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Abstract

This paper introduces the court's inability to discern different qualities of goods of the same kind into an overlapping generations model. This friction makes fiat money circulate not only as a means of payments for goods, but also as a means of liability repayments in an equilibrium. This result holds with or without dynamic inefficiency. In this equilibrium, a shortage of real money balances for liability repayments can cause underinvestment by borrowers, even if the money supply follows a Friedman rule. This problem can be resolved by an elastic money supply through an intraday discount window at a zero discount fee. In this case, supplying no fiat money overnight maximizes aggregate consumption in a monetary steady state in a dynamically efficient economy. This policy, however, can also lead to a self-fulfilling crunch of discount window lending if commercial banks intermediate the lending from the central bank with a collateral constraint, and if the discount window market is segregated. This equilibrium can be eliminated if the central bank is the monopolistic public issuer of fiat money that also acts as the lender of last resort.

Keywords: Nominal contract; Discount window; Credit crunch; Lender of last resort; Payment system.

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1 Introduction

Credit contracts are usually nominal contracts. Thus, money is not only a means of payments for goods, but also a means of liability repayments. This paper shows that the dual features of money can be replicated as an equilibrium phenomenon, if imperfect contract enforcement by the court is introduced into an overlapping generations model.

In the model, the court is assumed unable to distinguish among different qualities of goods of the same kind. Thus, the court cannot recognize it as a breach of contract if a borrower acquires an input from a lender by promising to deliver a high-quality product later, and then provides a low-quality product not expected by the lender, in order to save the cost of production. Ex ante, this inability of the court deters lenders from entering into real credit contracts with borrowers.

This problem can be circumvented if borrowers are liable to repay instruments that are recognizable by the court. Fiat money can be used as such an instrument if there is a competitive goods market in which borrowers can sell their products for fiat money. In this case, lenders can refuse to pay a sufficiently large amount of fiat money for borrowers to fulfill their liabilities, if borrowers sell low-quality products in the goods market. Borrowers have an incentive to fulfill their liabilities, because if they fail to repay fiat money to their lenders, then the court can dump all of their belongings in the competitive goods market to recover fiat money without determining which part of the belongings are high-quality products. Therefore, borrowers bring only high-quality products to the goods market.

In this case, however, underinvestment in borrowers' production can occur due to a shortage of real money balances for liability repayments, even if the money supply follows a Friedman rule. This problem can be resolved if lenders can borrow fiat money from the central bank against their holdings of borrowers' IOUs through an intraday discount window at a zero discount fee. This way, lenders can pay sufficiently large real money balances to borrowers in the goods market.

This result adds to the literature on elastic money supplies through discount windows and the payment system, such as Freeman (1996, 1999), Fujiki (2003, 2006), Martin (2004), Mills (2006), Gu et al. (2011), and Chapman and Martin (2013). Especially, this paper derives endogenous nominal credit contracts along with the need for discount window lending of fiat money, as Freeman (1996) does. The difference is that Freeman's model is based on spatial separation and limited communication across distant locations, while this paper features imperfect contract enforcement by the court. Thus, this paper demonstrates that the implication of Freeman's model is robust even in the absence of limited communication. Furthermore, this result does not require dynamic inefficiency, or a shortage of stores of value other than money. These features of the model are consistent with the advancement of information technology, which has been reducing the cost of remote communication, and financial innovation, which has been increasing the supply of liquid assets, in reality.

In the literature, Green (1999) raises a question regarding who should provide discount window lending, arguing that ordinary private entities can offer such lending on behalf of the central bank in Freeman's model.[1] Reflecting the fact that the central bank supplies fiat money to the public through commercial banks, this paper assumes that the central bank must incur a higher transaction cost to provide discount window lending than commercial banks. Furthermore, this paper incorporates limited commitment by commercial banks, whereby they do not repay fiat money borrowed from the central bank unless the central bank takes collateral from them. The central bank must prevent any default by commercial banks to observe a Friedman rule, as fiat money not repaid by commercial banks would cause an unintended increase in the overnight money supply.

Throughout this paper, commercial banks are assumed to have ample collateral; thus, there exists a monetary steady state in which a sufficiently large amount of discount window lending of fiat money prevents underinvestment in borrowers' production. In this steady

[1] Mills (2004) also provides an enforcement mechanism to make private IOUs circulate without a need for discount window lending in Freeman's model.

state, supplying no fiat money overnight maximizes aggregate consumption if the economy is dynamically efficient. However, this policy also leads to another equilibrium in which a self-fulfilling crunch of discount window lending occurs, if each commercial bank can provide discount window lending to only a subset of lenders holding borrowers' IOUs.

The collateral constraint plays a key role in this result. The amount of fiat money repaid on each borrower's IOU depends on the amount of fiat money each borrower receives in the goods market, which is, in turn, determined by the total amount of fiat money supplied by commercial banks through discount window lending. Thus, if a commercial bank expects that the other commercial banks do not discount borrowers' IOUs, then it, in turn, expects that borrowers do not obtain a sufficiently large amount of fiat money to repay their IOUs in the goods market. A commercial bank with this expectation does not borrow fiat money from the central bank to discount borrowers' IOUs, as doing so would result in default on its discount window lending, and hence the loss of collateral submitted to the central bank.

The self-fulfilling crunch of discount window lending can be prevented if the central bank can be committed to discounting borrowers' IOUs despite its inefficiency in doing so. The central bank does not suffer a coordination failure as commercial banks do, because it is the monopolistic issuer of fiat money. It does not extract a monopolistic rent either, because it is not a profit-seeking organization. Thus, the central bank has a role in preventing the fragility of discount window lending as the monopolistic public issuer of fiat money that also acts as the lender of last resort.

This result adds to the literature on elastic money supplies and runs on demand deposits, such as Champ, Smith, and Williamson (1996), Williamson (1998), Antinolfi, Huybens, and Keister (2001), Smith (2002), Antinolfi and Keister (2006), and Martin (2006). Also, Allen, Carletti, and Gale (2014) find that the lender-of-last-resort function of the central bank can prevent runs on demand deposits, if the demand deposits are nominal. This paper contributes to the literature by showing that both an elastic money supply from the central bank and the lender-of-last-resort function of the central bank are essential even in the absence of demand deposits.

More generally, there is an extensive literature on liquidity shortages in the financial market, such as Holmstrom and Tirole (1998), Allen and Gale (2004), Brunnermeier and Pedersen (2007), Gertler and Kiyotaki (2008), and Malherbe (2016). While these papers feature real economies, this paper analyzes the fragility of bank lending of fiat money in a monetary economy.

Finally, there is a literature on the optimality of nominal contracts for risk sharing, such as Jovanovic and Ueda (1997), Freeman and Tabellini (1998), and Doepke and Schneider (2017). In contrast to this literature, this paper focuses on the use of nominal contracts as a remedy for imperfect contract enforcement by the court. Fiat money in the model can be considered legal tender, as its primary function is characterized as the means of liability repayments recognizable by the court. This paper shows that this characteristic of fiat money leads to its use as a means of payments in the goods market as well.

The remainder of this paper is organized as follows. A baseline model without money is described in section 2. Fiat money and discount window lending are introduced in sections 3 and 4, respectively. The fragility of discount window lending by commercial banks is analyzed in section 5. Section 6 concludes the study.

2 Baseline model without money

Time is discrete, and has an infinite horizon. Agents are born in each period, and live for two periods. Call the set of agents born in period t "cohort t" for $t = 0, 1, 2, ...$ Also, call agents in their first period "young", and those in the second period "old".

In each cohort, agents are split into two types: lenders and borrowers, each of which are on a $[0, 1]$ continuum. For each type, the measure of agents is defined by the Lebesgue measure over $[0, 1]$. Each young lender is born with a unit of perishable goods, and can produce an amount ρ (> 0) of goods when old by investing a unit of goods when young. In contrast, each young borrower is born with no goods, but can produce an amount $f(x)$ of goods when old by investing an amount x of goods when young. The function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is strictly increasing and concave, and satisfies the Inada condition (i.e., $f(0) = 0$ and $\lim_{x \downarrow 0} f'(x) = \infty$). Assume that the marginal return on investment in a borrower's production diminishes sufficiently fast that it is never efficient to invest all the young lenders' goods endowments in borrowers' production:

$$f'(1) < \rho \tag{1}$$

Alternatively, each borrower can produce an amount $g(x)$ of low-quality goods when old by using an amount x of goods when young. Assume that the function $g : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies

$$g(x) \begin{cases} = 0 & \text{if } x = 0 \\ > f(x) & \text{if } x > 0 \end{cases} \tag{2}$$

For simplicity, assume low-quality goods does not provide any utility to any agent, and hereafter call them "wastes." Both borrowers and lenders can distinguish goods and wastes. Each agent maximizes the expected value of consumption of goods when old.

There exists a competitive credit market in which young lenders can exchange their goods for young borrowers' IOUs that promise to deliver goods in the next period. An equilibrium is characterized by the following conditions: each agent's utility maximization, given the competitive real interest rate in the credit market; and the value of the real interest rate that makes the amount of goods borrowed by each young borrower equal the amount of goods lent by each young lender in each period, given that borrowers and lenders in each cohort have the same measure. Throughout the paper, agents have perfect foresight, given no aggregate shock in the model.

2.1 Autarky due to imperfect contract enforcement

If the court can distinguish goods and wastes, then borrowers and lenders in each cohort can arrange real credit contracts, in which borrowers are liable to repay goods to lenders when old. This is because if an old borrower refuses to repay goods, then the court can seize the goods produced by the defaulting borrower and enforce the repayment of goods to lenders.[2]

Hereafter, introduce incomplete enforcement of real credit contracts by the court:

Assumption 1. The court cannot distinguish goods and wastes.

The underlying assumption is that goods and wastes are similar enough that the court cannot tell the difference between the two. This assumption can be interpreted as reflecting the difficulty in defining different qualities of goods in legal terms if the goods have the same physical features, because, in contrast to quantity, the evaluation of quality is inevitably subjective. Assumption 1 is a stylized assumption to incorporate this kind of difficulty for the court.

Given Assumption 1, a borrower repays only wastes to a lender in a real credit contract, because it is cheaper to produce wastes than goods, as implied by (2). This way, the borrower can spare more goods for investment in its production of goods for its own consumption. In this case, the lender cannot claim the borrower's default in court, because the borrower can successfully claim that the wastes delivered by the borrower fall into the same category of substance that the borrower is obliged to deliver in the credit contract. Expecting this consequence, no lender participates in a real credit market when young; thus agents live in autarky if Assumption 1 holds in the baseline model.

3 Introducing fiat money and nominal credit contracts

3.1 Environment

Let us introduce fiat money into the baseline model. Assume that there are a unit continuum of the initial old in period 0, each of which maximizes the consumption of goods in the period.

[2]Given (1), it is straightforward to derive the equilibrium in this case, in which the marginal gross rate of return on a borrower's production, $f'(x_{b,t})$, equals that on a lender's production, ρ, as well as the gross real interest rate in the credit market, $1 + r_t$, for $t = 0, 1, 2, ...$, where $x_{b,t}$ denotes the amount of goods invested in each borrower's production in period t. Note that the strict monotonicity of the function f and (1) ensure that there exists a unique value of $x_{b,t}$ such that $f'(x_{b,t}) = \rho$.

The initial old are endowed with an amount M_0 (≥ 0) of fiat money for each in period 0. The government imposes a lump-sum transfer of fiat money to, or from, each old lender from period 1 onward so that

$$M_t = \gamma M_{t-1} \tag{3}$$

for $t = 1, 2, 3, ...$, where M_t is the average amount of fiat money supplied per lender at the beginning of period t, and $\gamma > 0$. The court can identify fiat money.

Without loss of generality, consider a competitive market for goods produced by old borrowers and that for goods sold by young lenders separately in each period. Call the former a "morning goods market" and the latter an "afternoon goods market". Given the presence of these goods markets, agents can write a credit contract such that a lender transfers goods to a borrower when young in exchange for the borrower's IOU that promises to repay a specified amount of fiat money when old. It is feasible for old borrowers to acquire fiat money to repay, as they can sell their output for fiat money in the morning goods market. The buyers in this market are old lenders, which can obtain fiat money in the afternoon goods market when young by selling their goods endowments to old lenders in the previous cohort.

