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Abstract

We develop a series of small infinitary epistemic logics to study deductive inference involving intra/inter-personal beliefs/knowledge such as common knowledge, common beliefs, and infinite regress of beliefs. Specifically, propositional epistemic logics GL($\boldsymbol{L}_\alpha$) are presented for ordinal α up to a given α^o ($\alpha^o \geq \omega$) so that GL($\boldsymbol{L}_0$) is finitary KD^n with n agents and GL($\boldsymbol{L}_\alpha$) ($\alpha \geq 1$) allows conjunctions of certain countably infinite formulae. GL($\boldsymbol{L}_\alpha$) is small in that the language is countable and can be constructive. The set of formulae $\boldsymbol{L}_\alpha$ is increasing up to $\alpha = \omega$ but stops at ω. We present Kripke-completeness for GL($\boldsymbol{L}_\alpha$) for each $\alpha \leq \omega$, which is proved using the Rasiowa-Sikorski lemma and Tanaka-Ono lemma. GL($\boldsymbol{L}_\alpha$) has a sufficient expressive power to discuss intra/inter-personal beliefs with infinite lengths. As applications, we discuss the explicit definabilities of Axioms T (truthfulness), 4 (positive introspection), 5 (negative introspection), and of common knowledge in GL($\boldsymbol{L}_\alpha$). Also, we discuss the rationalizability concept in game theory in our framework. We evaluate where these discussions are done in the series GL($\boldsymbol{L}_\alpha$), $\alpha \leq \omega$.

Key Words: Infinitary Epistemic Logic, Completeness, Rasiowa-Sikorski Lemma, Tanaka-Ono Lemma, Common knowledge, Explicit Definability, Game Theory, Rationalizability

1 Introduction

We develop a series of infinitary epistemic logics to study deductive inference involving intra/inter-personal beliefs/knowledge in social situations. In these situations, people's beliefs may include infinitary components such as common knowledge, common beliefs, and infinite regress of beliefs. To approach such situations, we extend the finitary epistemic logic KD^n with n agents to infinitary logics, illustrated as

$$\mathrm{KD}^n = \mathrm{GL}(\boldsymbol{L}_0) \rightrightarrows \mathrm{GL}(\boldsymbol{L}_1) \rightrightarrows \cdots \rightrightarrows \mathrm{GL}(\boldsymbol{L}_\omega). \tag{1}$$

Each logic GL($\boldsymbol{L}_\alpha$) is "small" in that the set of formulae is countable and can be constructive. These logics are formulated in a Hilbert-style, and each is complete with respect to Kripke

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semantics. This implies that the logics in (1) are connected by the conservative extension relation $\rightrightarrows$, and the series can be used in various manners to evaluate infinitary concepts. Our approach offers a new framework, alternative to the existing literatures on related issues on infinitary epistemic concepts, with applications to evaluations of epistemic axioms and of decision-making processes in game theory.

First, we compare our approach with two literatures on infinitary epistemic concepts: the *infinitary logic* literature since Karp [19] (for epistemic logics, Kaneko-Nagashima [20], Tanaka-Ono [33], Tanaka [32], Heifetz [12]), and the *fixed-point logic* literature (for epistemic logics, Fagin *et al.* [8], Meyer-van der Hoek [25], and for μ-calculus, Enqvist, *et al.* [7], Jäger, *et al.* [16], and Jäger-Studer [17]). Both approaches have some merits and demerits; to discuss such merits and demerits, we note that the infinitary epistemic concepts we consider in applications are typically constructed by iterated substitution of the belief operators.

The infinitary logic approach is capable of discussing various infinitary concepts in an explicit and unified manner. However, the languages are very large (at least continuum) in terms of sets of formulae. A large language is not only unnecessary but also sometimes imposes an obstacle for a precise study of targeted infinitary concepts. The fixed-point logic approach has a merit to be specific to targeted infinitary concepts, but has the inconvenience that targeted concepts are indirectly expressed by a fixed-point argument. In contrast to these approaches, ours allows for explicit and unified treatments of targeted concepts and enables us to evaluate, as in (1), how large (i.e., infinitarily small) a given targeted concept requires. The key to our approach is a syntactical concept of *germinal forms*, upon which we build a series of languages, as explained below.

Our base logic is a finitary KD^n with language $\boldsymbol{L}_0$ (the set of formula); the agents have classical logical abilities and contradiction-free beliefs, described by the belief operators $\mathbf{B}_i(\cdot)$ for agents $1, ..., n$. We extend the finitary language $\boldsymbol{L}_0$ by adding conjunctions of certain infinite sequences of formulae in $\boldsymbol{L}_0$. Specifically, we take a countable number of infinite sequences $\langle C^\nu(p) : \nu \geq 0\rangle = \langle C^0(p), C^1(p), ...\rangle$ from $\boldsymbol{L}_0$, which we call *germinal forms.* A typical example is common knowledge. The germinal form for it is given as $\langle C^\nu(p) : \nu \geq 0\rangle = \langle \mathbf{B}_N^\nu(p) : \nu \geq 0\rangle$;

$$C^0(p) = p,\ C^1(p) = \wedge_{i\in N}\mathbf{B}_i(p), ...,\ C^{\nu+1}(p) = \wedge_{i\in N}\mathbf{B}_i C^\nu(p), ... \tag{2}$$

The conjunction $\mathbf{B}_N^\omega(p) := \wedge\langle \mathbf{B}_N^\nu(p) : \nu \geq 0\rangle$ is the common knowledge of p, meaning that p holds, all agents believe p, all agents believe all believe p, and so on. This is not in $\boldsymbol{L}_0$, and we extend $\boldsymbol{L}_0$ to $\boldsymbol{L}_1$ to have $\mathbf{B}_N^\omega(A)$ as a targeted formula.

