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Relationally equal treatment of equals characterizes combinations of values for cooperative games

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Abstract

We axiomatize the set of affine combinations between the Shapley value, the equal surplus division value, and the equal division value in cooperative games with transferable utilities. The set is characterized by efficiency, linearity, the balanced contributions property for equal contributors and outsiders, and the differential null player out property. The balanced contributions property for equal contributors and outsiders requires the balance of contributions between two players who contribute the same amount to the grand coalition and whose singleton coalitions earn the same worth. The differential null player out property requires that an elimination of a null player affects the other players identically. These two relational axioms are obtained by investigating Myerson's (1980) balanced contributions property and Derks and Haller's (1999) null player out property, respectively, from the perspective of a principle of Aristotle's distributive justice, whereby "equals should be treated equally".

Keywords: Axiomatization, Balanced contribution, Null player out, Shapley value, Equal surplus division value, Equal division value

JEL Classification: D63, C71

1 Introduction

The problems of distributing something among multiple agents have been discussed since early times. For instance, in Book V of *Nicomachean Ethics*, the ancient Greek philosopher Aristotle noted that "*if they are not equal, they will not have what is equal, but this is the origin of quarrels and complaints—when either equals have and are awarded unequal shares, or unequals equal shares*" (p.112 in Aristotle et al. 1980). A principle we observe here is that equals should

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be treated equally.[1] This principle, often called equal treatment of equals, is also emphasized in the modern literature[2] on cooperative games with transferable utilities (TU games). The purpose of this study is to incorporate this principle into existing relational[3] axioms and provide new axiomatizations of affine/convex combinations of values in TU games.

Among numerous values for TU games, the Shapley value (Shapley 1953), the equal surplus division value (Driessen and Funaki 1991), and the equal division value are basic ones. The Shapley value assigns each player the expectation of his/her marginal contribution. The equal surplus division value allocates each player the worth of his/her singleton coalition and an equal share of the remainder. The equal division value offers every player an equal share of the worth of the grand coalition. These three values share several common properties such as efficiency and linearity. A value is efficient if the worth of the grand coalition is fully allocated among players. A value is linear if it is suitable for addition or constant multiplication of games.

One of the properties that distinguish the Shapley value from the other two values is the balanced contributions property introduced by Myerson (1980). This property requires that for any pair of players i, j in a game, an effect of a player i's departure from a game on the other player j is equal to that of j's departure on i. Although the Shapley value is the only efficient value that possesses this property, its weaker properties clarify the similarities between the three values. For instance, the balanced cycle contributions property (Kamijo and Kongo 2010) that requires the balance of the sum of effects of a player's departure on the next player between a cyclical order of players and its reverse order is a common property of the three values. The balanced contributions property for equal contributors (Yokote et al. 2017) that requires the same condition as the balanced contributions property only for players who contribute the same amount to the grand coalition is satisfied by both the Shapley value and the equal division value, but not by the equal surplus division value.

Another property that distinguishes the Shapley value from the other two values is the null player out property introduced by Derks and Haller (1999). This property also considers the effects of a player's departure from a game on the others. Null players are those who contribute nothing to the worth of coalitions; hence, we expect that the effect of a null player's departure from a game on each remaining player to be zero. This is the requirement of the null player out property. This property is satisfied by the Shapley value, but not by the equal surplus division value or the equal division value.

In light of these existing results, we characterize the set of affine combinations between the Shapley value, the equal surplus division value, and the equal division value. We introduce two relational axioms related to equal treatment of equals: One is a property that further weakens the balanced contributions property for equal contributors. The other is a weaker null player out property. These two properties can be seen as intrinsic to the three basic values because they characterize the affine combinations between the three values together with

[1] More precisely, here, we also observe another principle that unequals should be treated unequally; the two principles together imply "*The just, then, is a species of the proportionate*" (p.113 in Aristotle et al. 1980).

[2] See also Moulin (2003).

[3] Thomson (2012) categorizes axioms into punctual and relational as follows: "*... a relational axiom relates choices made across problems that are related in certain ways.*"

efficiency and linearity.

Our new weaker balanced contributions property requires the balance of contributions only for those who contribute the same amount to the grand coalition and those whose singleton coalitions yield the same worth. With respect to a player's deletion from a game, such players can be considered equivalent from the two viewpoints of the remaining players and the player itself. Hence, we treat the players equally following the balanced contributions property. Our weaker null player out property requires the effect of a null player's departure from a game on each remaining player to be uniform, though not always zero. To reiterate, with respect to a player's deletion from a game, all of the remaining players can be considered equivalent, and thus, we treat the players equally with respect to the effects of the player deletion. In other words, two properties trace back to a principle of Aristotle's distributive justice, that "equals should be treated equally".

The remainder of the paper is structured as follows. Section 2 introduces notation and definitions. Section 3 provides axioms. Section 4 presents our results. Section 5 provides concluding remarks. Section 6 proves our results.