Old lenders can enforce old borrowers to repay fiat money, because if an old borrower fails to repay fiat money as promised, then old lenders can ask the court to seize all the belongings of the defaulting borrower, including both goods and wastes, and sell them in the afternoon goods market for fiat money in the same period.[3] Because agents can distinguish goods and wastes, the court can dump the defaulting borrower's belongings in the competitive goods

[3]There is a timing issue such that borrowers can repay fiat money to lenders only after they sell goods in the morning goods market; thus, lenders can call default only after the end of the market. As shown below, the nominal price level in the morning goods market is never higher than that in the afternoon goods market in a monetary equilibrium. Assume that a borrower and a lender can write a nominal credit contract such that the repayable amount of fiat money increases if a borrower repays fiat money after the beginning of the afternoon goods market, so that the amount of goods that a borrower must sell for fiat money stays constant regardless of whether the goods are sold in the morning or the afternoon goods market. Such an increase in the repayable amount of fiat money can be regarded as intraday nominal interest.

Table 1: Sequence of events in each period in the presence of fiat money

Birth of a new cohort	Young borrowers and lenders are born. Lenders are born with a unit of goods for each.
Returns on investments	Old borrowers and lenders obtain goods from their investments made in the previous period.
Lump-sum money transfer	The government imposes a lump-sum transfer of fiat money from, or to, old lenders.
Morning goods market	Old lenders pay fiat money for goods sold by old borrowers.
Repayments of borrowers' IOUs	Old borrowers repay fiat money to old lenders to fulfill liabilities in nominal credit contracts written in the previous period.
Afternoon goods market	Old lenders pay fiat money for goods sold by young lenders.
Credit market	Young borrowers obtain goods from young lenders in exchange for their IOUs that promise to repay fiat money in the next period.
Investments	Young borrowers and lenders invest goods in their production.
Consumption and exit	Old borrowers and lenders consume goods, and exit the economy.

market blindly to recover the repayable amount of fiat money.[4] The flows of goods and fiat money are summarized in Table 1.

Because the circulation of fiat money is not imposed as an exogenous assumption in the model, it is necessary to define the court's action when there is not a sufficiently large amount of fiat money in the morning goods market for old borrowers to repay their liabilities. In such a case, assume that the court simply transfers the ownership of defaulting borrowers' belongings to old lenders on a pro rata basis.

[4] A borrower and a creditor can write an enforceable real credit contract such that the borrower is obliged to give up all belongings to the lender when old. Assume that there is an infinitesimally small utility cost for a borrower to engage in the production of goods, so that a borrower does not have incentive to enter into such a contract.

3.2 Each agent's utility maximization problem

The utility maximization problem for a young lender in cohort t is specified as follows:

$$
\begin{aligned}
\max_{\{x_{\ell,t}, b_{\ell,t}, m_{\ell,t}, m'_{\ell,t+1}\}} \quad & c_{\ell,t+1} \\
\text{s.t.} \quad & x_{\ell,t} + q_t b_{\ell,t} + p_{A,t} m_{\ell,t} = 1 \\
& c_{\ell,t+1} = \rho x_{\ell,t} + p_{M,t+1} m'_{\ell,t+1} + p_{A,t+1}(b_{\ell,t} + m_{\ell,t} + \tau_{\ell,t+1} - m'_{\ell,t+1}) \\
& m'_{\ell,t+1} \in [0,\ m_{\ell,t} + \tau_{\ell,t+1}] \\
& b_{\ell,t} + m_{\ell,t} + \tau_{\ell,t+1} - m'_{\ell,t+1} \geq 0 \\
& x_{\ell,t}, m_{\ell,t} \geq 0
\end{aligned}
\tag{4}
$$

where $c_{\ell,t+1}$ is the amount of goods consumed by an old lender in period $t+1$; $x_{\ell,t}$ is the amount of goods invested in a young lender's production in period t; $b_{\ell,t}$ is the nominal face value of borrowers' IOUs held by a young lender at the end of period t; q_t is the competitive real discount price of borrowers' IOUs in the credit market in period t; $m_{\ell,t}$ is the amount of fiat money held by a young lender at the end of period t; $m'_{\ell,t+1}$ is the amount of fiat money spent by an old lender in the morning goods market in period $t+1$; $\tau_{\ell,t+1}$ is a lump-sum transfer of fiat money from, or to, an old lender at the beginning of period $t+1$, which is a tax if it is negative and a subsidy if it is positive; and $p_{M,t}$ and $p_{A,t}$ are the real values of fiat money in terms of goods in the morning and the afternoon goods market, respectively, in period t. Each lender takes as given the values of q_t, $p_{A,t}$, $p_{A,t+1}$, and $p_{M,t+1}$.

The first and the second constraint in (4) are flow-of-funds constraints for a young and an old lender, respectively. The third constraint is a feasibility constraint on $m'_{\ell,t+1}$, which implies that an old lender can spend money in the morning goods market up to the lender's money holding at the beginning of the period. The fourth constraint is a non-negativity constraint on the amount of money spent by an old lender in the afternoon goods market. The last constraint implies that the values of $x_{\ell,t}$ and $m_{\ell,t}$ must be non-negative by definition.

The utility maximization problem for a young borrower in cohort t is specified as follows:

$$\begin{aligned} \max_{\{x_{b,t},b_{b,t}\}} \quad & c_{b,t+1} \\ \text{s.t.} \quad & x_{b,t} = q_t b_{b,t} \\ & c_{b,t+1} = f(x_{b,t}) - p_{M,t+1} b_{b,t} \\ & x_{b,t} \geq 0 \end{aligned} \tag{5}$$

where $c_{b,t+1}$ is the amount of goods consumed by an old borrower in period $t+1$; $x_{b,t}$ is the amount of goods invested in a young borrower's production in period t; and $b_{b,t}$ is the nominal face value of an IOU issued by a young borrower in period t. The first and the second constraint are the flow-of-funds constraints for a young and an old borrower, respectively. Note that $p_{M,t+1}b_{b,t}$ in the second constraint is the amount of goods that an old borrower must sell to repay an amount $b_{b,t}$ of fiat money to old lenders. The last constraint implies that $x_{b,t}$ is non-negative by definition.

3.3 Market clearing conditions

Given each type of agent in each cohort having a unit measure, the market clearing conditions are specified as

$$b_{b,t} = m'_{\ell,t+1} \tag{6}$$

$$m_{\ell,t} = \begin{cases} M_0 & \text{if } t = 0 \\ b_{\ell,t-1} + m_{\ell,t-1} + \tau_{\ell,t} - m'_{\ell,t} & \text{if } t = 1,2,3,... \end{cases} \tag{7}$$

$$b_{\ell,t} = b_{b,t} \tag{8}$$

for $t = 0, 1, 2, ...$, where (6) implies that the amount of fiat money repaid by old borrowers equals that paid by old lenders in the morning goods market; (7) implies that young lenders receive the amount of fiat money paid by old lenders in the afternoon goods market; and (8) implies that supply and demand are equal in the credit market.

An equilibrium is defined as follows:

Definition 1. For $t = 0, 1, 2, ...$, an equilibrium is characterized by the solutions to (4) and (5), given q_t, $p_{A,t}$, $p_{A,t+1}$, and $p_{M,t+1}$, and the values of q_t, $p_{A,t}$, and $p_{M,t+1}$ that satisfy (6)-(8).

Substituting (6) and (8) into (7) yields

$$m_{\ell,t} = m_{\ell,t-1} + \tau_{\ell,t} \tag{9}$$

for $t = 1, 2, 3, ...$ Set the following assumption:

Assumption 2. $\tau_{\ell,t} = (\gamma - 1)M_{t-1}$ for $t = 1, 2, 3, ...$

This assumption ensures (3) and

$$m_{\ell,t} = M_t \tag{10}$$

for $t = 1, 2, 3, ...$, given $m_{\ell,0} = M_0$ in an equilibrium.

3.4 Monetary steady state

Given Assumption 2, let us focus on a monetary steady state:

Definition 2. Given the definition of an equilibrium, a monetary steady state is an equilibrium such that $x_{\ell,t}$ and $x_{b,t}$ are constant across time periods, and

$$\frac{p_{A,t+1}}{p_{A,t}} = \frac{p_{M,t+1}}{p_{M,t}} = \frac{q_{t+1}}{q_t} = \frac{1}{\gamma} \tag{11}$$

in each period.

Let us define social welfare in a monetary steady state by aggregate consumption by each cohort.[5] This definition of social welfare implies that it is the first-best if $f'(x_{b,t}) = \max\{\rho, 1\}$. Note that if $\rho < 1$, then the economy is dynamically inefficient; thus, fixed-supplied fiat money is the best store of value for lenders, allowing a Pareto-improving inter-generational

[5]This assumption is equivalent to giving an equal weight to the utility of each cohort in the definition of social welfare, as the relative utility weight for the initial old compared with subsequent cohorts is zero.

transfer of goods in each period. If $\rho \geq 1$, then $f'(x_{b,t}) = \rho$ is necessary for a Pareto-efficient resource allocation within each cohort, given the assumption that each young lender is endowed with a sufficiently large amount of goods to achieve this equality, as implied by (1). Accordingly, it is referred to as underinvestment if $f'(x_{b,t}) > \max\{\rho, 1\}$—that is, the marginal gross rate of return on investment in a borrower's production, $f'(x_{b,t})$, exceeds that on a lender's production, ρ, or on fixed-supplied fiat money, 1, whichever is greater.

3.5 Underinvestment due to the opportunity cost of an overnight money holding in a dynamically efficient economy

The gross rate of return on an IOU incurred by a borrower is $p_{M,t+1}/q_t$, because if a young borrower issues a nominal IOU whose face value equals $b_{b,t}$, then the borrower receives an amount $q_t b_{b,t}$ of goods when young and must sell an amount $p_{M,t+1} b_{b,t}$ of goods in the morning goods market when old. Thus, the first-order condition for $b_{b,t}$ in (5) implies that

$$f'(x_{b,t}) = \frac{p_{M,t+1}}{q_t} \tag{12}$$

in a monetary steady state.

On the other hand, the gross rate of return on borrowers' IOUs for a lender is $p_{A,t+1}/q_t$, because an old lender can spend fiat money repaid by borrowers only in the afternoon goods market. Because a young lender must hold both borrowers' IOUs and fiat money until becoming old, or "overnight", in a monetary steady state, the following equation must hold:

$$\frac{p_{A,t+1}}{q_t} = \frac{p_{M,t+1}}{p_{A,t}} \geq \rho \tag{13}$$

where $p_{M,t+1}/p_{A,t}$ is the gross rate of return on an overnight money holding, as a lender can acquire fiat money in the afternoon goods market when young, and spend it in the morning goods market when old. Note that ρ is the gross rate of return on investment in a lender's production. Thus, the weak inequality in (13) holds in equality if lenders invest goods in their own production, and no lenders invest goods in their production if the strict inequality holds in (13).

In addition, the real value of fiat money in the morning goods market is never smaller than that in the afternoon goods market in a monetary steady state:

$$p_{M,t} \geq p_{A,t} \tag{14}$$

This is because if $p_{M,t} < p_{A,t}$, then no money holder would spend fiat money in the morning goods market, which would prevent old borrowers from acquiring fiat money to repay to old lenders.

Now, suppose $\rho > 1$ and $1/\gamma < \rho$. In this case, the economy is dynamically efficient, but the central bank does not retire the overnight money supply fast enough to implement a Friedman rule. Thus, the rate of return on an overnight money holding is smaller than that on a lender's production. Given (11) and $1/\gamma < \rho$, $p_{M,t+1}/p_{A,t+1} > 1$ because $p_{M,t+1}/p_{A,t} = (p_{M,t+1}/p_{A,t+1})(p_{A,t+1}/p_{A,t}) = (p_{M,t+1}/p_{A,t+1})(1/\gamma) \geq \rho$, as implied by (13). Given $p_{M,t+1}/p_{A,t+1} > 1$, (12) and (13) in turn imply $f'(x_{b,t}) = p_{M,t+1}/q_t = (p_{M,t+1}/p_{A,t+1})(p_{A,t+1}/q_t) > \rho$, i.e., underinvestment in borrowers' production. There occurs a wedge between the marginal gross rate of return on a borrower's production, $f'(x_{b,t})$, and that on a lender's production, ρ, in this case, because borrowers must compensate lenders for the opportunity cost of holding fiat money overnight, which is necessary for borrowers to fulfill their liabilities in credit contracts.

