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Axiomatizations of Coalition Aggregation Functions

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Abstract

We axiomatize Hart and Kurz's (1983) two coalition aggregation functions known as the γ-function and the δ-function. A coalition aggregation function is a mapping that assigns a partition to each coalition profile, where a coalition profile is a vector of coalitions selected by all players. Through our axiomatization results, we observe that neither the γ-function nor the δ-function satisfies monotonicity. We propose a monotonic function and axiomatically characterize it. An impossibility result on monotonicity is also provided.

Keywords: axiomatization; coalition formation; coalition structure; monotonicity

JEL Classification: C71

1 Introduction

In this paper, we attempt to answer the following question: What coalition structure should be assigned to a profile of coalitions selected by players? To clarify our question, we begin with a simple example. Suppose that there are three players, $N = \{1, 2, 3\}$. Every player chooses a coalition that she/he wants to form: For example, each student submits a list of students with whom she wants to share a room. We suppose that players 1 and 2 want to form the three-person coalition $\{1, 2, 3\}$, while player 3 dislikes player 1 and wants to form the two-person coalition with player 2, namely, $\{2, 3\}$. Their choices are summarized as

$$\sigma = (123, 123, 23),$$

where we omit the parentheses and write, for example, 123 to denote coalition $\{1, 2, 3\}$. We call such a vector a *coalition profile*. What coalition structure is "optimal" for this coalition profile?[*1] The first attempts to address this question were made by Hart and Kurz (1983, 1984). They defined two aggregation rules known as the γ-function $\mathcal{B}^\gamma$ and the δ-function $\mathcal{B}^\delta$. These functions assign a coalition structure to each coalition profile. In general, we call such a function a *coalition aggregation function*. Since the δ-function is slightly simpler than γ, we first introduce the δ-function. Its formal definition is provided in Section 2.

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*1 A coalition structure means a partition of the player set N. Moreover, the formal definition of a coalition profile is provided in Section 2.

The δ-function focuses on "who chooses the same coalition." In the coalition profile above, players 1 and 2 choose the same coalition 123. Therefore, the δ-function assigns the two-person coalition to players 1 and 2. Player 3 is the only player who chooses coalition 23. Hence, the δ-function assigns his one-person coalition to player 3. As a result, the δ-function groups these three players into coalition structure $\{12, 3\}$.

The γ-function features unanimous agreements. Let $S \subseteq N$ be a coalition of players. If *all* players in S choose the coalition S, then it might be natural for the players to form and belong to the coalition S. However, if someone in S chooses a different coalition, the γ-function partitions the players in S into singletons. In this sense, the γ-function strictly requires unanimous agreement. In the example, player 3 chooses coalition 23, which is different from the choice of the other two players. Hence, the proposed coalition 123 lacks unanimous agreement, and the proposers 1 and 2 are partitioned into two one-person coalitions. As a result, the γ-function assigns partition $\{1, 2, 3\}$, namely, three one-person coalitions, to the coalition profile $(123, 123, 23)$.

Each of these two rules can be seen as a function that assigns a partition of the player set to a coalition profile. Our purpose is to axiomatically analyze such functions. We first axiomatize the γ-function and the δ-function. Through the axiomatizations, we observe that these two functions do not satisfy *monotonicity*. Monotonicity is a basic property: If a player changes his/her choice to a larger coalition in the sense of superset, then he/she should belong to a larger coalition or at least the same coalition. For example, consider a coalition profile $\sigma = (12, 23, 23)$. Both $\mathcal{B}^\gamma$ and $\mathcal{B}^\delta$ assign $\{1, 23\}$ to this profile. Therefore, player 3 belongs to the two-person coalition 23. Now, player 3 changes his mind and accepts player 1. Let $\sigma' = (12, 23, 123)$. For this new coalition profile σ', it holds that $\mathcal{B}^\gamma(\sigma') = \mathcal{B}^\delta(\sigma') = \{1, 2, 3\}$, namely, three one-person coalitions. Therefore, player 3 belongs to his one-person coalition. In other words, accepting player 1 caused player 3 to belong to a smaller coalition. This violates monotonicity. In this paper, we provide and axiomatically characterize a monotonic rule. In addition, we provide an impossibility result on monotonicity. Our results are summarized in Table 1 in Section 5.

Our motivation for studying coalition aggregation functions is to establish a connection between players' coalition choices and coalition structures. Coalition formation has been mainly studied in the field of cooperative game theory. In general, cooperative game theory addresses two topics: (i) What coalition is to be formed? and (ii) What allocation is to be chosen? However, the first topic is often avoided by implicitly or explicitly assuming that players form the grand coalition.[*2] Moreover, in many models including cooperative games, hedonic games, and matching problems, players can form a coalition by simply "agreeing" to join a coalition. This simplification allows us to introduce the concept of a coalition into these models in a straightforward way, while it omits the formulation of the step of consensus building among players to form a coalition. Therefore, in this paper, through analyzing coalition aggregation functions, we revisit a theoretical foundation of coalition formation among players.

In addition to the theoretical perspective, this paper may facilitate experimental analyses of coalition

[*2] One of the leading works that address stable coalition structures without such an assumption is the class of *hedonic games* introduced by Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002). Moreover, Greenberg (1994) and Kóczy and Lauwers (2004) introduce the notion of a *coalition structure core*, which can be seen as a set of pairs of a payoff allocation and a coalition structure.

formation. As mentioned above, coalition formation has been mainly studied in the context of cooperative game theory. Therefore, coalition formation theory does not contain a framework in which players, namely, participants of an experiment, make a decision in some way. In other words, players do not choose anything in cooperative game theory. This fact has prevented researchers from performing a coalition formation experiment. Since in this paper we offer the axiomatic rationale for the connection between players' coalition choices and coalition structures, our attempt can be a preliminary step toward experiments on coalition formation theory. In Section 5, we propose an approach to derive a non-cooperative game for experiments from our model.

The rest of this paper is organized as follows. In Section 2, we introduce basic definitions and the notion of a coalition aggregation function. In Section 3, we introduce and axiomatize the δ-function and the γ-function. In Section 4, we observe that these two functions do not satisfy monotonicity. We offer an impossibility result on monotonicity and define a new coalition aggregation function that obeys monotonicity. Its axiomatization is also provided in this section. In Section 5, we summarize our results. Table 1 in Section 5 shows the axiomatic systems of the coalition aggregation functions we study in this paper. All the proofs and the independence of the axioms are provided in the appendix.