Related literature

The differences between the Shapley value, the equal surplus division value, and the equal division value are explored in more detail through their axiomatic characterizations. Considering the class of games on a fixed player set, van den Brink (2007) and Casajus and Huettner (2014a) describe the differences between the Shapley value, the equal surplus division value, and the equal division value by differences between the null, dummifying, and nullifying players. van den Brink (2009) explains the differences between the Shapley value and the equal division value by differences between the pair of linearity and the null player property and collusion neutrality. Casajus and Huettner (2014b) ascribe the differences between the two values to the difference between axioms on monotonicity. Considering the class of games on valuable player sets, Kamijo and Kongo (2010, 2012) attribute the differences between the Shapley value and the equal division value to differences between players whose deletion does not affect the others' payoffs. Béal et al. (2015) illustrate the difference between the equal surplus division and the equal division value due to differences between players whose deletion does not affect the others' payoffs in the presence of a special player.

To reconcile these differences, certain mixtures of the three values are also introduced, and their properties are investigated. Considering the class of games on a fixed player set, van den Brink et al. (2013) and Casajus and Huettner (2013, 2014b) study convex combinations of the Shapley value and the equal division value, often called egalitarian Shapley values (Joosten 1996). Ju et al. (2007) examine convex combinations of the Shapley value and the equal surplus division value, called consensus values. Ferrières (2016, 2017) and Kongo (2017) study convex combinations of the equal surplus division value and the equal division value. Yokote and Funaki (2017) investigate convex combinations of the three values (for games with six or more players). Considering the class of games on valuable player sets, van den Brink et al. (2013) examine egalitarian Shapley values. van den Brink and Funaki (2009) and van den Brink et al. (2016) study convex combinations of the equal surplus division value and the equal division value.

2 Preliminaries

Let $|\cdot|$ be a cardinality of a set. Let $N \subseteq \mathbb{N}$, with $|N| = n$, be a finite **set of players**. For any $i \in N$, we use i to represent a singleton coalition $\{i\} \subseteq N$. For any $N \subseteq \mathbb{N}$, a **TU game** is a pair (N, v), where $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Let Γ be a set of all TU games. For any $(N, v) \in \Gamma$ and any $i \in N$, a **subgame of** (N, v) **on** $N \setminus i$ is a pair $(N \setminus i, v|_{N\setminus i})$, where for any $S \subseteq N \setminus i$, $v|_{N\setminus i}(S) = v(S)$. For simplicity, we use $(N \setminus i, v)$ to represent $(N \setminus i, v|_{N\setminus i})$. For any $N \subseteq \mathbb{N}$ and any $S \subseteq N$, the **S-unanimity game** is a pair (N, u_S), where $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ if $T \not\supseteq S$. Given $(N, v) \in \Gamma$, $i \in N$ is a **null player** in (N, v) if for any $S \subseteq N \setminus i$, $v(S \cup i) = v(S)$.

A value is a function from Γ to $\mathbb{R}^n$. The following are three well-known values on Γ.

- The **Shapley value** (Shapley 1953) $Sh : \Gamma \to \mathbb{R}^n$: for any $(N, v) \in \Gamma$ and any $i \in N$,

$$Sh_i(N, v) = \sum_{S \ni i} \frac{(|S| - 1)!(n - |S|!)}{n!}(v(S) - v(S \setminus i)).$$

- The **equal surplus division value** (Driessen and Funaki 1991) $ESD : \Gamma \to \mathbb{R}^n$: for any $(N, v) \in \Gamma$ and any $i \in N$,

$$ESD_i(N, v) = \frac{v(N) - \sum_{j\in N} v(j)}{n} + v(i).$$

- The **equal division value** $ED : \Gamma \to \mathbb{R}^n$: for any $(N, v) \in \Gamma$ and any $i \in N$,

$$ED_i(N, v) = \frac{v(N)}{n}.$$

3 Axioms

An axiom is a property on values. Let φ be a value of games. All of the above-mentioned three values satisfy the following two basic axioms.

Efficiency (E): For any $(N, v) \in \Gamma$, $\sum_{i\in N} \varphi_i(N, v) = v(N)$.

Linearity (L): For any $(N, v), (N, w) \in \Gamma$, and any $\lambda \in \mathbb{R}$, $\varphi(N, v + \lambda w) = \varphi(N, v) + \lambda\varphi(N, w)$, where for any $S \subseteq N$, $(v + \lambda w)(S) = v(S) + \lambda w(S)$.

Among the above three values, only the Shapley value satisfies the following property.

Balanced contributions (BC, Myerson 1980): For any $(N, v) \in \Gamma$, any $i \in N$ and any $j \in N \setminus i$, it holds that $\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v)$.

We introduce a weaker property than BC, satisfied by not only the Shapley value but also the equal division value. Let $i, j \in N$ with $v(N \setminus i) = v(N \setminus j)$. Each such player contributes the same amount to the worth of the grand coalition; hence, we call them **equal contributors**. Yokote et al. (2017) introduce a property weaker than **BC** by requiring the balance of contributions only for pairs of equal contributors.[4]

Balanced contributions of equal contributors (BC$^-$, Yokote et al. 2017): For any $(N, v) \in \Gamma$ and any $i, j \in N$ such that $v(N \setminus i) = v(N \setminus j)$, it holds that $\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v)$.

We introduce a much weaker property than **BC$^-$**, satisfied by all of the above-mentioned three values. Let $i, j \in N$ with $v(i) = v(j)$. Each such player can obtain the same worth by only himself/herself if he/she leaves the game; thus, we call them **equal outsiders**. We say that $i, j \in N$ are **equal contributors and outsiders** if they are both equal contributors and equal outsiders, i.e., $v(N \setminus i) = v(N \setminus j)$ and $v(i) = v(j)$. The much weaker property than **BC$^-$** is defined by requiring the balance of contributions only for pairs of equal contributors and outsiders.

