta_{21}\kappa)I].
\end{aligned}
$$

As is seen from the above, the two effects are a decreasing function of $I$. But, as income increases from $I_{\min}$ to $I_{\max}$, the substitution effect decreases from zero to a negative value, whereas the income effect decreases from a positive value to zero. It means that there exists an income level $I_2^G$ between $I_{\min}$ and $I_{\max}$ at which the sum of the two effects becomes zero.

Next consider another decomposition of the price effect due to Sasakura (2016):

$$
\underbrace{\left.\frac{\partial q_1}{\partial p_1}\right|_{p_2,I=\text{const}}}_{\text{price effect}} = \underbrace{\left.\frac{\partial q_1}{\partial p_1}\right|_{I,\theta=\text{const}}}_{\text{unit-elasticity effect}} + \underbrace{\frac{\frac{\partial\theta}{\partial p_1}I}{p_1}}_{\text{ratio effect}},
$$

where

$$
\theta=\frac{p_1q_1^*}{I},\ \left.\frac{\partial q_1}{\partial p_1}\right|_{I,\theta=\text{const}}=-\frac{q_1^*}{p_1}<0,\ \frac{\frac{\partial\theta}{\partial p_1}I}{p_1}=-\frac{u_1^*}{p_1}\frac{(u_2^*)^2}{|U^*|}+q_1^*\frac{u_2^*(u_1^*u_{12}^*-u_2^*u_{11}^*)}{p_1|U^*|}.
$$

$\theta$ is the ratio of the expenditure on good 1 to income. The unit-elasticity effect represents a response of the demand for good 1 to a change in the price of good 1 if the ratio $\theta$ remains unchanged (or equivalently if the expenditure on good 1 remains unchanged). It is always negative. The ratio effect represents a further response of the demand for good 1 resulting from a change in the ratio $\theta$. As the right-hand side of the above formula shows, the ratio effect is the sum of the two terms. The first is none other than the substitution effect, while the second is called the transfer effect which is interpreted to be a transfer of demand from good 2 to good 1 in response to a change in real income. Since good 2 is a normal good, the transfer effect is positive. But the substitution effect is negative. Thus the sign of the ratio effect depends on the values of the two terms.[10] In order for good 1 to be a Giffen good, the ratio effect must be positive and so large that it dominates the unit-elasticity effect.

The unit-elasticity effect is calculated at once as

$$
-\frac{q_1^*}{p_1}=-\frac{(1-\kappa)\beta_3p_2-(\beta_{22}-\beta_{21}\kappa)I}{p_1D},
$$

because of (6). The unit-elasticity effect is an increasing function of $I$, and as income increases from $I_{\min}$ to $I_{\max}$, it increases from a negative value to zero. The substitution effect has

[10] For the details, see Sasakura (2016). Using such a decomposition of the price effect, the CES utility function as well as the Cobb-Douglas utility function are analyzed there. The CRRA utility function can also be analyzed in a similar way.

already been obtained above. As for the transfer effect,

$$
\begin{aligned}
&q_1^* \frac{u_2^*(u_1^* u_{12}^* - u_2^* u_{11}^*)}{p_1|U^*|} \\
&= \frac{q_1^*}{p_1|U^*|}\chi^{-\gamma}(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})\alpha_1\alpha_2\beta_1^2\gamma(\beta_{21} - \beta_{22})\chi^{-2\gamma-1}(1-\kappa)\kappa^{-\gamma-1} \\
&= \frac{\chi^{-3\gamma-1}}{p_1|U^*|}(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})\alpha_1\alpha_2\beta_1^2\gamma(\beta_{21} - \beta_{22})q_1^*(1-\kappa)\kappa^{-\gamma-1} \\
&= \frac{\kappa^{-\gamma-1}(1-\kappa)(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})}{p_1(\beta_{21} - \beta_{22})(\alpha_1\kappa^{-\gamma-1} + \alpha_2\kappa^{-2\gamma})D}[(1-\kappa)\beta_3 p_2 - (\beta_{22} - \beta_{21}\kappa)I].
\end{aligned}
$$