2 Preliminaries

Let $N = \{1, ..., n\}$ be a set of players. A *coalition* is a subset $S \subseteq N$. We denote the cardinality of coalition S by $|S|$. We use n to denote $|N|$. We assume $n \geq 3$. For every $i \in N$, $\mathcal{A}_i$ is the set of coalitions that contain i, formally, $\mathcal{A}_i = \{S \subseteq N | i \in S\}$. For example, let $N = \{1, 2, 3\}$. We have

$$\mathcal{A}_1 = \{1, 12, 13, 123\},$$
$$\mathcal{A}_2 = \{2, 12, 23, 123\},$$
$$\mathcal{A}_3 = \{3, 13, 23, 123\},$$

where we omit the parentheses and write, for example, 123 to denote coalition $\{1, 2, 3\}$. Henceforth, we use this notation for coalitions in examples and tables. For every nonempty coalition $S \subseteq N$, let $\mathcal{A}_S = \times_{i \in S} \mathcal{A}_i$. We use σ to denote an element of $\mathcal{A}_N$. We call σ a *coalition profile*. We typically use $\mathcal{P}$ to denote a partition (or a coalition structure). For every coalition $S \subseteq N$, let $\Pi(S)$ be the set of all partitions of S. For any $S \subseteq N$, any $\mathcal{P} \in \Pi(S)$, and any $i \in S$, let $\mathcal{P}_i$ denote the coalition in partition $\mathcal{P}$ that contains player i. A *coalition aggregation function* (*CA-function*) is a mapping that assigns a partition to each coalition profile, $\mathcal{B} : \mathcal{A}_N \to \Pi(N)$. Since $\mathcal{B}(\sigma)$ is a partition, let $\mathcal{B}_i(\sigma)$ denote the coalition in partition $\mathcal{B}(\sigma)$ that contains player i.

3 The δ-function and the γ-function

Now, we introduce the two CA-functions proposed by Hart and Kurz (1983). The δ-function is given as

$$\mathcal{B}^\delta(\sigma) = \{T \subseteq N | i, j \in T \iff \sigma_i = \sigma_j\}.$$

As mentioned in Section 1, the δ-function focuses on players who choose the same coalition. To see this, let $N = \{1, 2, 3, 4\}$ and $\sigma = (12, 12, 234, 234)$. Since players 1 and 2 choose the same coalition 12, they belong to coalition 12 in $\mathcal{B}^\delta(\sigma)$, namely, $\mathcal{B}^\delta_1(\sigma) = \mathcal{B}^\delta_2(\sigma) = 12$. Similarly, since players 3 and 4 choose the same coalition 234, they belong to coalition 34 in $\mathcal{B}^\delta(\sigma)$, namely, $\mathcal{B}^\delta_3(\sigma) = \mathcal{B}^\delta_4(\sigma) = 34$. Note that the proposed coalition (*i.e.*, 234) does not have to coincide with the set of players who choose it (*i.e.*, players 3 and 4). The resulting coalition structure is $\mathcal{B}^\delta(\sigma) = \{12, 34\}$.

In contrast, the γ-function requires players to make a unanimous agreement. Formally, $\mathcal{B}^\gamma(\sigma) = \{T^i_\sigma | i \in N\}$, where

$$T^i_\sigma = \begin{cases} \sigma_i & \text{if } \sigma_j = \sigma_i \text{ for every } j \in \sigma_i, \\ \{i\} & \text{otherwise.} \end{cases}$$

Consider the coalition profile $\sigma = (12, 12, 234, 234)$ again. The list of coalitions proposed in σ is $\{12, 234\}$. Since all members in coalition 12 agree to form the coalition 12, the γ-function assigns coalition 12 to them. However, for coalition 234, there is a player who does not agree to join it, namely, player 2. Hence, in the sense of the γ-function, coalition 234 lacks unanimous agreement and is not formed. As a result, players 3 and 4 are not contained in any unanimously agreed coalition. The γ-function assigns a one-person coalition to such a player. Therefore, the resulting coalition structure is $\mathcal{B}^\gamma(\sigma) = \{12, 3, 4\}$. Note that, in general, a player who chooses his/her one-person coalition $\sigma_i = \{i\}$, if any, achieves unanimous agreement by himself/herself.

The two CA-functions for $n = 3$ are fully described in Tables $\mathcal{B}^\delta$ and $\mathcal{B}^\gamma$. To fit each table on a page, we use symbols N, X, Y, Z, I to denote partitions as follows:

$$\text{N} = \{123\},\ \text{X} = \{12, 3\},\ \text{Y} = \{13, 2\},\ \text{Z} = \{23, 1\},\ \text{I} = \{1, 2, 3\}.$$

For example, $\mathcal{B}^\delta(123, 123, 123) = \{123\}(= \text{N})$, $\mathcal{B}^\delta(12, 123, 123) = \{23, 1\}(= \text{Z})$, $\mathcal{B}^\delta(123, 12, 123) = \{13, 2\}(= \text{Y})$, and $\mathcal{B}^\delta(123, 123, 13) = \{12, 3\}(= \text{X})$.

Table $\mathcal{B}^\delta$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	Y	Y	Y	X	I	I	I	X	I	Z	I	X	I	I	I
12	Z	X	I	I	I	X	I	I	I	X	Z	I	I	X	I	I
13	Z	I	I	I	Y	Y	Y	Y	I	I	Z	I	I	I	I	I
1	Z	I	I	I	I	I	I	I	I	I	Z	I	I	I	I	I

Table $\mathcal{B}^\gamma$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	I	I	I	I	I	I	I	I	I	Z	I	I	I	I	I
12	I	X	I	I	I	X	I	I	I	X	Z	I	I	X	I	I
13	I	I	I	I	Y	Y	Y	Y	I	I	Z	I	I	I	I	I
1	I	I	I	I	I	I	I	I	I	I	Z	I	I	I	I	I

We now introduce the axioms to characterize these two CA-functions.

Axiom 1 (Unanimity, **UN**)**.** For any $\sigma \in \mathcal{A}_N$ and any $\emptyset \neq S \subseteq N$, if $\sigma_j = S$ for every $j \in S$, then $S \in \mathcal{B}(\sigma)$.

This axiom states that if all members of a coalition agree on the formation of the coalition, then the coalition should be formed. For convenience, we call player i a *unanimous member* if $\sigma_j = \sigma_i$ for every $j \in \sigma_i$ (or equivalently, if there exists $S \subseteq N$ such that $i \in S$ and $\sigma_j = S$ for every $j \in S$). All CA-functions that we discuss in this paper satisfy this axiom.

Axiom 2 (Disagreement, **DA**)**.** For any $\sigma \in \mathcal{A}_N$ and any $i, j \in N$, if $\sigma_i \neq \sigma_j$, then $\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma)$.

Axiom **DA** describes a specific form of disagreement. If the list of the players with whom player i wants to form a coalition, σ_i, is different from that of player j, then these two players cannot reach an agreement in this sense. For example, player i wants to invite player k, while player j does not. However, player i wants to invite player j and vice versa. In this case, their proposals clearly conflict over player k. A CA-function that obeys **DA** assigns different coalitions to such players.

Table **UN** and Table **DA** describe these axioms for $n = 3$. Table **UN+DA** similarly shows the restriction derived from the combination of both axioms. In the tables, for example, YI means that both $\{13, 2\}$(=Y) and $\{1, 2, 3\}$(=I) are possible, and the other coalition structures (N, X, and Z) are ruled out. In the same manner, "any" means all partitions are possible.