Balanced contributions for equal contributors & outsiders (BC^{--}): For any $(N, v) \in \Gamma$, any $i \in N$ and any $j \in N \setminus i$ such that $v(N \setminus i) = v(N \setminus j)$ and $v(i) = v(j)$, it holds that $\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v)$.

Again, among the above three values, only the Shapley value possesses the following axiom.

Null player out (NPO, Derks and Haller 1999): For any $(N, v) \in \Gamma$, any null player $i \in N$ in (N, v), and any $j \in N \setminus i$, $\varphi_j(N, v) = \varphi_j(N \setminus i, v)$.

We introduce a weaker property than **NPO**, satisfied by all of the above-mentioned three values. In the following weaker axiom, we require that the effect of a null player's departure on any remaining player not necessarily be zero but be common to all others.[5]

Differential null player out (NPO$^-$): For any $(N, v) \in \Gamma$, any null player $k \in N$ in (N, v), any $i \in N \setminus k$, and any $j \in N \setminus \{i, k\}$, it holds that $\varphi_i(N, v) - \varphi_i(N \setminus k, v) = \varphi_j(N, v) - \varphi_j(N \setminus k, v)$.[6]

As mentioned in the Introduction, the basic idea behind BC^{--} and NPO$^-$ can be traced back to Aristotle's distributive justice: equals should be treated equally. To apply this principle to TU games, we should clarify two questions: (i) What players are equal? (ii) What does "being treated equally" mean? In BC^{--}, (i) two players are considered to be equal if deleting either identically

[4] Weaker balanced contributions properties requiring the same condition as **BC** only on specific players are also studied by Calvo and Santos (2006), Lorenzo-Freire et al. (2007), Lorenzo-Freire (2016), and Yokote and Kongo (2017).

[5] The corresponding conditions on the class of games on a fixed player set are examined by Ferrières (2016, 2017) and Kongo (2017) under the name of nullified equal loss or equal effect of a player's nullification on others. Their properties focus on the effect of a player's nullification (i.e., a player becoming a null player) on others, instead of that of a null player's departure because the player set is fixed.

[6] This axiom is also known as weak null player out in van den Brink and Funaki (2009).

affects the deleted player itself and the entirety of the remaining players, and (ii) the effects of a player's deletion on the other should be balanced between two equal players. In NPO$^-$, (i) all players except a deleted player are considered to be equal, and (ii) the effects of a player's deletion on the others should be the same.

4 Results

4.1 Primary result

The following is our primary result that characterizes the class of all affine combinations of Sh, ESD, and ED by four axioms.

Theorem 1. *On the class of games on valuable player sets, a value φ satisfies* **E**, **L**, **BC**$^{--}$, *and* **NPO**$^-$ *if and only if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha+\beta+\gamma = 1$ and the value φ represented as follows: for any $(N, v) \in \Gamma$,*

$$\varphi(N, v) = \alpha Sh(N, v) + \beta ESD(N, v) + \gamma ED(N, v). \tag{1}$$

4.2 Independence of axioms in Theorem 1

4.2.1 Without E

The value $\varphi_i^1(N, v) = 0$ for any $i \in N$ and any $(N, v) \in \Gamma$ satisfies **L**, **BC**$^{--}$, and **NPO**$^-$, but not **E**.

4.2.2 Without L

The value $\varphi_i^2(N, v) = \begin{cases} \frac{v(i)}{\sum_{k\in N} v(k)} v(N) & \text{if } \sum_{k\in N} v(k) \neq 0 \\ ED_i(N, v) & \text{otherwise} \end{cases}$ for any $i \in N$ and any $(N, v) \in \Gamma$ satisfies **E**, **BC**$^{--}$, and **NPO**$^-$, but not **L**.

4.2.3 Without BC^{--}

The value $\varphi_i^3(N, v) = \frac{v(N) - \sum_{k\in N}(v(k)+v(N\setminus k))}{n} + v(i) + v(N \setminus i)$ for any $i \in N$ and any $(N, v) \in \Gamma$ satisfies **E**, **L**, and **NPO**$^-$, but not **BC**$^{--}$.

4.2.4 Without NPO$^-$

The value $\varphi^4(N, v) = \begin{cases} ESD(N, v) & \text{if } n = 3 \\ ED(N, v) & \text{otherwise} \end{cases}$ for any $(N, v) \in \Gamma$ satisfies **E**, **L**, and **BC**$^{--}$, but not **NPO**$^-$.

5 Concluding remarks

Theorem 1 characterizes a large class of solutions in the sense that parameters α, β, and γ have 2 degrees of freedom (i.e., they can be obtained from a 2-dimensional plane). Introducing additional axioms narrows down possible parameters, whereby we can axiomatize a restricted class of solutions. We exemplify three such axiomatizations.

Desirability, D For any $(N, v) \in \Gamma$ and $i, j \in N$ such that $v(S \cup i) - v(S) \geq v(S \cup j) - v(S)$ for all $S \subseteq N \backslash \{i, j\}$, it holds that $\varphi_i(N, v) \geq \varphi_j(N, v)$.

