WINPEC Working Paper Series No.E1813
January 2019

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January 14, 2019

Abstract

This paper presents a three-period model to analyze why central-bank notes, i.e., payment instruments, and bank deposits, i.e., saving instruments, must be separated from each other as is the case in the modern banking system. The model shows that credit creation by note-issuing commercial banks improves risk sharing in the economy, because private bank notes can serve as payment instruments backed by a diversified pool of commercial bills issued by payers. If there are sunk costs of production, however, this characteristic of private bank notes can cause a self-fulfilling mass refusal of private bank notes by payees. Commercial banks can reduce the amount of reserves necessary to prevent such a bank-note run if they set up a conduit that issues only payment instruments, which corresponds to the central bank. The model replicates short-term re-discounting of commercial bills by the central bank as the optimal policy endogenously.

Keywords: Bank notes; Credit creation; Trade credit; Central bank; Discount window; Bank run

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1 Introduction

Historically, commercial banks had been issuing private bank notes; however, central banks subsequently monopolized bank-note issues in their countries. As a result, central-bank notes, i.e., payment instruments, and bank deposits, i.e., saving instruments, are issued by separate entities in the modern banking system.

This feature of the modern banking system is inconsistent with the standard result in theoretical literature that liquid stores of wealth are not only held as saving instruments, but also transferred as payment instruments.[1] It also poses a question as to whether the central bank should refrain from issuing both bank notes and bank deposits for households and non-financial firms, even if technological progress lowers physical transaction costs to do so. To provide an explanation for these observations, this paper presents a parsimonious model that endogenizes the separation between central-bank notes and bank deposits.

The key feature of the model is imperfect synchronization of different markets. In this environment, if a producer buys an input from a supplier by issuing a commercial bill, then there occurs an inevitable time lag before the supplier can access a financial market to share idiosyncratic default risk in commercial bills with other suppliers. Because idiosyncratic default risk in commercial bills can be updated any time, the presence of the time lag makes risk sharing among suppliers imperfect. As a result, suppliers charge a risk premium on commercial bills paid by producers.

This problem can be remedied if there exist commercial banks issuing private bank notes in exchange for commercial bills before producers make payments to suppliers. In this case, suppliers do not charge a risk premium on private bank notes paid by producers, because private bank notes are backed by a diversified pool of commercial bills through commercial banks' balance sheets; thus, an update on idiosyncratic default risk in commercial bills does not affect the total return on a pool of commercial bills, or private bank notes, given the law

[1]For example, see Woodford (1990) and Kiyotaki and Moore (2005) for such models.

of large numbers. This arrangement can be regarded as credit creation, because commercial banks provide bank credit to producers before suppliers become the ultimate creditors of producers via commercial banks' balance sheets by accepting private bank notes.

This result contrasts with the standard theory of money, such as Samuelson (1958), Townsend (1980), and Kiyotaki and Wright (1989), which derives the need for money from a lack of credit. This paper adds to this literature by presenting an alternative environment in which bank notes are essential despite the availability of trade credit. In addition, characterizing commercial banks as the providers of credit creation clarifies the difference between commercial banks and money market mutual funds, the latter only pool securities. This implication of the model contrasts with Corrigan's (1982) classical analysis, which defines a bank as any organization that is eligible to issue transaction accounts.

The model, however, shows that note-issuing commercial banks are subject to a self-fulfilling run on private bank notes, if producers must install inputs supplied by multiple suppliers sequentially, and if installed inputs become sunk costs. In this case, if suppliers refuse to sell inputs for private bank notes presented by producers, then producers cannot produce goods or retrieve inputs installed earlier, and hence must default on commercial bills held by commercial banks. As a result, commercial banks must default on private bank notes, justifying suppliers' refusals of private bank notes.

This result is related to Diamond and Dybvig's (1983) model of a self-fulfilling run on demand deposits.[2] In contrast to this type of bank run, a self-fulfilling run on private bank notes does not rely on such an assumption that private bank notes are redeemable on demand, or that bank assets can be liquidated at some liquidation cost before maturity. It does not require a sequential service constraint for the repayments of deposits either.[3] Thus,

[2]See Ennis and Keister (2010) for a survey of this type of model.

[3]These features of private bank notes in the model are similar to those of bank shares in Jacklin's (1987) model, which shows that a self-fulfilling bank run on demand deposits can be prevented without any efficiency loss if banks issue shares instead of demand deposits. In this regard, this paper presents an alternative environment in which banks suffer a self-fulfilling run despite issuing no demand deposits. In related literature, Allen and Gale (2004) present a model in which a run on demand deposits occurs

the suspension of payments does not prevent a self-fulfilling run on private bank notes. This result holds because a self-fulfilling run on private bank notes occurs due to endogenous default on bank assets, rather than a costly liquidation of bank assets. This feature of the model is consistent with the fact that banking crises often occurred among note-issuing commercial banks in England and the U.S. in the 19th century despite the suspension of payments being a standard response in the banking sector (see Calomiris and Gorton 1991, Dwyer 1996, and Quinn 2004, for example).

In the literature, Kobayashi and Nakajima (2014) also consider a self-fulfilling bank run due to endogenous default on bank assets to analyze the misallocation of capital and labor during a banking crisis in a dynamic general equilibrium model.[4] This paper adds to their work by showing that a self-fulfilling run on private bank notes is inherent in credit creation by note-issuing commercial banks.

This paper further shows that the instability of private bank notes leads to a need for central-bank notes. The model demonstrates that a self-fulfilling run on private bank notes can be prevented if commercial banks hold a sufficiently large amount of independent safe assets as reserves. Nonetheless, the amount of reserves necessary to prevent a self-fulfilling run can be reduced if commercial banks set up a conduit that issues only payment instruments, which corresponds to the central bank.

In this case, commercial banks pool commercial bills received from producers at the central bank, and supply to producers central-bank notes backed by a pool of commercial bills. Commercial banks also pool their reserves at the central bank. Then, once producers pay central-bank notes to suppliers, commercial banks collect these central-bank notes

if and only if it is unavoidable given the market price of bank assets. Allen, Carletti, and Gale (2009) show that the central bank's open market operation can prevent the excess volatility of asset prices due to cash-in-the-market asset pricing in this environment.

[4]In their model, households' lending to firms must be intermediated by banks, and firms need to obtain from banks long-term loans to install capital stock, and then short-term loans to finance their use of factors of production, i.e., working capital. They show the existence of an equilibrium in which households refuse to make short-term deposits at banks on self-fulfilling expectations that firms cannot obtain short-term loans from banks and thus must default on long-term loans held by banks.

from suppliers by issuing deposit certificates in exchange. At the same time, they also retrieve commercial bills from the central bank, so that deposit certificates are fully backed by commercial bills.

This arrangement ensures that there are no incumbent holders of central-bank notes when the central bank issues new central-bank notes to producers. As a result, if suppliers' refusals of central-bank notes caused producers to default on commercial bills, then all the reserves pooled at the central bank would be allocated to protect central-bank notes held by producers. This feature of central-bank notes reduces suppliers' incentive to refuse central-bank notes paid by producers, helping to prevent a self-fulfilling run. Private bank notes do not have such a feature, as they are both paid to current suppliers and saved by past suppliers, given the absence of the central bank.

This result confirms that separating central-bank notes and bank deposits improves the efficiency of the banking system. Furthermore, the round-trip transfer of commercial bills in the process of central-bank note supply replicates short-term re-discounting of commercial bills by the central bank in practice. This result adds to the literature on elastic money supply by the central bank or a private clearing house, such as Freeman (1996a, 1999), Green (1997), Fujiki (2003, 2006), Martin (2004), Mills (2006), Gu et al. (2011), and Chapman and Martin (2013), and also the literature on elastic money supply and bank runs, such as Champ, Smith, and Williamson (1996), Williamson (1998), Smith (2002), Antinolfi and Keister (2006), Martin (2006), and Allen, Carletti, and Gale (2014).[5] In this regard, the contribution of this paper is to derive an endogenous role of the central bank in elastic money supply without assuming the special ability for the central bank to issue generally acceptable fiat money or spatial separation.

Finally, historical analysis points out that there were inefficiencies in private bank-note issues in the past, such as discounts due to physical distances from the branches of issuing banks (Gorton 1999, 2014) and risky reserves, such as state bonds, that caused banking crises

[5] Also, see Ennis (2016) for a survey of models on discount window lending.

during the free banking era in the U.S. (Rolnick and Weber 1984). Also, there is a strand of literature that analyzes private bank-note issues with agency problems and externality, such as Freeman (1996b), Monnet and Sanches (2015), and Sanches (2016). These problems play no role in this paper. Thus, this paper adds to the literature by showing that even if these problems of private bank notes disappear, introducing a central bank as the issuer of payment instruments still improves the efficiency of the banking system.

The remainder of the paper is organized as follows. The baseline model is defined in section 2. Commercial banks are introduced in section 3. The instability of private bank notes is analyzed in section 4. The benefit of central-bank notes in the presence of reserve assets is described in sections 5 and 6. Section 7 discusses related issues. Section 8 concludes.

2 Baseline model

Time is discrete and indexed by 0, 1, and 2. There exist two types of agents, manufacturers and loggers, each of which is a unit continuum. Each agent is indexed by $(i, j) \in \{M, L\} \times [0, 1]$, where $i = M$ if the agent is a manufacturer and $i = L$ if the agent is a logger, and j is the subindex among agents of the same type. For each type, the measure of agents is defined by the Lebesgue measure of the set of the agents' subindices on $[0, 1]$. A logger is risk-averse, maximizing expected utility, $Eu(c_{L,j})$, where E is the expectation operator in period 0; the function u is strictly increasing and concave, and twice differentiable; and $c_{L,j}$ denotes logger j's (i.e., agent (L, j)'s) consumption of goods in period 2 for $j \in [0, 1]$. A manufacturer is risk-neutral, maximizing the expected value of consumption of goods in period 2.[6]

Each logger is endowed with a unit of wood in period 0, and can produce an amount $\underline{\alpha}$ of goods in period 2 by using a unit of wood in the period. In contrast, each manufacturer receives no endowment of wood, but can produce in period 2 an amount $\bar{\alpha}$ of goods with probability $\mu_{M,j}$ and no goods with probability $1 - \mu_{M,j}$ from each unit of wood used in

[6]This assumption simplifies the model by obviating the need for considering risk sharing among manufacturers throughout the paper.

period 0, where $\mu_{M,j}$ is specific to manufacturer j (i.e., agent (M, j)) for $j \in [0, 1]$. For all $j \in [0, 1]$, $\mu_{M,j}$ is an i.i.d. random variable whose value is revealed publicly in period 1, and the probability distribution of $\mu_{M,j}$ is uniform over $[0, 1]$. Assume that

$$\bar{\alpha} > 4\underline{\alpha} > 0 \tag{1}$$

This assumption ensures that it is socially optimal for manufacturers to use loggers' wood for their production in any case considered below.