Table **UN**

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	any	any	YI	any	any	any	YI	any	any	Z	YI	XI	XI	XI	I
12	any	X	any	YI	any	X	any	YI	any	X	Z	YI	XI	X	XI	I
13	any	any	any	YI	Y	Y	Y	Y	any	any	Z	YI	XI	XI	XI	I
1	ZI	ZI	ZI	I	ZI	ZI	ZI	I	ZI	ZI	Z	I	I	I	I	I

Table **DA**

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	any	YI	YI	YI	XI	I	I	I	XI	I	ZI	I	XI	I	I	I
12	ZI	XI	I	I	I	XI	I	I	I	XI	ZI	I	I	XI	I	I
13	ZI	I	I	I	YI	YI	YI	YI	I	I	ZI	I	I	I	I	I
1	ZI	I	I	I	I	I	I	I	I	I	ZI	I	I	I	I	I

Table **UN+DA**

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	YI	YI	YI	XI	I	I	I	XI	I	Z	I	XI	I	I	I
12	ZI	X	I	I	I	X	I	I	I	X	Z	I	I	X	I	I
13	ZI	I	I	I	Y	Y	Y	Y	I	I	Z	I	I	I	I	I
1	ZI	I	I	I	I	I	I	I	I	I	Z	I	I	I	I	I

As Table **UN+DA** shows, the two axioms **UN** and **DA** are not enough to specify a unique CA-function. We now consider the remaining players' reaction to one player's deviation from an agreement.

Axiom 3 (Integrative Reaction, **IR**)**.** For any $\sigma \in \mathcal{A}_N$, any $i \in N$, and any $\sigma'_i \in \mathcal{A}_i$ with $\sigma'_i \neq \sigma_i$, if $|\mathcal{B}_i(\sigma)| \geq 3$, then $\mathcal{B}_j(\sigma'_i, \sigma_{-i}) = \mathcal{B}_k(\sigma'_i, \sigma_{-i})$ for any $j, k \in \mathcal{B}_i(\sigma) \setminus \{i\}$ with $j \neq k$.

Let σ be a current coalition profile and $\mathcal{B}(\sigma)$ be the corresponding coalition structure. In the coalition structure $\mathcal{B}(\sigma)$, if $|\mathcal{B}_i(\sigma)| \geq 3$, then player i shares a coalition with at least two players, say j and k (hence, $\mathcal{B}_i(\sigma)$ is $\{i, j, k\}$ or its superset). Now, player i tries to deviate from the coalition $\mathcal{B}_i(\sigma)$ by changing his choice from σ_i to σ'_i. The axiom guarantees that players j and k can keep their coalition even after player i's deviation.

The following axiom is a variation of **IR**.

Disintegrative Reaction, **DR**. For any $\sigma \in \mathcal{A}_N$, any $i \in N$, and any $\sigma'_i \in \mathcal{A}_i$ with $\sigma'_i \neq \sigma_i$, if $|\mathcal{B}_i(\sigma)| \geq 3$, then $\mathcal{B}_j(\sigma'_i, \sigma_{-i}) \neq \mathcal{B}_k(\sigma'_i, \sigma_{-i})$ for any $j, k \in \mathcal{B}_i(\sigma) \setminus \{i\}$ with $j \neq k$.

This axiom is the counterpart of **IR**: It no longer allows players j and k to stay in the same coalition after i's deviation because the coalition lacks player i's agreement. Moreover, we can consider the following axiom to be a coalitional version of **DR**. In this extended axiom, in addition to individual deviations, coalitional deviations are incorporated.[*3] For $n = 3$, **DR** is equivalent to **DR**$^+$.

Axiom 4 (Coalitional Disintegrative Reaction, **DR**$^+$)**.** For any $\sigma \in \mathcal{A}_N$, any $\emptyset \neq S \subseteq N$, and any $\sigma'_S \in \mathcal{A}_S$ with $\sigma'_j \neq \sigma_j$ for all $j \in S$, if there exists $T \in \mathcal{B}(\sigma)$ such that $T \supseteq S$ and $|T \setminus S| \geq 2$, then $\mathcal{B}_j(\sigma'_S, \sigma_{-S}) \neq \mathcal{B}_k(\sigma'_S, \sigma_{-S})$ for any $j, k \in T \setminus S$ with $j \neq k$.

Because of the constraints $|\mathcal{B}_i(\sigma)| \geq 3$ (in **IR**) and $|T \setminus S| \geq 2$ (in **DR**$^+$), the restriction of these axioms is modest: For $n = 3$, only if $\mathcal{B}(\sigma) = 123$ for some σ can these axioms restrict CA-functions.

We obtain the following two characterizations.

Proposition 3.1. CA-function $\mathcal{B}$ satisfies **UN**, **DA**, **IR** if and only if it is $\mathcal{B}^\delta$.

Proposition 3.2. CA-function $\mathcal{B}$ satisfies **UN**, **DA**, **DR**$^+$ if and only if it is $\mathcal{B}^\gamma$.

The proofs and the independence of the axioms are provided in the appendix. Here, we briefly demonstrate the outline of the proofs for $n = 3$. In view of Table **UN+DA**, the coalition profiles whose coalition structure is not determined are

$$\begin{aligned} &\{(\sigma_1, 123, 123) | \sigma_1 = 12, 13, 1\} \\ \cup &\{(123, \sigma_2, 123) | \sigma_2 = 12, 23, 2\} \\ \cup &\{(123, 123, \sigma_3) | \sigma_3 = 13, 23, 3\}. \end{aligned} \tag{3.1}$$

Since $\mathcal{B}(123, 123, 123) = 123$ holds by **UN**, a CA-function that obeys **IR** assigns partition $\{1, 23\}$(=Z)

[*3] In this sense, the relationship between **DR** and **DR**$^+$ is similar to that between a Nash equilibrium and a strong Nash equilibrium.

to every $\sigma \in \{(\sigma_1, 123, 123) | \sigma_1 = 12, 13, 1\}$. This similarly holds for players 2 and 3. As a result, we obtain Table $\mathcal{B}^\delta$. If it satisfies $\mathbf{DR}^+$ instead, then partition $\{1,2,3\}$(=I) is assigned to every coalition profile in (3.1), which results in Table $\mathcal{B}^\gamma$.

4 Monotonicity

We first show that neither $\mathcal{B}^\delta$ nor $\mathcal{B}^\gamma$ satisfies monotonicity. Monotonicity is given as follows.

Axiom 5 (Monotonicity, **MO**)**.** For any $\sigma \in \mathcal{A}_N$, any $i \in N$, and any $\sigma_i' \in \mathcal{A}_i$, if $\sigma_i \subseteq \sigma_i'$, then $\mathcal{B}_i(\sigma_i, \sigma_{-i}) \subseteq \mathcal{B}_i(\sigma_i', \sigma_{-i})$.