Positivity, P For any $(N, v) \in \Gamma$ and $i \in N$ such that (N, v) is monotonic,[7] it holds that $\varphi_i(N, v) \geq 0$.

Self duality, SD For any $(N, v) \in \Gamma$, it holds that $\varphi(N, v) = \varphi(N, v^d)$, where for any $S \subseteq N$ with $S \neq \emptyset$, $v^d(S) = v(N) - v(N \backslash S)$.

Covariance, COV For any $(N, v) \in \Gamma$, any $\alpha \in \mathbb{R}$ and any $(\beta_i)_{i \in N} \in \mathbb{R}^n$, it holds that $\varphi(N, \alpha v + \beta) = \alpha \varphi(N, v) + \beta$, where for any $S \subseteq N$, $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{i \in S} \beta_i$.

Positivity for zero-worth player, PZ For any $(N, v) \in \Gamma$ such that $v(i) = 0$ for all $i \in N$ and $v(N) \geq 0$, it holds that $\varphi_i(N, v) \geq 0$ for all $i \in N$.

Note that **P** and **PZ** are independent.

Theorem 2. *On the class of games on valuable player sets, a value φ satisfies* **E**, **L**, **BC^{--}**, **NPO^{-}**, **D**, **P** *and* **SD** *if and only if there exist $\alpha \in [0, 1]$ and the value φ represented as follows: for any $(N, v) \in \Gamma$,*

$$\varphi(N, v) = \alpha Sh(N, v) + (1 - \alpha) ED(N, v).$$

Theorem 3. *On the class of games on valuable player sets, a value φ satisfies* **E**, **L**, **BC^{--}**, **NPO^{-}**, **D**, **P** *and* **COV** *if and only if there exist $\alpha \in [0, 1]$ and the value φ represented as follows: for any $(N, v) \in \Gamma$,*

$$\varphi(N, v) = \alpha Sh(N, v) + (1 - \alpha) ESD(N, v).$$

Theorem 4. *On the class of games on valuable player sets, a value φ satisfies* **E**, **L**, **BC^{--}**, **NPO^{-}**, **D**, **P** *and* **PZ** *if and only if there exist $\alpha \in [0, 1]$ and the value φ represented as follows: for any $(N, v) \in \Gamma$,*

$$\varphi(N, v) = \alpha ESD(N, v) + (1 - \alpha) ED(N, v).$$

Theorems 2, 3, and 4 axiomatize the egalitarian Shapley values, the consensus values, and convex combinations of ESD and ED, respectively. We remark that ESD, ED, and Sh violate SD, COV, and PZ, respectively. Hence, Theorems 2, 3, and 4 can be proven by combining our Theorem 1 and Theorem 2 of Radzik and Driessen (2013).

6 Proofs

In several mathematical expressions in the following proofs, we use the following abbreviations: L3 stands for Lemma 3, and IH stands for the induction hypothesis.

[7] A game (N, v) is monotonic if $v(S) \geq v(T)$ for all $S, T \subseteq N$ such that $S \supseteq T$.

6.1 Preparation for the proof of Theorem 1

First, we discuss certain notation and two axioms used by the proof. Given $(N, v) \in \Gamma$, $i \in N$ and $j \in N \setminus i$ are **symmetric** in (N, v) if for any $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) = v(S \cup j)$.

Symmetry (S): For any $(N, v) \in \Gamma$, if $i \in N$ and $j \in N \setminus i$ are symmetric in (N, v), then $\varphi_i(N, v) = \varphi_j(N, v)$.

Anonymity (A): For any $(N, v) \in \Gamma$ and any $\pi : N \to \mathbb{N}$, $\varphi_i(N, v) = \varphi_{\pi(i)}(\pi(N), \pi v)$, where $\pi v(S) = v(\pi^{-1}(S))$ for any $S \subseteq \pi(N)$.

Note that each of the Shapley value, the equal surplus division value, and the equal division value satisfies the above two axioms.

Let $N \subseteq \mathbb{N}$ with $|N| \geq 3$ and $T \subseteq N$ with $3 \leq |T| \leq n - 1$. We define $(N, w_T) \in \Gamma$ by

$$w_T = u_T + \sum_{k \in T} \sum_{\ell \in N \setminus T} \frac{1}{2|T| - n} u_{\{k, \ell\}}.$$

Lemma 1. *For any $i \in N$, $w_T(i) = 0$.*

Proof. This immediately follows from the fact that $|T| \geq 3$. □

Lemma 2. *For any $i \in N$, $w_T(N \setminus i) = \frac{(|T|-1)(n-|T|)}{2|T|-n}$.*

Proof. Let $i \in N$. If $i \in T$,

$$w_T(N \setminus i) = u_T(N \setminus i) + \sum_{\ell \in N \setminus T} \frac{1}{2|T| - n} u_{\{i, \ell\}}(N \setminus i) + \sum_{k \in T \setminus i} \sum_{\ell \in N \setminus T} \frac{1}{2|T| - n} u_{\{k, \ell\}}(N \setminus i)$$

$$= 0 + \sum_{\ell \in N \setminus T} \frac{1}{2|T| - n} \cdot 0 + \sum_{k \in T \setminus i} \sum_{\ell \in N \setminus T} \frac{1}{2|T| - n} \cdot 1 = \frac{(|T| - 1)(n - |T|)}{2|T| - n}.$$