3.6 Underinvestment due to a shortage of real money balances for liability repayments

If $\rho > 1$ and $1/\gamma = \rho$—that is, the central bank implements a Friedman rule—then there is no opportunity cost of an overnight money holding. This is the same if $\rho \leq 1$ and $\gamma = 1$, as an inter-generational transfer of goods from young agents to old agents in exchange for fixed-supplied fiat money is weakly Pareto-improving in this case. In both cases, the first-best investment in borrowers' production, i.e., $x_{b,t} = f'^{-1}(\max\{\rho, 1\})$, can be realized in a monetary steady state if $f'^{-1}(\max\{\rho, 1\})$ is not too large.

To confirm this result, note that a young lender cannot spend more than a unit of goods on acquiring borrowers' IOUs and fiat money due to a flow-of-funds constraint:

$$x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} \geq 0 \tag{15}$$

as implied by the first and the last constraint in (4). Moreover, the market clearing conditions, (6)-(8), and Assumption 2 indicate that the amount of fiat money that each old borrower can repay to lenders is capped by the amount of fiat money held by each old lender in a monetary steady state:

$$b_{\ell,t} = b_{b,t} = m'_{\ell,t+1} \leq m_{\ell,t} + \tau_{\ell,t+1} = \gamma m_{\ell,t} \tag{16}$$

where the weak inequality is the upper bound on $m'_{\ell,t+1}$ as implied by the third constraint in (4).

Given $\rho > 1$ and $1/\gamma = \rho$, or $\rho \leq 1$ and $\gamma = 1$, (12) and (13) imply that $p_{M,t+1}/p_{A,t+1} = 1$ if $f'(x_{b,t}) = \max\{\rho, 1\}$.[6] In this case, the real value of fiat money in the morning goods market, $p_{M,t+1}$, does not have to be so high to compensate lenders for the opportunity cost of overnight money holdings, which is zero. Given $p_{M,t+1}/p_{A,t+1} = 1$, (13) in turn implies

$$q_t = p_{A,t} \tag{17}$$

Thus, given $f'(x_{b,t}) = \max\{\rho, 1\}$, (16) and (17) can be combined into

$$f'^{-1}(\max\{\rho, 1\}) = x_{b,t} = q_t b_{b,t} = q_t b_{\ell,t} \leq \gamma p_{A,t} m_{\ell,t} \tag{18}$$

where the second equality holds because the real discount value of a borrower's IOU, $q_t b_{b,t}$, equals the amount of goods invested by a borrower, $x_{b,t}$, as implied by the first constraint in (5). The weak inequality in (18) is compatible with the flow-of-funds constraint for a young

[6] Note that $f'(x_{b,t}) = p_{M,t+1}/q_t = (p_{M,t+1}/p_{A,t+1})(p_{A,t+1}/q_t) \geq (p_{M,t+1}/p_{A,t+1})\rho$, as implied by (12) and (13). Thus, if $\rho > 1$ and $f'(x_{b,t}) = \rho$, then $p_{M,t+1}/p_{A,t+1} \leq 1$. Hence, $p_{M,t+1}/p_{A,t+1} = 1$ in this case, given (14). Also, if $\gamma = 1 \geq \rho$ and $f'(x_{b,t}) = 1$, then $(p_{M,t+1}/p_{A,t+1})(p_{A,t+1}/q_t) = (p_{M,t+1}/p_{A,t+1})(p_{M,t+1}/p_{A,t}) = (p_{M,t+1}/p_{A,t+1})^2 = 1$, as implied by (11) and (13).

lender, (15), if and only if $f'^{-1}(\max\{\rho, 1\})$ is sufficiently small. In this case, the first-best investment in borrowers' production is feasible in a monetary steady state.

If $f'^{-1}(\max\{\rho, 1\})$ is too large to satisfy (15) and (18) simultaneously, then underinvestment in borrowers' production is inevitable in a monetary steady state. In this case, each borrower cannot issue a sufficiently large real value of an IOU to finance the first-best investment, because the amount of fiat money that each old borrower can promise to repay to lenders is capped by the overnight supply of fiat money in a monetary steady state, as implied by (10) and (16).

The results described above can be formalized by the following proposition, given a single-crossing condition on a borrower's production function, f, that ensures the uniqueness of a monetary steady state:[7]

Assumption 3. The function f satisfies the following properties:

$$\lim_{x\to 0} \frac{\gamma(x^{-1}-1)^2}{f'(x)} > 1 \tag{19}$$

$$\frac{d}{dx}\left\{\gamma\left(\frac{1}{x}-1\right)^2\right\} < f''(x) \quad \text{for all } x \in \left(0, \frac{\gamma}{1+\gamma}\right) \tag{20}$$

Proposition 1. Suppose $M_0 > 0$ and Assumptions 1-3 hold. Given Definitions 1 and 2, there exists a monetary steady state such that

$$x_{b,t} = x_b^* \equiv f'^{-1}\left(\frac{1}{\gamma}\right) \tag{21}$$

$$q_t = p_{A,t} = p_{M,t} = \begin{cases} \frac{1-x_b^*}{M_t} & \text{if } \frac{1}{\gamma} > \rho \\ \text{any real number in } \left[\frac{x_b^*}{\gamma M_t}, \frac{1-x_b^*}{M_t}\right] & \text{if } \frac{1}{\gamma} = \rho \end{cases} \tag{22}$$

$$x_{\ell,t} = \begin{cases} 0 & \text{if } \frac{1}{\gamma} > \rho \\ 1 - x_b^* - p_{A,t}M_t & \text{if } \frac{1}{\gamma} = \rho \end{cases} \tag{23}$$

[7] Assumption 3 is a single-crossing condition between $\gamma(x^{-1}-1)^2$ and $f'(x)$ for $x \in \left(0, \frac{\gamma}{1+\gamma}\right)$. For example, if $f(x) = Ax^{\sigma}$ where $\sigma \in (0,1)$, then (19) is satisfied for all $A > 0$, and (20) is satisfied if $A \in (0, 2^{1+\sigma}/[\sigma(1-\sigma)])$.

if $\rho\gamma \leq 1$ and $x_b^* \leq \gamma(1+\gamma)^{-1}$;

$$x_{b,t} = x_b^{**} \text{ such that } f'(x_b^{**}) = \gamma\left(\frac{1}{x_b^{**}} - 1\right)^2 \tag{24}$$

$$(q_t,\ p_{A,t},\ p_{M,t}) = \left(\frac{x_b^{**}}{\gamma M_t},\ \gamma\left(\frac{1}{x_b^{**}} - 1\right)q_t,\ \gamma^2\left(\frac{1}{x_b^{**}} - 1\right)^2 q_t\right) \tag{25}$$

$$x_{\ell,t} = 0 \tag{26}$$

where $x_b^{**} \in (0, \gamma(1+\gamma)^{-1})$, if $\rho\gamma \leq 1$ and $x_b^* > \gamma(1+\gamma)^{-1}$;

$$x_{b,t} = x_b^{***} \equiv f'^{-1}\left(\rho^2\gamma\right) \tag{27}$$

$$(q_t,\ p_{A,t},\ p_{M,t}) = \left(\frac{x_b^{***}}{\gamma M_t},\ \rho\gamma q_t,\ \rho^2\gamma^2 q_t\right) \tag{28}$$

$$x_{\ell,t} = 1 - x_b^{***} - p_{A,t}M_t \tag{29}$$

if $\rho\gamma > 1$ and $x_b^{***} \leq (1+\rho)^{-1}$; and (24)-(26) hold where $x_b^{**} \in (0, (1+\rho)^{-1})$, if $\rho\gamma > 1$ and $x_b^{***} > (1+\rho)^{-1}$. In all four cases,

$$b_{\ell,t} = b_{b,t} = m'_{\ell,t+1} = \frac{x_{b,t}}{q_t} \tag{30}$$

There is no other monetary steady state.

Proof. See Appendix A. □

4 Introducing an elastic money supply through an intraday discount window

4.1 Environment

A shortage of real money balances for liability repayments described above can be resolved if fiat money is supplied through discount window lending within each period. To confirm this result, assume that at the beginning of each period, each old lender can borrow fiat money from the central bank at a zero discount fee up to the face value of borrowers' IOUs the lender holds. The maturity of the borrowing comes at the end of the same period. Note that old lenders' liabilities to the central bank are nominal debt; thus, the court can enforce the repayments of fiat money from old lenders to the central bank by seizing the repayments of borrowers' IOUs to old lenders.

With this assumption, the upper bound in the feasibility constraint on $m'_{\ell,t+1}$, $m'_{\ell,t+1} \in [0,\ m_{\ell,t} + \tau_{\ell,t+1}]$, disappears in the utility maximization problem for a young lender, (4), because the amount of fiat money that an old lender can spend in the morning goods market is no longer limited to the lender's money holding at the beginning of the period, $m_{\ell,t} + \tau_{\ell,t+1}$. The other part of (4) remains the same. Note that the non-negativity constraint on the amount of fiat money spent by an old lender in the afternoon goods market in (4), $b_{\ell,t} + m_{\ell,t} + \tau_{\ell,t+1} - m'_{\ell,t+1} \geq 0$, ensures that the amount of fiat money that an old lender borrows from the central bank through discount window lending, $m'_{\ell,t+1} - m_{\ell,t} - \tau_{\ell,t+1}$, does not exceed the face value of borrowers' IOUs held by the lender, $b_{\ell,t}$, as assumed above.

The utility maximization problem for a young borrower, (5), and the market clearing conditions, (6)-(8), remain the same. The central bank accommodates old lenders' demand for discount window lending passively at a zero discount fee. Overall, the definition of an equilibrium is revised as follows:

Definition 3. An equilibrium with discount window lending is characterized as in Definition 1, except that $m'_{\ell,t+1} \in [0,\ m_{\ell,t} + \tau_{\ell,t+1}]$ in the utility maximization problem for a young lender, (4), is replaced by $m'_{\ell,t+1} \geq 0$.

4.2 No underinvestment with an elastic money supply through an intraday discount window

The following proposition summarizes the properties of a monetary steady state with discount window lending from the central bank:

Proposition 2. Suppose Assumptions 1-2 hold. Given Definitions 2 and 3, (30) holds in any monetary steady state. If $M_0 > 0$ and $\rho\gamma \leq 1$, then there exists a monetary steady state such that

$$f'(x_{b,t}) = \frac{1}{\gamma} \tag{31}$$

$$q_t = p_{A,t} = p_{M,t} = \frac{1 - x_{b,t} - x_{\ell,t}}{M_t} \tag{32}$$

$$x_{\ell,t} = \begin{cases} 0 & \text{if } \frac{1}{\gamma} > \rho \\ \text{any real number in } [0,\ 1 - x_{b,t}) & \text{if } \frac{1}{\gamma} = \rho \end{cases} \tag{33}$$

Alternatively, if $M_0 = 0$, then there exists a monetary steady state such that

$$f'(x_{b,t}) = \frac{p_{M,t+1}}{q_t} = \rho \tag{34}$$

$$x_{\ell,t} = 1 - x_{b,t} > 0 \tag{35}$$

where $p_{A,t}$ can be any real number in $[q_t,\ p_{M,t}]$ satisfying $p_{A,t+1}/p_{A,t} \leq \rho$, given (11); and the values of q_t, $b_{b,t}$, $b_{\ell,t}$, and $m'_{\ell,t+1}$ can be any set of positive real numbers satisfying (30). There is no other monetary steady state.

Proof. See Appendix B. □

Proposition 2 demonstrates that if the central bank adopts a Friedman rule, i.e., $1/\gamma = \max\{\rho,\ 1\}$, and offers discount window lending to old lenders, then it can induce the first-best investment in borrowers' production, i.e., $x_{b,t} = f'^{-1}(\max\{\rho,\ 1\})$, in a monetary steady state. In this case, the amount of fiat money that old lenders can pay for borrowers' output in the morning goods market can be different from the overnight supply of fiat money, as the central bank can supply an additional amount of fiat money through discount window lending within each period. Therefore, borrowers can set a sufficiently high value of $b_{b,t}$ to finance the first-best investment in their production, given the real value of fiat money in the morning goods market.[8]

[8]Regarding Proposition 2, note that (3) implies that $M_t = 0$ for $t = 1, 2, 3, ...$ if $M_0 = 0$. Thus, $p_{A,t+1}/p_{A,t} \leq \rho$ must be satisfied in a monetary steady state if $M_0 = 0$, because otherwise young lenders would demand overnight money holdings.