In period 0, each manufacturer is randomly matched with a logger, and vice versa. Thus, every agent has one match. In each match, a manufacturer can make a take-it-or-leave-it offer to buy wood from a logger with an IOU that promises to deliver goods in period 2. A manufacturer can be committed to delivering any share of goods produced in period 2 upon successful production, and involuntarily defaults on its IOU if it fails to produce goods in the period. Hereafter, a manufacturer's IOU is referred to as a "commercial bill".

In period 1, there exists a bill market, in which loggers can swap commercial bills received from manufacturers for bill dealers' IOUs. There are N symmetric bill dealers in the market, where N is an integer not less than 2. Bill dealers offer loggers an exchange rate between their IOUs and commercial bills through Bertrand competition.

Assume that loggers can arrive at the bill market only after the realization of $\mu_{M,j}$ for $j \in [0, 1]$ in period 1. The underlying assumption is that agents cannot synchronize different market transactions perfectly. Thus, there exists an inevitable time lag between a goods market transaction and a bill market transaction. Given this time lag, it is natural to assume that there can be a revision of the idiosyncratic default probability for each commercial bill any time while the holders of commercial bills transit from the goods market to the bill market. The realization of $\mu_{M,j}$ for $j \in [0, 1]$ in period 1 is a stylized assumption to incorporate such a continuous update on idiosyncratic default risk in a discrete-time set-up.

An equilibrium is characterized by a zero profit condition for each bill dealer in the bill market in period 1 due to Bertrand competition; a take-it-or-leave-it-offer of a commercial bill by a manufacturer to a logger in each pairwise meeting in period 0; and rational expectations held by agents and bill dealers. See Table 1 for a summary of the model.

Table 1: Chronological order of events in the baseline model

Period 0	Each manufacturer can make a take-it-or-leave-it offer of its commercial bill for a logger's wood in a pairwise meeting.
Period 1	The idiosyncratic default probability for each commercial bill is revealed publicly.
	Loggers can swap commercial bills for bill dealers' IOUs in a bill market with Bertrand competition among bill dealers.
Period 2	Manufacturers repay their commercial bills upon successful production of goods. Bill dealers repay their IOUs by the repayments on commercial bills they hold.

2.1 Equilibrium

The zero profit condition for bill dealers implies that each commercial bill is priced fairly in the bill market in period 1:

$$p_{M,j} = \mu_{M,j} b_{TC} \tag{2}$$

for $j \in [0, 1]$, where $p_{M,j}$ denotes the face value of a bill dealer's IOU that is exchanged for manufacturer j's commercial bill in period 1, given $\mu_{M,j}$; and b_{TC} denotes the face value of a commercial bill issued by each manufacturer to a logger in a pairwise meeting in period 0.[7] Both face values, $p_{M,j}$ and b_{TC}, are in terms of the amounts of goods repayable in period 2. The value of b_{TC} is the same across manufacturers, given the symmetry of manufacturers in period 0. There is no risk premium in $p_{M,j}$, as each bill dealer can diversify idiosyncratic default risk by holding commercial bills issued by a positive measure of manufacturers.

[7]The subscript "TC" stands for "trade credit".

Given (2), each manufacturer chooses the value of b_{TC} that just satisfies a logger's participation constraint for selling wood in a pairwise meeting in period 0:

$$\int_0^1 u(\mu b_{TC})\, d\mu = u(\underline{\alpha}) \tag{3}$$

where μ denotes the realized success probability for a manufacturer's production in period 2. The left-hand side and the right-hand side of (3) are a logger's expected utilities when selling wood for a commercial bill and when retaining wood for the logger's own production, respectively. In the equilibrium, a manufacturer can issue a commercial bill whose face value equals b_{TC} if and only if the amount of goods that a manufacturer can produce upon successful production, $\bar{\alpha}$, is not less than b_{TC}. Otherwise, loggers retain wood.

The baseline model can be regarded as a primitive economy relying on trade credit without a banking service. This interpretation is consistent with the fact that trade credit from the wholesalers of inputs to manufacturers had become common in various industries in England by the late 17th century, preceding the development of commercial banking.[8] Also, the goods market is assumed to be decentralized, which makes it impossible for a logger to diversify idiosyncratic default risk in commercial bills by selling wood to a continuum of manufacturers simultaneously. This assumption can be interpreted as the need for each producer to use a specific supplier's input. Only the modern feature of the baseline model is the presence of bill dealers for asset pooling, which serve the same function as mutual funds in practice without offering a payment service. Thus, the baseline model can be interpreted as inserting mutual funds into a primitive economy without banks. In the next section, commercial banks are introduced into this economy to show that they can improve social welfare because they issue payment instruments.

[8]For example, manufacturers in the 18th-century West Riding textile industry often got wool from wholesalers on credit. Big wholesalers in other industries like linen, iron, and groceries were also major sources of credit. See Quinn (2004) for more details.

3 Credit creation by note-issuing commercial banks

In the baseline model, (3) implies that the face value of a commercial bill that a manufacturer must issue to buy wood, b_{TC}, involves a risk premium, because a logger faces uncertainty on the resale price of a commercial bill in period 1, $p_{M,j}$ for $j \in [0,1]$, when selling wood for a commercial bill in period 0. This incomplete risk sharing due to a time lag between a goods market transaction and a bill market transaction can be remedied if bill dealers can pool commercial bills in advance, so that manufacturers can pay loggers bill dealers' IOUs backed by a diversified pool of commercial bills. Hereafter, such bill dealers are referred to as "commercial banks", and their IOUs are referred to as "private bank notes", given their use as payment instruments. See Table 2 for the classification of IOUs when commercial banks are introduced into the model.

Table 2: Classification of IOUs in the model with commercial banks

Issuer	Name of IOU
Manufacturer	Commercial bill
Commercial bank	Private bank note

3.1 Introducing note-issuing commercial banks

Suppose that there exist N symmetric commercial banks, where N is an integer not less than 2. At the beginning of period 0, manufacturers and commercial banks can swap commercial bills and private bank notes with each other in a competitive market, where commercial banks offer an exchange rate between commercial bills and private bank notes through Bertrand competition.[9] This market opens before manufacturers and loggers are matched pairwise in period 0. The subsequent events are as same as in the baseline model. The definition of an equilibrium is also as same as in the baseline model, except that it includes a zero profit

[9] This assumption is consistent with the fact that banks issued private bank notes mainly by discounting promissory notes or bills presented by customers during the free banking era in the U.S. See Weber (2015a) for more details.

condition for each commercial bank due to Bertrand competition. See Table 3 for a summary of events in the model with commercial banks. The bill market in period 1 is omitted in Table 3 and thereafter, as it does not play any role in the presence of commercial banks.

Table 3: Chronological order of events in the model with commercial banks

Period 0	Manufacturers can swap their commercial bills for private bank notes in a competitive market with Bertrand competition among commercial banks. Each manufacturer can make a take-it-or-leave-it offer of its commercial bills and private bank notes for a logger's wood in a pairwise meeting.
Period 1	The idiosyncratic default probability for each commercial bill is revealed publicly.
Period 2	Manufacturers repay commercial bills upon successful production of goods. Commercial banks repay private bank notes by the repayments on commercial bills they hold.

3.2 Equilibrium with note-issuing commercial banks

In the equilibrium, the zero profit condition for each commercial bank implies that commercial bills are exchanged for private bank notes at a fair price in period 0:

$$q = E[\mu_i b_{BK}] = \frac{b_{BK}}{2} \tag{4}$$

where b_{BK} denotes the face value of a commercial bill issued by each manufacturer to a commercial bank; and q denotes the face value of private bank notes that are exchanged for a commercial bill.[10] Both face values, b_{BK} and q, are in terms of the amount of goods repayable in period 2. As is the case with bill dealers in the baseline model, there is no risk premium in q because each commercial bank can diversify idiosyncratic default risk by holding commercial bills issued by a positive measure of manufacturers.

[10]The subscript "BK" stands for "bank".

Given (4), each manufacturer chooses the value of b_{BK} that just satisfies a logger's participation constraint for selling wood in a pairwise meeting in period 0:

$$u\left(\frac{b_{BK}}{2}\right) = u(\underline{\alpha}) \tag{5}$$

The left-hand side and the right-hand side of (5) are a logger's expected utilities when selling wood for private bank notes paid by a manufacturer and when retaining wood, respectively. Thus, $b_{BK} = 2\underline{\alpha}$, given $u' > 0$. Because $\bar{\alpha} > b_{BK}$ as implied by (1), it is feasible for each manufacturer to issue a commercial bill whose face value equals b_{BK}.

Given $u'' < 0$, applying Jensen's inequality to (3) and (5) implies that

$$b_{BK} < b_{TC} \tag{6}$$

This difference is due to a risk premium on an individual commercial bill paid to a logger. Thus, manufacturers have incentive to swap their commercial bills for private bank notes in period 0 to avoid incurring a risk premium.[11] Note that manufacturers receive all the social surplus, because they keep loggers indifferent by making take-it-or-leave-it offers for wood, while bill dealers and commercial banks earn no profit in the equilibrium. Hence, a higher expected profit for each manufacturer is equivalent to an improvement in social welfare in terms of Pareto efficiency.

The exchange between commercial bills and private bank notes in period 0 can be interpreted as commercial banks' lending of private bank notes. It can be also regarded as credit creation, because the extension of bank credit to manufacturers takes place before loggers become the ultimate creditors of manufacturers via commercial banks' balance sheets by accepting private bank notes.