As mentioned in Section 1, this axiom states that if a player changes his/her choice to a larger coalition in the sense of superset, then he/she should belong to a larger coalition or at least the same coalition. It is clear that the two functions violate this axiom: Table $\mathcal{B}^\delta$ shows that we have $\mathcal{B}^\delta(12, 23, 23) = \{1, 23\}$(=Z). However, if player 3 changes his choice to 123, then $\mathcal{B}^\delta(12, 23, 123) = \{1, 2, 3\}$(=I). Therefore, $\mathcal{B}_3^\delta(12, 23, 23) = 23 \not\subseteq 3 = \mathcal{B}_3^\delta(12, 23, 123)$. The same holds for $\mathcal{B}^\gamma$. This example shows that accepting player 1 caused player 3 to lose player 2 and belong to his one-person coalition. In general, violating **MO** means that by accepting more players, a player can be assigned to a smaller coalition.

A natural question that arises from this observation is what function satisfies **MO** together with the basic axioms **UN** and **DA**. To answer this question, we introduce the following notion and axiom. Players i and j are said to be a (direct) *pair* in σ if $i \in \sigma_j$ and $j \in \sigma_i$. We denote a pair by $i \overset{\sigma}{\sim} j$. In words, a pair describes two players who accept each other.

Axiom 6 (Pairwise Disagreement, **PD**)**.** For any $\sigma \in \mathcal{A}_N$ and any $i, j \in N$, if not $i \overset{\sigma}{\sim} j$ (namely, i is not a pair with j), then $\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma)$.

This is equivalent to the following statement: if $j \notin \sigma_i$ then $j \notin \mathcal{B}_i(\sigma)$. Therefore, a CA-function that obeys **PD** interprets σ_i as player i's message that he/she does not want to share a coalition with the players in $N \setminus \sigma_i$. Table **PD** describes this axiom for $n = 3$. Note that **DA** implies **PD**. The following result shows the difficulty of the question above.

Proposition 4.1. No CA-function simultaneously satisfies **UN**, **PD**, and **MO**.

The proof is provided in the appendix. Here, we demonstrate its outline through Table **UN+PD**. The underlined partitions in Table **UN+PD** violate **MO**. To see this, we focus on $\mathcal{B}(123, 12, 13)$ in Table **UN+PD**. Since $\mathcal{B}(12, 12, 13) = \{12, 3\}$(=X), **MO** implies $\mathcal{B}_1(123, 12, 13) \supseteq 12$. In the same manner, since $\mathcal{B}(13, 12, 13) = \{13, 2\}$(=Y), **MO** implies $\mathcal{B}_1(123, 12, 13) \supseteq 13$. Hence, it holds that $\mathcal{B}(123, 12, 13) = \{123\}$(=N). However, as described in the table, **UN** and **PD** jointly require that $\mathcal{B}(123, 12, 13)$ is either X or Y or I. This contradicts **MO**.

Since **PD** is a weaker version of **DA**, Proposition 4.1 implies that even if we replace **IR** ($\mathbf{DR}^+$) in Proposition 3.1 (Proposition 3.2) by **MO**, we can obtain no CA-function. As long as we require a CA-function to satisfy **UN**, the only approach left is to weaken **PD**. To achieve this, we introduce the following notion and axiom. Players i and j are said to be an *indirect pair* in σ if there is a sequence of

Table **PD**

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	any	XYI	YZI	YI	XYI	XYI	YI	YI	XZI	XI	ZI	I	XI	XI	I	I
12	XZI	XI	ZI	I	XI	XI	I	I	XZI	XI	ZI	I	XI	XI	I	I
13	YZI	YI	YZI	YI	YI	YI	YI	YI	ZI	I	ZI	I	I	I	I	I
1	ZI	I	ZI	I	I	I	I	I	ZI	I	ZI	I	I	I	I	I

Table **UN+PD**

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	XYI	YZI	YI	XYI	$\underline{\text{XYI}}$	YI	YI	XZI	XI	Z	I	XI	XI	I	I
12	XZI	X	ZI	I	XI	X	I	I	$\underline{\text{XZI}}$	X	Z	I	XI	X	I	I
13	YZI	YI	$\underline{\text{YZI}}$	YI	Y	Y	Y	Y	ZI	I	Z	I	I	I	I	I
1	ZI	I	ZI	I	I	I	I	I	ZI	I	Z	I	I	I	I	I

players $k_1, ..., k_M$ such that $k_1 = i$, $k_M = j$, and $k_m \overset{\sigma}{\sim} k_{m+1}$ for $m = 1, ..., M-1$. We denote an indirect pair by $i \overset{\sigma}{\approx} j$.

Axiom 7 (Weak Pairwise Disagreement, $\mathbf{PD}^-$)**.** For any $\sigma \in \mathcal{A}_N$ and any $i, j \in N$, if not $i \overset{\sigma}{\approx} j$ (namely, i is not an indirect pair with j), then $\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma)$.

An indirect pair is adopted instead of a direct pair that is used in **PD**. This is the only difference between $\mathbf{PD}^-$ and **PD**. Note that if i and j are a direct pair, then they are an indirect pair. Therefore, **PD** implies $\mathbf{PD}^-$. Hence, we have

$$\mathbf{DA} \Rightarrow \mathbf{PD} \Rightarrow \mathbf{PD}^-.$$

Table $\mathbf{PD}^-$ describes the restriction that $\mathbf{PD}^-$ requires.

Table $\mathbf{PD}^-$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	any	any	any	YI	any	any	YI	YI	any	XI	ZI	I	XI	XI	I	I
12	any	XI	ZI	I	XI	XI	I	I	any	XI	ZI	I	XI	XI	I	I
13	any	YI	any	YI	YI	YI	YI	YI	ZI	I	ZI	I	I	I	I	I
1	ZI	I	ZI	I	I	I	I	I	ZI	I	ZI	I	I	I	I	I

The following proposition states that $\mathbf{PD}^-$, together with **UN** and **MO**, specifies a unique CA-function. We define

$$\mathcal{B}^{\approx}(\sigma) = \{T \subseteq N | i, j \in T \iff i \overset{\sigma}{\approx} j\}.$$

Proposition 4.2. CA-function $\mathcal{B}$ satisfies **UN**, $\mathbf{PD}^-$, **MO** if and only if it is $\mathcal{B}^{\approx}$.

The proof and the independence of the axioms are provided in the appendix.*4 We call $\mathcal{B}^{\approx}$ the *pairwise*

*4 To be more specific, we can use a technical axiom that is a slight variant of $\mathbf{PD}^-$ and is weaker than $\mathbf{PD}^-$ as follows:

function. The pairwise function for $n = 3$ is described in Table $\mathcal{B}^{\approx}$. Interpreting $\mathcal{B}^{\approx}$ is straightforward: If two players form an indirect pair, then we group them. We can consider $\approx$ as a binary relation. Binary relation $\approx$ is an equivalence relation and, hence, partitions the player set into equivalence classes (namely, coalitions). This is the partition $\mathcal{B}^{\approx}(\sigma)$.