The next layer $\boldsymbol{L}_2$ is obtained from $\boldsymbol{L}_1$ by adding the infinite conjunctions $\wedge\langle C^\nu(A) : \nu \geq 0\rangle$ for $A \in \boldsymbol{L}_1$, e.g., $\mathbf{B}_N^\omega(\mathbf{B}_N^\omega(A))$; roughly speaking, each formula in $\boldsymbol{L}_2$ includes infinitary conjunctions nested at most twice. Assuming that the set of germinal forms are unchanged, we define $\boldsymbol{L}_0, \boldsymbol{L}_1, ..., \boldsymbol{L}_\alpha, ...$ up to some ordinal $\alpha^o \geq \omega := \{0, 1, ...\}$. We show that this extension stops at $\boldsymbol{L}_\omega = \cup_{\alpha<\omega}\boldsymbol{L}_\alpha = \boldsymbol{L}_{\omega+1} = ... = \boldsymbol{L}_{\alpha^o}$. The language $\boldsymbol{L}_\alpha$ is kept countable for all $\alpha \leq \omega$. Also, we show that the ordinal depth of each formula in $\boldsymbol{L}_\omega$ is less than ω^2.

Infinitary concepts such as common knowledge are typically constructed by iteration of substitutions. Our formulation of a germinal form is rich enough to capture these infinitary concepts. In our approach, however, germinal forms are more generally defined even to allow nonconstructive sequences $\langle C^\nu(p) : \nu \geq 0\rangle$. This implies that our theory is quite flexible and could go beyond our current applications.

The proof systems in the series (1) are uniform; they share the same logical axiom schemata and inference rules only with the restriction to each $\boldsymbol{L}_\alpha$.[1] The Kripke semantics is defined also in a uniform manner over α. Each GL($\boldsymbol{L}_\alpha$) is proved to be sound and complete with respect to Kripke semantics. It follows from this result that GL($\boldsymbol{L}_{\alpha+1}$) is a conservative extension of GL($\boldsymbol{L}_\alpha$), i.e., for any formula $A \in \boldsymbol{L}_\alpha$, A is provable in GL($\boldsymbol{L}_\alpha$) if and only if it is provable in GL($\boldsymbol{L}_{\alpha+1}$). In (1), the double arrow $\rightrightarrows$ describes the conservative extension relation.[2]

To prove Kripke-completeness, we adopt the $\mathcal{Q}$-filter method developed in Tanaka-Ono [33]. $\mathcal{Q}$-filters play the corresponding role to that of maximal consistent sets of formulae in the standard construction of a canonical model. The $\mathcal{Q}$-filter method is crucial, since GL($\boldsymbol{L}_\alpha$) deals with both particular infinitary conjunctions and modality. To treat these aspects, our proof relies upon two lemmas, the Rasiowa-Sikorski lemma and Tanaka-Ono lemma; the countability of the language $\boldsymbol{L}_\alpha$ is crucial in applications of these lemmas. Although we use various algebraic concepts, our model theory is Kripke semantics, but not algebraic semantics. Our completeness theorem can be modified to systems including additional epistemic axioms, Axioms T (truthfulness – $\mathbf{B}_i(A) \supset A$), 4 (positive introspection – $\mathbf{B}_i(A) \supset \mathbf{B}_i\mathbf{B}_i(A)$), and/or 5 (negative introspection – $\neg\mathbf{B}_i(A) \supset \mathbf{B}_i(\neg\mathbf{B}_i(A))$).

We deliberately choose the base logic $\mathrm{KD}^n = \mathrm{GL}(\boldsymbol{L}_0)$. In the literature of epistemic logic, all, some, or none of Axioms T, 4, and 5 for $\mathbf{B}_i(\cdot)$ are adopted depending upon purposes/environments. Axioms 4 and 5 include infinitary aspects, though they are expressed in a finitary way. In our approach, we can study these axioms in terms of explicit definability in GL($\boldsymbol{L}_\alpha$) in the series in (1), that is, we ask whether there is a formula in GL($\boldsymbol{L}_\alpha$) such that it is an extension of $\mathbf{B}_i(\cdot)$ and satisfies each of T, 4, and 5. For T, it is affirmatively answered in all α, for 4, we need $\alpha = \omega$, and for 5, the answer is entirely negative. Also, we consider faithful embedding of the logics added T and/or 4 in GL($\boldsymbol{L}_\alpha$). Axiom D is included as a basic axiom in our framework, since it is crucial in proving (20) for playability in Section 5.

We also consider the faithful embedding of the common knowledge logic, denoted CK($\boldsymbol{L}^{\mathbf{C}}$), which is the fixed-point extension of KD^n, to GL($\boldsymbol{L}_\alpha$). As a whole, CK($\boldsymbol{L}^{\mathbf{C}}$) is faithfully embedded into GL($\boldsymbol{L}_\omega$). Logic CK($\boldsymbol{L}^{\mathbf{C}}$) is also a fragment of modal μ-calculus (Alberucci [1]). In this context, we show that a comparison between the *rank function* given in Alberucci, *et al.* [2] and our ordinal depth for $\boldsymbol{L}_\omega$ coincide.