If $i \notin T$,

$$w_T(N \setminus i) = u_T(N \setminus i) + \sum_{k \in T} \frac{1}{2|T| - n} u_{\{k, i\}}(N \setminus i) + \sum_{k \in T} \sum_{\ell \in (N \setminus T) \setminus i} \frac{1}{2|T| - n} u_{\{k, \ell\}}(N \setminus i)$$

$$= 1 + \sum_{k \in T} \frac{1}{2|T| - n} \cdot 0 + \sum_{k \in T} \sum_{\ell \in (N \setminus T) \setminus i} \frac{1}{2|T| - n} \cdot 1$$

$$= 1 + \frac{|T|(n - |T| - 1)}{2|T| - n} = \frac{2|T| - n + n|T| - |T|^2 - |T|}{2|T| - n} = \frac{(|T| - 1)(n - |T|)}{2|T| - n}.$$

□

Lemma 3. *Let φ satisfy **E** and **BC**$^{--}$. Let $t \in \mathbb{N}$ with $t \geq 2$. If φ satisfies **A** on the class of games with $t - 1$ players, then φ satisfies **S** on the class of games with t players.*

Proof. Let $(N, v) \in \Gamma$ with $n = t$. Let i, j be symmetric players in (N, v). Then,

$$\varphi_i(N, v) - \varphi_j(N, v) \overset{\mathbf{BC}^{--}}{=} \varphi_i(N \setminus j, v) - \varphi_j(N \setminus i, v) = 0,$$

where the second equality follows from **A** on the class of games with $t - 1$ players. □

6.2 Proof of Theorem 1

First, we show the "if" part. It is clear that Sh, ESD, and ED satisfy **E**, **L**, $\mathbf{BC}^{--}$, and $\mathbf{NPO}^{-}$. Hence, for any $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma = 1$, φ satisfies them.

Below, we consider the "only if" part. If $n = 1$, **E** completes the proof.

If $n = 2$, let $\{i, j\} \subseteq \mathbb{N}$. Set $\gamma = 2\varphi_j(\{i, j\}, u_i)$. Let α,β be arbitrary real numbers such that $\alpha + \beta = 1 - \gamma$. Set $\phi^{\alpha,\beta,\gamma} = \alpha Sh + \beta ESD + \gamma ED$. Then,

$$\varphi_j(\{i, j\}, u_i) = \frac{\gamma}{2} = \phi_j^{\alpha,\beta,\gamma}(\{i.j\}, u_i).$$

By **E**, $\varphi_i(\{i, j\}, u_i) = \phi_i^{\alpha,\beta,\gamma}(\{i.j\}, u_i)$. Hence, it follows that

$$\varphi(\{i, j\}, u_i) = \phi^{\alpha,\beta,\gamma}(\{i.j\}, u_i). \tag{2}$$

Next, we consider a game $(\{i.j\}, u_j)$. Now,

$$\varphi_i(\{i.j\}, u_i) + \varphi_i(\{i.j\}, u_j) \overset{\mathbf{L}}{=} \varphi_i(\{i.j\}, u_i + u_j) \overset{\mathbf{L3,E}}{=} 1,$$

and hence,

$$\varphi_i(\{i, j\}, u_j) = 1 - \varphi_i(\{i, j\}, u_i) \overset{(2)}{=} 1 - \phi_i^{\alpha,\beta,\gamma}(\{i.j\}, u_i) = \phi_i^{\alpha,\beta,\gamma}(\{i.j\}, u_j).$$

By **E**,

$$\varphi(\{i, j\}, u_j) = \phi^{\alpha,\beta,\gamma}(\{i.j\}, u_j). \tag{3}$$

By Lemma 3,

$$\varphi(\{i, j\}, u_{\{i,j\}}) = \phi^{\alpha,\beta,\gamma}(\{i, j\}, u_{\{i,j\}}). \tag{4}$$

By (2)-(4) and **L**, we obtain that $\varphi(\{i, j\}, v) = \phi^{\alpha,\beta,\gamma}(\{i, j\}, v)$ for all v.

Let $k \in \mathbb{N} \setminus \{i, j\}$. Consider $(\{i, j, k\}, u_j)$. Set $\varphi(\{i, j, k\}, u_j) = (x, y, z)$. By $\mathbf{NPO}^{-}$, there exist $d, d' \in \mathbb{R}$ such that

$$\varphi(\{j, k\}, u_j) = (y + d, z + d), \text{ and } \varphi_i(\{i, j\}, u_j) = (x + d', y + d').$$

By $\mathbf{BC}^{--}$ applied to i and k in $(\{i, j, k\}, u_j)$,

$$x - z = x + d' - (z + d) \iff d = d'.$$

Hence, $\varphi_j(\{i, j\}, u_j) = \varphi_j(\{j, k\}, u_j)$. By the fact that $\varphi_j(\{i, j\}, u_j) = \phi_j^{\alpha,\beta,\gamma}(\{i, j\}, u_j)$, it holds that $\phi_j^{\alpha,\beta,\gamma}(\{j, k\}, u_j) = \phi_j^{\alpha,\beta,\gamma}(\{i, j\}, u_j) = \varphi_j(\{j, k\}, u_j)$. By **E**,

$$\phi^{\alpha,\beta,\gamma}(\{j, k\}, u_j) = \varphi(\{j, k\}, u_j). \tag{5}$$

Next, we consider $(\{j, k\}, u_k)$. Now,

$$\varphi_k(\{j.k\}, u_k) + \varphi_k(\{j.k\}, u_j) \overset{\mathbf{L}}{=} \varphi_k(\{j.k\}, u_k + u_j) \overset{\mathbf{L3,E}}{=} 1,$$

and hence,

$$\varphi_k(\{j, k\}, u_k) = 1 - \varphi_k(\{j, k\}, u_j) \overset{(5)}{=} 1 - \phi_k^{\alpha,\beta,\gamma}(\{j.k\}, u_j) = \phi_k^{\alpha,\beta,\gamma}(\{j.k\}, u_k).$$