Table $\mathcal{B}^{\approx}$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	N	N	Y	N	N	Y	Y	N	X	Z	I	X	X	I	I
12	N	X	Z	I	X	X	I	I	N	X	Z	I	X	X	I	I
13	N	Y	N	Y	Y	Y	Y	Y	Z	I	Z	I	I	I	I	I
1	Z	I	Z	I	I	I	I	I	Z	I	Z	I	I	I	I	I

5 Concluding remarks

Table 1 summarizes our axiomatization results. In the table, symbol $\oplus$ means that the axiom is used for the axiomatization of the CA-function; $+$ means that the CA-function satisfies the axiom; $-$ means that the CA-function does not satisfy the axiom. Propositions 3.1, 3.2, and 4.2 are axiomatization results. Proposition 4.1 shows that no function simultaneously satisfies **UN**, **PD**, and **MO**. Note that **DA** $\Rightarrow$ **PD** $\Rightarrow$ **PD**$^-$ holds. For $\mathcal{B}^{\delta}$ and $\mathcal{B}^{\gamma}$, we cannot use **PD** instead of **DA**, which is not sufficient to specify a unique CA-function.

Table 1 Axioms and CA-functions

		UN	**DA**	**PD**	**PD**$^-$	**IR**	**DR**$^+$	**MO**
Prop.3.1	$\mathcal{B}^{\delta}$	$\oplus$	$\oplus$	$+$	$+$	$\oplus$	$-$	$-$
Prop.3.2	$\mathcal{B}^{\gamma}$	$\oplus$	$\oplus$	$+$	$+$	$-$	$\oplus$	$-$
Prop.4.2	$\mathcal{B}^{\approx}$	$\oplus$	$-$	$-$	$\oplus$	$-$	$-$	$\oplus$
Prop.4.1	$\nexists \mathcal{B}$	$+$		$+$				$+$

In this paper, we considered **UN** to be an axiom that all CA-functions should satisfy. If we are not restricted to the CA-functions satisfying **UN** and consider a weak form of unanimity, then, in view of Proposition 4.1, an axiomatic system consisting of **PD**, **MO**, and such weak unanimity might specify a CA-function. Although violating **UN** can be a clear drawback, such a system may deserve further investigation in the future.

As mentioned in the introduction, a CA-function can play a key role to introduce decision making and non-cooperative games for experiments into coalition formation theory. Below, we propose an approach to achieve this. Let $(N, \mathcal{B})$ be given.[*5] A non-cooperative game consists of three factors: A player set,

if i is not an indirect pair with j and neither i nor j is a unanimous member, then $\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma)$. Proposition 4.2 holds even with this axiom instead of **PD**$^-$. Even if we adopt this technical axiom, the proof is almost unchanged.

[*5] One can choose a CA-function $\mathcal{B}$ in view of the axiomatic rationale we offered in this paper.

strategy sets, and a payoff function that assigns a payoff vector to each strategy profile. We use the same player set N. We consider $\mathcal{A}_i$ to be player i's strategy set: Player i chooses a coalition to join. Since CA-function $\mathcal{B}$ connects a strategy profile and a partition, we need a function, say ϕ, that connects a partition and a payoff profile to build a payoff function $\phi \circ \mathcal{B}$. Formally, $\mathcal{B} : \times_{i\in N}\mathcal{A}_i \to \Pi(N)$ and $\phi : \Pi(N) \to \mathbb{R}^N$. Such a function ϕ is also studied by, for example, Hart and Kurz (1984), Casajus (2009), and Abe (2018). As a result, we obtain a non-cooperative game $(N, (\mathcal{A}_i)_{i\in N}, \phi \circ \mathcal{B})$. Our axiomatic rationale serves to choose $\mathcal{B}$. One can choose ϕ that represents a game-theoretic or economic situation she/he is interested in. For example, Hart and Kurz (1984) and Abe (2018) provide ϕ that represents a symmetric majority game, and Casajus (2009) formulates a gloves game. By deriving such a non-cooperative game from a cooperative game, one can apply the notions of non-cooperative game theory such as Nash equilibrium and dominant strategy to situations formulated as cooperative games.

Appendix

In the appendix, let

$$D_i(\sigma) := \{j \in N | j \overset{\sigma}{\sim} i\}$$

for any $i \in N$ and any $\sigma \in \mathcal{A}_N$. Note that $i \in D_i(\sigma)$ for any $i \in N$ and any $\sigma \in \mathcal{A}_N$.

Proof of Proposition 3.1 (Axiomatization of $\mathcal{B}^\delta$)

Proof. The if-part follows from Lemma A.2. We show the only-if-part.

Step 1. Let $\sigma \in \mathcal{A}_N$. For any nonempty $S \subseteq N$, define $H^\sigma_S = \{i \in N | \sigma_i = S\}$. We first prove that $H^\sigma := \{H^\sigma_S | S \subseteq N, H^\sigma_S \neq \emptyset\}$ is a partition of N. For any two different coalitions $S, T \subseteq N$, we have $H^\sigma_S \cap H^\sigma_T = \emptyset$ because if there is a player i such that $i \in H^\sigma_S \cap H^\sigma_T$ then $S = \sigma_i = T$. This contradicts $S \neq T$. Moreover, it holds that $\bigcup_{S\subseteq N} H^\sigma_S = N$ because if there is a player $i \in N \setminus \bigcup_{S\subseteq N} H^\sigma(S)$ then $\sigma_i \neq S$ for any $S \subseteq N$. This contradicts $\sigma_i \in \mathcal{A}_i \subseteq 2^N$. Hence, H^σ is a partition of N. We now define $K^\sigma := \{S \in H^\sigma | \sigma_j = S \text{ for all } j \in S\}$ and $L^\sigma := H^\sigma \setminus K^\sigma$.

Step 2. By **UN**, for each coalition $S \in K^\sigma$, $S \in \mathcal{B}(\sigma)$.

Step 3. For each coalition $T \in L^\sigma$ with $|T| = 1$, say $\{i\} = T$, we have $\sigma_i \neq \sigma_j$ for any $j \in N \setminus \{i\}$. Hence, by **DA**, $B_i(\sigma) = \{i\}$. Now, we fix a coalition $T \in L^\sigma$ with $|T| \geq 2$ and consider a player $i \in T$. Let $R := \sigma_i$. Note that it follows from $T \in L^\sigma$ that

$$\begin{array}{l} \sigma_j = R \text{ for every } j \in T, \\ \sigma_j \neq R \text{ for every } j \in N \setminus T. \end{array} \tag{A.1}$$

If $R = T$, then $T \in K^\sigma$, which contradicts $T \in L^\sigma = H^\sigma \setminus K^\sigma$; if $R \subsetneq T$, then for some player k in $T \setminus R$, $k \notin R = \sigma_k$, which contradicts $\sigma_k \in \mathcal{A}_k$. Hence, we have $R \supsetneq T$. It follows that $|R| \geq 3$.

For the strategy profile σ and the coalition R, define the following strategy profile $\sigma^0 \in \mathcal{A}_N$: for every $j \in N$,

$$\sigma^0_j = \begin{cases} R & \text{if } j \in R, \\ \sigma_j & \text{otherwise.} \end{cases}$$

By **UN**, $R \in \mathcal{B}(\sigma^0)$. Let $R \setminus T = \{i_1, ..., i_m\}$ for some natural number m. Note that for every $i \in R \setminus T$, $\sigma^0_i = R$.