Although CK($\boldsymbol{L}^{\mathbf{C}}$) can be regarded as being in the intersection of our approach and modal μ-calculus, these two approaches differ from each other not only in that the former is infinitary while the latter is finitary, but also in that the differences are substantive. We make a small summary of comparisons between our approach and modal μ-calculus in the end of Section 4.3.

Using our framework, we study a decision making process in game theory, called "rationalizability" (cf., Osborne-Rubinstein [27]). In this theory, an agent "rationalizes" his possible decision by looking for a prediction about his opponent's decision, assuming that the opponent uses the same criterion. This leads to an infinite regress of such rationalization. We show that the full discourse from a consideration of decision-making to the stage of playing a final decision can be given in logic GL($\boldsymbol{L}_2$). Thus, our framework allows for explication of game theoretic decision making with a clear-cut notion of depths of infinitary reasoning.

The paper format is as follows: Section 2 gives the definition of the sets of formulae. Section 3 formulates the system GL($\boldsymbol{L}_\alpha$) and the Kripke semantics, and states the completeness result.

[1] Below KD^n, a hierarchy of logics of shallow epistemic depths is developed in Kaneko-Suzuki [22]. Each system is a fragment of KD^n with a finite epistemic structure, and continues to KD^n.

[2] These lemmas require the set of permissible infinitary conjunctions to be countable

In Sections 4 and 5, we give discussions on applications of our framework and the completeness result to the definability problems of various epistemic concepts, and also on an application to the rationalizability concept in game theory. A proof of Kripke-completeness is given in Section 6. Section 7 concludes the paper.

2 Small Infinitary Languages $\boldsymbol{L}_\alpha$

We fix an ordinal α^o with $\alpha^o \geq \omega = \{0, 1, ...\}$. We define the class of infinitary languages $\{\boldsymbol{L}_\alpha : \alpha \leq \alpha^o\}$. For each α, $\boldsymbol{L}_\alpha$ is constructed from $\cup_{\beta<\alpha}\boldsymbol{L}_\beta$ in an inductive manner, and we will show that $\boldsymbol{L}_\alpha$ becomes constant after $\alpha = \omega$. We also evaluate the depths of formulae in $\boldsymbol{L}_\alpha$, and show that the depth of the entire set $\boldsymbol{L}_\omega$ is ω^2. In the end of this section, we make brief comparisons with the set of formulae in the literatures of infinitary logics. We stipulate that Greek letters α, β, γ are ordinals up to α^o, but Greek ν runs over the natural numbers $0, 1, ...$

We adopt the following list of primitive symbols:

propositional variables: $\mathbf{p}_0, \mathbf{p}_1, ...$; *logical connectives*: $\neg$ (not), $\supset$ (implies), $\wedge$ (and);

unary belief operators: $\mathbf{B}_1(\cdot)$, ..., $\mathbf{B}_n(\cdot)$ $(1 \leq n < \omega)$; *parentheses*: (,); *brackets*: $\langle\ ,\rangle$.

The conjunction symbol $\wedge$ is applied to a finite set of formulae and some infinite sequences of formulae. An infinitary conjunction is written as $\wedge\langle C^\nu : \nu \geq 0\rangle$ and will be specified below. We denote $\mathcal{P}_0 = \{\mathbf{p}_0, \mathbf{p}_1, ...\}$, and the set of agents (the subscripts for the beliefs operators) by $N = \{1, ..., n\}$. We may abbreviate the parentheses (,) and use different brackets when they cause no confusions.

Let α be an ordinal with $\alpha \leq \alpha^o$. Let $\mathcal{F}_\alpha$ be a given set of formulae with $\mathcal{F}_0 = \emptyset$, which is the source of infinitary conjunctions and is specified below. We define the set $\boldsymbol{L}_\alpha$ for $\alpha \geq 0$ by a double induction. Specifically, when $\alpha = 0$, $\mathcal{P}_0 = \{\mathbf{p}_0, \mathbf{p}_1, ...\}$, and when $\alpha > 0$, $\mathcal{P}_\alpha = \cup_{\beta<\alpha}\boldsymbol{L}_\beta$, provided that the set of formulae $\boldsymbol{L}_\beta$ is already defined for all $\beta < \alpha$. We define the set $\boldsymbol{L}_\alpha$ for each $\alpha \geq 0$ by the following three steps:

Iα0: all formulae in $\mathcal{P}_\alpha \cup \mathcal{F}_\alpha$ belong to $\boldsymbol{L}_\alpha$;

Iα1 (finitary extension): if A, B are formulae in $\boldsymbol{L}_\alpha$, so are $(A \supset B)$, $(\neg A)$, $\mathbf{B}_i(A)$ $(i \in N)$; and if Φ is a nonempty finite set of formulae in $\boldsymbol{L}_\alpha$, then $(\wedge\Phi)$ is a formula in $\boldsymbol{L}_\alpha$;

Iα2 (infinitary extension): if $\wedge\langle C^\nu : \nu \geq 0\rangle, \wedge\langle D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ and $A \in \boldsymbol{L}_\alpha$, then

(*i*) $\wedge\langle A \supset C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$;

(*ii*) $\wedge\langle \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ for all $i \in N$;

(*iii*) $\wedge\langle\wedge\{C^\nu, D^\nu\} : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$.

When $\alpha = 0$, step Iα2 is vacuous since $\mathcal{F}_0 = \emptyset$; thus, $\boldsymbol{L}_0$ is the set of all finitary formulae.