[11] Note that commercial banks can diversify idiosyncratic default risk in commercial bills, even if they issue private bank notes in exchange for some commercial bills in period 0, and then buy the remaining commercial bills in the bill market in period 1. Thus, if a manufacturer pays a commercial bill to a logger in the presence of commercial banks, then the logger can sell the commercial bill to a commercial bank at a fair price in the bill market in period 1, as shown in (2). As a result, the face value of a commercial bill that a manufacturer must issue to buy wood from a logger in this case equals b_{TC}, as shown in (3).

To clarify, the result of the model does not change even if there is an update on the idiosyncratic default probability for each commercial bill, $1 - \mu_{M,j}$ for $j \in [0, 1]$, during a time lag between the supply of private bank notes and goods market transactions in period 0. This is because such an update has no effect on the return on a pool of commercial bills held by each commercial bank, or the return on private bank notes, given the law of large numbers. Hence, the model is robust to any timing of an update on idiosyncratic default risk in commercial bills after commercial banks issue private bank notes.[12]

4 Self-fulfilling run on private bank notes

4.1 Introducing sunk costs of production

Now introduce sunk costs of production into the model. In this environment, note-issuing commercial banks can be subject to a self-fulfilling run.

In addition to manufacturers and loggers, assume there is a $[0, 1]$ continuum of miners, each of which is endowed with a unit of clay in period 1. Like a logger, a miner can produce an amount $\underline{\alpha}$ of goods in period 2 by using a unit of clay in period 1, and also maximizes the same concave utility function as a logger. With the introduction of clay, assume that a manufacturer needs to install not only a unit of wood in period 0, but also a unit of clay in period 1, to conduct its production. Once installed, wood cannot be retrieved for any alternative use. The other features of a manufacturer's production technology, i.e., the probability of successful production and the amount of goods produced in period 2 upon successful production, remain the same as in the baseline model.

Assume that the markets in period 1 are organized in the same way as the markets in period 0: manufacturers can swap commercial bills for private bank notes in a competitive

[12]It is possible to assume that loggers cannot observe an update on the idiosyncratic default probability for each commercial bill held by commercial banks when they enter the goods market. Even in such a case, the result of the model does not change if each commercial bank can be committed to not unwinding the pool of commercial bills acquired at the beginning of period 0, so that adverse selection does not occur. See Dang et al. (2017) and Tomura (2014) for further analysis of this type of banking service.

market at the beginning of period 1, whereby commercial banks offer an exchange rate between commercial bills and private bank notes through Bertrand competition; and then manufacturers are matched pairwise with miners in that period.[13]

Assume that the value of $\mu_{M,j}$ for $j \in [0,1]$ is realized after the pairwise meetings in period 1, which represents a continuous update on idiosyncratic default risk in commercial bills as described above. The other part of the model remains the same as described in the previous section. See Table 4 for a summary of the model.

Table 4: Chronological order of events in the model with sunk costs of production

Period 0	(∗) Manufacturers can swap their commercial bills for private bank notes in a competitive market with Bertrand competition among commercial banks. (∗∗) Each manufacturer can make a take-it-or-leave-it offer of its commercial bills and private bank notes for a logger's wood in a pairwise meeting.
Period 1	The same events as (∗) and (∗∗) in period 0 take place, except that manufacturers are matched with miners endowed with clay, rather than loggers. The idiosyncratic default probability for each commercial bill is revealed publicly.
Period 2	Manufacturers repay commercial bills upon successful production of goods. Commercial banks repay private bank notes by the repayments on commercial bills they hold.

4.2 Equilibrium without a self-fulfilling run on private bank notes

There exists an equilibrium in which each manufacturer repeats twice the same set of transactions described in the previous section. In this equilibrium, each manufacturer issues to a commercial bank a commercial bill whose face value equals b_{BK} in exchange for private bank notes whose face value equals $b_{BK}/2$ in each of periods 0 and 1. Then, each manufacturer

[13]For this set-up, assume that miners are anonymous, so that miners cannot be committed to delivering clay to manufacturers in period 1 even if they write forward contracts with manufacturers in period 0. As a result, manufacturers must buy clay in pairwise meetings in period 1 when miners are endowed with clay.

pays the private bank notes to a logger and a miner to buy wood in period 0 and clay in period 1, respectively. These transactions satisfy the zero profit condition for commercial banks as well as participation constraints for a logger and a miner to sell their inputs, as implied by (4) and (5). Also, (1) ensures that each manufacturer can repay the two commercial bills issued in periods 0 and 1, the face values of which sum up to $2b_{BK}$, or $4\underline{\alpha}$ as implied by (5), upon successful production in period 2.

Without loss of generality, the remainder of this paper focuses on an equilibrium in which N commercial banks are symmetric in each period; thus, each commercial bank deals with one-Nth of agents of each type (i.e., manufacturers, loggers, and miners). Given this assumption, Table 5 summarizes the evolution of each commercial bank's balance sheet in periods 0 and 1 in the equilibrium described above. The value of a commercial bank's balance-sheet item is defined by the product of the value of securities per agent and the measure of the agents included in the item.

4.3 Self-fulfilling run on private bank notes

In the aforementioned equilibrium, private bank notes serve not only as payment instruments for manufacturers, but also as saving instruments for loggers. As a result, commercial banks have incumbent creditors when they issue new private bank notes in period 1. This characteristic of private bank notes can cause a self-fulfilling mass refusal of private bank notes by new payees, i.e., miners.

To confirm this result, consider the same actions as described in Table 5 up to pairwise meetings in period 1. Then, suppose that a miner expects the other miners' refusals to sell clay to manufacturers in those meetings. If this expectation is correct, then the manufacturers meeting the other miners in period 1 must default on their commercial bills, because they cannot produce any goods in period 2 without clay.

Following the standard bankruptcy rule in reality, assume a pro rata distribution of default losses to the holders of private bank notes at each commercial bank.

Table 5: Evolution of each commercial bank's balance sheet with sunk costs of production in case of no self-fulfilling run on private bank notes

	Events	Each commercial bank's balance sheet (assets)	Each commercial bank's balance sheet (liabilities)
Period 0	Manufacturers and commercial banks swap IOUs (commercial bills and private bank notes) with each other.	Commercial bills issued in period 0 ($b_{BK}/(2N)$)	Private bank notes supplied to manufacturers ($b_{BK}/(2N)$)
	Manufacturers pay private bank notes for loggers' wood in pairwise meetings.	Commercial bills issued in period 0 ($b_{BK}/(2N)$)	Private bank notes paid to loggers in period 0 ($b_{BK}/(2N)$)
Period 1	Manufacturers and commercial banks swap IOUs with each other.	Commercial bills issued in period 0 ($b_{BK}/(2N)$) Commercial bills issued in period 1 ($b_{BK}/(2N)$)	Private bank notes paid to loggers in period 0 ($b_{BK}/(2N)$) Private bank notes supplied to manufacturers ($b_{BK}/(2N)$)
	Manufacturers pay private bank notes for miners' clay in pairwise meetings.	Commercial bills issued in period 0 ($b_{BK}/(2N)$) Commercial bills issued in period 1 ($b_{BK}/(2N)$)	Private bank notes paid to loggers in period 0 ($b_{BK}/(2N)$) Private bank notes paid to miners in period 1 ($b_{BK}/(2N)$)

Notes: The third column shows the balance sheet of each commercial bank, given the symmetry of N commercial banks. In parentheses are the expected returns on commercial bills if they are on the asset side, and the face values of private bank notes if they are on the liability side. Whether it is an expected return or a face value, the value of a commercial bank's balance-sheet item is defined by the product of the value of securities per agent (i.e., $b_{BK}/2$) and the measure of the agents included in the item (i.e., $1/N$).

Assumption 1. *If the returns on assets held by a commercial bank in period 2 are less than the total face value of liabilities issued by the commercial bank, then they are distributed to the creditors of the commercial bank on a pro rata basis in period 2.*

Also assume that a manufacturer can cancel an exchange between its commercial bill and private bank notes in period 1 if its payment of private bank notes is refused by a miner, because there is no gain for a manufacturer and a commercial bank to maintain cross liabilities between them until period 2 in such a case. This assumption can be interpreted as allowing an early repayment of a commercial bill before maturity.[14]

Assumption 2. *A manufacturer can cancel an exchange between its commercial bill and a commercial bank's private bank notes in period 1 by returning the private bank notes to the commercial bank in the same period.*

Table 6 shows a commercial bank's balance sheet in period 1 when all miners refuse to sell clay to manufacturers in the period, given Assumption 2. It implies that a measure $1/N$ of loggers remain the holders of private bank notes at each commercial bank, even after manufacturers return refused private bank notes to commercial banks in period 1.

Table 6 further implies that if a miner sells clay to a manufacturer in period 1 while the other miners refuse to do so, then at most only one manufacturer can repay commercial bills held by commercial banks in period 2, and that this manufacturer's repayments will be shared with a positive measure of loggers at the commercial banks holding the manufacturer's commercial bills, given Assumption 1. Because the measure of a manufacturer is zero, the return on private bank notes is also zero in this case. Thus, if a miner expects the other miners' refusals to sell clay to manufacturers in period 1, then this miner in turn expects a zero return on private bank notes in period 2, regardless of whether it sells clay to a

[14]The difference between the face value of private bank notes lent by a commercial bank, $b_{BK}/2$, and the face value of commercial bills issued by a manufacturer, b_{BK}, can be interpreted as interest payable at maturity. A manufacturer does not have to incur this interest expense if it returns borrowed private bank notes to a commercial bank before maturity.

Table 6: Each commercial bank's balance sheet in period 1 in case of a self-fulfilling run on private bank notes

Commercial bills issued in period 0 (0)	Private bank notes paid to loggers in period 0 ($b_{BK}/(2N)$)
Commercial bills issued in period 1 (canceled)	Private bank notes supplied to manufacturers in period 1 (canceled)

Notes: The table shows each commercial bank's balance sheet after manufacturers repay refused private bank notes to commercial banks in period 1 to retire their commercial bills issued to the commercial banks in the same period. In parentheses are the expected returns on commercial bills if they are on the asset side, and the face values of private bank notes if they are on the liability side. Whether it is an expected return or a face value, the value of a commercial bank's balance-sheet item is defined by the product of the value of securities per agent and the measure of the agents included in the item. "canceled" in a parenthesis indicates that the balance-sheet item is canceled within period 1.

manufacturer in period 1 or not.[15]

A miner with this expectation still sells clay to a manufacturer in period 1 if it can receive an additionally issued commercial bill that satisfies its participation constraint for selling clay in the period:

$$\int_0^1 \mu u(\hat{b}) + (1-\mu)u(0)\, d\mu = u(\underline{\alpha}) \tag{7}$$

where μ denotes the realized success probability for a manufacturer's production in period 2, and $\hat{b}$ denotes the minimum face value of an additional commercial bill that a manufacturer must issue to buy clay from the miner. The left-hand side and the right-hand side of (7) are the values of the miner's expected utilities when selling clay and when retaining clay, respectively. The left-hand side of (7) reflects the miner's expectation that if the miner sells clay for an additional commercial bill issued by a manufacturer, then it cannot share idiosyncratic default risk in the commercial bill with the other miners, because the other miners do not hold any commercial bills, given their refusals to sell clay to manufacturers.