Now, define $\sigma^1 \in \mathcal{A}_N$ as

$$\sigma^1_j = \begin{cases} \sigma_j & \text{if } j = i_1, \\ \sigma^0_j & \text{otherwise.} \end{cases}$$

In view of $i_1 \notin T$ and (A.1), we have $\sigma_{i_1} \neq R$. By **IR**, for any $j, k \in R \setminus \{i_1\}$, we have $\mathcal{B}_j(\sigma^1) = \mathcal{B}_k(\sigma^1)$, which implies $\mathcal{B}_j(\sigma^1) \supseteq R \setminus \{i_1\}$ for every $j \in R \setminus \{i_1\}$. Since $\sigma^1_j \neq R$ for any $j \in (N \setminus R) \cup \{i_1\}$ and $\sigma^1_j = R$ for any $j \in R \setminus \{i_1\}$, **DA** implies that $\mathcal{B}_j(\sigma^1) \subseteq R \setminus \{i_1\}$ for every $j \in R \setminus \{i_1\}$. Hence, we obtain $\mathcal{B}_j(\sigma^1) = R \setminus \{i_1\}$ for every $j \in R \setminus \{i_1\}$. In the same manner, given σ and σ^1, we define $\sigma^2 \in \mathcal{A}_N$ as

$$\sigma^2_j = \begin{cases} \sigma_j & \text{if } j = i_2, \\ \sigma^1_j & \text{otherwise.} \end{cases}$$

Similarly, by **IR** and **DA**, we obtain $\mathcal{B}_j(\sigma^2) = R \setminus \{i_1, i_2\}$ for every $j \in R \setminus \{i_1, i_2\}$. Repeating this procedure until i_m, we have $\sigma^m = \sigma$ and, hence, $\mathcal{B}_j(\sigma) = \mathcal{B}_j(\sigma^m) = R \setminus \{i_1, ..., i_m\}$ for every $j \in R \setminus \{i_1, ..., i_m\}$. As $T = R \setminus \{i_1, ..., i_m\}$, we obtain $\mathcal{B}_j(\sigma) = T$ for every $j \in T$. Hence, in view of Step 2, $\mathcal{B}(\sigma) = K^\sigma \cup L^\sigma = H^\sigma$.

Step 4. We show $H^\sigma = \{S \subseteq N | i, j \in S \iff \sigma_i = \sigma_j\}$. Let $S \in H^\sigma$. There exists $T \subseteq N$ such that for any $j \in S$, $\sigma_j = T$. Hence, it holds that $i, j \in S \Rightarrow \sigma_i = \sigma_j$. Now consider $j \in N \setminus S$. There exists $T' \subseteq N$ such that $T' \neq T$ and $\sigma_j = T'$. Hence, it follows that $i \in S$ and $j \notin S \Rightarrow \sigma_i \neq \sigma_j$. Thus $S \in \mathcal{B}^\delta$. Now let $S \in \mathcal{B}^\delta$. Let $T := \sigma_j = \sigma_k$ for any $j, k \in S$. Then we have $H^\sigma(T) = S$ and $H^\sigma(T) \in H^\sigma$, which means $S \in H^\sigma$.

In view of Steps 3 and 4, $\mathcal{B}(\sigma) = H^\sigma = B^\delta(\sigma)$. This completes the proof. □

Proof of Proposition 3.2 (Axiomatization of $\mathcal{B}^\gamma$)

Proof. The if-part follows from Lemma A.3. We show the only-if-part. Step 1 and Step 2 are the same as Proposition 3.1.

Step 3. For any coalition $T \in L^\sigma$ with $|T| = 1$, say $\{i\} = T$, we have $\sigma_i \neq \sigma_j$ for any $j \in N \setminus \{i\}$. Hence, by **DA**, $B_i(\sigma) = \{i\}$. Now, we fix a coalition $T \in L^\sigma$ with $|T| \geq 2$ and consider a player $i \in T$. Let $R := \sigma_i$. In the same manner as Proposition 3.1, we have $R \supsetneq T$ and $|R| \geq 3$. For the given strategy profile σ, similarly define strategy profile $\sigma^0 \in \mathcal{A}_N$ as

$$\sigma^0_j = \begin{cases} R & \text{if } j \in R, \\ \sigma_j & \text{otherwise.} \end{cases}$$

By **UN**, $R \in \mathcal{B}(\sigma^0)$. Let $R \setminus T = \{i_1, ..., i_m\}$ for some natural number m. In view of $T \in H^\sigma$, we have $\sigma_j \neq R$ for every $j \in N \setminus T$. Hence, $\sigma^0_j = R \neq \sigma_j$ for every $j \in R \setminus T$. It follows from $\mathbf{DR}^+$that

$$\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma) \text{ for any different } i, j \in T. \tag{A.2}$$

Since $\sigma_j \neq R$ for every $j \in N \setminus T$ and $\sigma_j = R$ for every $j \in T$, in view of **DA**, we have

$$\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma) \text{ for any } i \in T \text{ and any } j \in N \setminus T. \tag{A.3}$$

From (A.2) and (A.3), it follows that $\mathcal{B}_i(\sigma) \neq \mathcal{B}_j(\sigma)$ for any $i \in T$ and any $j \in N \setminus \{i\}$. Hence, $\mathcal{B}_i(\sigma) = \{i\}$ for each $i \in T$. This holds for every coalition $T \in L^\sigma$ with $|T| \geq 2$. Hence, we have $\mathcal{B}(\sigma) = K^\sigma \cup \{\{i\} | i \in \widehat{L^\sigma}\}$, where $\widehat{L^\sigma} = \bigcup_{T \in L^\sigma} T$. Thus, $\mathcal{B}(\sigma) = \mathcal{B}^\gamma(\sigma)$, which completes the proof. □

Proof of Proposition 4.1 (No CA-function simultaneously satisfies **UN**, **PD**, **MO**)

Proof. Let $\mathcal{B}$ satisfy **UN**, **PD**, and **MO**. Fix a player $i \in N$. Let $\sigma^1 \in \mathcal{A}_N$ be

$$\sigma_j^1 = \begin{cases} \{1, i\} & \text{if } j = i, \\ \{j, i\} & \text{otherwise,} \end{cases}$$

namely, $\sigma^1 = (\{1,i\}, \{2,i\}, ..., \{i-1,i\}, \{1,i\}, \{i+1,i\}, ..., \{n,i\})$. In view of **UN**, $\{1,i\} \in \mathcal{B}(\sigma^1)$. In the same manner, for $k = 1, ..., i-1, i+1, ..., n$, we define

$$\sigma_j^k = \begin{cases} \{k, i\} & \text{if } j = i, \\ \{j, i\} & \text{otherwise} \end{cases}$$

and obtain $\{k,i\} \in \mathcal{B}(\sigma^k)$ by **UN**. Now, let σ^* be

$$\sigma_j^* = \begin{cases} N & \text{if } j = i, \\ \{j, i\} & \text{otherwise.} \end{cases}$$