In Iα1, the conjunction symbol $\wedge$ is applied to finite sets of formulae. We write $A \wedge B, A \wedge B \wedge C$ for $\wedge\{A, B\}$ and $\wedge\{A, B, C\}$, etc., and $A \equiv B$ for $(A \supset B) \wedge (B \supset A)$. **I**$\alpha$1 and **I**$\alpha$2 are interactive since formulae generated by Iα2 may be used in Iα1, and *vice versa*.

The set $\mathcal{F}_\alpha$ is determined by a given set of germinal forms specified as follows. A sequence $\langle C^\nu : \nu \geq 0\rangle$ is called a *germinal form* iff $C^\nu \in \boldsymbol{L}_0$ for all $\nu \geq 0$ and a finite number of propositional variables occur in $\langle C^\nu : \nu \geq 0\rangle$. Let $p_1, ..., p_m$ be the propositional variables occurring in $\langle C^\nu : \nu \geq 0\rangle$. We often denote each C^ν in $\langle C^\nu : \nu \geq 0\rangle$ by $C^\nu(p_1, ..., p_m)$, though some of them may not be included in C^ν. Let $A_1, ..., A_m$ be formulae in $\mathcal{P}_\alpha = \cup_{\beta<\alpha}\boldsymbol{L}_\beta$, which are called *germs*. By substituting A_t for each occurrence of p_t in $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle$, we obtain the sequence $\langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle$. We say that $\Phi = \langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle$ *is generated by a germinal form* $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle$ and *germs* $A_1, ..., A_m$ in $\mathcal{P}_\alpha$. This generation is illustrated as follows:

$$\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle \quad \underset{\text{substituting } A_t \text{ for } p_t}{\longrightarrow} \quad \Phi = \langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle \tag{3}$$

For example, $\langle C^\nu(p) : \nu \geq 0\rangle = \langle \mathbf{B}_N^\nu(p) : \nu \geq 0\rangle$ is the germinal form for common knowledge. We remark that germinal forms do not require $(p_1, ..., p_m)$ to enter $C^\nu(p_1, ..., p_m)$ positively, e.g., $\langle C^\nu(p) : \nu \geq 0\rangle = \langle \neg p, \neg\neg p, ...\rangle$ is a germinal form, and a less trivial one will be given later.

Let $\mathcal{G}$ be a nonempty countable (possibly finite) set of germinal forms. We define:

$$\mathcal{F}_\alpha = \{\wedge\Phi : \Phi \text{ is generated some germinal form in } \mathcal{G} \text{ and germs in } \mathcal{P}_\alpha\}. \tag{4}$$

Since $\mathcal{G}$ is at most countable and used uniformly for all $\alpha \leq \alpha_0$, we can see that the sets $\mathcal{F}_\alpha$ and $\boldsymbol{L}_\alpha$ remain countable for each $\alpha \leq \alpha^o$.

In addition, Iα0 to Iα2 generate the other infinite conjunctions. We call $\wedge\langle C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ an *α-infinite conjunction,* and $\langle C^\nu : \nu \geq 0\rangle$ an *α-permissible sequence*. Sometimes, we simply call $\wedge\langle C^\nu : \nu \geq 0\rangle$ an infinite conduction. We stipulate that $A \in \langle C^\nu : \nu \geq 0\rangle$ iff $A \in \{C^\nu : \nu \geq 0\}$. We use the same expression, $\wedge\Phi$, for a finite conjunction or an infinite conjunction. We write $\mathbf{B}_i(\Phi)$ for $\langle \mathbf{B}_i(C) : C \in \Phi\rangle$ if Φ is an α-permissible sequence or $\{\mathbf{B}_i(C) : C \in \Phi\}$ if Φ is a finite set of formulae in $\boldsymbol{L}_\alpha$.

A series of languages $\{\boldsymbol{L}_\alpha : \alpha \leq \alpha^o\}$ is determined by a given set of germinal forms $\mathcal{G}$; we may write $\boldsymbol{L}_\alpha = \boldsymbol{L}_\alpha(\mathcal{G})$ to emphasize the choice of $\mathcal{G}$ for $\boldsymbol{L}_\alpha$. Each $\boldsymbol{L}_\alpha$ serves a language for an epistemic logic GL($\boldsymbol{L}_\alpha$) to be given in Section 3. Thus, $\{\boldsymbol{L}_\alpha : \alpha \leq \alpha^o\} = \{\boldsymbol{L}_\alpha(\mathcal{G}) : \alpha \leq \alpha^o\}$ is not only a series of languages but also determines a series of epistemic logics. When $\mathcal{G}$ is changing with fixed α, we have another series of languages and logics. Using these series, we discuss the required depth α and germinal forms $\mathcal{G}$ for a discourse involving infinitary concepts.

In Section 1, we gave the germinal form $\langle C^\nu(p) : \nu \geq 0\rangle = \langle \mathbf{B}_N^\nu(p) : \nu \geq 0\rangle$ for common knowledge, which is defined by (2).[3] As emphasized in Section 1, this is generated by iterations of substitutions. Here, we give a few more examples; the last one is not based on iterations of substitutions.