4.3 Optimality of the use of a notional unit of account and legal tender

Proposition 2 implies that if the gross rate of return on investment in a lender's production, ρ, exceeds one, then supplying no fiat money overnight, i.e., $M_0 = 0$, achieves the first-best investment in borrowers' production. This policy also maximizes aggregate consumption by each cohort in a monetary steady state. Even though a positive amount of the overnight money supply with a Friedman rule, $1/\gamma = \rho$, can also achieve the first-best investment in borrowers' production as implied by (31), it requires an inter-generational transfer of goods from young lenders to old lenders in exchange for fiat money. Because lenders in each cohort give and receive the same amount of goods over the course of their lives in this case, each cohort of agents can increase their aggregate consumption if they invest goods in lenders' production in the same cohort rather than transferring the goods to lenders in the previous cohort in exchange for fiat money.[9]

No overnight money supply implies that fiat money does not serve as a store of value. In this case, fiat money exists only as a notional unit of account when young borrowers and lenders write nominal credit contracts in each period, as fiat money to be repaid by borrowers is not supplied yet. This feature of fiat money can be regarded as that of legal tender, which is defined as "a debtor cannot successfully be sued for non-payment if he pays into court in legal tender."[10] The result of the model implies that adopting a notional unit of account, such as an inconvertible currency unit, with an ex-post elastic supply of legal tender increases aggregate output, as it obviates a need for storing some real asset, such as gold, as a means of payments intertemporally.

[9] Proposition 2 implies that if the central bank supplies fiat money overnight with a Friedman rule, i.e., $\rho = 1/\gamma$, then the real value of fiat money acquired by young lenders in each period, $p_{A,t}m_{\ell,t}$, can be arbitrarily small, as it is indeterminate in a monetary steady state. However, it cannot be set to zero, given $m_{\ell,t} = M_t > 0$.

[10] This definition is provided by the U.K. Royal Mint. See `https://www.royalmint.com/help/trm-faqs/legal-tender-amounts/` (accessed on April 6, 2019).

4.4 Price indeterminacy in a dynamically efficient economy with an elastic money supply

Proposition 2 confirms that if the economy is dynamically inefficient (i.e., $\rho < 1$), then the first-best investment in borrowers' production can be achieved by a fixed amount of the overnight money supply (i.e., $M_0 > 0$ and $\gamma = 1$). In this case, the nominal price level, i.e., the inverse of $p_{M,t}$ or $p_{A,t}$, is proportional to the fixed overnight supply of fiat money, M_0. Thus, the overnight money supply functions as the nominal anchor, as is the case in the quantity theory of money.

Proposition 2 also implies that if there is no overnight money supply in a dynamically efficient economy (i.e., $M_0 = 0$ and $\rho \geq 1$), then the elastic money supply through discount window lending causes an indeterminacy of the nominal price level, while it can still achieve the first-best investment in borrowers' production. This result is similar to price indeterminacy in Smith's model (2002).[11] The nominal price level can be fixed if the government and the central bank can be committed to intervening into goods markets if the nominal price level deviates from the targeted value.[12]

[11] In Smith's (2002) model, discount window lending is necessary due to the possibility of an aggregate liquidity shock that can cause a run on demand deposits, given the need for money due to spatial separation and limited communication. This feature of the model is based on Champ, Smith, and Williamson's (1996) model.

[12] The price indeterminacy cannot be prevented even if the central bank fixes the amount of fiat money supplied through discount window lending. To see this result, denote by B_t the fixed average supply of fiat money per lender through discount window lending in period t, given a zero discount fee. Given $M_0 = 0$, the market clearing condition for discount window lending is $m'_{\ell,t} = B_t$, where $m'_{\ell,t}$ is the amount of fiat money paid by an old lender in the morning goods market in period t. Given (30), the face value of a borrower's IOU, $b_{b,t}$, equals B_{t+1} in a monetary steady state. This condition pins down the real discount price of borrowers' IOUs in the current period, q_t, given $q_t = x_{b,t}/B_{t+1}$ as implied by (30). Given $b_{b,t-1}$, however, the real value of fiat money in the morning goods market in the current period, $p_{M,t}$, can deviate from the expectation in the previous period, $E_{t-1}p_{M,t} = \rho q_{t-1}$, which is implied by (34). This is because once investment in a borrower's production in the previous period, $x_{b,t-1}$, is made, $p_{M,t}$ only needs to satisfy $p_{A,t} \in [q_t,\ p_{M,t}]$ in the current period, as implied by Proposition 2. The government and the central bank can eliminate this ex-post price indeterminacy by making joint commitments such that the government collects goods endowments from young lenders by a lump-sum tax and sells the collected goods to young lenders for fiat money in the goods markets if $p_{M,t}$ or $p_{A,t}$ is less than ρq_{t-1}, while the central bank issues new fiat money to buy goods in the goods markets if $p_{M,t}$ or $p_{A,t}$ exceeds ρq_{t-1}. Alternatively, the central bank can supply a positive amount of fiat money overnight, i.e., $M_t > 0$, by retiring money at such a rate that the gross rate of return on an overnight money holding, $1/\gamma$, is arbitrarily close to, but greater than, ρ, while maintaining a zero discount fee in discount window lending. In this case, (32) and (33) hold; thus,

5 Fragility of competitive discount window lending by commercial banks

In reality, the central bank does not extend commercial loans directly. To incorporate this feature of the central bank, assume that a unit continuum of agents called commercial banks enter the economy in each period. For simplicity, assume that each commercial bank maximizes the expected value of its consumption of goods in the period of its entry, and exits the economy at the end of the period. Commercial banks can distinguish goods and wastes. Then introduce the following assumption:

Assumption 4. The central bank incurs a real transaction cost to lend fiat money to borrowers and lenders in each period, whereas it incurs no transaction cost to lend fiat money to commercial banks. Commercial banks incur no transaction cost to lend fiat money to lenders, or to borrow fiat money from the central bank. The government can finance the real transaction cost incurred by the central bank by imposing a lump-sum tax on young lenders to collect part of their goods endowments.

This assumption makes commercial banks more efficient suppliers of fiat money to lenders than the central bank. It reflects the inefficiency of the central bank in commercial transactions relative to commercial banks in reality. The assumption of no transaction cost for interbank transactions reflects the fact that the central bank transacts with financial institutions on a regular basis.

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the nominal price level is determinate. Also, (31) implies that the value of $x_{b,t}$ is arbitrarily close to the first-best. This result is similar to Antinolfi and Keister's (2006) finding in Smith's (2002) model. Contrary to their finding, this policy does not make the overnight money holding, $p_{A,t}M_t$, arbitrarily close to zero, because a lender's production technology is linear in this paper's model. As a result, this policy replaces investment in lenders' production completely with a constant inter-generational transfer of goods in each period, i.e., $x_{\ell,t} = 0$ and $p_{A,t}M_t = 1 - x_{b,t}$, despite the social gross rate of return on the former, ρ, being greater than that on the latter, 1, for each cohort.

5.1 No monetary equilibrium with private money issued by competitive commercial banks

If commercial banks can issue their own private money, they buy an arbitrarily large amount of goods with their own money unless the competitive real value of money, which they take as given, becomes zero in the morning and the afternoon goods market, i.e., $p_{M,t} = p_{A,t} = 0$. Thus, there is no monetary equilibrium in this case. Even if the government can impose a finite limit on the amount of money that commercial banks can issue to buy goods in each period, the overnight money supply increases at a constant pace. This case corresponds to the case in which $\gamma > 1$ in Proposition 1; thus underinvestment in borrowers' production occurs, i.e., $f'(x_{b,t}) > \max\{\rho, 1\}$, regardless of the value of ρ, as shown in the proposition.

5.2 Introducing competitive discount window lending of central-bank money by commercial banks

To prevent an unintended increase in the overnight money supply, the central bank must require commercial banks to lend and retrieve fiat money within the same period. This can be done by allowing only the central bank to issue fiat money, and requiring commercial banks to repay any fiat money borrowed from the central bank within the same period. Assume that the central bank maintains a zero discount fee for discount window lending to commercial banks. Also assume that there is a competitive discount window market between commercial banks and old lenders. Given this assumption, the sequence of events in each period is as described in Table 2.

If there is no friction, the supply of fiat money from the central bank is simply channeled through commercial banks to old lenders without changing any result in Proposition 2. In this case, competition among a unit continuum of commercial banks leads to a zero profit condition for each commercial bank in an equilibrium, which ensures a zero discount fee for discount window lending to old lenders.

Table 2: Sequence of events in each period with competitive discount window lending by commercial banks

Birth of a new cohort	Young borrowers and lenders are born. Lenders are born with a unit of goods for each.
Returns on investments	Old borrowers and lenders obtain goods from their investments made in the previous period.
Lump-sum money transfer	The government imposes a lump-sum transfer of fiat money from, or to, old lenders.
Discount window market	The central bank lends fiat money to commercial banks at a zero discount fee, and then each old lender can borrow fiat money from commercial banks up to the face value of borrowers' IOUs that the lender holds in a discount window market.
Morning goods market	Old lenders pay fiat money for goods sold by old borrowers.
Repayments of borrowers' IOUs	Old borrowers repay fiat money to old lenders to fulfill liabilities in nominal credit contracts written in the previous period.
Repayments of discount window lending	Old lenders repay discount window lending of fiat money to commercial banks, and then commercial banks pass on the fiat money to the central bank as the repayments of the central bank's discount window lending to them.
Afternoon goods market	Old lenders can pay fiat money for goods sold by young lenders.
Credit market	Young borrowers obtain goods from young lenders in exchange for their IOUs that promise to repay fiat money in the next period.
Investments	Young borrowers and lenders invest goods in their production.
Consumption and exit	Old borrowers and lenders consume goods and exit the economy.

5.3 Collateral constraint on commercial banks

Now introduce a friction due to limited commitment by commercial banks:

Assumption 5. If a commercial bank borrows fiat money from the central bank, then it can spend borrowed fiat money to buy goods for its own consumption in the morning goods market anonymously, and default on the loan from the central bank at no cost.

On this assumption, note that commercial banks are never better off by spending fiat money in the afternoon goods market, because the real value of money is weakly higher in the morning goods market, as implied by (14).

Assumption 5 implies that, to avoid an increase in the overnight money supply due to commercial banks' default, the central bank must lend fiat money to commercial banks in exchange for some assets worth not less than the real value of the fiat money lent. Assume that commercial banks own such collateral:

Assumption 6. For $j \in [0, 1]$ and $t = 0, 1, 2, ...$, commercial bank j in period t is endowed with an amount $\bar{k}$ of assets that yield an amount $\theta_{j,t}$ of goods per unit at the end of the period, where j is the index of each commercial bank born in each period, and $\theta_{j,t}$ is an i.i.d. random variable following a $[\underline{\theta}, 1]$ uniform distribution with $\underline{\theta} \in (0, 1)$. The value of $\theta_{j,t}$ is revealed publicly during the time lag between discount window lending of fiat money from the central bank to commercial banks, and that from commercial banks to old lenders before the morning goods market opens in each period.

These assets can be interpreted as government and private securities that the central bank can accept in practice. The random asset return for each commercial bank, $\theta_{j,t}$, aims to capture fluctuations in the market values of these securities. For simplicity, $\theta_{j,t}$ is assumed to be i.i.d. across commercial banks to abstract from an aggregate shock. Also, it is assumed that there is a time lag between transactions between the central bank and commercial banks, and those between commercial banks and old lenders. Assumption 6 implies that commercial banks can decide whether to spend borrowed fiat money for their own consumption or lend the fiat money to old lenders after the revelation of $\theta_{i,t}$ for each. This assumption can be interpreted such that commercial banks make their decisions on the latest possible information on the value of their collateral.

Hereafter, $k_{bk,j,t}$ denotes the amount of assets that commercial bank j in period t submits to the central bank as collateral, whereas $b_{bk,j,t}$ denotes the amount of fiat money that the commercial bank borrows from the central bank. Assumptions 5 and 6 imply the following incentive-compatibility constraint for each commercial bank to repay fiat money borrowed from the central bank:

$$\theta_{j,t} k_{bk,j,t} \geq p_{M,t} b_{bk,j,t} \tag{36}$$

given the realization of $\theta_{j,t}$. The left-hand side of (36) is the realized value of collateral submitted to the central bank, and the right-hand side is the real value of borrowed fiat money in the morning goods market. Note that this constraint is sufficient to incentivize each commercial bank to repay borrowed fiat money to the central bank when they receive the repayments of fiat money from old lenders after the morning goods market, because the real value of borrowed fiat money in the afternoon goods market is less than that in the morning goods market, i.e., the right-hand side of (36).