From **MO**, it follows that

$$\{1,i\} \subseteq \mathcal{B}_i(\sigma^*), \{2,i\} \subseteq \mathcal{B}_i(\sigma^*), ..., \{n,i\} \subseteq \mathcal{B}_i(\sigma^*).$$

Hence, $\mathcal{B}_i(\sigma^*) = N$. This holds for every $i \in N$, namely, $\mathcal{B}_i(\sigma^*) = N$ for every $i \in N$. However, since it does not hold that $2 \in \sigma_1^*$ and $1 \in \sigma_2^*$, players 1 and 2 are not a pair. Hence, in view of **PD**, $\mathcal{B}_1(\sigma^*) \neq \mathcal{B}_2(\sigma^*)$. This is a contradiction. □

Lemma for Proposition 4.2

Lemma A.1. Let $\mathcal{B}$ be an CA-function and $\sigma \in \mathcal{A}_N$. If $\mathcal{B}$ satisfies **UN** and **MO**, then $\mathcal{B}_i(\sigma) \supseteq D_i(\sigma)$ for every $i \in N$.

Proof. Assume that there exists $k \in D_i(\sigma)$ such that $k \notin \mathcal{B}_i(\sigma)$. Since $k \in D_i(\sigma)$, we have $k \in \sigma_i$ and $i \in \sigma_k$. We define σ' as follows: for every $j \in N$,

$$\sigma_j' = \begin{cases} \sigma_j & \text{if } j \in N \setminus \{i, k\}, \\ \{i, k\} & \text{otherwise (namely, } j = i \text{ or } k). \end{cases}$$

By **UN**, $\mathcal{B}_i(\sigma') = \mathcal{B}_k(\sigma') = \{i,k\}$. Note that $\sigma_i \supseteq \{i,k\} = \sigma_i'$ because $i \in \sigma_i$ and $k \in \sigma_i$. Now define σ^* as follows: for every $j \in N$,

$$\sigma_j^* = \begin{cases} \sigma_j & \text{if } j \in N \setminus \{k\}, \\ \{i, k\} & \text{otherwise (namely, } j = k). \end{cases}$$

By **MO**, $\mathcal{B}_i(\sigma^*) \supseteq \mathcal{B}_i(\sigma') = \{i,k\}$. Since $k \in \mathcal{B}_k(\sigma^*)$, $\mathcal{B}_i(\sigma^*) = \mathcal{B}_k(\sigma^*)$. Hence, $\mathcal{B}_k(\sigma^*) \supseteq \{i,k\}$. In a similar manner, by **MO**, $\mathcal{B}_k(\sigma) \supseteq \mathcal{B}_k(\sigma^*) \supseteq \{i,k\}$. Since $i \in \mathcal{B}_i(\sigma)$, $\mathcal{B}_i(\sigma) = \mathcal{B}_k(\sigma)$. Hence, $\mathcal{B}_i(\sigma) \supseteq \{i,k\}$. This contradicts $k \notin \mathcal{B}_i(\sigma)$. □

Proof of Proposition 4.2 (Axiomatization of $\mathcal{B}^{\approx}$)

Proof. The if-part follows from Lemma A.4. We show the only-if-part. Let $\sigma \in \mathcal{A}_N$. Let $K^\sigma := \{\emptyset \neq S \subseteq N | \sigma_j = S \text{ for all } j \in S\}$. In view of **UN**, $K^\sigma \subseteq \mathcal{B}(\sigma)$. Let $\widehat{K^\sigma} := \bigcup_{S \in K^\sigma} S$. Every player i in $N \setminus \widehat{K^\sigma}$ is not a unanimous member: there is a player $j \in \sigma_i$ such that $\sigma_j \neq \sigma_i$. We partition $N \setminus \widehat{K^\sigma}$ into $\{S_1, ..., S_M\}$ for some natural number M by indirect pairs: for each $m = 1, ..., M$,

$$i \approx j \text{ for any } i, j \in S_m,$$
$$i \not\approx j \text{ for any } i \in S_m \text{ and any } j \notin S_m.$$

We fix an arbitrary coalition $S \in \{S_1, ..., S_M\}$. From **PD**$^-$, it follows that

$$\mathcal{B}_i(\sigma) \subseteq S \text{ for every } i \in S. \tag{A.4}$$

Claim. We now prove that $\mathcal{B}_i(\sigma) \supseteq S$ for every $i \in S$. We first fix $i \in S$ and $j \in D_i(\sigma)$. We have $\{i, j\} \subseteq D_i(\sigma)$ and $\{i, j\} \subseteq D_j(\sigma)$. Lemma A.1 implies that

$$\mathcal{B}_i(\sigma) \supseteq D_i(\sigma) \supseteq \{i, j\}, \text{ and}$$
$$\mathcal{B}_j(\sigma) \supseteq D_j(\sigma) \supseteq \{i, j\}.$$

Hence, $\mathcal{B}_i(\sigma) = \mathcal{B}_j(\sigma)$, which implies that $\mathcal{B}_i(\sigma) = \mathcal{B}_j(\sigma) \supseteq (D_i(\sigma) \cup D_j(\sigma))$. We now consider the same j and fix $k \in D_j(\sigma)$. In the same manner, we have $\mathcal{B}_j(\sigma) = \mathcal{B}_k(\sigma) \supseteq (D_j(\sigma) \cup D_k(\sigma))$. For the player j, we have $\mathcal{B}_i(\sigma) = \mathcal{B}_j(\sigma) = \mathcal{B}_k(\sigma) \supseteq (D_i(\sigma) \cup D_j(\sigma) \cup D_k(\sigma))$. Applying this procedure to every $i' \in S$ and every $j' \in D_{i'}(\sigma)$, we have $\mathcal{B}_i(\sigma) \supseteq \bigcup_{j \in S} D_j(\sigma) = S$ for every $i \in S$. This completes the claim.//

In view of (A.4) and the claim, we have $\mathcal{B}_i(\sigma) = S$ for every $i \in S$. Hence, we have $\mathcal{B}_i(\sigma) = \{j \in N | i \overset{\sigma}{\approx} j\}$ for every $i \in N \setminus \widehat{K^\sigma}$. For every $i \in \widehat{K^\sigma}$, since i is a unanimous member, $\mathcal{B}_i(\sigma) = \sigma_i = \{j \in N | i \overset{\sigma}{\approx} j\}$. Thus, $\mathcal{B}(\sigma) = \{T \subseteq N | i, j \in T \iff i \overset{\sigma}{\approx} j\}$. □

Lemmas for the if-parts of Propositions 3.1, 3.2, and 4.2

Lemma A.2. CA-function $\mathcal{B}^\delta$ satisfies **UN**, **DA**, **IR**.