Example 2.1 (1) Positive introspection: Let $i \in N$ be fixed. We define

$$\mathbf{B}_i^0(p) = \mathbf{B}_i(p) \text{ and } \mathbf{B}_i^{\nu+1}(p) = \mathbf{B}_i(\mathbf{B}_i^\nu(p)) \text{ for } \nu \geq 0. \tag{5}$$

The sequence $\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$ is a possible germinal form. Then, we denote $\mathbf{B}_i^\omega(p) := \wedge\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$. For $A \in \mathcal{P}_\alpha$, $\mathbf{B}_i^\omega(A)$ belongs to $\mathcal{F}_\alpha$ as long as $\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle \in \mathcal{G}$. We will see in Section 4 that the formula $\mathbf{B}_i^\omega(A)$ is regarded as the infinitary extension of finitary $\mathbf{B}_i(A)$ in that $\mathbf{B}_i^\omega(A)$ enjoys the positive introspection property (Axiom 4) in GL($\boldsymbol{L}_\omega$).

For both common knowledge and positive introspection, the germinal forms are obtained by substituting for one propositional variable. The next example needs two propositional variables.

[3] The common belief of A is defined by plugging germ $\wedge_{i\in N}\mathbf{B}_i(A)$ to p in $\langle \mathbf{B}_N^\nu(p) : \nu \geq 0\rangle$, that is, $\mathbf{B}_N^\omega(\wedge_{i\in N}\mathbf{B}_i(A))$.

Game theoretical examples may involve more propositional variables; one example is given in Section 5.

(2) Infinite regress: Let $n = 2$. We prepare two formulae $\mathbf{B}_i(p_j)$ and $\mathbf{B}_j(p_i)$ with $\{i, j\} = \{1, 2\}$. Then, the germinal forms $\langle \mathbf{Ir}_i^\nu[p_1, p_2] : \nu \geq 0\rangle$, $i = 1, 2$, are generated as follows: for $i, j = 1, 2$ $(i \neq j)$,

$$\mathbf{Ir}_i^0[p_1, p_2] = \mathbf{B}_i(p_i);\ \text{and}\ \mathbf{Ir}_i^{\nu+1}[p_1, p_2] = \mathbf{B}_i(\mathbf{Ir}_j^\nu[p_1, p_2])\ \text{for}\ \nu \geq 0. \tag{6}$$

We write the conjunction $\mathbf{Ir}_i[p_1, p_2] := \wedge\langle \mathbf{Ir}_i^\nu[p_1, p_2] : \nu \geq 0\rangle$ for $i = 1, 2$. Let $A_1, A_2 \in \mathcal{P}_\alpha$. The epistemic infinite regress for agent i from A_i and A_j is given $\mathbf{Ir}_i[A_1, A_2] = \wedge\langle \mathbf{B}_i(A_i), \mathbf{B}_i\mathbf{B}_j(A_j), \mathbf{B}_i\mathbf{B}_j\mathbf{B}_i(A_i), ...\rangle$.

Epistemic infinite regress is a subjective concept in that each formula for i occurs in the scope of $\mathbf{B}_i(\cdot)$, and is an extension of common belief. When $A_1 = A_2 = A$, $\mathbf{Ir}_1[A_1, A_2] \wedge \mathbf{Ir}_2[A_1, A_2]$ is equivalent to the common belief of A. The epistemic infinitary regress takes subjectivity (and individuality) more seriously than common knowledge and common belief.

(3) More general germinal forms: We do not assume positivity for germinal forms. The example already given is $\langle \neg p, \neg\neg p, ...\rangle$, which is generated by iterated substitutions with $\neg p$. This is inconsistent, but still allowed in our theory. A consistent example is the germinal forms

$$\begin{aligned} &\langle \mathbf{B}_1(p_1), \neg\mathbf{B}_1\mathbf{B}_2(p_2), \neg\mathbf{B}_1\neg\mathbf{B}_2\mathbf{B}_1(p_1), \neg\mathbf{B}_1\neg\mathbf{B}_2\neg\mathbf{B}_1\mathbf{B}_2(p_2), ...\rangle; \\ &\langle \mathbf{B}_2(p_2), \neg\mathbf{B}_2\mathbf{B}_1(p_1), \neg\mathbf{B}_2\neg\mathbf{B}_1\mathbf{B}_2(p_2), \neg\mathbf{B}_2\neg\mathbf{B}_1\neg\mathbf{B}_2\mathbf{B}_1(p_1), ...\rangle, \end{aligned} \tag{7}$$

each of which is obtained by $C_1^\nu(p_1, p_2) = \neg\mathbf{B}_1 C_2^{\nu-1}(p_1, p_2)$ and $C_2^\nu(p_1, p_2) = \neg\mathbf{B}_2 C_2^{\nu-1}(p_1, p_2)$ for each $\nu \geq 1$ with $C_1^0(p_1, p_2) = \mathbf{B}_1(p_1)$ and $C_2^0(p_1, p_2) = \mathbf{B}_2(p_2)$. Their conjunctions are consistent in our logic containing them in the language.

The above examples are constructed by iteration of substitutions. However, our formulation also allows for infinite conjunctions that cannot be obtained by iterated substitutions. For example, let $\{k_\nu : \nu \geq 0\}$ be the sequence of Fibonacci numbers and define $C^\nu(p) = \mathbf{B}_1^{k_0}\mathbf{B}_2^{k_1}...\mathbf{B}_i^{k_\nu}(p)$, where $i = 1$ if ν is even and $i = 2$ otherwise. This sequence $\langle C^\nu(p) : \nu \geq 0\rangle$ is a germinal form but cannot be generated by iteration of substitutions. Moreover, germinal forms defined by uncomputable $\{k_\nu : \nu \geq 0\}$ are also allowed.