To give all commercial banks an incentive to repay borrowed fiat money to the central bank, (36) must be satisfied for any possible realization of $\theta_{j,t}$, because the central bank must lend fiat money to commercial banks before the realization of $\theta_{j,t}$ as assumed in Assumption 6. Thus, the central bank must set the following collateral constraint on each commercial bank:

$$\underline{\theta} k_{bk,j,t} \geq p_{M,t} b_{bk,j,t} \tag{37}$$

given $\theta_{j,t} \geq \underline{\theta}$ for all $j \in [0, 1]$, as assumed in Assumption 6. This constraint implies that the expected value of collateral submitted to the central bank is larger than the real value of fiat money lent by the central bank. This feature of (37) is consistent with the existence of a haircut in a collateralized discount window or a repo offered by the central bank to commercial banks in practice. Throughout the paper, assume no collateral shortage:

Assumption 7. The value of $\bar{k}$ is arbitrarily large.

5.4 Self-fulfilling crunch of competitive discount window lending in a dynamically efficient economy

Now suppose $\rho > 1$. In this case, supplying no fiat money overnight, i.e., $M_0 = 0$, is necessary to maximize aggregate consumption by each cohort in a monetary steady state, as described in section 4.3. Given the collateral constraint, (37), this policy can lead to another equilibrium in which discount window lending disappears due to a self-fulfilling expectation, if the discount window market is segregated.

This result can be confirmed by introducing the possibility of default on borrowers' IOUs and discount window lending into the model. See Appendix C for the formal description of the result. Here, let us consider specific behavior of each commercial bank to illustrate the result. This behavior can be derived from a formal profit maximization problem for each commercial bank.

Introduce limited participation into the discount window market such that each commercial bank can contact only a finite number of old lenders, while each old lender can contract more than one commercial banks. This assumption can be interpreted as due to each commercial bank's special knowledge about its clients. Commercial banks contacting the same old lender have Bertrand competition to offer a discount fee for the lender. This way, the discount window market is segregated, but still competitive if all commercial banks offer a discount fee to old lenders.

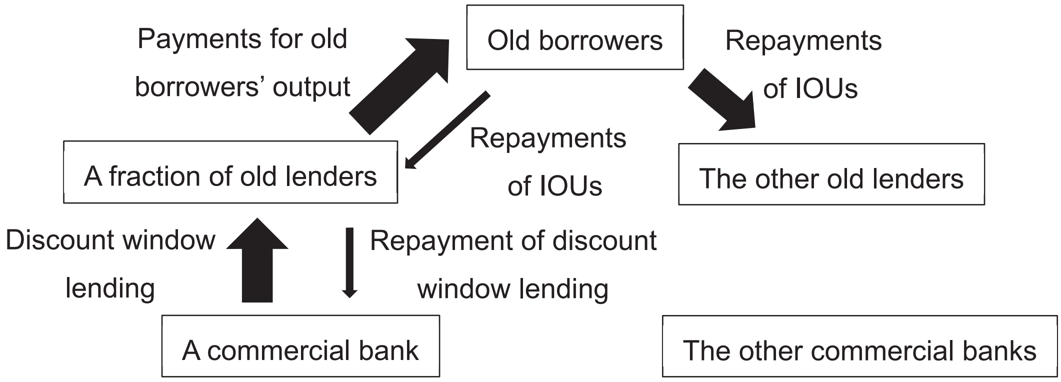
Given the collateral constraint, (37), suppose that a commercial bank lends fiat money to an old lender only up to the amount that the old lender can repay. This is because a commercial bank must over-collateralize the borrowing of fiat money from the central bank, as implied by (37). Therefore, if an old lender fails to repay fiat money lent by a commercial bank, then the commercial bank has to lose collateral worth more than the real value of the fiat money lent. Hence, there is no trade surplus in discount window lending between a commercial bank and an old lender, if the older lender is expected to default.

In contrast to the segregated discount window market, the goods and the credit market remain Warlasian. They are also integrated such that each young lender holds every borrower's IOU in the credit market, and that each old borrower sells its output to every old lender in the morning goods market. This assumption can be interpreted such that a young borrower needs the variety of inputs supplied by each young lender, while the consumption of an old lender is a composite of all varieties of output produced by old borrowers.

Given this environment and no collateral shortage as assumed in Assumption 7, there exists a monetary steady state in which competition among commercial banks leads to a zero discount fee for each old lender. In this case, the utility maximization problem for each young lender remains the same as that described in the previous section. Thus, Proposition 2 holds in this monetary steady state, including the first-best investment in borrowers' production.

Alternatively, suppose that all the commercial banks except one do not lend fiat money to old lenders in the discount window market in a period. If the remaining commercial bank still lends fiat money to a finite number of old lenders, then these old lenders' payments of fiat money for old borrowers' output in the morning goods market are dispersed among a continuum of old borrowers, and then repaid to a continuum of old lenders, given the integrated goods and credit market as assumed above. Because the measure of a finite number of old lenders is zero, the repayment of borrowers' IOUs to these old lender is smaller than the amount of fiat money these old lender pay in the morning goods market. Therefore, these old lenders must default if they receive discount window lending of fiat money. Thus, each commercial bank follows suit if it expects that the other banks do not extend discount window lending to old lenders. See Figure 1 for the illustration of flows of

Figure 1: Flows of fiat money if only one commercial bank extends discount window lending to old lenders in the absence of an overnight money supply



Note: The thicker the arrow, the larger the quantity of fiat money represented.

fiat money in this case.

Note that these results hold despite Assumption 7. Thus, a credit crunch can occur even if commercial banks are arbitrarily well-capitalized. This feature of the model is consistent with the sudden nature of a banking crisis and a credit crunch in reality. Also, a self-fulfilling crunch of discount window lending by commercial banks can persist for any number of periods. In this case, young borrowers do not enter into credit contracts with young lenders ex ante, given the expected lack of fiat money for liability repayments in the future. Thus, the economy is reduced to autarky in this case.

5.5 Need for a monopolistic public issuer of fiat money that also acts as the lender of last resort

The central bank can prevent a self-fulfilling crunch of discount window lending by pledging to lend fiat money directly to old lenders if a self-fulfilling crunch occurs. In this case, each commercial bank expects old borrowers to receive a sufficiently large amount of fiat money to repay their IOUs, even if the other commercial banks do not lend to old lenders in the discount window market. This consideration eliminates the self-fulfilling expectation of disappearance of discount window lending by commercial banks.

A question remains as to whether the central bank can be credibly committed to lending to old lenders when necessary, given the presence of a transaction cost to do so, as assumed in Assumption 4. In this regard, the central bank does not have to incur any transaction cost on an equilibrium path, once it eliminates the self-fulfilling crunch of discount window lending by its commitment. This effect of the central bank's commitment is similar to that of a deposit insurance in Diamond and Dybvig's (1983) model. Furthermore, the central bank is the monopolistic issuer of fiat money; thus it can internalize the aggregate amount of fiat money lent to old lenders without any coordination failure. It does not extract a monopolistic rent if it is a public organization, which is usually the case in reality. Thus, the central bank has a role in preserving the stability of discount window lending as the monopolistic public issuer of fiat money that also acts as the lender of last resort.

5.6 No self-fulfilling crunch of discount window lending in the presence of an overnight money supply

Alternatively, it can be shown that the self-fulfilling crunch of discount window lending can be prevented if the overnight money supply is positive. In this case, even if all the commercial banks refuse to lend fiat money to old lenders in the discount window market, old lenders can still pay their overnight money holdings to old borrowers in the morning goods market. Given this consideration, each commercial bank can lend a positive amount of fiat money to old lenders without any concern about default, even if the other commercial banks do not lend. Because each commercial bank's lending of fiat money to old lenders increases the amount of fiat money that old borrowers receive in the morning goods market, each commercial bank expects the repayment of each borrower's IOU to be of a higher value, and thus lends more. Ultimately, commercial banks can supply a sufficiently large amount of fiat money for old borrowers to repay their IOUs fully in any monetary equilibrium. See Appendix C for more details.

If $\rho > 1$, however, it is necessary to set the overnight money supply to zero to maximize aggregate consumption by each cohort in a monetary steady state, as described in section 4.3. Thus, a positive overnight money supply is inferior to no overnight money supply, if the central bank can be committed to be the lender of last resort.

5.7 Interpretation of the economy with no overnight money supply

Fiat money in the model is issued by the central bank; thus, it corresponds to base money. The presence of bank deposits can be introduced into the model without any contradiction to no overnight supply of fiat money. For example, suppose that young lenders swap borrowers' IOUs for bank deposits at commercial banks in each period. They can pay deposit balances, or send bank transfers, for borrowers' output when old, whereas borrowers can repay their lOUs by received deposit balances. To prevent deposit balances from being private money, assume that the central bank requires commercial banks to settle bank transfers by fiat money it issues, i.e., bank reserves, as is the case in reality. Thus, if a borrower repays its IOU by a deposit balance received from an old lender, then the old lender's commercial bank must submit collateral to the central bank to borrow bank reserves, and then send the bank reserves to the borrowers bank, which passes on the bank reserves to the commercial banks holding the borrower's IOU. The old lender's commercial bank can repay bank reserves to the central bank within the same period if it receives bank reserves from other banks for the repayments of borrowers' IOUs it holds.

In this case, bank reserves are perfect substitutes to fiat money directly paid by old lenders. Thus, the monetary steady state with the first-best investment in borrowers' production can exist as described in Proposition 2. Furthermore, to avoid losing collateral submitted to the central bank, each commercial bank follows suit if it expects that the other commercial banks do not issue bank deposits in exchange for borrowers' IOUs when there is no overnight supply of bank reserves.[13] Hence, the possibility of a self-fulfilling crunch of

[13] There is no run on bank deposits by depositors, given no collateral shortage at each commercial bank

discount window lending also remains.

In reality, no overnight supply of bank reserves has been observed in a country adopting the so-called channel, or corridor, system. In this system, the central bank sets a narrow band between the lending rate for commercial banks in need of bank reserves and the deposit rate for commercial banks in excess of bank reserves, so that the interbank overnight interest rate comes in the middle of the two rates without any change in the supply of bank reserves. While a central bank adopting this system usually supplies a small amount of bank reserves to commercial banks overnight, the Bank of Canada run this system targeting a zero overnight supply of bank reserves for March 2006 to May 2007, and succeeded in hitting the target in several months. Thus, the economy with no overnight supply of fiat money in the model can be interpreted as an economy adopting such a policy while the advancement of electronic retail payments eliminates the use, and hence the overnight holdings, of physical central-bank notes.

6 Conclusions

Using an overlapping generations model, this paper shows that if the court cannot discern different qualities of goods of the same kind, fiat money circulates not only as a means of payment for goods, but also as a means of liability repayments in an equilibrium. In such an equilibrium, underinvestment by borrowers can occur due to a shortage of real money balances for liability repayments, even if the money supply follows a Friedman rule. This problem can be resolved if the central bank provides an elastic money supply through a discount window at a zero discount fee. This result replicates the implication of Freeman's (1996) model without spatial separation and limited communication across distant locations. This paper also replicates Smith's (2002) finding of price indeterminacy due to an elastic money supply.

as assumed in Assumption 7.

This paper also shows that if the central bank must provide discount window lending indirectly through commercial banks with a collateral constraint, and if the discount window market is segregated, then there can be a self-fulfilling crunch of discount window lending by commercial banks in a dynamically efficient economy. This equilibrium can be eliminated if the central bank is the monopolistic public issuer of fiat money that also acts as the lender of last resort.

One of the remaining issues is to incorporate more functions of commercial banks into the model. Another issue is to introduce more frictions into the bankruptcy process in the model. Currently, the model assumes that borrowers repay all the fiat money received in the morning goods market passively to the holders of their IOUs, regardless of whether they go bankrupt or not. If borrowers stopped repaying fiat money in case of bankruptcy, then a self-fulfilling crunch of discount window lending may be possible even in the presence of an overnight money supply. This paper abstracts from frictions in the bankruptcy process to clarify that discount window lending can be fragile even without such a friction. Further investigation into these issues is left for future research.