[15]This result does not depend on whether a manufacturer's commercial bills issued in periods 0 and 1 are held by two commercial banks or one commercial bank. In either case, if a miner sells clay to a manufacturer in period 1, then the manufacturer's repayment of a commercial bill upon successful production in period 2 is shared with a measure $1/N$ of loggers at the commercial bank holding the commercial bill. Also note that if no miner sells clay to manufacturers in period 1, then all manufacturers default on commercial bills. Thus, the return on private bank notes is zero in any case.

The miner does not exchange the commercial bill for private bank notes either, given a zero expected return on private bank notes. Thus, if the miner sells clay for an additionally issued commercial bill in period 1, then it receives the direct repayment of the commercial bill, $\hat{b}$, upon successful production and no repayment otherwise, as indicated by (7).

To issue an additional commercial bill whose face value equals $\hat{b}$, the manufacturer meeting the miner in period 1 must be able to repay all the commercial bills it owes upon successful production of goods in period 2.[16] Because the manufacturer cannot buy clay from the miner with private bank notes, it can repay refused private bank notes to retire its commercial bill issued to a commercial bank in period 1, given Assumption 2. Thus, the manufacturer can issue an additional commercial bill whose face value equals $\hat{b}$ if and only if[17]

$$\bar{\alpha} \geq b_{BK} + \hat{b} \tag{8}$$

where b_{BK} is the face value of a commercial bill issued to a commercial bank in period 0.

If (8) is satisfied, then each miner sells clay to a manufacturer in period 1 even if it expects the other miners' refusals to do so; thus this expectation cannot be self-fulfilling. If (8) is violated, then each miner's refusal to sell clay to a manufacturer in period 1, and hence a zero expected return on private bank notes, can be a self-fulfilling rational expectation.[18]

[16] It is assumed that law prohibits a borrower from issuing debt to a new lender in order to be bankrupt and reduce incumbent lenders' shares of the borrower's bankruptcy estate. It is also assumed that the bankruptcy process is so slow that a manufacturer cannot reorganize commercial bills issued to commercial banks before issuing an additional commercial bill to a miner in period 1.

[17] The difference between the left-hand side and the right-hand side of (8) equals the manufacturer's profit upon successful production in period 2. Thus, if (8) is satisfied, then it is incentive-compatible for the manufacturer to issue an additional commercial bill to buy clay in period 1, because the manufacturer can earn a non-negative profit from successful production in period 2, whereas the manufacturer's profit is zero if the manufacturer defaults on commercial bills due to no production of goods in period 2.

[18] In the equilibrium with this expectation, each agent expects a self-fulfilling run on private bank notes if private bank notes are paid to a positive measure of loggers in period 0. Thus, commercial banks can issue private bank notes only at the beginning of period 1. As a result, each manufacturer buys wood from a logger in period 0 by issuing a commercial bill whose face value equals b_{TC}, as in the baseline model.

5 Introducing reserve assets

In the following two sections, it is shown that commercial banks can avoid a self-fulfilling run on private bank notes if they have a sufficiently large amount of safe assets as reserves; however, they can reduce the amount of reserves necessary to prevent a self-fulfilling run, if they set up a conduit that issues only payment instruments, i.e., the central bank.

Let us start from introducing reserves into the model. Suppose that each commercial bank is endowed with an amount g (> 0) of goods in period 0 and can store the goods as reserves until period 2.[19] The other assumptions remain the same as described in section 4.1. Because the presence of reserves does not alter any outcome if commercial banks do not fail, there exists an equilibrium without a self-fulfilling run on private bank notes as described in section 4.2.

To investigate the possibility of a self-fulfilling run, consider the same actions as described in Table 5 up to pairwise meetings in period 1. Then suppose that a miner expects the other miners' refusals to sell clay to manufacturers in those meetings. If this expectation is correct, then the manufacturers meeting the other miners in period 1 return refused private bank notes to commercial banks in the period, as described in section 4.3. In this case, the remaining holders of private bank notes are the manufacturer meeting the miner with this expectation in period 1 and a measure $1/N$ of loggers receiving each commercial bank's private bank notes in period 0, as indicated by Table 6. Given Assumption 2, the miner with this expectation expects that the recovery value of private bank notes paid by the manufacturer that it meets in period 1 is gN, unless gN exceeds the face value of the private bank notes, $b_{BK}/2$, because the reserves held by the commercial bank issuing these private bank notes, g, is equally divided by the remaining bank-note holders at the commercial bank,

[19]This assumption is consistent with the fact that note-issuing commercial banks had to acquire state or federal government bonds by shareholders' capital, and then deposit them at the state or the federal government as reserves during the free banking and the national banking era in the U.S. before the foundation of the U.S. Federal Reserve in 1913. See Weber (2015a, b) for more details.

whose measures sum up to $1/N$.[20]

Suppose that $gN < b_{BK}/2$, or $gN < \underline{\alpha}$ given (5). In this case, the manufacturer meeting the miner with the aforementioned expectation has two options: to pay private bank notes to the miner; or to repay private bank notes to retire its commercial bill issued to a commercial bank in period 1. If the manufacturer chooses the latter option, then the miner's participation constraint for selling clay remains (7), because the miner receives no private bank note in this case, as in the case described in section 4.3. As a result, the total face value of commercial bills owed by the manufacturer becomes $b_{BK} + \hat{b}$, as implied by (8).

If the manufacturer chooses the former option instead, then the miner expects to receive an amount gN of goods from private bank notes paid by the manufacturer in any case in period 2, as described above. In this case, the miner's participation constraint for selling clay in period 1, (7), is modified to

$$\int_0^1 \mu u(gN + \hat{b}') + (1 - \mu)u(gN)\, d\mu = u(\underline{\alpha}) \tag{9}$$

where μ is the realized success probability for the manufacturer's production in period 2, and $\hat{b}'$ denotes the face value of an additional commercial bill that the manufacturer must issue to buy clay from the miner in period 1. The feasibility condition for the manufacturer to issue such an additional commercial bill is

$$\bar{\alpha} \geq 2b_{BK} + \hat{b}' \tag{10}$$

where $2b_{BK}$ equals the total face value of two commercial bills issued to commercial banks in periods 0 and 1.

For $g \in (0, \underline{\alpha}/N)$, there exists a positive threshold such that $2b_{BK} + \hat{b}' < b_{BK} + \hat{b}$ if and only if g is greater than this threshold.[21] Thus, to minimize the total face value of

[20]Note that the measure of the manufacturer meeting the miner with the aforementioned expectation in period 1 is zero. Also, this manufacturer's repayments of commercial bills upon successful production are measured as zero on a commercial bank's balance sheet.

[21]Note that $\hat{b}'$ is decreasing in gN, and converges to 0 as gN approaches to $\underline{\alpha}$ from below, as implied by

commercial bills, the manufacturer pays private bank notes to the miner in period 1 if g is sufficiently large, and repays private bank notes to a commercial bank in the period otherwise. The condition for no self-fulfilling run on private bank notes remains (8) in the latter case, whereas it becomes (10) in the former case. Hence, the introduction of reserves shrinks the parameter space in which a self-fulfilling run on private bank notes can occur, but does not eliminate it entirely.

6 Separating the central bank from commercial banks

6.1 Introducing the central bank

Now make three modifications to the model described in section 5. First, commercial banks set up a separate entity, which is just a conduit, at the beginning of period 0, and transfer all of their goods endowments to this entity in return for the entity's equity, which is subordinated to the entity's IOUs. The entity retains these goods as reserves until period 2. Second, at the beginnings of periods 0 and 1, commercial banks supply the entity's IOUs to manufacturers in exchange for the manufacturers' commercial bills, while passing on the received commercial bills to the entity, so that the entity's IOUs are fully backed by commercial bills. Third, once manufacturers pay the entity's IOUs to loggers in period 0, commercial banks replace the entity's IOUs with their own IOUs at par value at the end of the period, so that no loggers retain the entity's IOUs in period 1. At the same time, they retrieve commercial bills from the central bank by returning the entity's IOUs collected at the end of period 0, so that their own IOUs issued to loggers are fully backed by commercial bills.

Given the entity's specialization in issuing payment instruments, call the entity "the central bank", and the entity's IOUs "central-bank notes". Also, call commercial banks' IOUs in the presence of central-bank notes "bank deposits", as they are supplied only as

(9). Also, $u(\hat{b}) - u(0) = 2(u(\underline{\alpha}) - u(0))$, as implied by (7). Given $\underline{\alpha} > 0$ and $u'' < 0$, this equality implies $\hat{b} > 2\underline{\alpha}$. Because $2\underline{\alpha} = b_{BK}$ as implied by (5), the sign of $b_{BK} + \hat{b} - (2b_{BK} + \hat{b}')$ is the same as the sign of g minus the threshold.

saving instruments for loggers. See Table 7 for the classification of IOUs.

Table 7: Classification of IOUs in the model with the central bank

Issuer	Name of IOU
Manufacturer	Commercial bill
Commercial bank	Bank deposit
Central bank	Central-bank note

The only effect of the three modifications described above is to make loggers and miners hold different entities' IOUs, i.e., bank deposits and central-bank notes, respectively, in period 1. Note that both types of IOUs are fully backed by a diversified pool of commercial bills as private bank notes are. Thus, there exists an equilibrium in which each agent's production and consumption remain the same as in the equilibrium without a self-fulfilling run on private bank notes described in section 4.2.[22] See Table 8 for the evolution of the balance sheets of the central bank and each commercial bank in this equilibrium.