The subformulae of $A \in \boldsymbol{L}_\alpha = \boldsymbol{L}_\alpha(\mathcal{G})$ are defined in the standard manner. Then, $\boldsymbol{L}_\alpha$ is subformula-closed. It is proved by the double induction over ordinals α and over Iα0 - Iα2.

Lemma 2.1. *Any subformula of $A \in \boldsymbol{L}_\alpha$ belongs to $\boldsymbol{L}_\alpha$.*

The set of formulae $\boldsymbol{L}_\alpha$ is increasing up to $\alpha = \omega$, but it becomes constant after $\alpha = \omega$.

Theorem 2.1. ***(Stopping at** ω**)*** *Let $\mathcal{G}$ be a fixed nonempty set of germinal forms. If $\alpha < \omega$, then $\boldsymbol{L}_\alpha \subsetneq \boldsymbol{L}_{\alpha+1}$; and if $\omega \leq \alpha \leq \alpha^o$, then $\boldsymbol{L}_\alpha = \boldsymbol{L}_\omega = \mathcal{P}_\omega$ $(= \cup_{\beta<\omega}\boldsymbol{L}_\beta)$.*

Proof. Let $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle$ be a germinal form in $\mathcal{G}$. Since $\wedge\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle \in \boldsymbol{L}_1 - \boldsymbol{L}_0$, we have $\boldsymbol{L}_0 \subsetneq \boldsymbol{L}_1$. Let $1 \leq \alpha < \omega$. Suppose $\boldsymbol{L}_{\alpha-1} \subsetneq \boldsymbol{L}_\alpha$. By I$(\alpha+1)$0, $\boldsymbol{L}_\alpha \subseteq \boldsymbol{L}_{\alpha+1}$. Take $A_1, ..., A_m \in \boldsymbol{L}_\alpha - \boldsymbol{L}_{\alpha-1}$. Then, $\wedge\langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle$ is in $\mathcal{F}_{\alpha+1}$ but not in $\mathcal{F}_\alpha$; so, it is not in $\boldsymbol{L}_\alpha$. Hence, $\boldsymbol{L}_\alpha \subsetneq \boldsymbol{L}_{\alpha+1}$.

Consider the latter assertion of the theorem. By Iω0-Iω2, $\mathcal{P}_\omega \subseteq \boldsymbol{L}_\omega$. Now, we show $\boldsymbol{L}_\omega \subseteq \mathcal{P}_\omega$. Take germs $A_1, ..., A_m \in \mathcal{P}_\omega$. These germs belong to $\boldsymbol{L}_\gamma$ for some $\gamma < \omega$. Hence, $\wedge\langle C^\nu(A_1, ..., A_m) :$

$\nu \geq 0\rangle$ belongs to $\mathcal{F}_{\gamma+1}$. Thus, any formulae generated by Iω0- Iω2 belong to $\boldsymbol{L}_\beta$ for some $\beta < \omega$. Hence, $\boldsymbol{L}_\omega \subseteq \mathcal{P}_\omega = \cup_{\beta<\omega}\boldsymbol{L}_\beta$. Now, by induction over α up to α^o, we have $\mathcal{P}_\omega = \boldsymbol{L}_\omega = \boldsymbol{L}_\alpha$ for all α ($\omega \leq \alpha \leq \alpha^o$).■

The set $\boldsymbol{L}_\alpha = \boldsymbol{L}_\alpha(\mathcal{G})$ ($0 \leq \alpha \leq \omega$, a countable $\mathcal{G}$) is small in the sense that it remains countable. Also, the depths of formulae in $\boldsymbol{L}_\alpha$ are relevant to evaluations of infinitary concepts such as common knowledge. We introduce the *depth function* δ over $\boldsymbol{L}_\omega$, which assigns an ordinal number to each formula in $\boldsymbol{L}_\omega$. We define δ inductively along the definition of formulae as follows:

d0: $\delta(p) = 0$ for all propositional variables p;

d1: $\delta(\neg A) = \delta(A) + 1$, and $\delta(A \supset B) = \max(\delta(A), \delta(B)) + 1$;

d2: $\delta(\mathbf{B}_i(A)) = \delta(A) + 1$ for all $i \in N$;

d3: $\delta(\wedge\Phi) = \sup\{\delta(C) + 1 : C \in \Phi\}$.

Step d3 have several cases; Φ may be a finite set of formulae in Iα1 and Φ may be an α-permissible sequence in $\mathcal{F}_\alpha$ or generated by Iα2. If $\sup\{\delta(C) + 1 : C \in \Phi\}$ is a limit ordinal, then $\delta(\wedge\Phi) = \sup\{\delta(C) : C \in \Phi\}$, and otherwise, $\delta(\wedge\Phi) = \sup\{\delta(C) : C \in \Phi\} + 1$. For any set of formulae Γ, we define $\delta(\Gamma) = \sup\{\delta(A) : A \in \Gamma\}$. Since $\boldsymbol{L}_0$ consists only of finitary formulae, we have $\delta(\boldsymbol{L}_0) = \sup\{\delta(A) : A \in \boldsymbol{L}_0\} = \omega$. It follows from d0-d3 that for any $A \in \boldsymbol{L}_\omega$, $\delta(C) < \delta(A)$ for any proper subformula C of A.