By **E**,

$$\varphi(\{j, k\}, u_k) = \phi^{\alpha,\beta,\gamma}(\{j.k\}, u_k). \tag{6}$$

By Lemma 3,

$$\varphi(\{j,k\},u_{\{j,k\}}) = \phi^{\alpha,\beta,\gamma}(\{j,k\},u_{\{j,k\}}). \tag{7}$$

By (5)-(7) and **L**, we obtain that $\varphi(\{j,k\},v) = \phi^{\alpha,\beta,\gamma}(\{j,k\},v)$ for all v.

Starting from the fact that $\varphi = \phi^{\alpha.\beta,\gamma}$ on the class of games defined for $\{i,j\}$, we derive that $\varphi = \phi^{\alpha.\beta,\gamma}$ on the class of games defined for $\{j,k\}$. By repeating this argument, we obtain that $\varphi = \phi^{\alpha.\beta,\gamma}$ for all 2-person games.

If $n = 3$, let $\{i,j,k\} \subseteq \mathbb{N}$. Set $\beta = 3\varphi_k(\{i,j,k\},u_{\{i,j\}}) - \gamma$, $\alpha = 1-\beta-\gamma$, and $\phi^{\alpha,\beta,\gamma} = \alpha Sh + \beta ESD + \gamma ED$. By **L**,

$$\varphi(\{i,j,k\},u_{\{i,j\}}) + \varphi(\{i,j,k\},u_{\{i,k\}}) = \varphi(\{i,j,k\},u_{\{i,j\}} + u_{\{i,k\}}). \tag{8}$$

By Lemma 3,

$$\varphi_j(\{i,j,k\},u_{\{i,j\}} + u_{\{i,k\}}) = \varphi_k(\{i,j,k\},u_{\{i,j\}} + u_{\{i,k\}}). \tag{9}$$

By **E** and Lemma 3, there exist $a,b \in \mathbb{R}$ such that

$$\varphi(\{i,j,k\},u_{\{i,j\}}) = (a,a,1-2a), \text{and} \tag{10}$$

$$\varphi(\{i,j,k\},u_{\{i,k\}}) = (b,1-2b,b). \tag{11}$$

By (8) and (9),

$$\varphi_j(\{i,j,k\},u_{\{i,j\}})+\varphi_j(\{i,j,k\},u_{\{i,k\}}) = \varphi_k(\{i,j,k\},u_{\{i,j\}})+\varphi_k(\{i,j,k\},u_{\{i,k\}}).$$

By (10) and (11),

$$1+a-2b = 1-2a+b \iff a = b. \tag{12}$$

Then,

$$\varphi_k(\{i,j,k\},u_{\{i,j\}}) \overset{(10)}{=} 1-2a \overset{(12)}{=} 1-2b \overset{(11)}{=} \varphi_j(\{i,j,k\},u_{\{i,k\}}). \tag{13}$$

Similarly, we obtain

$$\varphi_k(\{i,j,k\},u_{\{i,j\}}) = \varphi_i(\{i,j,k\},u_{\{j,k\}}). \tag{14}$$

By (13) and (14),

$$\phi_k^{\alpha,\beta,\gamma}(\{i,j,k\},u_{\{i,j\}}) = \frac{\beta+\gamma}{3} = \varphi_k(\{i,j,k\},u_{\{i,j\}}) \overset{(13)}{=} \varphi_j(\{i,j,k\},u_{\{i,k\}}) \overset{(14)}{=} \varphi_i(\{i,j,k\},u_{\{j,k\}}).$$

By the fact that $\phi_j^{\alpha,\beta,\gamma}(\{i,j,k\},u_{\{i,k\}}) = \phi_i^{\alpha,\beta,\gamma}(\{i,j,k\},u_{\{j,k\}}) = \frac{\beta+\gamma}{3}$, it holds that $\phi_j^{\alpha,\beta,\gamma}(\{i,j,k\},u_{\{i,k\}}) = \varphi_j(\{i,j,k\},u_{\{i,k\}})$ and $\phi_i^{\alpha,\beta,\gamma}(\{i,j,k\},u_{\{j,k\}}) = \varphi_i(\{i,j,k\},u_{\{j,k\}})$. By **E** and Lemma 3,

$$\phi^{\alpha,\beta,\gamma}(\{i,j,k\},u_T) = \varphi(\{i,j,k\},u_T) \text{ for all } T \subseteq \{i,j,k\} \text{ with } |T| = 2. \tag{15}$$