6.2 Reserve-saving effect of central-bank notes

Now let us investigate the possibility of a self-fulfilling run on central-bank notes. Assume the same bankruptcy rule for both commercial banks and the central bank, so that the result of this investigation is not driven by a special treatment of the central bank in the bankruptcy process:

Assumption 3. Assumptions 1 and 2 are applied to both commercial banks and the central bank.

[22]This result does not change even if commercial banks do not transfer commercial bills to the central bank at the beginning of period 0, but instead supply their own IOUs to manufacturers in the period. As shown below, introducing the central bank shrinks the parameter space in which a self-fulfilling run can occur, if loggers do not hold central-bank notes at the beginning of period 1. Nonetheless, it is assumed that commercial banks supply central-bank notes in period 0, because this behavior makes the effect of central-bank notes described below robust in a general case in which the economy starts from period $-x$, where x is an arbitrary positive integer, and manufacturers must install inputs sequentially in each period before periods 0 and 1.

Table 8: Evolution of the balance sheets of the central bank and each commercial bank without a self-fulfilling run on central-bank notes

(a) After commercial banks supply central-bank notes to manufacturers in period 0

The central bank's balance sheet

Goods held as reserves (gN)	Shares owned by N commercial banks (gN)
Commercial bills issued in period 0 ($b_{BK}/2$)	Central-bank notes supplied to manufacturers ($b_{BK}/2$)

Each commercial bank's balance sheet

A share in the central bank (g)	

(b) After manufacturers pay central-bank notes for loggers' wood in period 0

The central bank's balance sheet

Goods held as reserves (gN)	Shares owned by N commercial banks (gN)
Commercial bills issued in period 0 ($b_{BK}/2$)	Central-bank notes paid to loggers ($b_{BK}/2$)

Each commercial bank's balance sheet

A share in the central bank (g)	

(c) After commercial banks replace central-bank notes with bank deposits in period 0

The central bank's balance sheet

Goods held as reserves (gN)	Shares owned by N commercial banks (gN)

Each commercial bank's balance sheet

A share in the central bank (g)	
Commercial bills issued in period 0 ($b_{BK}/(2N)$)	Bank deposits held by loggers ($b_{BK}/(2N)$)

(d) After commercial banks supply central-bank notes to manufacturers in period 1

The central bank's balance sheet

Goods held as reserves (gN)	Shares owned by N commercial banks (gN)
Commercial bills issued in period 1 ($b_{BK}/2$)	Central-bank notes supplied to manufacturers ($b_{BK}/2$)

Each commercial bank's balance sheet

A share in the central bank (g)	
Commercial bills issued in period 0 ($b_{BK}/(2N)$)	Bank deposits held by loggers ($b_{BK}/(2N)$)

(e) After manufacturers pay central-bank notes for miners' clay in period 1

The central bank's balance sheet

Goods held as reserves (gN)	Shares owned by N commercial banks (gN)
Commercial bills issued in period 1 ($b_{BK}/2$)	Central-bank notes paid to miners ($b_{BK}/2$)

Each commercial bank's balance sheet

A share in the central bank (g)	
Commercial bills issued in period 0 ($b_{BK}/(2N)$)	Bank deposits held by loggers ($b_{BK}/(2N)$)

Notes: In parentheses are the expected returns on the balance-sheet items if they are on the asset side, and the face values of bank deposits and central-bank notes and the expected returns on shares in the central bank if they are on the liability side. The value of a balance-sheet item is defined by the product of the value of securities per agent and the measure of the agents included in the item.

Given the same balance sheets of commercial banks and the central bank as described in panel (d) of Table 8, suppose that a miner expects the other miners' refusals to sell clay to manufacturers in pairwise meetings in period 1. If this expectation is correct, then the manufacturers meeting the other miners in period 1 repay refused central-bank notes to retire their commercial bills held by the central bank in the period, given Assumptions 2 and 3. Thus, a miner with this expectation expects that the manufacturer that it meets in period 1 is the sole holder of central-bank notes, given loggers holding bank deposits in the period, as indicated by panel (d) of Table 8. The central bank can fully guarantee the face value of central-bank notes paid by this manufacturer, because the measure of a manufacturer is zero while the value of reserves held by the central bank, gN, is positive.[23] Hence, each miner sells clay for central-bank notes paid by a manufacturer in period 1, regardless of the miner's expectation of the other miners' behavior in the period.[24] See Table 9 for the central bank's balance sheet expected by each miner if the other miners are expected to refuse to sell clay to manufacturers in period 1.

This result implies that introducing the central bank into the banking system blocks a self-fulfilling run on bank notes for any positive value of reserves held by each commercial bank, g. This effect of central-bank notes is beneficial when the value of g is not large enough to prevent a self-fulfilling run on private bank notes, or to satisfy (10). Hence, it is optimal to separate payment instruments, i.e., central-bank notes, from saving instruments, i.e., bank deposits, as observed in the modern banking system.[25]

[23]Note that central-bank notes are senior to shares in the central bank, as assumed in section 6.1.

[24]More generally, if a miner expects that all but a finite number of miners refuse to sell clay to manufacturers in period 1, then it is also rational to expect that the central bank can guarantee central-bank notes accepted by the finite number of miners, because the measure of these miners is zero. If all but a positive measure of miners refuse to sell clay to manufacturers in period 1, then a positive measure of manufacturers can obtain wood and clay to produce goods in period 2; thus, the total return on these manufacturers' commercial bills held by the central bank becomes certain by the law of large numbers, and hence sufficient to repay the face value of the central-bank notes held by these manufacturers in period 2.

[25]Introducing the central bank without reserves does not affect the parameter space in which a self-fulfilling run on bank notes can occur, because central-bank notes cannot have a reserve-saving effect without reserves. See Appendix A for more details.

Table 9: The central bank's balance sheet expected by each miner in case of a run on central-bank notes in period 1

Goods held as reserves (gN)	Shares owned by N commercial banks (gN)
Commercial bill issued by the manufacturer that the miner meets in period 1 (0)	Central-bank notes supplied to the manufacturer that the miner meets in period 1 (0)
Commercial bills issued by the other manufacturers in period 1 (canceled)	Central-bank notes supplied to the other manufacturers in period 1 (canceled)

Notes: The table shows the central bank's balance sheet after manufacturers repay refused central-bank notes to the central bank to retire their commercial bills issued in period 1. In parentheses are the expected returns on the balance-sheet items if they are on the asset side, and the face value of central-bank notes and the expected return on shares in the central bank if they are on the liability side. Whether it is an expected return or a face value, the value of a balance-sheet item is defined by the product of the value of securities per agent and the measure of the agents included in the item. "canceled" in a parenthesis indicates that the balance-sheet item is canceled within period 1.

6.3 Robustness of the reserve-saving effect of central-bank notes

This implication of the model is robust even if Assumption 2 does not hold for either commercial banks or the central bank. In this case, manufacturers cannot retire commercial bills before maturity by repaying refused bank notes in period 1. As a result, the central bank has a unit measure of creditors, i.e., the sum of miners accepting central-bank notes from manufacturers and the other manufacturers retaining refused central-bank notes, in any case in period 1. On the other hand, if commercial banks issue private bank notes without the central bank, then they have a unit measure of loggers as another set of creditors in period 1. Thus, given the total value of reserves in the banking system, gN, the central bank only needs to protect a unit measure of creditors, while commercial banks issuing private bank notes without the central bank must protect a measure two of creditors. This feature of the central bank reduces the amount of reserves necessary to prevent a self-fulfilling run on bank notes.[26]

[26]More precisely, if Assumption 2 does not hold, then the recovery value of central-bank notes held by a manufacturer in case of a self-fulfilling run on central-bank notes equals $gN/1$, or gN, because the central bank has a unit measure of creditors in period 1 as described above. Thus, the value of an additional commercial bill that a manufacturer must issue to buy clay from a miner in case of a self-fulfilling run in

6.4 Optimality of short-term re-discounting of commercial bills by the central bank

Table 8 shows that to keep both central-bank notes and bank deposits fully backed by commercial bills, commercial banks must transfer commercial bills to the central bank when they obtain central-bank notes, and then retrieve commercial bills from the central bank when they replace central-bank notes with bank deposits. This round-trip transfer of commercial bills in exchange for central-bank notes can be interpreted as the central bank's re-discounting of commercial bills. Furthermore, Table 8 implies that it must be completed within the same period, as otherwise loggers would hold central-bank notes as saving instruments. Hence, the model replicates short-term re-discounting of commercial bills by the central bank, which has been a traditional channel for central-bank note supply in practice, as the optimal policy.

7 Discussion

7.1 Incorporating a maturity mismatch between bank deposits and bank assets

While the maturities of commercial bills and bank deposits coincide in the model, a large proportion of bank deposits are demand deposits in reality. It is possible to incorporate this feature of bank deposits into the model without changing the main result of the model.

For example, suppose there are another type of agents called sellers in period 1 that are endowed with an arbitrarily large amount of goods in the period and are also indifferent to whether to consume goods in period 1 or 2. Also suppose that loggers consume goods in period 1 rather than in period 2. In this case, if there is a market for exchanges between goods and bank deposits in period 1, then loggers can buy goods from sellers by paying their

period 1 equals the value of b' that satisfies (9). In contrast, if there is no central bank, then the recovery value of private bank notes held by a manufacturer in case of a self-fulfilling run on private bank notes equals $g/(2/N)$, or $gN/2$, as each commercial bank has a measure $2/N$ of creditors in period 1. Hence, a manufacturer must issue a larger value of an additional commercial bill than b' to obtain a miner's clay in this case.

bank deposits.[27] Thus, besides a change in the identities of depositors in period 1, there is no change in the result of the model.

In this regard, note that the value of commercial bills held by commercial banks is not affected even if loggers fail to buy goods from sellers. Thus, a seller accepts the payment of bank deposits, i.e., bank transfers, from a logger even if the other sellers refuse to do so. Hence, there is no self-fulfilling mass refusal of bank transfers by sellers.

This property of the model hinges on the assumption that bank deposits are not redeemable into goods in period 1. If bank deposits must be redeemable on demand, then it is possible to consider an environment in which bank deposits are subject to a self-fulfilling run, as shown by Diamond and Dybvig (1983). The result of the model does not require such a feature of bank deposits.