Consider the formula $\mathbf{B}_i^\omega(p) = \wedge\langle\mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$ in Example 2.1.(1). Then, $\delta(\mathbf{B}_i^\omega(A)) = \omega + 1$ and $\mathbf{B}_i^\omega(p) \in \boldsymbol{L}_1 - \boldsymbol{L}_0$, provided $\langle\mathbf{B}_i^\nu(p) : \nu \geq 0\rangle \in \mathcal{G}$. Any formula D in $\boldsymbol{L}_1$ including $\mathbf{B}_i^\omega(A)$ takes the form $\omega + k$ for some finite k, and this k may be arbitrary large; thus, $\delta(D) < \omega + \omega = \omega 2$ and $\delta(\boldsymbol{L}_1) = \omega 2$. The following theorem generalizes this observation.

Theorem 2.2. ***(Depths of formulae)*** *Suppose that $\mathcal{G}$ has a germinal form $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle$ such that $\sup\{\delta(C^\nu(p_1, ..., p_m)) : \nu \geq 0\} = \omega$.*

(1): *If $0 \leq \alpha < \omega$, then $\delta(A) < \omega(\alpha + 1)$ for all $A \in \boldsymbol{L}_\alpha$; and $\delta(\boldsymbol{L}_\alpha) = \omega(\alpha + 1)$.*

(2): *$\delta(A) < \omega^2$ for all $A \in \boldsymbol{L}_\omega$; and $\delta(\boldsymbol{L}_\omega) = \omega^2$.*

Proof. (1): As mentioned above, $\delta(A) < \omega$ for all $A \in \boldsymbol{L}_0$ and $\delta(\boldsymbol{L}_0) = \omega$. Let $1 \leq \alpha < \omega$, and suppose the induction hypothesis that $\delta(A) < \omega\alpha$ for all $A \in \boldsymbol{L}_{\alpha-1}$ and $\delta(\boldsymbol{L}_{\alpha-1}) = \omega\alpha$. Then, we prove the assertions for α. First, we show that $\delta(A) < \omega(\alpha + 1)$ for all $A \in \boldsymbol{L}_\alpha$.

Let $\wedge\Phi \in \mathcal{F}_\alpha$. Since $\delta(A) < \omega\alpha$ for all $A \in \Phi$ by the induction hypothesis, we have $\delta(\wedge\Phi) \leq \omega\alpha$ by d3. Thus, $\delta(A) \leq \omega\alpha$ for any $A \in \mathcal{P}_\alpha \cup \mathcal{F}_\alpha$. Now, consider I$\alpha$1. Suppose the other induction hypothesis that for any immediate subformula C of A generated by Iα1, $\delta(C) \leq \omega\alpha + k$ for some $k < \omega$. Then, by d1-d3, we have $\delta(A) \leq \omega\alpha + k'$ for some $k' < \omega$.

Consider Iα2. The induction hypothesis is that $\delta(D) \leq \omega\alpha + k$ and $\delta(\wedge\Phi) \leq \omega\alpha + k$ for some $k < \omega$. Then, $\delta(D \supset C) \leq (\omega\alpha + k) + 1$ for any $C \in \Phi$; and so $\delta(\wedge\langle D \supset C : C \in \Phi\rangle) \leq (\omega\alpha + k) + 1$. Also, $\delta(\mathbf{B}_i(C)) \leq (\omega\alpha + k) + 1$ for any $C \in \Phi$; and so $\delta(\wedge\langle\mathbf{B}_i(C) : C \in \Phi\rangle) \leq (\omega\alpha + k) + 1$. The case of I$\alpha$2.$(iii)$ is similar. Thus, for a formula A generated by Iα2, it still holds that $\delta(A) \leq \omega\alpha + k'$ for some $k' < \omega$. By these two paragraphs and induction, it holds that $\delta(A) < \omega(\alpha + 1)$ for all $A \in \boldsymbol{L}_\alpha$.

For $\delta(\boldsymbol{L}_\alpha) = \omega(\alpha + 1)$, we show that for any $k < \omega$, there is a formula $C \in L_\alpha$ so that $\delta(C) \geq \omega\alpha + k$. Now, since $\delta(\boldsymbol{L}_{\alpha-1}) = \omega\alpha$, there are formulae $A_1, ..., A_m \in \boldsymbol{L}_{\alpha-1}$ such that $\delta(A_t) \geq \omega(\alpha - 1)$ for $t = 1, ..., m$. Let $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle$ be a germinal form given in the assumption of the theorem. Consider $\wedge\langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle$. Since $\sup\{\delta(C^\nu) : \nu \geq 0\} = \omega$, there is a ν for any $k < \omega$ such that $\delta(C^\nu(A_1, ..., A_m)) \geq \omega(\alpha-1)+k$. Hence, $\delta(\wedge\langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle) = \omega(\alpha - 1) + \omega = \omega\alpha$. Then, using I$\alpha$1, for any $k < \omega$, we find a formula $F \in L_\alpha$ so that $\delta(F) \geq \omega\alpha + k$. Thus, $\delta(L_\alpha) \geq \sup_k(\omega\alpha + k) = \omega\alpha + \omega = \omega(\alpha + 1)$, and by the conclusion of the previous paragraph, we have $\delta(L_\alpha) = \omega(\alpha + 1)$.