Next, let $\ell \in \mathbb{N}\backslash\{i,j,k\}$ and consider $(\{i,j,\ell\},u_{\{i,j\}})$. Set $\varphi(\{i,j,k,\ell\},u_{\{i,j\}}) = (w,x,y,z)$. By $\mathbf{NPO}^-$, there exist $d,d' \in \mathbb{R}$ such that

$$\varphi(\{i,j,k\},u_{\{i,j\}}) = (w+d,x+d,y+d) \text{ and } \varphi(\{i,j,\ell\},u_{\{i,j\}}) = (w+d',x+d',z+d').$$

By $\mathbf{BC}^{--}$ applied to k, ℓ in $(\{i,j,k,\ell\}, u_{i,j})$,

$$y - z = y + d - (z + d') \iff d = d'.$$

Hence, $\varphi_i(\{i,j,\ell\}, u_{\{i,j\}}) = \varphi_i(\{i,j,k\}, u_{\{i,j\}}) = \phi_i^{\alpha,\beta,\gamma}(\{i,j,k\}, u_{\{i,j\}}) = \phi_i^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,j\}})$, and $\varphi_j(\{i,j,\ell\}, u_{\{i,j\}}) = \varphi_j(\{i,j,k\}, u_{\{i,j\}}) = \phi_j^{\alpha,\beta,\gamma}(\{i,j,k\}, u_{\{i,j\}}) = \phi_j^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,j\}})$. By **E**,

$$\varphi(\{i,j,\ell\}, u_{\{i,j\}}) = \phi^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,j\}}). \tag{16}$$

Next, consider $\varphi(\{i,j,\ell\}, u_{\{i,\ell\}})$. By Lemma 3,

$$\varphi_j(\{i,j,\ell\}, u_{\{i,j\}} + u_{\{i,\ell\}}) = \varphi_\ell(\{i,j,\ell\}, u_{\{i,j\}} + u_{\{i,\ell\}}). \tag{17}$$

By **E** and Lemma 3, there exist $a, b \in \mathbb{R}$ such that

$$\varphi(\{i,j,\ell\}, u_{\{i,j\}}) = (a, a, 1-2a), \text{and} \tag{18}$$
$$\varphi(\{i,j,\ell\}, u_{\{i,\ell\}}) = (b, 1-2b, b). \tag{19}$$

By **L**

$$\varphi(\{i,j,\ell\}, u_{\{i,j\}}) + \varphi(\{i,j,\ell\}, u_{\{i,\ell\}}) = \varphi(\{i,j,\ell\}, u_{\{i,j\}} + u_{\{i,\ell\}}). \tag{20}$$

By (17) and (20),

$$\varphi_j(\{i,j,\ell\}, u_{\{i,j\}}) + \varphi_j(\{i,j,\ell\}, u_{\{i,\ell\}}) = \varphi_\ell(\{i,j,\ell\}, u_{\{i,j\}}) + \varphi_\ell(\{i,j,\ell\}, u_{\{i,\ell\}}).$$

By (18) and (19),

$$1 + a - 2b = 1 - 2a + b \iff a = b. \tag{21}$$

Then,

$$\varphi_\ell(\{i,j,\ell\}, u_{\{i,j\}}) \overset{(18)}{=} 1 - 2a \overset{(21)}{=} 1 - 2b \overset{(19)}{=} \varphi_j(\{i,j,\ell\}, u_{\{i,\ell\}}). \tag{22}$$

Similarly, we obtain

$$\varphi_\ell(\{i,j,\ell\}, u_{\{i,j\}}) = \varphi_i(\{i,j,\ell\}, u_{\{j,\ell\}}). \tag{23}$$

By (22) and (23),

$$\phi_\ell^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,j\}}) \overset{(16)}{=} \varphi_\ell(\{i,j,\ell\}, u_{\{i,j\}}) \overset{(22)}{=} \varphi_j(\{i,j,\ell\}, u_{\{i,\ell\}}) \overset{(23)}{=} \varphi_i(\{i,j,\ell\}, u_{\{j,\ell\}}).$$

By the fact that $\phi_\ell^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,j\}}) = \phi_j^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,\ell\}}) = \phi_i^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{j,\ell\}})$, we obtain $\phi_j^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{i,\ell\}}) = \varphi_j(\{i,j,\ell\}, u_{\{i,\ell\}})$, and $\phi_i^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_{\{j,\ell\}}) = \varphi_i(\{i,j,\ell\}, u_{\{j,\ell\}})$. By **E** and Lemma 3,

$$\phi^{\alpha,\beta,\gamma}(\{i,j,\ell\}, u_T) = \varphi(\{i,j,\ell\}, u_T) \text{ for all } T \subseteq \{i,j,\ell\} \text{ with } |T| = 2. \tag{24}$$

Starting from the fact that $\varphi = \phi^{\alpha.\beta,\gamma}$ on the class of T-unanimity games with $|T| = 2$ defined for $\{i,j,k\}$, we derive that $\varphi = \phi^{\alpha.\beta,\gamma}$ on the class of T-unanimity games with $|T| = 2$ defined for $\{i,j,\ell\}$. By repeating this argument, we obtain that $\varphi = \phi^{\alpha.\beta,\gamma}$ for all T-unanimity games with $|T| = 2$ defined for any $N \subseteq \mathbb{N}$ with $|N| = 3$. In other words, by (15) and (24),

$$\phi^{\alpha,\beta,\gamma}(N, u_T) = \varphi(N, u_T) \text{ for all } N \subseteq \mathbb{N} \text{ with } n = 3 \text{ and any } T \subseteq N \text{ with } |T| = 2. \tag{25}$$