7.2 Reconciling the model with check payments and bank transfers, and the overnight supply of central-bank liabilities

In the modern payment system, people often make payments by checks and bank transfers, instead of physical central-bank notes. Note, however, that payees receiving checks and bank transfers obtain deposit balances at their banks, rather than those at payers' banks. The transfer of deposit balances between commercial banks is settled by an interbank transfer of bank reserves, i.e., current account balances of central-bank notes at the central bank. Thus, payment instruments between commercial banks are separated from saving instruments held by depositors, even when people pay by checks and bank transfers. Moreover, to obtain bank reserves, commercial banks must give up eligible securities to the central bank through open market operations and discount windows. These features of payments by checks and bank transfers are consistent with the model, whereby payers pay central-bank liabilities backed

[27]Loggers pay all of their bank deposit balances in this case, as they do not consume goods in period 2. If the market is perfectly competitive, then loggers can buy the amount of goods worth the face value of bank deposits they pay. If the market is bilateral and the result of each transaction is determined by Nash bargaining, then the amount of goods that loggers receive from sellers is less than the face value of bank deposits they pay, unless loggers have all the bargaining power.

by some assets to payees, and payees end up holding bank deposits at their own banks after the transfer of central-bank liabilities.[28]

This interpretation of the model may seem inconsistent with the existence of overnight supply of central-bank notes and bank reserves in practice, as the key feature of central-bank notes in the model is the absence of central-bank notes held as saving instruments. The implication of the model still holds, however, because the total size of commercial banks' liabilities would be larger than that of the central bank if the central bank were merged with commercial banks. As described in section 6.3, a bank needs more reserves to prevent a self-fulfilling run on its bank notes as the size of its liabilities gets larger. Thus, the model implies that even if part of the central bank's liabilities are held as saving instruments, separating the central bank from commercial banks still reduces the need for reserves in the banking system.[29]

7.3 Can senior private bank notes substitute for central-bank notes?

Given no aggregate shock in the model, private bank notes can replicate the reserve-saving effect of central-bank notes described in section 6.2, if they are senior to bank deposits. To confirm this result, consider the model described in section 4.2, in which only commercial banks exist, and then assume that private bank notes paid to miners in period 1 have seniority over those paid to loggers in period 0. Given this assumption, call private bank notes paid to miners in period 1 "senior private bank notes", and rephrase private bank notes held by loggers in period 1 "subordinated deposits".

[28]Note that this interpretation of checks and bank transfers is robust even if bank reserves are not convertible into central-bank notes. The key feature of bank reserves that is consistent with the model is that commercial banks must give up some assets to acquire bank reserves, so that bank reserves are backed by those assets.

[29]Demand for the overnight supply of central-bank liabilities in practice may be due to special protections for central-bank liabilities, such as the legal-tender status of central-bank notes and a loss sharing rule between the central bank and the government. Also, it takes time for depositors to bring central-bank notes to banks physically to make deposits in reality. The model implies that if there were no such privilege of central-bank liabilities or time lag, then the most efficient way to supply central-bank liabilities is elastic intraday supply on demand without an overnight supply.

Now suppose that a minor expects the other minors' refusals to sell clay to manufacturers in period 1. As described in section 4.3, the manufacturers meeting the other miners return refused senior private bank notes to commercial banks in such a case, given Assumption 2. Therefore, the miner expects that a positive amount of reserves held by each commercial bank can fully guarantee senior private bank notes held by the manufacturer that it meets in period 1, given the measure of a manufacturer being zero. Thus, each miner sells clay for senior private bank notes paid by a manufacturer, regardless of the other miners' behavior in period 1.

The equivalence between central-bank notes and senior private bank notes, however, does not hold if an aggregate shock is introduced into the model. For example, suppose that the idiosyncratic success probability for each manufacturer's production, $\mu_{M,j}$ for $j = [0, 1]$, follows a $[0, \bar{\mu}]$ uniform distribution, where the value of $\bar{\mu}$ is a common random variable for all manufacturers. Thus, the value of repayments on a pool of commercial bills in period 2 is also random. Assume that each commercial bank is risk-neutral. It can be shown that if the realized value of $\bar{\mu}$ is unverifiable, that is, senior private bank notes must be debt, then social welfare with central-bank notes dominates that with senior private bank notes.

This result holds because, even though senior private bank notes are immune from a self-fulfilling run, new payees receiving senior private bank notes, i.e., miners, receive a larger share of commercial banks' assets than past payees receiving subordinated deposits, i.e., loggers, if the realization of a low value of $\bar{\mu}$ makes commercial banks default on subordinated deposits in period 2. Thus, risk sharing between new and past payees becomes imperfect.[30] In contrast, central-bank notes can prevent a self-fulfilling run without sacrificing risk sharing between new and past payees, because the introduction of the central bank simply separates

[30] In this case, an increase in a risk premium on subordinated deposits is larger than a decrease in a risk premium on senior private bank notes, given the strict concavity of the utility function u. As a result, commercial banks have to require each manufacturer to issue a larger face value of commercial bills. Because all the social surplus goes to manufacturers, a lower expected profit for each manufacturer implies a welfare loss.

new and past payees into different entities holding the same amount of commercial bills. See Appendix B for more details.

7.4 Comparison with historical observations

The model shows that a self-fulfilling run on private bank notes can occur even if private bank notes are not redeemable on demand, as described in section 7.1. Thus, the suspension of payments does not prevent a self-fulfilling run on private bank notes. This implication of the model is consistent with the fact that banking crises often occurred among note-issuing commercial banks in England and the U.S. in the 19th century despite the suspension of payments being a standard response in the banking sector (see Calomiris and Gorton 1991, Dwyer 1996, and Quinn 2004 for more details).

The model also shows that private bank notes can be free of a self-fulfilling run if note-issuing commercial banks are sufficiently well capitalized to have adequate reserves. This result is consistent with the British experience that Scotland had a stable free banking era before the enactment of Peel's Act of 1844 and the Scottish Bank Act of 1845, whereas England and Wales suffered the instability of note-issuing country banks during the same period. Indeed, one of the reasons for this contrast between the two banking systems was attributed to the poor capitalization of country banks in England and Wales, which led the British Parliament to permit the foundation of joint-stock note-issuing banks in England and Wales outside London in 1826 after the 1825 bank panic (see Collins 1988, White 1995, and Quinn 2004 for more details).

Furthermore, the model implies that making private bank notes senior to bank deposits reduces the necessary amount of reserves to prevent a self-fulfilling run, as discussed in section 7.3. This feature of the model is consistent with the U.S. banking regulation during both the free banking and the national banking era before 1913, in which state or federal law required note-issuing commercial banks to deposit state or federal government bonds as reserves at the state or the federal government, while guaranteeing the seniority of bank notes (see Breckenridge 1899, Weber 2015a and 2015b).

These implications of the model confirm the feasibility of stable free banking. Yet, the model also demonstrates that introducing a central bank as the issuer of payment instruments still reduces the total amount of reserves necessary to maintain the stability of the banking system. This result is obtained without assuming the inconvenient features of private bank notes observed in the past, such as discounts due to physical distances from the branches of issuing banks (Gorton 1999, 2014) or risky reserves, such as state bonds, that caused banking crises during the free banking era in the U.S. (Rolnick and Weber 1984). There is no agency problem or externality caused by commercial banks in the model, either. Thus, the model implies that even in the absence of these problems of private bank notes, introducing a central bank improves the efficiency of the banking system.

8 Conclusions

This paper presents a three-period model to endogenize the separation of payment and saving instruments as observed in the modern banking system. The key environment in the model is imperfect synchronization of different markets, which prevents the suppliers of production inputs from perfectly diversifying idiosyncratic default risk in commercial bills paid by producers. To circumvent this friction, commercial banks can issue private bank notes to producers in exchange for commercial bills in advance, so that producers can pay suppliers payment instruments that are already backed by a diversified pool of commercial bills. Private bank notes, however, are subject to a self-fulfilling mass refusal by suppliers if producers must install inputs supplied by multiple suppliers sequentially, and if installed inputs become sunk costs. While commercial banks can prevent such a self-fulfilling run on private bank notes by holding a sufficiently large amount of reserves, they can reduce the necessary amount of reserves if they set up a conduit that issues only payment instruments, i.e., the central bank. Moreover, the model replicates short-term re-discounting of commercial bills by the central bank as the optimal policy for central-bank note supply endogenously.

A question remains as to whether the issuer of payment instruments must be a public institution, or can be a privately-owned institution. In the model, commercial banks can set up the issuer of payment instruments by themselves without any efficiency loss, given no externality or agency problem caused by them. To investigate into this question further, it is necessary to introduce more frictions into the model, such as oligopoly in the banking sector, agency problems and externalities caused by commercial banks, and inefficiencies associated with public ownership. This issue is left for future research.

Appendices

A The model with the central bank without reserve assets

Consider the model described in section 6.1. Then, suppose that commercial banks receive no goods endowment in period 0, i.e., $g = 0$. Also suppose that the behavior of commercial banks and agents before pairwise meetings in period 1 remains the same as shown in panels (a)-(d) of Table 8, and that a miner expects the other miners' refusals to sell clay to manufacturers in those pairwise meetings.

As described in section 4.3, the manufacturers meeting the other miners in period 1 repay refused central-bank notes to the central bank in such an event, given Assumptions 2 and 3. Thus, if a miner expects the other miners' refusals to sell clay to manufacturers in period 1, then this miner in turn expects that the central bank holds only the commercial bill issued by the manufacturer that it meets in period 1. If this expectation is correct, then the return on the remaining central-bank notes held by this manufacturer never exceeds the return on this manufacturer's commercial bill held by the central bank in any case. Thus, this manufacturer is weakly better off if it repays central-bank notes to the central bank to retire its commercial bill issued in period 1, and then pays its commercial bill directly to the miner with the aforementioned expectation. Given no payment of central-bank notes by this manufacturer, the miner's participation constraint for selling clay in period 1 remains the same as that in the case without the central bank, i.e., (7). Therefore, the condition for no self-fulfilling run also remains the same as in the case without the central bank, i.e., (8).