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Appendix

A Proof of Proposition 1

First of all, (11) implies $q_t > 0$. Given $q_t > 0$, the two market clearing conditions, (6) and (8), and the first constraint in (5), i.e., $x_{b,t} = q_t b_{b,t}$, imply (30) immediately. Given $M_0 > 0$ and (2), $m_{\ell,t} = M_t = \gamma M_{t-1}$ for $t = 1, 2, 3, \ldots$ Thus, (11) holds in a monetary steady state, as otherwise $p_{A,t} m_{\ell,t} = 1 - x_{b,t} - x_{\ell,t}$, which is implied by (30) and the first constraint in (4), would not be constant.

Given $q_t > 0$ and the Inada condition satisfied by the function f, the first-order condition for $b_{b,t}$ in (5) implies

$$f'(q_t b_{b,t}) = \frac{p_{M,t+1}}{q_t} \tag{A.1}$$

in an equilibrium, which in turn implies $b_{b,t} > 0$ as $p_{M,t+1}$ is finite in an equilibrium.

Given $q_t > 0$ and $b_{b,t} = b_{\ell,t} > 0$ as implied by (8), the first-order conditions for $x_{\ell,t}$, $b_{\ell,t}$, $m_{\ell,t}$, and $m'_{\ell,t+1}$ in (4) are respectively

$$-\eta_{1,t} + \rho + \underline{\lambda}_{x,\ell,t} = 0 \tag{A.2}$$

$$-\eta_{1,t} q_t + p_{A,t+1} + \eta_{4,t} = 0 \tag{A.3}$$

$$-\eta_{1,t} p_{A,t} + p_{A,t+1} + \bar{\lambda}_{m',\ell,t+1} + \eta_{4,t} + \underline{\lambda}_{m,\ell,t} = 0 \tag{A.4}$$

$$p_{M,t+1} - p_{A,t+1} + \underline{\lambda}_{m',\ell,t+1} - \bar{\lambda}_{m',\ell,t+1} + \eta_{4,t} = 0 \tag{A.5}$$

where $\eta_{1,t}$, $\bar{\lambda}_{m',\ell,t+1}$, $\underline{\lambda}_{m',\ell,t+1}$, $\eta_{4,t}$, $\underline{\lambda}_{x,\ell,t}$, and $\underline{\lambda}_{m,\ell,t}$ are Lagrange multipliers for $x_{\ell,t} + q_t b_{\ell,t} + p_{A,t} m_{\ell,t} = 1$, $m'_{\ell,t+1} \geq 0$, $m'_{\ell,t+1} \leq m_{\ell,t} + \tau_{\ell,t+1}$, $b_{\ell,t} + m_{\ell,t} - m'_{\ell,t+1} + \tau_{\ell,t+1} \geq 0$, $x_{\ell,t} \geq 0$, and $m_{\ell,t} \geq 0$.

Because Assumption (2) implies $m_{\ell,t} = M_t = \gamma M_{t-1}$, $m_{\ell,t} > 0$, and thus $\underline{\lambda}_{m,\ell,t} = 0$, given $M_0 > 0$. Also, (7) implies $m_{\ell,t} = b_{\ell,t} + m_{\ell,t} - m'_{\ell,t+1} + \tau_{\ell,t+1} > 0$, and (6) implies that $m'_{\ell,t+1} > 0$, given $b_{b,t} > 0$. Hence, $\eta_{4,t} = 0$ and $\underline{\lambda}_{m',\ell,t+1} = 0$. Substituting these results into

(A.2)-(A.5) yields

$$\underline{\lambda}_{x,\ell,t} = \eta_{1,t} - \rho \geq 0 \tag{A.6}$$

$$\eta_{1,t} = \frac{p_{A,t+1}}{q_t} = \frac{p_{M,t+1}}{p_{A,t}} \tag{A.7}$$

$$\bar{\lambda}_{m',\ell,t+1} = p_{M,t+1} - p_{A,t+1} \geq 0 \tag{A.8}$$

Now split the parameter space into two regions: $\rho \leq \gamma^{-1}$ and $\rho > \gamma^{-1}$. Suppose $\rho \leq \gamma^{-1}$ and $p_{M,t+1} = p_{A,t+1}$. In this case,

$$\eta_{1,t} = \frac{p_{M,t+1}}{p_{A,t}} = \frac{p_{A,t+1}}{p_{A,t}} = \frac{1}{\gamma} \tag{A.9}$$

given (11). Thus,

$$\frac{p_{A,t}}{q_t} = \frac{p_{A,t+1}}{q_t} \cdot \frac{p_{A,t}}{p_{A,t+1}} = \eta_{1,t}\gamma = 1, \tag{A.10}$$

$$f'(q_t b_{b,t}) = \frac{p_{M,t+1}}{q_t} = \frac{p_{A,t+1}}{q_t} = \frac{1}{\gamma} \tag{A.11}$$

as implied by (11) and (A.1). Because $f'' < 0$, the inverse function of f', f'^{-1}, exists. Therefore, the steady state value of $q_t b_{b,t}$ is unique. Given $\underline{\lambda}_{x,\ell,t} \geq 0$ and $\bar{\lambda}_{m',\ell,t+1} = 0$, $x_{\ell,t} \geq 0$ and $b_{b,t} = m'_{\ell,t+1} \leq m_{\ell,t} + \tau_{\ell,t+1}$. These constraints are satisfied if and only if

$$q_t b_{b,t} = q_t m'_{\ell,t+1} \leq q_t(m_{\ell,t} + \tau_{\ell,t+1}) = \gamma q_t m_{\ell,t} \tag{A.12}$$

$$x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} = 1 - q_t b_{b,t} - q_t m_{\ell,t} \geq 0 \tag{A.13}$$

given Assumption 2 and (A.10). Denote $f'^{-1}(\gamma^{-1})$ by x_b^*. These conditions are equivalent to

$$q_t m_{\ell,t} \in \left[\frac{x_b^*}{\gamma}, 1 - x_b^*\right] \tag{A.14}$$

This range is non-empty if and only if $x_b^* \leq \gamma(1+\gamma)^{-1}$. If $\rho = \gamma^{-1}$, then $\underline{\lambda}_{x,\ell,t} \geq 0 = \eta_{1,t} - \rho = 0$. Thus, $x_{\ell,t}$ is indeterminate, and any value of $q_t m_{\ell,t}$ in this range can be a steady state value. If $\rho < \gamma^{-1}$, then $\eta_{1,t} > \rho$. Hence, $x_{\ell,t} = 0$, which implies $q_t m_{\ell,t} = q_t M_t = 1 - x_b^*$. The results described in this paragraph are sufficient for (21)-(23).

If $\rho \le \gamma^{-1}$ and $p_{M,t+1} > p_{A,t+1}$, then $\bar{\lambda}_{m',\ell,t+1} > 0$. Thus, $b_{b,t} = m'_{\ell,t+1} = m_{\ell,t} + \tau_{\ell,t+1} = \gamma m_{\ell,t}$, given Assumption 2. Also,

$$\eta_{1,t} = \frac{p_{M,t+1}}{p_{A,t}} = \frac{p_{M,t+1}}{p_{A,t+1}}\frac{p_{A,t+1}}{p_{A,t}} > \frac{1}{\gamma} \ge \rho \tag{A.15}$$

Hence, $\underline{\lambda}_{x,\ell,t} > 0$ and $x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} = 0$. Therefore,

$$\eta_{1,t} = \frac{p_{A,t+1}}{q_t} = \frac{p_{A,t}}{q_t}\frac{p_{A,t+1}}{p_{A,t}} = \frac{1}{x_{b,t}} - 1 \tag{A.16}$$

given $b_{b,t} = \gamma m_{\ell,t}$. As a result, (A.1) implies that

$$f'(x_{b,t}) = \frac{p_{M,t+1}}{q_t} = \frac{p_{A,t+1}}{q_t} \cdot \frac{p_{A,t}}{p_{A,t+1}} \cdot \frac{p_{M,t+1}}{p_{A,t}} = \left(\frac{p_{A,t+1}}{q_t}\right)^2 \frac{p_{A,t}}{p_{A,t+1}} = \gamma \left(\frac{1}{x_{b,t}} - 1\right)^2 \tag{A.17}$$

given (11), (A.7), and (A.16). As implied by (A.15) and (A.16), the solution for this equation must satisfy $\frac{1}{x_{b,t}} - 1 > \frac{1}{\gamma}$, or $x_{b,t} < \gamma(1+\gamma)^{-1}$. Given Assumption 3, $f'(x_{b,t})$ and $\gamma(x_{b,t}^{-1} - 1)^2$ can have only one intersection for $x_{b,t} \in (0, \gamma(1+\gamma)^{-1})$ at most, and they do have one if and only if $f'(\gamma(1+\gamma)^{-1}) > \gamma[\gamma^{-1}(1+\gamma) - 1]^2 = \gamma^{-1}$, or $x_b^* \equiv f'^{-1}(\gamma^{-1}) > \gamma(1+\gamma)^{-1}$. Given $m_{\ell,t} = M_t$ and (11), the results described in this paragraph are sufficient for (24)-(26).

Suppose $\rho > \gamma^{-1}$. Because $\eta_{1,t} = p_{M,t+1} p_{A,t+1}^{-1} \gamma^{-1}$ as implied by (A.7), (A.6) requires $p_{M,t+1} \ge \rho\gamma p_{A,t+1}$. Thus, $\bar{\lambda}_{m',\ell,t+1} > 0$ and $m'_{\ell,t+1} = m_{\ell,t} + \tau_{\ell,t+1}$, given $\rho > \gamma^{-1}$ and (A.8). Hence,

$$b_{b,t} = b_{\ell,t} = \gamma m_{\ell,t} \tag{A.18}$$

given (6), (8), and Assumption 2 in this case.

If $\rho > \gamma^{-1}$ and $p_{M,t+1} = \rho\gamma p_{A,t+1}$, then $\eta_{1,t} = p_{M,t+1} p_{A,t}^{-1} = \rho > \gamma^{-1}$, given (11) and (A.7). Thus, $\underline{\lambda}_{x,\ell,t} = 0$ and $x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} \ge 0$, as implied by (A.6). In this case, (A.1) implies that

$$f'(x_{b,t}) = \frac{p_{M,t+1}}{q_t} = \frac{p_{A,t+1}}{q_t} \cdot \frac{p_{A,t}}{p_{A,t+1}} \cdot \frac{p_{M,t+1}}{p_{A,t}} = \rho^2\gamma \tag{A.19}$$

given (11) and (A.7). Because $\eta_{1,t} = p_{A,t+1} q_t^{-1} = \rho$ and (A.7) also imply that $p_{A,t} = \rho\gamma q_t$, $x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} \ge 0$ implies that $x_{b,t} \le 1 - \rho\gamma q_t m_{\ell,t}$, given the first constraint in (5).

Also, $x_{b,t} = q_t b_{b,t} = \gamma q_t m_{\ell,t}$ as implied by (A.18). Thus, the unique root for $x_{b,t}$ in (A.19) must be in $(0, (1+\rho)^{-1})$. The results described in this paragraph are sufficient for (27)-(29).

If $\rho > \gamma^{-1}$ and $p_{M,t+1} > \rho\gamma p_{A,t+1}$, then $\underline{\lambda}_{x,\ell,t} > 0$ and $x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} = 0$. Thus, (A.16) holds, given $b_{b,t} = b_{\ell,t} = \gamma m_{\ell,t}$. Hence, (A.17) holds. Also, $\eta_{1,t} = p_{M,t+1} p_{A,t+1}^{-1} \gamma^{-1} > \rho$ and (A.16) imply that (A.17) must have a root for $x_{b,t}$ in $(0, (1+\rho)^{-1})$ if there exists a monetary steady state in this case. Because $\rho > \gamma^{-1}$ implies that $(1+\rho)^{-1} < \gamma(1+\gamma)^{-1}$, Assumption 3 implies that $f'(x_{b,t})$ and $\gamma(x_{b,t}^{-1} - 1)^2$ can have only one intersection for $x_{b,t} \in (0, (1+\rho)^{-1})$ at most, and that they do have one if and only if $f'((1+\rho)^{-1}) > \gamma[(1+\rho)-1]^2 = \rho^2\gamma$. The results described in this paragraph are sufficient for (24)-(26).

B Proof of Proposition 2

In any monetary steady state, $q_t > 0$, $x_{b,t} = q_t b_{b,t}$, and (30), given (6), (8), (11), and the first constraint in (5). Also, $q_t > 0$ and the Inada condition satisfied by the function f imply (A.1) and $b_{b,t} > 0$ in an equilibrium, as described in Appendix A.