Next, let $N \subseteq \mathbb{N}$ with $n = 3$ and $i \in N$. Set $N = \{i, j, k\}$. Now,

$$\begin{aligned}\varphi_i(N, u_i) - \varphi_j(N, u_i) &\overset{\mathbf{NPO}^-}{=} \varphi_i(N \setminus k, u_i) - \varphi_j(N \setminus k, u_i) \\ &\overset{IH}{=} \phi_i^{\alpha,\beta,\gamma}(N \setminus k, u_i) - \phi_j^{\alpha,\beta,\gamma}(N \setminus k, u_i) = \phi_i^{\alpha,\beta,\gamma}(N, u_i) - \phi_j^{\alpha,\beta,\gamma}(N, u_i).\end{aligned}$$

Similarly, $\varphi_i(N, u_i) - \varphi_k(N, u_i) = \phi_i^{\alpha,\beta,\gamma}(N, u_i) - \phi_k^{\alpha,\beta,\gamma}(N, u_i)$. Together with **E**, $\varphi(N, u_i) = \phi^{\alpha,\beta,\gamma}(N, u_i)$. As N and i are arbitrarily chosen, we obtain

$$\phi^{\alpha,\beta,\gamma}(N, u_i) = \varphi(N, u_i) \text{ for any } N \subseteq \mathbb{N} \text{ with } n = 3 \text{ and } i \in N. \tag{26}$$

By **E** and Lemma 3,

$$\phi^{\alpha,\beta,\gamma}(N, u_N) = \varphi(N, u_N) \text{ for any } N \subseteq \mathbb{N} \text{ with } n = 3. \tag{27}$$

By (25)-(27) and **L**, $\phi^{\alpha,\beta,\gamma}(N, v) = \varphi(N, v)$ for any $N \subseteq \mathbb{N}$ and for any v.

Lastly, if $n \geq 4$, let $t \geq 3$. Suppose that $\phi^{\alpha,\beta,\gamma}(N, v) = \varphi(N, v)$ for all $N \subseteq \mathbb{N}$ with $n = t$ and all v. We prove that $\phi^{\alpha,\beta,\gamma}(N, v) = \varphi(N, v)$ for all $N \subseteq \mathbb{N}$ with $n = t + 1$ and all v.

Let $N \subseteq \mathbb{N}$ with $n = t + 1$ and $T \subseteq \mathbb{N}$ with $3 \leq |T| \leq n - 1$. By Lemmas 1 and 2, $w_T(i) = w_T(j)$ and $w_T(N \setminus i) = w_T(N \setminus j)$ for any $i, j \in N$, By $\mathbf{BC}^{--}$ applied to two arbitrarily chosen players, together with **E**,

$$\phi^{\alpha,\beta,\gamma}(N, w_T) = \varphi(N, w_T) \text{ for any } N \subseteq \mathbb{N} \text{ with } n = t{+}1 \text{ and any } T \subseteq N \text{ with } 3 \leq |T| \leq n{-}1. \tag{28}$$

Let $N \subseteq \mathbb{N}$ with $n = t + 1$ and $T \subseteq \mathbb{N}$ with $|T| \leq 2$. Let $i, j \in N$. If $i, j \in T$ or $i, j \notin T$, by Lemma 3, $\varphi_i(N, u_T) - \varphi_j(N, u_T) = \phi_i^{\alpha,\beta,\gamma}(N, u_T) - \phi_j^{\alpha,\beta,\gamma}(N, u_T)$. Otherwise, by $n \geq 4$, there exists $k \notin T$ with $k \in N \setminus \{i, j\}$ and

$$\begin{aligned}\varphi_i(N, u_T) - \varphi_j(N, u_T) &\overset{\mathbf{NPO}^-}{=} \varphi_i(N \setminus k, u_T) - \varphi_j(N \setminus k, u_T) \\ &\overset{IH}{=} \phi_i^{\alpha,\beta,\gamma}(N \setminus k, u_T) - \phi_j^{\alpha,\beta,\gamma}(N \setminus k, u_T) = \phi_i^{\alpha,\beta,\gamma}(N, u_T) - \phi_j^{\alpha,\beta,\gamma}(N, u_T).\end{aligned}$$

Hence, for any $i, j \in N$, $\varphi_i(N, u_T) - \varphi_j(N, u_T) = \phi_i^{\alpha,\beta,\gamma}(N, u_T) - \phi_j^{\alpha,\beta,\gamma}(N, u_T)$. By **E**,

$$\varphi(N, u_T) = \phi^{\alpha,\beta,\gamma}(N, u_T) \text{ for any } N \subseteq \mathbb{N} \text{ with } n = t{+}1 \text{ and any } T \subseteq N \text{ with } |T| \leq 2. \tag{29}$$

By **E** and Lemma 3,

$$\varphi(N, u_N) = \phi^{\alpha,\beta,\gamma}(N, u_N) \text{ for any } N \subseteq \mathbb{N} \text{ with } n = t + 1. \tag{30}$$

As the set $\{u_T : |T| \leq 2\} \cup \{w_T : 3 \leq |T| \leq n - 1\} \cup \{u_N\}$ forms a basis, (28)-(30) and **L** complete the proof. $\square$

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