B The model with sunk costs of production and aggregate risk

B.1 Introducing an unverifiable aggregate shock

Consider the model described in section 4.1. Modify the model to assume that the idiosyncratic probability of successful production in period 2 for each manufacturer, $\mu_{M,j}$, follows a uniform distribution over $[0, \bar{\mu}]$ for all $j \in [0, 1]$. The upper bound, $\bar{\mu}$, is also a random variable whose value is the same for all $j \in [0, 1]$:

$$\bar{\mu} = \begin{cases} \mu_H & \text{with probability } \eta \\ \mu_L & \text{with probability } 1 - \eta \end{cases} \tag{A.1}$$

where $\eta \in (0, 1)$ and $0 < \mu_L < \mu_H < 1$. The value of $\bar{\mu}$ is realized in period 1 at the same time as the realization of $\mu_{M,j}$ for $j \in [0, 1]$. Set an assumption similar to (1):

$$\bar{\alpha} > \frac{4\underline{\alpha}}{\eta\mu_H + (1 - \eta)\mu_L} \tag{A.2}$$

Assume that the realized value of $\bar{\mu}$ in period 1 is unverifiable in court. This assumption implies that if an entity issues a senior liability to agents that is contingent on the realized value of $\bar{\mu}$, then the shareholders of the entity can renege on the repayment of the senior liability. Thus, a senior liability must be debt. Also, assume that commercial banks can be committed to sharing the profit of an entity equally among them if they found an entity with an equal equity share for each of them. This assumption enables them to be joint shareholders of an entity without any friction. The underlying assumption is that commercial banks can penalize one another if any of them deviates from the commitment. For simplicity, the model abstracts from modelling repeated interactions among commercial banks in a formal infinite-time horizon environment.

Also, assume that each commercial bank maximizes the expected value of its profit in period 2. The other assumptions remain the same as in the model described in section 4.1.

In addition, note that because manufacturers are risk-neutral and can make take-it-or-leave-it offers to loggers and miners, and also because expected profit for commercial banks is zero in any equilibrium, all the social surplus goes to manufacturers. Thus, social welfare in terms of Pareto efficiency can be measured by expected profit for each manufacturer.

B.2 Equilibrium with central-bank notes

Now introduce the central bank into this environment in the same way as described in section 6.1, except that each commercial bank transfers half of its goods endowment to the central bank in period 0, retaining the other half as its own reserves until period 2. This allocation of goods endowment is socially optimal, because it makes central-bank notes and bank deposits identical in terms of risk, so that the sum of risk premia charged by a logger and a miner, which reflects their disutility, is minimized. Denote by v_{AG} the face value of central-bank notes that a manufacturer pays to a logger and a miner in periods 0 and 1, respectively, and denote by b_{AG} the face value of commercial bills issued by a manufacturer in each of periods 0 and 1. Because central-bank notes received by each logger are replaced with bank deposits at par value at the end of period 0, v_{AG} also equals the face value of bank deposits per depositor. See Table A.1 for the balance sheet of each commercial bank and the central bank at the end of period 1.

Table A.1: Balance sheets of each commercial bank and the central bank at the end of period 1 in the model with sunk cost of production and aggregate risk

The central bank's balance sheet		Each commercial bank's balance sheet	
Goods held as reserves ($gN/2$)	Shares owned by N commercial banks ($gN/2$)	A share in the central bank ($g/2$)	
Commercial bills issued in period 1 ($\bar{\mu}b_{AG}/2$)	Central-bank notes paid to miners (v_{AG})	Goods held as reserves ($g/2$)	
		Commercial bills issued in period 0 ($\bar{\mu}b_{AG}/(2N)$)	Bank deposits held by loggers (v_{AG}/N)

Notes: In parentheses are the expected returns on the balance-sheet items if they are on the asset side, and the face values of bank deposits and central-bank notes and the expected return on shares in the central bank if they are on the liability side. The value of a balance-sheet item is defined by the product of the value of securities per agent and the measure of the agents included in the item.

Because it remains the same that the central bank holds a positive amount of reserves while the measure of a manufacturer is zero, there is no self-fulfilling run on central-bank notes in this case as described in section 6.2.

Given this result and the identical risk profile of central-bank notes and bank deposits, the binding participation constraints for a logger and a miner to sell a unit of wood in period 0 and a unit of clay in period 1, respectively, are identical:

$$\eta u(v_{AG}) + (1-\eta) u\left(\min\left\{\frac{\mu_L}{2}\cdot b_{AG} + \frac{gN}{2},\ v_{AG}\right\}\right) = u(\underline{\alpha}) \tag{A.3}$$

The minimum operator exists in the second term on the left-hand side of (A.3), because it is possible that the central bank and commercial banks cannot repay the face value of their IOUs fully if $\bar{\mu} = \mu_L$. This event occurs with probability $1-\eta$ as assumed above. In this case, the holders of central-bank notes and bank deposits receive all the assets held by the central bank and commercial banks, respectively, which equals $(\mu_L/2)\cdot b_{AG} + gN/2$ per agent.

The zero-profit condition for each commercial bank due to Bertrand competition is as follows:

$$\eta\left(\frac{\mu_H}{2}\cdot b_{AG} + \frac{gN}{2} - v_{AG}\right) + (1-\eta)\left(\frac{\mu_L}{2}\cdot b_{AG} + \frac{gN}{2} - \min\left\{\frac{\mu_L}{2}\cdot b_{AG} + \frac{gN}{2},\ v_{AG}\right\}\right) = \frac{gN}{2} \tag{A.4}$$

The left-hand side is expected profit for each of the central bank and commercial banks in period 2, whereas the right-hand side is the amount of reserves supplied by each of the central bank and commercial banks. Both sides are per agent holding central-bank notes or bank deposits. The expected profit of the central bank is zero, because the central bank is owned by commercial banks as shareholders. If the central bank earned a positive expected profit, then a subset of commercial banks would set up a new central bank to offer manufacturers central-bank notes with a higher rate of return in Bertrand competition.

The equilibrium values of b_{AG} and v_{AG} are determined by (A.3) and (A.4), provided that it is feasible for a manufacturer to issue twice a commercial bill whose face value equals b_{AG}.

It can be shown that

$$\frac{\mu_L}{2} \cdot b_{AG} + \frac{gN}{2} \geq v_{AG} \tag{A.5}$$

$$b_{AG} = \frac{2\underline{\alpha}}{\eta\mu_H + (1-\eta)\mu_L} \tag{A.6}$$

$$v_{AG} = \underline{\alpha} \tag{A.7}$$

if and only if

$$gN \geq \frac{2\underline{\alpha}\eta(\mu_H - \mu_L)}{\eta\mu_H + (1-\eta)\mu_L} \tag{A.8}$$

By continuity, this result implies that if $\mu_H - \mu_L$, i.e., the size of aggregate risk, is sufficiently small, then $\bar{\alpha} \geq 2b_{AG}$, that is, it is feasible for each manufacturer to issue a commercial bill whose face value equals b_{AG} in each of periods 0 and 1, given (A.2).

On the other hand, if $\mu_H - \mu_L$ is so large that gN cannot satisfy the condition in (A.8), then it is possible that the values of b_{AG} and v_{AG} solving (A.3) and (A.4) become too large to satisfy $\bar{\alpha} \geq 2b_{AG}$ due to the strict concavity of the utility function u. In this case, manufacturers cannot issue a sufficiently large face value of commercial bills to buy wood and clay from loggers and miners, respectively; thus, loggers and miners retain their inputs for their own production.

B.3 Equilibrium with senior private bank notes

Instead of introducing the central bank, suppose that commercial banks issue senior private bank notes and subordinated deposits. Assume that commercial banks issue subordinated private bank notes to manufacturers in period 0, so that loggers are indifferent to swapping subordinated private bank notes for subordinated deposits at the end of the period. Denote by q_{AG} the face value of senior private bank notes that a manufacturer pays to a miner in period 1 and by d_{AG} the face value of subordinated private bank notes that a manufacturer pays to a logger in period 0. Also denote by $\tilde{b}_{AG}$ the face value of commercial bills issued by a manufacturer in each of periods 0 and 1. Note that d_{AG} also equals the face value of subordinated deposits per logger. See Table A.2 for each commercial bank's balance sheet at the end of period 1 in this case.

Table A.2: Balance sheet of each commercial bank issuing senior private bank notes and subordinated deposits at the end of period 1 in the model with sunk cost of production and aggregate risk

Goods held as reserves (g)	
Commercial bills issued in period 0 ($\bar{\mu}\tilde{b}_{AG}/(2N)$)	Senior private bank notes held by miners (q_{AG}/N)
Commercial bills issued in period 1 ($\bar{\mu}\tilde{b}_{AG}/(2N)$)	Subordinated deposits held by loggers (d_{AG}/N)

Notes: In parentheses are the expected returns on the balance-sheet items if they are on the asset side, and the face values of the commercial bank's IOUs if they are on the liability side. The value of a balance-sheet item is defined by the product of the value of securities per agent and the measure of the agents included in the item.

Because it remains the same that each commercial bank holds a positive amount of reserves while the measure of a manufacturer is zero, there is no self-fulfilling run on private bank notes in this case as described in section 7.3.

Given this result, the binding participation constraints for a miner and a logger to sell a unit of clay in period 1 and a unit of wood in period 0 are, respectively,

$$\eta u(q_{AG}) + (1-\eta)u\left(\min\left\{\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG} + gN,\ q_{AG}\right\}\right) = u(\underline{\alpha}) \tag{A.9}$$

$$\eta u(d_{AG}) + (1-\eta)u\left(\min\left\{\max\left\{\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG} + gN - q_{AG},\ 0\right\},\ d_{AG}\right\}\right) = u(\underline{\alpha}) \tag{A.10}$$

The minimum operator exists in the second term on the left-hand side of (A.9), because it is possible that commercial banks cannot repay the face value of senior private bank notes fully if $\bar{\mu} = \mu_L$. This event occurs with probability $1-\eta$ as assumed above. In this case, the holders of senior private bank notes receive all the assets held by commercial banks, which equals $(\mu_L/2)\cdot 2\tilde{b}_{AG} + gN$ per agent. Also, even if senior private bank notes are fully repaid, the residual of each commercial bank's assets can be insufficient to repay subordinated deposits fully. This is the case if $(\mu_L/2)\cdot 2\tilde{b}_{AG} + gN - q_{AG} < d_{AG}$, as indicated by the second term on the left-hand side of (A.10).