Adding the substitution effect and the transfer effect leads to the ratio effect:

$$
\begin{aligned}
&\frac{\frac{\partial\theta}{\partial p_1} I}{p_1} \\
&= \frac{\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}}{p_1\alpha_1\alpha_2\beta_1\gamma(\beta_{21} - \beta_{22})(\alpha_1\kappa^{-\gamma-1} + \alpha_2\kappa^{-2\gamma})D} \\
&\quad \times\Big\{-(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})(\beta_1 I - \beta_3 p_1) \\
&\quad +\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)[(1-\kappa)\beta_3 p_2 - (\beta_{22} - \beta_{21}\kappa)I]\Big\} \\
&= \frac{\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}}{p_1\alpha_1\alpha_2\beta_1\gamma(\beta_{21} - \beta_{22})(\alpha_1\kappa^{-\gamma-1} + \alpha_2\kappa^{-2\gamma})D} \\
&\quad \times\Big\{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})\beta_3 p_1 + \alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)^2\beta_3 p_2 \\
&\quad - \big[(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})\beta_1 + \alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22} - \beta_{21}\kappa)\big]\, I\Big\}.
\end{aligned}
$$

As is seen from the above, the ratio effect is a decreasing function of $I$, and as income increases from $I_{\text{min}}$ to $I_{\text{max}}$, it decreases from a positive value to a negative value. It implies that there exists an income level between $I_{\text{min}}$ and $I_{\text{max}}$ at which the ratio effect vanishes. Denote such an income level by $I_2^Z$. Then,

$$
I_2^Z = \frac{\beta_3}{\beta_1}\frac{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})p_1 + \alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)^2 p_2}{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}) + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22} - \beta_{21}\kappa)}. \tag{10}
$$

Note that $(\partial\theta/\partial p_1)I/p_1 \lesseqqgtr 0$ for $I \gtreqqless I_2^Z$. If the ratio effect is zero, the expenditure on good 1 remains unchanged after its price rises. It means that the expenditure on good 2 also remains unchanged, which in turn implies that $\partial q_2^*/\partial p_1 = 0$ for $I = I_2^Z$. In the case of a positive (negative) ratio effect, the expenditure on good 1 increases (decreases) after its price rises, and the demand for good 2 decreases (increases).

The price effect is obtained either by adding the substitution and income effects or by adding the unit-elasticity and ratio effects as follows:

$$
\begin{aligned}
&\left.\frac{\partial q_1}{\partial p_1}\right|_{p_2, I=\text{const}} \\
&= \frac{\alpha_1 + \alpha_2\kappa^{-\gamma}}{p_1\alpha_1\alpha_2\beta_1\gamma(\beta_{21} - \beta_{22})(\alpha_1\kappa^{-\gamma-1} + \alpha_2\kappa^{-2\gamma})D}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big\{-(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2(\beta_1 I-\beta_3 p_1)\\
&+\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)[(1-\kappa)\beta_3 p_2-(\beta_{22}-\beta_{21}\kappa)I]\Big\}\\
=\;&\frac{\alpha_1+\alpha_2\kappa^{-\gamma}}{p_1\alpha_1\alpha_2\beta_1\gamma(\beta_{21}-\beta_{22})(\alpha_1\kappa^{-\gamma-1}+\alpha_2\kappa^{-2\gamma})D}\\
&\times\Big\{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2\beta_3 p_1+\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)\beta_3 p_2\\
&-[(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2\beta_1+\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)^2]I\Big\}.
\end{aligned}
$$

As has already been pointed out, there exists an income level between $I_{\min}$ and $I_{\max}$ at which the price effect vanishes. Denote such an income level by $I_2^G$ again. Then,

$$
I_2^G=\frac{\beta_3}{\beta_1}\frac{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2 p_1+\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)p_2}{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2+\alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)^2}. \tag{11}
$$

Note that $(\partial q_1/\partial p_1)|_{p_2,I=\text{const}} \gtreqless 0$ for $I \lesseqgtr I_2^G$.

Although it has already been shown that $I_{\min}<I_2^G<I_{\max}$ and $I_{\min}<I_2^Z<I_{\max}$, the following lemma has more information.

*Lemma 2: Under Assumptions 1-4,* $I_{\min}<I_2^G<I_2^Z<I_{\max}$.
Proof: See Appendix B.

And the following proposition holds.

*Proposition 3: Under Assumptions 1-4, good 1 is a Giffen good for* $I_{\min}<I<I_2^G$.