Given $q_t > 0$ and $b_{b,t} = b_{\ell,t} > 0$ as implied by (8), the first-order conditions for $x_{\ell,t}$, $b_{\ell,t}$, $m_{\ell,t}$, and $m'_{\ell,t+1}$ in (4) without $m'_{\ell,t+1} \in [0,\ m_{\ell,t} + \tau_{\ell,t+1}]$ in the constraint set are (A.2)-(A.5) with $\bar{\lambda}_{m',\ell,t+1} = 0$. Also, (6) implies that $m'_{\ell,t+1} > 0$, given $b_{b,t} > 0$, and thus $\underline{\lambda}_{m',\ell,t+1} = 0$. Substituting these equalities into (A.2)-(A.5) yields

$$\underline{\lambda}_{x,\ell,t} = \eta_{1,t} - \rho \geq 0 \tag{A.20}$$

$$\eta_{1,t} = \frac{p_{M,t+1}}{q_t} \tag{A.21}$$

$$\underline{\lambda}_{m,\ell,t} = p_{M,t+1}\left(\frac{p_{A,t}}{q_t} - 1\right) \geq 0 \tag{A.22}$$

$$\eta_{4,t} = p_{M,t+1} - p_{A,t+1} \tag{A.23}$$

If $b_{\ell,t} + m_{\ell,t} + \tau_{\ell,t+1} - m'_{\ell,t+1} = 0$ in an equilibrium, then it contradicts (30) unless $m_{\ell,t} + \tau_{\ell,t+1} = 0$. Suppose $m_{\ell,t} + \tau_{\ell,t+1} > 0$. In this case, $b_{\ell,t} + m_{\ell,t} + \tau_{\ell,t+1} - m'_{\ell,t+1} > 0$, and thus $\eta_{4,t} = p_{M,t+1} - p_{A,t+1} = 0$; (11) holds because this case implies $M_0 > 0$ and

$m_{\ell,t} = M_t > 0$; and because (7) implies $m_{\ell,t+1} \geq 0$, $\underline{\lambda}_{m,\ell,t} = p_{M,t+1}(p_{A,t}/q_t - 1) = 0$ in a monetary steady state in this case. Hence,

$$\underline{\lambda}_{x,\ell,t} = \eta_{1,t} - \rho = 1/\gamma - \rho \geq 0 \tag{A.24}$$

Also, (31) and (33) holds, whereas $x_{\ell,t}$ and $p_{A,t}$ can be any pair of non-negative real numbers satisfying $x_{\ell,t} + p_{A,t}M_t = 1 - x_{b,t}$ if $\frac{1}{\gamma} = \rho$.

Next, suppose that $m_{\ell,t} + \tau_{\ell,t+1} = 0$. In this case, (7) and (30) imply that $m_{\ell,t+1} = b_{\ell,t} + m_{\ell,t} + \tau_{\ell,t+1} - m'_{\ell,t+1} = 0$. Thus,

$$x_{\ell,t} = 1 - q_t b_{\ell,t} - p_{A,t} m_{\ell,t} = 1 - x_{b,t} \tag{A.25}$$

given (30). Therefore, if $x_{\ell,t} = 0$, then it violates (1). Hence, $x_{\ell,t} > 0$ and $\underline{\lambda}_{x,\ell,t} = \eta_{1,t} - \rho = 0$. Substituting this result and (A.21) into (A.1) yields (34). Also, (11), (A.22), and (A.23) imply

$$p_{M,t+1} \geq p_{A,t+1} \tag{A.26}$$

$$p_{A,t} \geq q_t \tag{A.27}$$

Therefore, it must be the case that

$$\rho = \frac{p_{M,t+1}}{q_t} \geq \frac{p_{A,t+1}}{q_t} = \frac{p_{A,t+1}}{p_{A,t}} \cdot \frac{p_{A,t}}{q_t} \geq \frac{p_{A,t+1}}{p_{A,t}} \tag{A.28}$$

where the first equality is implied by (34).

C Model with the possibility of default on borrowers' IOUs and discount window lending

C.1 Environment

Hereafter, add a subscript i to variables specific to old lender i in period t for $i \in [0, 1]$ and $t = 0, 1, 2...$ Given the face value of borrowers' IOUs, $b_{\ell,i,t-1}$, and the amount of fiat money,

$m_{\ell,i,t-1}$, held from the previous period, the utility maximization problem for old lender i in period t is specified as follows:

$$
\begin{aligned}
&\max_{m'_{\ell,i,t}} c_{\ell,i,t} \\
&\text{s.t. } c_{\ell,i,t} = \rho x_{\ell,i,t-1} + p_{M,t} m'_{\ell,i,t} - h(d_{\ell,i,t}) p_{M,t} \max\{0,\ m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}\} \\
&\qquad\qquad + p_{A,t}[(1-d_{b,t}) b_{\ell,i,t-1} - (1-d_{\ell,i,t}) \max\{0, m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}\} \\
&\qquad\qquad\quad - \min\{0, m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}\}] \\
&\qquad d_{\ell,i,t} = \begin{cases} \max\left\{0,\ 1 - \frac{(1-d_{b,t}) b_{\ell,i,t-1}}{m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}}\right\} & \text{if } m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t} > 0 \\ 0 & \text{if } m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t} \leq 0 \end{cases} \\
&\qquad m'_{\ell,i,t} \geq 0
\end{aligned}
\tag{A.29}
$$

for $i \in [0,1]$, where $d_{\ell,i,t}$ is the fraction of discount window lending from a commercial bank that the old lender involuntarily defaults; the function h determines a commercial bank's offer of a proportional discount fee, given the expected value of $d_{\ell,i,t}$; and $d_{b,t}$ is the defaulted fraction of each old borrower's IOU in period t, which is taken as given.

The first constraint is the flow-of-funds constraint for the old lender, which adds the payment of a real discount fee for discount window lending, $h(d_{\ell,i,t}) p_{M,t}(m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t})$, to the second constraint in (4) if and only if the amount of discount window lending taken by the old lender, $m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}$, is positive. For simplicity, assume that the old lender pays a real discount fee from the goods that it buys in the morning goods market. The last term in the first constraint is the real value of fiat money spent by the old lender in the afternoon goods market. This term includes a maximum and a minimum operator so that the old lender's repayment of discount window lending, $(1 - d_{\ell,i,t})(m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t})$, enters in this term if and only if $m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t} > 0$.

The second constraint implies that the old lender must involuntarily default if the amount of fiat money it receives on borrowers' IOUs is less than the amount of fiat money it borrows from a commercial bank. The defaulted fraction of borrowers' IOUs, $d_{b,t}$, is the same across old lenders, given Assumption 9 as described below. The third constraint is the feasibility constraint on the amount of fiat money that the old lender spends in the morning goods market, $m'_{\ell,i,t}$.

Next, let us define the profit maximization problem for each commercial bank. Assume limited participation in the discount window market:

Assumption 8. For $j \in [0,1]$ and $t = 1, 2, 3, ...$, commercial bank j born in period t can contact old lender i in cohort $t-1$ in the discount window market in the period if and only if $i \in \{j, g_+(j), g_-(j)\}$, where i and j are the indices of an old lender and a commercial bank, respectively, in each period. The functions $g_+ : [0,1] \to (0,1]$ and $g_- : [0,1] \to [0,1)$ are modulo operations for real numbers defined by

$$g_+(j) \equiv \begin{cases} j+s & \text{if } j+s \leq 1 \\ j+s-1 & \text{if } j+s > 1 \end{cases}, \quad g_-(j) \equiv \begin{cases} j-s & \text{if } j-s \geq 0 \\ j-s+1 & \text{if } j-s < 0 \end{cases} \tag{A.30}$$

for $j \in [0,1]$, where s is a constant in $(0,1)$. Commercial banks contacting the same old lender are faced with Bertrand competition to offer a proportional discount fee, given the fraction of fiat money lent that the old lender is expected to repay. Without loss of generality, if old lender i in cohort $t-1$ receives the same offers from commercial banks in period t, then the lender chooses the offer from commercial bank i born in the period.

This assumption can be interpreted as reflecting each commercial bank's special knowledge about its clients. At the same time, it allows interbank competition. Given the symmetry among commercial banks, each lender ends up borrowing from the commercial bank sharing the same index number if Bertrand competition takes place.

In contrast, assume that the markets between lenders and borrowers are integrated:

Assumption 9. Each young lender holds an equal share of every borrower's IOU in the credit market, and each old borrower sells an equal share of its output to every old lender in the morning goods market.

A possible interpretation of this symmetry assumption is that a young borrower needs the variety of inputs supplied by each young lender, and the consumption of an old lender is a composite of all varieties of output produced by old borrowers.

With these assumptions, commercial bank j's profit maximization problem for $j \in [0,1]$ in period t is specified as follows:

$$\begin{aligned}
\max_{b_{bk,j,t},s_{bk,j,t}} \quad & h(d_{bk,j,t})p_{M,t}b_{bk,j,t} + E_t\left[(1-d_{bk,j,t}-s_{bk,j,t})p_{A,t}b_{bk,j,t} - \theta_{j,t}(1-s_{bk,j,t})\frac{p_{M,t}b_{bk,j,t}}{\underline{\theta}}\right] \\
\text{s.t.} \quad & \frac{p_{M,t}b_{bk,j,t}}{\underline{\theta}} \in [0,\bar{k}] \\
& s_{bk,j,t} \in [0,\ 1-d_{bk,j,t}]
\end{aligned} \tag{A.31}$$

where E_t is the expectation operator when commercial bank j borrows an amount $b_{bk,j,t}$ of fiat money from the central bank before the realization of $\theta_{j,t}$ in period t; $s_{bk,j,t}$ ($\in [0,1]$) is the fraction of borrowed fiat money that the commercial bank repays to the central bank in the period, given the realization of $\theta_{j,t}$; $d_{bk,j,t}$ is the commercial bank's expectation of the default rate on its discount window lending, given the symmetry among old lenders j, $g_+(j)$, and $g_-(j)$; and the function h determines the proportional real discount fee as defined below.

The objective function in (A.31) is expected profit for the commercial bank in period t. The first term is the fee revenue from discount window lending. Inside the expectation operator is the net balance of two terms: the expected amount of goods that the commercial bank consumes by not repaying fiat money to the central bank; and the expected loss from the confiscation of collateral by the central bank. In the former term, fiat money is evaluated by the real value of fiat money in the afternoon goods market, because commercial banks can receive the repayments of fiat money from old lenders only after the morning goods market. In the latter term, $p_{M,t}b_{bk,j,t}/\underline{\theta}$ equals the amount of collateral submitted to the central bank, $k_{bk,j,t}$, as implied by (37); thus, the amount of collateral that the central bank returns to a commercial bank is proportional to the amount of fiat money that the commercial bank repays to the central bank. This assumption ensures that the central bank confiscates no collateral if the fiat money lent to the commercial bank is fully repaid.

The first constraint in (A.31) is the feasibility constraint for $k_{bk,j,t}$, given (37). This constraint never binds, given Assumption 7. The second constraint implies that the commercial bank can repay fiat money to the central bank only up to the repayment of fiat money it receives from old lenders.

In (A.31), it is optimal for the commercial bank to set $s_{bk,j,t}$ to the upper bound, $1-d_{bk,j,t}$, for any realization of $\theta_{j,t}$, because $\theta_{j,t} \geq \underline{\theta}$ for all j and t as assumed in Assumption 6 and $p_{A,t} \leq p_{M,t}$ in any monetary steady state as described in (14). Thus, $b_{bk,j,t} > 0$ only if $h(d_{bk,j,t}) \geq d_{bk,j,t} E_t \theta_{j,t}/\underline{\theta}$. If there is Bertrand competition among commercial banks, then the value of the function h must be this lower bound. If there is no other commercial bank making a competing offer, then the commercial bank becomes the monopolist for the three old lenders that it can contact. Therefore, the commercial bank's offer to old lenders in the discount window market can be characterized by the following function h:

$$h(d_{bk,j,t}) \equiv \begin{cases} \frac{d_{bk,j,t}(1+\underline{\theta})}{2\underline{\theta}} & \text{if there is Bertrand competition} \\ \max\left\{\frac{p_{M,t}-p_{A,t}(1-d_{bk,j,t})}{p_{M,t}},\ \frac{d_{bk,j,t}(1+\underline{\theta})}{2\underline{\theta}}\right\} & \text{if the commercial bank is monopolistic} \end{cases} \tag{A.32}$$

On the first line, note that $E_t\theta_{j,t} = (1+\underline{\theta})/2$, as implied by Assumption 6. On the second line, the first term inside the max operator is the value of the function h that makes an old lender indifferent to discount window lending, as implied by the first constraint in an old lender's utility maximization problem, (A.29). The second term is the minimum value of the function h that makes the commercial bank break even.