Proof. <u>**UN**</u>: Assume that there is $S \subseteq N$ such that $\sigma_j = S$ for every $j \in S$ and $S \notin \mathcal{B}^\delta(\sigma)$. Since $S \notin \mathcal{B}^\delta(\sigma)$, there are $i \in S$ and $k \in S$ such that $\mathcal{B}^\delta_i(\sigma) \neq \mathcal{B}^\delta_k(\sigma)$: players i and k belong to different coalitions, while $\sigma_i = \sigma_k$. This contradicts the definition of $\mathcal{B}^\delta$. <u>**DA**:</u> This immediately follows from the definition. <u>**IR**:</u> Let $i \in N$ and $j, k \in \mathcal{B}^\delta_i(\sigma)$. They are three different players. Since $i, j, k \in \mathcal{B}^\delta_i(\sigma)$, we have $\sigma_i = \sigma_j = \sigma_k$. It follows from $\sigma_j = \sigma_k$ that $\mathcal{B}^\delta_j(\sigma'_i, \sigma_{-i}) = \mathcal{B}^\delta_k(\sigma'_i, \sigma_{-i})$. □

Lemma A.3. CA-function $\mathcal{B}^\gamma$ satisfies **UN**, **DA**, **DR**$^+$.

Proof. <u>**UN**</u>: This follows in the same manner as $\mathcal{B}^\delta$. <u>**DA**</u>: If $\sigma_i \neq \sigma_j$, then $\mathcal{B}^\gamma_i(\sigma) = \{i\} \neq \{j\} = \mathcal{B}^\gamma_j(\sigma)$. <u>**DR**$^+$:</u> Let $T \in \mathcal{B}^\delta(\sigma)$. Let $S \subseteq T$, $j \in T \setminus S$, and $k \in T \setminus S$. Since $\sigma'_i \neq \sigma_i = T$ for every $i \in S$, we have $\mathcal{B}^\gamma_j(\sigma'_S, \sigma_{-S}) = \{j\}$ and $\mathcal{B}^\gamma_k(\sigma'_S, \sigma_{-S}) = \{k\}$. □

Lemma A.4. CA-function $\mathcal{B}^{\approx}$ satisfies **UN**, **PD**$^-$, **MO**.

Proof. <u>**UN**:</u> If $\sigma_j = S$ for every $j \in S$, then $i \overset{\sigma}{\sim} j$ for any $i, j \in S$, which implies $i \overset{\sigma}{\approx} j$ for any $i, j \in S$. Hence, $S \in \mathcal{B}^{\approx}(\sigma)$. <u>**PD$^-$**:</u> This immediately follows from the definition. <u>**MO**:</u> Fix $i \in N$. In view of the definition, $\mathcal{B}_i^{\approx}(\sigma) = \{j \in N | i \overset{\sigma}{\approx} j\}$. Since $\sigma_i' \supseteq \sigma_i$, we have $\mathcal{B}_i^{\approx}(\sigma) = \{j \in N | i \overset{\sigma}{\approx} j\} \supseteq \{j \in N | i \overset{(\sigma_i', \sigma_{-i})}{\approx} j\} = \mathcal{B}_i^{\approx}(\sigma_i', \sigma_{-i})$. □

Independence

Define $\mathcal{B}^s$ as follows: $\mathcal{B}^s(\sigma) = \{\{i\} | i \in N\}$ for every $\sigma \in \mathcal{A}_N$. Functions $\mathcal{B}^1$ to $\mathcal{B}^4$ are provided in the following table, in which underlined partitions violate the corresponding axiom.

Independence for $\mathcal{B}^\delta$

- $\mathcal{B}^\gamma$ satisfies **UN** and **DA**, but violates **IR**.
- $\mathcal{B}^1$ satisfies **UN** and **IR**, but violates **DA**.
- $\mathcal{B}^s$ satisfies **DA** and **IR**, but violates **UN**.

Independence for $\mathcal{B}^\gamma$

- $\mathcal{B}^\delta$ satisfies **UN** and **DA**, but violates **DR$^+$**.
- $\mathcal{B}^2$ satisfies **UN** and **DR$^+$**, but violates **DA**.
- $\mathcal{B}^s$ satisfies **DA** and **DR$^+$**, but violates **UN**.

Independence for $\mathcal{B}^{\approx}$

- $\mathcal{B}^3$ satisfies **UN** and **PD$^-$**, but violates **MO**.
- $\mathcal{B}^4$ satisfies **UN** and **MO**, but violates **PD$^-$**.
- $\mathcal{B}^s$ satisfies **PD$^-$** and **MO**, but violates **UN**.

$\mathcal{B}^1$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	Y	Y	Y	X	I	I	$\underline{\text{Y}}$	X	I	Z	$\underline{\text{Y}}$	X	I	I	I
12	Z	X	I	$\underline{\text{Y}}$	I	X	I	$\underline{\text{Y}}$	I	X	Z	$\underline{\text{Y}}$	I	X	I	I
13	Z	I	X	$\underline{\text{Y}}$	Y	Y	Y	Y	I	I	Z	$\underline{\text{Y}}$	I	I	I	I
1	Z	$\underline{\text{Z}}$	$\underline{\text{Z}}$	I	$\underline{\text{Z}}$	$\underline{\text{Z}}$	$\underline{\text{Z}}$	I	$\underline{\text{Z}}$	$\underline{\text{Z}}$	Z	I	I	I	I	I

$\mathcal{B}^2$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	I	I	I	I	I	I	$\underline{\text{Y}}$	I	I	Z	$\underline{\text{Y}}$	I	I	I	I
12	I	X	I	$\underline{\text{Y}}$	I	X	I	$\underline{\text{Y}}$	I	X	Z	$\underline{\text{Y}}$	I	X	I	I
13	I	I	X	$\underline{\text{Y}}$	Y	Y	Y	Y	I	I	Z	$\underline{\text{Y}}$	I	I	I	I
1	I	$\underline{\text{Z}}$	$\underline{\text{Z}}$	I	$\underline{\text{Z}}$	$\underline{\text{Z}}$	$\underline{\text{Z}}$	I	$\underline{\text{Z}}$	$\underline{\text{Z}}$	Z	I	I	I	I	I

$\mathcal{B}^3$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	Y	Y	Y	I	I	I	I	I	I	Z	I	I	I	I	I
12	Z	X	I	I	I	X	I	I	I	X	Z	I	I	X	I	I
13	Z	I	X	I	Y	Y	Y	Y	I	I	Z	I	I	I	I	I
1	Z	I	I	I	I	I	I	I	I	I	Z	I	I	I	I	I

$\mathcal{B}^4$

$\mathcal{A}_3$	123				13				23				3			
$\mathcal{A}_1 \setminus \mathcal{A}_2$	123	12	23	2	123	12	23	2	123	12	23	2	123	12	23	2
123	N	N	N	Y	N	N	Y	Y	N	X	Z	I	X	X	I	I
12	N	X	N	I	X	X	I	I	N	X	Z	I	X	X	I	I
13	N	N	N	Y	Y	Y	Y	Y	Z	I	Z	I	I	I	I	I
1	Z	I	Z	I	I	I	I	I	Z	I	Z	I	I	I	I	I

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