# 4 Relationship between Two Analyses

The same utility maximization problem of a consumer has been analyzed in different ways in Sections 2 and 3. What is the relationship between the two? As is expected, they are the same in terms of economics. Concretely speaking, it is expected that $I_1^Z=I_2^Z$ and $I_1^G=I_2^G$. In fact, the proof of their equalities is not trivial, but it is possible if the following two facts are paid attention to. First, taking the natural logarithm of both sides of (5) yields

$$
\log\kappa=\frac{1}{\gamma}\left[\log\frac{\alpha_2}{\alpha_1}+\log(\beta_1 p_2-\beta_{22}p_1)-\log(\beta_{21}p_1-\beta_1 p_2)\right].
$$

Differentiating it with respect to $p_1$ and arranging gives

$$
\kappa'=-\frac{\kappa}{\gamma}\frac{(\beta_{21}-\beta_{22})\beta_1 p_2}{(\beta_1 p_2-\beta_{22}p_1)(\beta_{21}p_1-\beta_1 p_2)}(<0). \tag{12}
$$

Second, substituting $A^*$ and $B^*$ into the first-order condition (4) and arranging gives

$$
\beta_1 p_2=\frac{\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma}}{\alpha_1+\alpha_2\kappa^{-\gamma}}p_1. \tag{13}
$$

Using these facts, the following lemmas are proved.

*Lemma 3: Under Assumptions 1-4,.* $I_1^Z = I_2^Z$
Proof: See Appendix C.

*Lemma 4: Under Assumptions 1-4,* $I_1^G = I_2^G$.
Proof: See Appendix D.

So let us put $I^Z = I_1^Z = I_2^Z$, and $I^G = I_1^G = I_2^G$. Then, Propositions 2 and 3 are unified into one proposition:

*Proposition 4: Under Assumptions 1-4, good 1 is a Giffen good for* $I_{\text{min}} < I < I^G$.

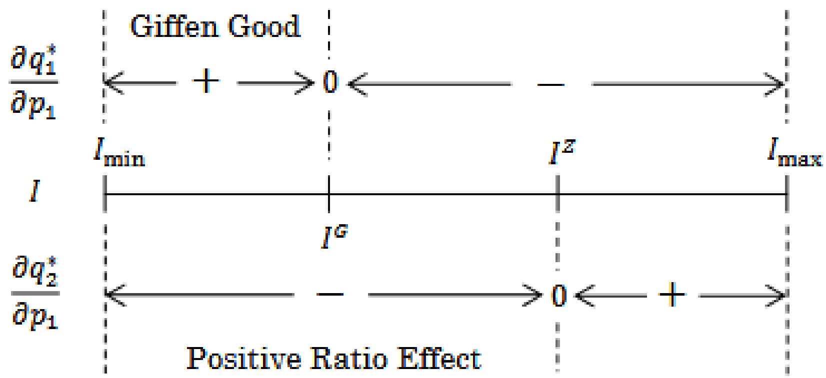


Figure 3. Responses of Demands to Price of Good 1

Also the results obtained so far can be summarized as in Figure 3. Roughly speaking, there occur three patterns of the signs of $\partial q_1^*/p_1$ and $\partial q_2^*/p_1$, i.e., $(-,+)$, $(-,-)$, and $(+,-)$ according as $I$ falls from $I_{\text{max}}$ to $I_{\text{min}}$. As is seen, there are two bifurcation points, $I^Z$ and $I^G$, until good 1 becomes a Giffen good.

When $I = I^Z$, the demands for goods 1 and 2 are calculated by substituting (9) and (10) into (6) and (7) as follows:

$$\begin{aligned}
q_1^* &= \frac{\beta_3}{\beta_1} \frac{-\kappa'(\beta_{21}-\beta_{22})p_1}{-\kappa'(\beta_{21}-\beta_{22})p_1 + (1-\kappa)(\beta_{22}-\beta_{21}\kappa)} \\
&= \frac{\beta_3}{\beta_1} \frac{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})}{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma}) + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)}, \\
q_2^* &= \frac{\beta_3(1-\kappa)^2}{-\kappa'(\beta_{21}-\beta_{22})p_1 + (1-\kappa)(\beta_{22}-\beta_{21}\kappa)} \\
&= \frac{\alpha_1\alpha_2\beta_3\gamma\kappa^{-\gamma-1}(1-\kappa)^2}{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma}) + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)}.
\end{aligned}$$