Also, the rational expectations of commercial banks require that

$$d_{bk,j,t} = \begin{cases} \max\left\{0,\ 1 - \frac{\sum_{i\in\{j,g_+(j),g_-(j)\}} \mathbf{1}_j(i)(1-d_{b,t})b_{b,t-1}}{b_{bk,j,t}}\right\} & \text{if } b_{bk,j,t} > 0 \\ 0 & \text{if } b_{bk,j,t} = 0 \end{cases} \tag{A.33}$$

for all $j \in [0,1]$, where $\mathbf{1}_j(i)$ is an indicator function that equals one if commercial bank j extends a positive amount of discount window lending to old lender i, and zero otherwise. Given (8), where $b_{\ell,t}$ is the value of $b_{\ell,i,t}$ for all $i \in [0,1]$, and the second constraint in (A.29), this condition makes $d_{bk,j,t}$ equal $d_{\ell,i,t}$ for $i \in \{j, g_+(j), g_-(j)\}$ if commercial bank j becomes monopolistic in the discount window market, and for $i = j$ if commercial bank j has Bertrand competition with other commercial banks.

Finally, the market clearing condition for the discount window market in period t must be added to equilibrium conditions:

$$b_{bk,j,t} = m'_{\ell,j,t} - m_{\ell,j,t-1} - \tau_{\ell,t} \tag{A.34}$$

for $j \in [0,1]$. Without loss of generality, let us focus on a symmetric equilibrium in which commercial banks are homogeneous in each period. Given Assumption 8, commercial bank j lends to old lender j, or extends no discount window lending, in case of which both sides of (A.34) are zero, in a symmetric equilibrium, because no commercial bank lends monopolistically in a symmetric equilibrium.

The market clearing condition for the morning goods market, (6), is modified to

$$(1 - d_{b,t})b_{b,t-1} = \int_0^1 m'_{\ell,i,t}\, di \tag{A.35}$$

$$m'_{\ell,i,t} = \min\{b_{b,t-1}, \bar{m}'_{\ell,i,t}\} \tag{A.36}$$

for all $i \in [0,1]$, where $\bar{m}'_{\ell,i,t}$ is the maximum value among the solutions to $m'_{\ell,i,t}$ in (A.29). In (A.36), it is assumed that if $p_{M,t} = p_{A,t}$—that is, old lenders are indifferent to buying goods in the morning goods market or the afternoon goods market—then they pay an amount of fiat money that is enough for old borrowers to repay the face value of their IOUs, $b_{b,t-1}$, whenever possible.

The market clearing condition for the afternoon goods market, (7), is modified to

$$\begin{aligned}\int_0^1 m_{\ell,i,t}\, di = \int_0^1 (1 - d_{b,t})b_{\ell,i,t-1} - (1 - d_{\ell,i,t}) \max\{0, m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}\} \\ - \min\{0, m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}\}\, di\end{aligned} \tag{A.37}$$

where the left-hand side is the total amount of overnight money holdings by young lenders in cohort t, and the right-hand side is the total amount of fiat money spent by old lenders in cohort $t-1$ in the afternoon goods market.

Except (A.37), the equilibrium conditions associated with young borrowers and lenders in period t are independent of variables associated with old lenders and borrowers in the period. Overall, the definition of an equilibrium is revised as follows:

Definition 4. In a symmetric equilibrium with the possibility of default on borrowers' IOUs and discount window lending, variables in period t are determined by (4) in which $m'_{\ell,t+1} \geq 0$ replaces $m'_{\ell,t+1} \in [0,\ m_{\ell,t} + \tau_{\ell,t+1}]$ in the constraint set, (5), (8), and (A.29)-(A.37), given the amount of goods invested in each borrower's production in period $t-1$, $x_{b,t-1}$, the face value of each borrower's IOU issued in period $t-1$, $b_{b,t-1}$, and the amount of fiat money held by each old lender at the beginning of period t, $m_{\ell,i,t-1} + \tau_{\ell,i,t}$, for $i \in [0,1]$, whereas variables from periods $t+1$ onward satisfy the equilibrium conditions characterized by Definition 3.

C.2 Self-fulfilling crunch of discount window lending in case of no overnight money supply

The following proposition confirms that if $\rho > 1$ and $M_0 = 0$, then discount window lending by commercial banks can disappear due to a self-fulfilling expectation among commercial banks in period t:

Proposition 3. Suppose Assumptions 1-2 and 4-9 hold. Also suppose $\rho > 1$ and $M_0 = 0$. Given Definition 4, if $b_{b,t-1} = f'^{-1}(\rho)/q_{t-1}$, then there exists a symmetric equilibrium that coincides with the monetary steady state described in Proposition 2. Also, for any positive value of $b_{b,t-1}$, there exists another symmetric equilibrium such that $d_{b,t} = 1$ and $b_{bk,j,t} = 0$ for all $j \in [0,1]$ in period t.

Proof. On one hand, the equilibrium conditions described by Definition 4 can be satisfied with $d_{b,t} = d_{bk,j,t} = d_{\ell,j,t} = 0$ and $b_{bk,j,t} = b_{b,t-1}$ for $j \in [0,1]$ in period t, given Assumption 7. In this equilibrium, the economy remains in the monetary steady state described in Proposition 2.

On the other hand, suppose that an arbitrary commercial bank indexed by j' expects the other commercial banks to stop lending to lenders in period t. If this expectation is correct, then $b_{bk,j,t} = 0$ for all $j \neq j'$. In this case, the other old lenders than lenders j', $g_+(j')$, and $g_-(j')$ cannot pay any fiat money to old borrowers in the morning goods market, because they do not have any overnight money holding, given $m_{\ell,i,t-1} + \tau_{\ell,t} = M_0 = 0$ for all $i \in [0,1]$ as implied by (6)-(8) up to period $t-1$. As a result, even if commercial bank j' lends fiat money to the three old lenders in period t, each old borrower receives only an infinitesimal amount of fiat money from these old lenders, because the measure of these old lenders is zero. Thus, each old borrower must involuntarily default on the borrower's IOU:

$$(1 - d_{b,t})b_{b,t-1} = 0 \tag{A.38}$$

as implied by (A.34) and (A.35).

Given (A.38), if commercial bank j' borrows fiat money from the central bank and lends the fiat money to old lenders in period t, then it expects $d_{bk,j',t} = 1$, as implied by (A.33). Thus, it charges the old lenders a real discount fee that is not less than the value of collateral submitted to, and confiscated by, the central bank:

$$h(d_{bk,j',t}) = \frac{d_{bk,j',t}(1+\underline{\theta})}{2\underline{\theta}} = \frac{1+\underline{\theta}}{2\underline{\theta}} > \frac{p_{M,t} - p_{A,t}(1 - d_{bk,j,t})}{p_{M,t}} = 1 \quad \text{if } b_{bk,j',t} > 0 \tag{A.39}$$

as implied by (A.32). Because the right-hand side is the value of the function h that makes the old lenders break even, the old lenders do not demand discount window lending, i.e., $m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t} = 0$ for all $i \in [j', g_+(j'), g_-(j')]$, in this case. Hence, $b_{bk,j',t} = 0$, as implied by (A.34).

The rest of equilibrium conditions are satisfied regardless of the value of $b_{bk,j,t}$ for $j \in [0,1]$ in period t. Thus, there exists a symmetric equilibrium in which $d_{b,t} = 1$ and $b_{bk,j,t} = 0$ for all $j \in [0,1]$ in period t. $\square$

Suppose that the definition of a symmetric equilibrium is modified to allow default in borrowers' IOUs and discount window lending from period $t+1$ onward. In this case, Proposition 3 implies that if $\rho > 1$ and $M_0 = 0$, then a self-fulfilling crunch of discount window lending by commercial banks can persist for any number of periods, as it can occur with an arbitrary value of $b_{b,t-1}$. In this regard, note that old lenders cannot take discount window lending of fiat money in period t if $b_{b,t-1} = 0$, as they do not receive any fiat money that they can repay to commercial banks in the period.

C.3 No self-fulfilling crunch of discount window lending in the presence of an overnight money supply

In contrast, there is no such fragility if there is an overnight money supply:

Proposition 4. Suppose Assumptions 1-2 and 4-9 hold. Also suppose $M_0 > 0$. Given Definition 4, $d_{b,t} = 0$ and $b_{bk,j,t} = \max\{0,\ b_{b,t-1} - M_t\}$ for all $j \in [0, 1]$ in period t in any symmetric equilibrium.

Proof. On one hand, the equilibrium conditions described by Definition 4 can be satisfied with $d_{b,t} = d_{bk,j,t} = d_{\ell,j,t} = 0$ and $b_{bk,j,t} = \max\{0,\ b_{b,t-1} - M_t\}$ for all $j \in [0, 1]$ in period t, given Assumption 7.

On the other hand, suppose $d_{bk,j,t} > 0$ for all $j \in [0, 1]$ in period t in a symmetric equilibrium. In this case, $d_{bk,i,t} > 0$ for all $i \in [0, 1]$ as implied by (A.32) and (A.34). Therefore, $m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t} > 0$ for all $i \in [0, 1]$, as implied by the second constraint in (A.29). Thus, the first part of the market clearing condition for the morning goods market, (A.35), implies that

$$(1 - d_{b,t})b_{b,t-1} = \int_0^1 m'_{\ell,i,t}\, d_i > \int_0^1 m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t}\, d_i \tag{A.40}$$

given $m_{\ell,i,t-1} + \tau_{\ell,t} > 0$ for all $i \in [0, 1]$. Therefore, (A.33) in turn implies that $d_{dk,j,t} = 0$ for all $j \in [0, 1]$, given (A.34) and the symmetry among old lenders. This is a contradiction. Hence, $d_{bk,j,t} = 0$ for all $j \in [0, 1]$ in period t in any symmetric equilibrium.

Suppose $d_{bk,j,t} = 0$ and $b_{bk,j,t} = 0$ for all $j \in [0, 1]$ in period t in a symmetric equilibrium. In this case, (A.40) implies that $(1 - d_{b,t})b_{b,t-1} > 0$, given $m_{\ell,i,t-1} + \tau_{\ell,t} > 0$ for all $i \in [0, 1]$; thus, for all $j \in [0, 1]$, commercial bank j can keep $d_{bk,j,t} = 0$, if $m'_{\ell,j,t} - m_{\ell,j,t-1} - \tau_{\ell,t}$, and hence $b_{bk,j,t}$, is positive but sufficiently small, as implied by (A.33) and (A.34). If $b_{b,t-1} > M_t$, then $m'_{\ell,j,t} - m_{\ell,j,t-1} - \tau_{\ell,t} > 0$ as implied by (A.36). Otherwise, $m'_{\ell,j,t} - m_{\ell,j,t-1} - \tau_{\ell,t} \leq 0$. In the former case, (A.34) implies $b_{bk,j,t} > 0$, which is a contradiction. In the latter case, there is no contradiction.

Therefore, if $b_{b,t-1} > M_t$, then $d_{bk,j,t} = 0$ and $b_{bk,j,t} > 0$ for all $j \in [0,1]$ in period t in any symmetric equilibrium. In this case, $m'_{\ell,i,t} = b_{b,t-1}$ for all $i \in [0,1]$, as otherwise $d_{\ell,i,t} > 0$, and thus $d_{bk,j,t} > 0$.

If $b_{b,t-1} \leq M_t$, then $m'_{\ell,i,t} - m_{\ell,i,t-1} - \tau_{\ell,t} \leq 0$ for all $i \in [0,1]$, as implied by (A.36). Thus, $b_{bk,j,t} > 0$ never occurs in a symmetric equilibrium in this case, as implied by (A.34). Hence, $d_{bk,j,t} = 0$ and $b_{bk,j,t} = 0$ for all $j \in [0,1]$ in period t in any symmetric equilibrium, if $b_{b,t-1} \leq M_t$. Overall, the equilibrium mentioned at the beginning of the proof is the unique symmetric equilibrium. $\square$