The zero-profit condition for each commercial bank due to Bertrand competition is as follows:

$$\begin{aligned}\eta\left(\frac{\mu_H}{2}\cdot 2\tilde{b}_{AG}+gN-q_{AG}-d_{AG}\right)\\ +(1-\eta)\left(\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG}+gN-\min\left\{\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG}+gN,\ q_{AG}\right\}\right.\\ \left.-\min\left\{\max\left\{\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG}+gN-q_{AG},\ 0\right\},\ d_{AG}\right\}\right)=gN \quad \text{(A.11)}\end{aligned}$$

The left-hand side is the expected value of each commercial bank's profit in period 2, whereas the right-hand side is the amount of reserves supplied by each commercial bank. Both sides are per agent holding each commercial bank's IOUs. In this equation, $\min\left\{\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG}+gN,\ q_{AG}\right\}$ and $\min\left\{\max\left\{\frac{\mu_L}{2}\cdot 2\tilde{b}_{AG}+gN-q_{AG},\ 0\right\},\ d_{AG}\right\}$ equal the repayments to senior private bank notes held by a miner and subordinated deposits held by a logger, respectively, when the realized value of $\bar{\mu}$ equals μ_L, as in (A.9) and (A.10). The minimum and the maximum operator in these terms control for the cases in which commercial banks default on only subordinated deposits, or both subordinated deposits and senior private bank notes.

The equilibrium values of $\tilde{b}_{AG}$, q_{AG}, and d_{AG} are jointly determined by (A.9), (A.10), and (A.11), provided that it is feasible for a manufacturer to issue twice a commercial bill whose face value equals $\tilde{b}_{AG}$. As is the case with (A.8), it can be shown that

$$\tilde{b}_{AG}=\frac{2\underline{\alpha}}{\eta\mu_H+(1-\eta)\mu_L} \quad \text{(A.12)}$$

$$q_{AG}=d_{AG}=\underline{\alpha} \quad \text{(A.13)}$$

if and only if (A.8) holds. By continuity, if $\mu_H-\mu_L$ is sufficiently small, then $\bar{\alpha}\geq 2\tilde{b}_{AG}$, given (A.2).

B.4 Comparison between the equilibrium with central-bank notes and the equilibrium with senior private bank notes

Compare social welfare in the equilibrium with central-bank notes described in section B.2 and the equilibrium with senior private bank notes described in section B.3. As described in section B.1, manufacturers receive all the social surplus; thus, social welfare can be measured by the expected value of each manufacturer's profit, which is $(\eta\mu_H + (1-\eta)\mu_L)(\bar{\alpha} - 2b_{AG})$ in the equilibrium with central-bank notes, and $(\eta\mu_H + (1-\eta)\mu_L)(\bar{\alpha} - 2\tilde{b}_{AG})$ in the equilibrium with senior private bank notes. Thus, it is sufficient to compare b_{AG} and $\tilde{b}_{AG}$ to compare social welfare between the two equilibria.

As shown above, $b_{AG} = \tilde{b}_{AG}$ if (A.8) holds. Now suppose that

$$gN < \frac{2\underline{\alpha}\eta(\mu_H - \mu_L)}{\eta\mu_H + (1-\eta)\mu_L} \tag{A.14}$$

In this case, the central bank defaults on central-bank notes in the equilibrium described in section B.2 and commercial banks default on subordinated deposits in the equilibrium described in section B.3, if $\bar{\mu} = \mu_L$. Thus, substituting (A.4) for v_{AG} in (A.3) yields

$$\eta u\left(\frac{\mu_H}{2} \cdot b_{AG} - \frac{1-\eta}{\eta} \cdot \frac{gN}{2}\right) + (1-\eta)u\left(\frac{\mu_L}{2} \cdot b_{AG} + \frac{gN}{2}\right) = u(\underline{\alpha}) \tag{A.15}$$

which is the equilibrium condition for b_{AG}.

Then consider two cases. First, suppose that

$$\frac{\mu_L}{2} \cdot 2\tilde{b}_{AG} + gN \geq q_{AG} \tag{A.16}$$

In this case, commercial banks repay senior private bank notes fully even if $\bar{\mu} = \mu_L$. Thus, (A.9) implies that $q_{AG} = \underline{\alpha}$. Substituting this equation and (A.11) into (A.10) yields

$$\eta u\left(\frac{\mu_H}{2} \cdot 2\tilde{b}_{AG} - \underline{\alpha} - \frac{1-\eta}{\eta} \cdot gN\right) + (1-\eta)u\left(\frac{\mu_L}{2} \cdot 2\tilde{b}_{AG} + gN - \underline{\alpha}\right) = u(\underline{\alpha}) \tag{A.17}$$

Then introduce the following notation:

$$f_1 \equiv \frac{\mu_H}{2} \cdot b_{AG} - \frac{1-\eta}{\eta} \cdot \frac{gN}{2} \tag{A.18}$$

$$f_2 \equiv \frac{\mu_L}{2} \cdot b_{AG} + \frac{gN}{2} \tag{A.19}$$

$$g_1 \equiv \frac{\mu_H}{2} \cdot 2b_{AG} - \underline{\alpha} - \frac{1-\eta}{\eta} \cdot gN \tag{A.20}$$

$$g_2 \equiv \frac{\mu_L}{2} \cdot 2b_{AG} + gN - \underline{\alpha} \tag{A.21}$$

Note that (A.17) implies that $\eta u(f_1) + (1-\eta)u(f_2) = u(\underline{\alpha})$. As this result implies that $f_1 > \underline{\alpha} > f_2$, it can be shown that $g_1 - f_1 = f_1 - \underline{\alpha} > 0$ and $f_2 - g_2 = \underline{\alpha} - f_2 > 0$; thus, $g_1 > f_1 > \underline{\alpha} > f_2 > g_2 > 0$, where the last inequality holds given (A.16) and $q_{AG} = \underline{\alpha}$. By the strict concavity of the utility function u,

$$\frac{u(g_1) - u(\underline{\alpha})}{g_1 - \underline{\alpha}} < \frac{u(f_1) - u(\underline{\alpha})}{f_1 - \underline{\alpha}} \tag{A.22}$$

$$\frac{u(\underline{\alpha}) - u(g_2)}{\underline{\alpha} - g_2} > \frac{u(\underline{\alpha}) - u(f_2)}{\underline{\alpha} - f_2} \tag{A.23}$$

which implies $\eta u(g_1) + (1-\eta)u(g_2) < u(\underline{\alpha})$. Thus, (A.15) implies that $b_{AG} < \tilde{b}_{AG}$, given $u' > 0$.

Given (A.15), consider the second case:

$$\frac{\mu_L}{2} \cdot 2\tilde{b}_{AG} + gN < q_{AG} \tag{A.24}$$

In this case, commercial banks default on both senior private bank notes and subordinated deposits if $\bar{\mu} = \mu_L$. Thus, (A.10) implies that

$$\eta u(d_{AG}) + (1-\eta)u(0) = u(\underline{\alpha}) \tag{A.25}$$

Also, substituting (A.11) into (A.9) yields

$$\eta u\left(\frac{\mu_H}{2} \cdot 2\tilde{b}_{AG} - d_{AG} - \frac{1-\eta}{\eta} \cdot gN\right) + (1-\eta)u\left(\frac{\mu_L}{2} \cdot 2\tilde{b}_{AG} + gN\right) = u(\underline{\alpha}) \tag{A.26}$$

Then introduce the following notation:

$$h_1 \equiv \frac{\mu_H}{2} \cdot 2b_{AG} - d_{AG} - \frac{1-\eta}{\eta} \cdot gN \tag{A.27}$$

$$h_2 \equiv \frac{\mu_L}{2} \cdot 2b_{AG} + gN \tag{A.28}$$

Comparison between (A.25) and $\eta u(f_1) + (1-\eta)u(f_2) = u(\underline{\alpha})$ implied by (A.17) indicates that $d_{AG} > f_1$, given $f_2 > 0$. Given this result, it can be shown that $f_1 - h_1 = d_{AG} - f_1 > 0$ and $h_2 - f_2 = f_2 > 0$.

If $h_1 \leq h_2$, then $(\mu_H - \mu_L)b_{AG} - d_{AG} - gN/\eta \leq 0$. Because (A.9), (A.11), and (A.24) imply that $(\mu_L/2) \cdot 2\tilde{b}_{AG} + gN < q_{AG} = (\mu_H/2) \cdot 2\tilde{b}_{AG} - d_{AG} - (1-\eta)gN/\eta$, $\tilde{b}_{AG} > b_{AG}$ in this case.

If $h_1 > h_2$, then

$$\frac{u(d_{AG}) - u(f_1)}{d_{AG} - f_1} < \frac{u(f_1) - u(h_1)}{f_1 - h_1} \tag{A.29}$$

$$\frac{u(f_2) - u(0)}{f_2 - 0} > \frac{u(h_2) - u(f_2)}{h_2 - f_2} \tag{A.30}$$

and $d_{AG} > f_1 > h_1 > h_2 > f_2 > 0$, given that $f_1 - h_1 = d_{AG} - f_1$ and $h_2 - f_2 = f_2$ as shown above. Also, (A.25) implies

$$\frac{u(d_{AG}) - u(f_1)}{u(f_2) - u(0)} = \frac{1-\eta}{\eta} \tag{A.31}$$

given $\eta u(f_1) + (1-\eta)u(f_2) = u(\underline{\alpha})$ as implied by (A.17). Thus, (A.29) and (A.30) imply that $\eta u(h_1) + (1-\eta)u(h_2) < u(\underline{\alpha})$, given that $f_1 - h_1 = d_{AG} - f_1$ and $h_2 - f_2 = f_2$ as shown above. Therefore, $\tilde{b}_{AG} > b_{AG}$ given $u'' < 0$, regardless of whether $h_1 \leq h_2$ or $h_1 > h_2$.

Overall, it is proven that $\tilde{b}_{AG} > b_{AG}$ if (A.14) holds, provided that $\bar{\alpha} \geq 2\tilde{b}_{AG}$. If $\bar{\alpha} \in [2b_{AG}, 2\tilde{b}_{AG})$, then manufacturers cannot buy wood and clay from loggers and miners, respectively, if they use senior private bank notes and subordinated deposits. If $\bar{\alpha} < 2b_{AG}$, then they cannot do so even if they can use central-bank notes. Because manufacturers receive all the social surplus as described above, this result is sufficient to show that social welfare in the presence of central-bank notes weakly dominates that in the presence of senior private bank notes without the central bank if (A.14) holds. Social welfare is the same in both cases if (A.8) holds.

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