Using the above result, the ratio $\theta$ of the expenditure on good 1 to income for $I = I^Z$ is obtained at once as follow:

$$\begin{aligned}\frac{p_1 q_1^*}{I^Z} &= \frac{-\kappa'(\beta_{21}-\beta_{22})p_1^2}{-\kappa'(\beta_{21}-\beta_{22})p_1^2+(1-\kappa)^2\beta_1 p_2} \\ &= \frac{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})p_1}{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})p_1+\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)^2 p_2}.\end{aligned}$$

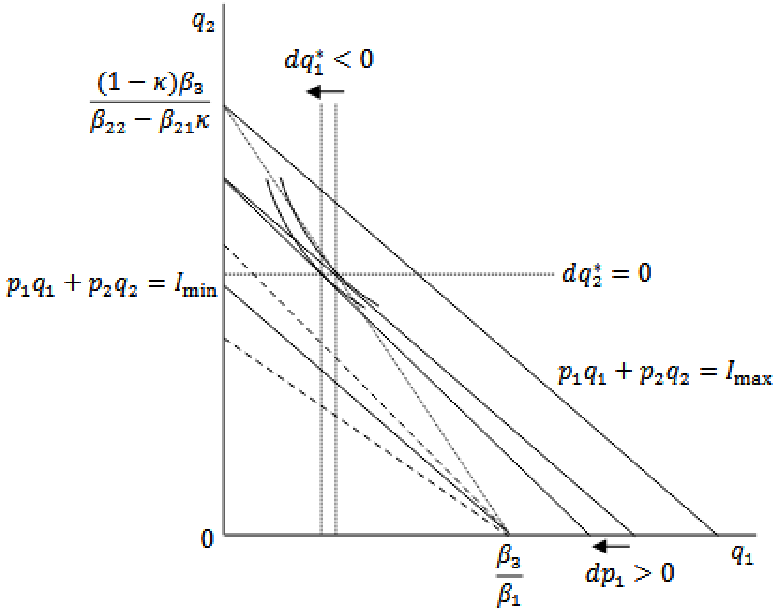


Figure 4. Zero Ratio Effect ($I = I^Z(< I_{\max})$)

Figure 4 describes responses of demands to a rise in the price of good 1 on the $q_1$-$q_2$ plane which is the same as Figure 2. Two budget lines for $I = I^Z$ are newly added. An initial optimal point is where an upper budget line touches an upper indifference curve. Remember that the optimal point is on the expansion path already explained (which is a downward sloping dotted straight line in Figure 4). The coordinates of the optimal point are represented by $(q_1^*, q_2^*)$ the values of which are shown above. A rise in the price of good 1 by $dp_1 > 0$ makes the budget line pivot clockwise on the $q_2$-intercept. Hence the lower straight line represents a new budget line. A new optimal point is found as the point where the lower budget line is tangent to a lower indifference curve. As is seen from Figure 4, the demand for good 1 decreases by $-dq_1^*(> 0)$ as a leftward arrow shows, while the demand for good 2 remains unchanged ($dq_2^* = 0$), which implies zero ratio effect. In this case the price effect is equal to the unit-elasticity effect, and it is calculated as

$$\begin{aligned}\frac{\partial q_1^*}{\partial p_1} &= -\frac{q_1^*}{p_1} \\ &= -\frac{\beta_3}{\beta_1}\frac{-\kappa'(\beta_{21}-\beta_{22})}{-\kappa'(\beta_{21}-\beta_{22})p_1+(1-\kappa)(\beta_{22}-\beta_{21}\kappa)} \\ &= -\frac{\beta_3}{\beta_1 p_1}\frac{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})}{(\alpha_1+\alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})+\alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)}.\end{aligned}$$

If $I$ crosses $I^Z$ from above to below in the neighborhood of $I^Z$, the ratio effect changes from negative to positive (or the sign of $\partial q_2^*/\partial p_1$ changes from *positive* to *negative*), but the sign of $\partial q_1^*/\partial p_1$ stays negative as Figure 3 shows.

When $I = I^G$, the coordinates of an optimal point are calculated by substituting (8) and (11) into (6) and (7) as follows:

$$
\begin{aligned}
q_1^* &= \frac{-\kappa'(\beta_{21}-\beta_{22})\beta_3 p_2}{-\kappa'(\beta_{21}-\beta_{22})\beta_1 p_2 + (\beta_{22}-\beta_{21}\kappa)^2} \\
&= \frac{\beta_3}{\beta_1}\frac{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2}{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2 + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)^2}, \\
q_2^* &= \frac{\beta_3(1-\kappa)(\beta_{22}-\beta_{21}\kappa)}{-\kappa'(\beta_{21}-\beta_{22})\beta_1 p_2 + (\beta_{22}-\beta_{21}\kappa)^2} \\
&= \frac{\alpha_1\alpha_2\beta_3\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)}{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2 + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)^2}.
\end{aligned}
$$

Using the above result, the expenditure-income ratio $\theta$ for $I = I^Z$ is obtained as follow:

$$
\begin{aligned}
\frac{p_1 q_1^*}{I^G} &= \frac{-\kappa'(\beta_{21}-\beta_{22})p_1}{-\kappa'(\beta_{21}-\beta_{22})p_1 + (1-\kappa)(\beta_{22}-\beta_{21}\kappa)} \\
&= \frac{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2 p_1}{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2 p_1 + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)p_2}.
\end{aligned}
$$

Compared with the case of $I^Z$, the demand $q_1^*$ for good 1 rises whereas the demand $q_2^*$ for good 2 falls along the expansion path. And also the ratio $\theta$ rises despite a decrease in $I$. When $I = I^G$, by definition the (positive) ratio effect exactly cancels the unit-elasticity effect out. In other words, it is not so large that good 1 becomes a Giffen good. But, once $I$ is below $I^G$, the ratio effect dominates the unit-elasticity effect, and the world of a Giffen good appears.

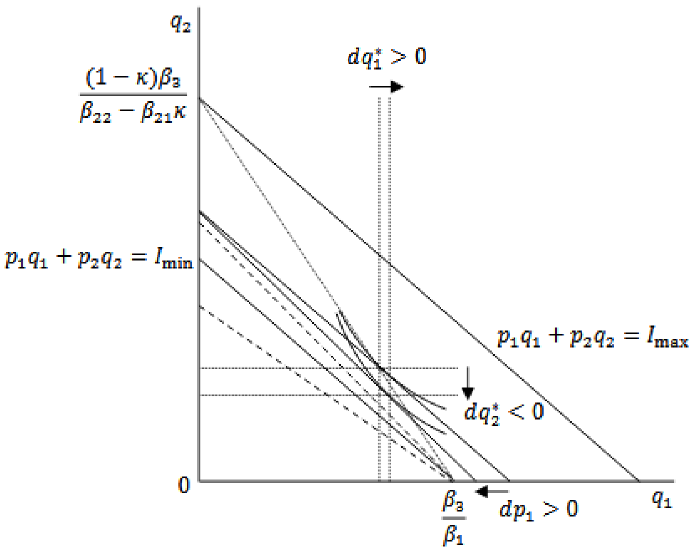


Figure 5. Very Large Ratio Effect and Giffen Good ($I_{\min} < I < I^G$)

Figure 5 describes the last pattern $(+, -)$ in Figure 3 on the $q_1$-$q_2$ plane. Two budget lines for $I_{\min} < I < I^G$ are added to Figure 2. As in Figure 4, demand changes due to a rise in the price of good 1 by $dp_1 > 0$ are observed between the contact point of an upper budget line and an upper indifference curve and the contact point of a lower budget line and a lower indifference curve.[11] Such a shift of an optimal point in this case leads to an increase in the demand for good 1 by $dq_1^* > 0$ and a decrease in the demand for good 2 by $-dq_2^*(> 0)$ as a rightward arrow and a downward arrow show, respectively. It implies that the ratio effect is so large that good 1 becomes a Giffen good. It is interesting to notice here that the unit-elasticity effect becomes larger in absolute value as $I$ falls. It may induce the price effect to become negative. In fact, however, the price effect remains positive and becomes even larger. Why? Because the ratio effect, and correctly speaking the transfer effect, gets larger than the absolute value of the unit-elasticity effect. It can be said, therefore, good 1 is a Giffen good for $I_{\min} < I < I^G$ at the sacrifice of good 2.

# 5 Concluding Remark

To be honest, I had been rather skeptical about the "existence" of a Giffen good for lack of an example which can be solved analytically for it before I encountered the paper by Kubler, Selden, and Wei (2013). It seemed that they submitted an example that produced Giffen behavior under clear conditions. It was based on the utility function belonging to the HARA (hyperbolic absolute risk aversion) family with decreasing absolute risk aversion (DARA) and decreasing relative risk aversion (DRRA). This paper reinterpreted their two-*asset* model as a two-*good* model as in usual microeconomics, and made a further analysis. The result is that economists will cast doubt on the existence of a Giffen good no longer for the same reason as mine, I believe. This paper focused on a Giffen good. As was pointed out, Assumption 3 is crucial for the existence of it. But, if the inequality sign is reversed as $\beta_{22} - \beta_{21}\kappa < 0$, good 1 becomes a normal good at once. In sum, the utility function proposed in this paper can be used for the analysis of either a normal good or an inferior good, of course, including a Giffen good.

# Appendix

# A Proof of Lemma 1

Here are the results of subtraction between two incomes in Lemma 1.

$$\begin{aligned} I_1^G - I_{\min} &= \frac{\beta_3}{\beta_1} \frac{(\beta_{22} - \beta_{21}\kappa)D}{-\kappa'(\beta_{21} - \beta_{22})\beta_1 p_2 + (\beta_{22} - \beta_{21}\kappa)^2} > 0, \\ I_1^Z - I_1^G &= \frac{\beta_3}{\beta_1} \frac{-\kappa'(\beta_{21} - \beta_{22})D^2}{-\kappa'(\beta_{21} - \beta_{22})p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)} \\ &\quad \times \frac{1}{-\kappa'(\beta_{21} - \beta_{22})\beta_1 p_2 + (\beta_{22} - \beta_{21}\kappa)^2} > 0, \end{aligned}$$

[11] Figures 4 and 5 are so drawn that the intercept of each budget line is above that of the $B = 0$ line ($\beta_3/\beta_{22}$). In fact it can be shown that if $(0 <)\gamma \leq 1$, the intercepts of both budget lines are below that of the $B = 0$ line. So a relatively large value of $\gamma$ is assumed in Figures 4 and 5.

$$
I_{\max} - I_1^Z = \frac{\beta_3}{\beta_1} \frac{-\kappa'(\beta_{21} - \beta_{22})p_1 D}{(\beta_{22} - \beta_{21}\kappa)[-\kappa'(\beta_{21} - \beta_{22})p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)]} > 0,
$$

because of Assumptions 1-3. Therefore, $I_{\min} < I_1^G < I_1^Z < I_{\max}$. Q.E.D.

# B Proof of Lemma 2

Here are the results of subtraction between two incomes in Lemma 2.

$$
\begin{aligned}
I_2^G - I_{\min} &= \frac{\beta_3}{\beta_1} \frac{\alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{22} - \beta_{21}\kappa)D}{(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2 + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{22} - \beta_{21}\kappa)^2} > 0, \\
I_2^Z - I_2^G &= \frac{\beta_3}{\beta_1} \frac{\alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{21} - \beta_{22})(\alpha_1 + \alpha_2\kappa^{-\gamma+1})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})D}{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}) + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22} - \beta_{21}\kappa)} \\
&\quad \times \frac{1}{(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2 + \alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(\beta_{22} - \beta_{21}\kappa)^2} > 0, \\
I_{\max} - I_2^Z &= \frac{\beta_3}{\beta_1} \frac{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})D}{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}) + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22} - \beta_{21}\kappa)} \\
&\quad \times \frac{1}{\beta_{22} - \beta_{21}\kappa} > 0,
\end{aligned}
$$

because of Assumptions 1-3. Therefore, $I_{\min} < I_2^G < I_2^Z < I_{\max}$. Q.E.D.

# C Proof of Lemma 3

Using (13), it can be shown that

$$
\frac{[(\beta_{21} - \beta_{22})\beta_1 p_2]^2}{(\beta_{21}p_1 - \beta_1 p_2)(\beta_1 p_2 - \beta_{22}p_1)} = \frac{(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2}{\alpha_1\alpha_2\kappa^{-\gamma}}. \tag{14}
$$

Substituting (12) and (14) into (9) yields

$$
\begin{aligned}
I_1^Z &= \frac{\beta_3}{\beta_1} \frac{-\kappa'(\beta_{21} - \beta_{22})p_1^2 + (1-\kappa)^2\beta_1 p_2}{-\kappa'(\beta_{21} - \beta_{22})p_1 + (1-\kappa)(\beta_{22} - \beta_{21}\kappa)} \\
&= \frac{\beta_3}{\beta_1} \left\{ \frac{\kappa}{\gamma} \frac{[(\beta_{21} - \beta_{22})\beta_1 p_2]^2}{(\beta_{21}p_1 - \beta_1 p_2)(\beta_1 p_2 - \beta_{22}p_1)} \frac{p_1}{\beta_1 p_2} p_1 + (1-\kappa)^2\beta_1 p_2 \right\} \\
&\quad \times \left\{ \frac{\kappa}{\gamma} \frac{[(\beta_{21} - \beta_{22})\beta_1 p_2]^2}{(\beta_{21}p_1 - \beta_1 p_2)(\beta_1 p_2 - \beta_{22}p_1)} \frac{p_1}{\beta_1 p_2} + (1-\kappa)(\beta_{22} - \beta_{21}\kappa) \right\}^{-1} \\
&= \frac{\beta_3}{\beta_1} \left\{ \frac{\kappa}{\gamma} \frac{(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2}{\alpha_1\alpha_2\kappa^{-\gamma}} \frac{\alpha_1 + \alpha_2\kappa^{-\gamma}}{\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}} p_1 + (1-\kappa)^2\beta_1 p_2 \right\} \\
&\quad \times \left\{ \frac{\kappa}{\gamma} \frac{(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})^2}{\alpha_1\alpha_2\kappa^{-\gamma}} \frac{\alpha_1 + \alpha_2\kappa^{-\gamma}}{\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}} + (1-\kappa)(\beta_{22} - \beta_{21}\kappa) \right\}^{-1} \\
&= \frac{\beta_3}{\beta_1} \frac{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma})p_1 + \alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)^2 p_2}{(\alpha_1 + \alpha_2\kappa^{-\gamma})(\alpha_1\beta_{21} + \alpha_2\beta_{22}\kappa^{-\gamma}) + \alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22} - \beta_{21}\kappa)} \\
&= I_2^Z,
\end{aligned}
$$

due to (10). Q.E.D.

## D Proof of Lemma 4

Substituting (12) and (14) obtained in the proof of Lemma 3 above into (8) yields

$$\begin{aligned}
I_1^G &= \frac{\beta_3 p_2[-\kappa'(\beta_{21}-\beta_{22})p_1+(1-\kappa)(\beta_{22}-\beta_{21}\kappa)]}{-\kappa'(\beta_{21}-\beta_{22})\beta_1 p_2+(\beta_{22}-\beta_{21}\kappa)^2} \\
&= \left\{\frac{\kappa}{\gamma}\frac{[(\beta_{21}-\beta_{22})\beta_1 p_2]^2}{(\beta_{21}p_1-\beta_1 p_2)(\beta_1 p_2-\beta_{22}p_1)}\frac{\beta_3 p_1}{\beta_1}+(1-\kappa)(\beta_{22}-\beta_{21}\kappa)\beta_3 p_2\right\} \\
&\quad\times\left\{\frac{\kappa}{\gamma}\frac{[(\beta_{21}-\beta_{22})\beta_1 p_2]^2}{(\beta_{21}p_1-\beta_1 p_2)(\beta_1 p_2-\beta_{22}p_1)}+(\beta_{22}-\beta_{21}\kappa)^2\right\}^{-1} \\
&= \left\{\frac{\kappa}{\gamma}\frac{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2}{\alpha_1\alpha_2\kappa^{-\gamma}}\frac{\beta_3 p_1}{\beta_1}+(1-\kappa)(\beta_{22}-\beta_{21}\kappa)\beta_3 p_2\right\} \\
&\quad\times\left\{\frac{\kappa}{\gamma}\frac{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2}{\alpha_1\alpha_2\kappa^{-\gamma}}+(\beta_{22}-\beta_{21}\kappa)^2\right\}^{-1} \\
&= \frac{\beta_3}{\beta_1}\frac{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2 p_1+\alpha_1\alpha_2\beta_1\gamma\kappa^{-\gamma-1}(1-\kappa)(\beta_{22}-\beta_{21}\kappa)p_2}{(\alpha_1\beta_{21}+\alpha_2\beta_{22}\kappa^{-\gamma})^2+\alpha_1\alpha_2\gamma\kappa^{-\gamma-1}(\beta_{22}-\beta_{21}\kappa)^2} \\
&= I_2^G,
\end{aligned}$$

due to (11). Q.E.D.

## References


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