The first part of (2) follows (1), since $\boldsymbol{L}_\omega = \cup_{\alpha<\omega}\boldsymbol{L}_\alpha$ by Theorem 2.1. The second part follows $\boldsymbol{L}_\omega = \cup_{\alpha<\omega}\boldsymbol{L}_\alpha$ and (1); indeed, $\delta(\boldsymbol{L}_\omega) = \delta(\cup_{\alpha<\omega}\boldsymbol{L}_\alpha) = \sup\{\delta(\boldsymbol{L}_\alpha) : \alpha \geq 0\} = \sup\{\omega(\alpha + 1) : \omega > \alpha \geq 0\} = \omega^2$.■

Theorem 2.2 is summarized in Table 2.1; our infinitary languages $\boldsymbol{L}_\alpha$ $(1 \leq \alpha \leq \omega)$ include infinitary conjunctions but are not much larger than the finitary language $\boldsymbol{L}_0$. These extensions are large enough for treatments of infinitary concepts mentioned above.

Table 2.1: Depths and cardinalities

	$\boldsymbol{L}_0$	$\subsetneq$	$\boldsymbol{L}_1$	$\subsetneq \cdot \quad \cdot \quad \cdot \subsetneq$	$\boldsymbol{L}_\omega$
#	$\aleph_0$		$\aleph_0$		$\aleph_0$
depth	ω	$<$	$\omega 2$	$<$	ω^2

Let us compare the above theorem with the infinitary logic approach. Following Kaneko-Nagashima [20], we construct $\mathcal{L}_\alpha$ $(0 \leq \alpha \leq \alpha^o)$ as follows. Let $\mathcal{F}_0 = \emptyset$, $\mathcal{P}_0 = \{\mathbf{p}_0, \mathbf{p}_1, ...\}$. Let $\mathcal{L}_0 = \boldsymbol{L}_0$. For any α $(1 \leq \alpha \leq \alpha^o)$, assuming that $\mathcal{L}_\beta$ are defined for any $\beta < \alpha$, we define

KNα : $\mathcal{F}_\alpha = \{\wedge\Phi : \Phi$ is a countable subset of $\cup_{\beta<\alpha}\mathcal{L}_\beta\}$,

and then $\mathcal{L}_\alpha$ is defined by Iα0 with $\mathcal{F}_\alpha$ and $\cup_{\beta<\alpha}\mathcal{L}_\beta$ and by Iα1-Iα2. We denote the set of formulae for step α by $\mathcal{L}_\alpha$. The set $\mathcal{L}_1$ is already uncountable. Also, $\mathcal{L}_\alpha$ does not stop at $\alpha = \omega$, e.g., $\cup_{\beta<\alpha}\mathcal{L}_\beta \subsetneq \mathcal{L}_\omega \subsetneq \mathcal{L}_{\omega+1}$ for all $\alpha \leq \omega$. Then, $\delta(\cup_{\alpha<\omega}\mathcal{L}_\alpha) = \omega^2$ but $\delta(\mathcal{L}_\omega) = \omega^2 + \omega$. This sequence $\mathcal{L}_\alpha$ increases up to the first uncountable ordinal ω_1, where we assume $\alpha^o \geq \omega_1$. Tanaka-Ono [33] considered the smallest set, $\mathcal{L}^{TO}$, that is closed with respect to finitary operations on $\neg$, $\supset$, $\mathbf{B}_i(\cdot)$ and countable conjunctions:

TO: for any countable subset Φ of $\mathcal{L}^{TO}$, $\wedge\Phi$ belongs to $\mathcal{L}^{TO}$.

Then, it holds that $\mathcal{L}^{TO} = \cup_{\beta<\omega_1}\mathcal{L}_\beta$. This $\mathcal{L}^{TO}$ is the *smallest* infinitary language in the sense of Karp [19].

3 Epistemic Logics GL($\boldsymbol{L}_\alpha$) $(0 \leq \alpha \leq \omega)$

We formulate a Hilbert-style proof theory and Kripke-semantics for epistemic logic GL($\boldsymbol{L}_\alpha$) = GL($\boldsymbol{L}_\alpha(\mathcal{G})$) with $0 \leq \alpha \leq \omega$ and a countable set of germinal forms $\mathcal{G}$. We state the soundness-completeness theorem (Theorem 3.1), which will be proved in Section 6. We discuss the hierarchy of GL($\boldsymbol{L}_\alpha(\mathcal{G})$) with respect to both α and $\mathcal{G}$, and provide four meta-lemmas to be used in Section 4.

3.1 Hilbert-style proof theory

The base logic for epistemic logic GL($\boldsymbol{L}_\alpha$) is an infinitary classical logic defined by the following four axiom schemata and two inference rules: for all formulae $A, B, C, \wedge\Phi$ in $\boldsymbol{L}_\alpha$,

L1: $A \supset (B \supset A)$;

L2: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

L3: $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

L4: $\wedge\Phi \supset C$, where $C \in \Phi$;

Modus Ponens: $\dfrac{A \supset B \quad A}{B}$; and **$\wedge$-rule:** $\dfrac{\{A \supset C : C \in \Phi\}}{A \supset \wedge\Phi}$.

We add the following epistemic axiom schemata and inference rule: for any $A, C, \wedge\Phi$ in $\boldsymbol{L}_\alpha$ and $i \in N$,

K: $\mathbf{B}_i(A \supset C) \supset (\mathbf{B}_i(A) \supset \mathbf{B}_i(C))$;

D: $\neg\mathbf{B}_i(\neg A \wedge A)$;

$\wedge$-Barcan: $\wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$;

Necessitation: $\dfrac{A}{\mathbf{B}_i(A)}$.

The above axiomatization is an infinitary version of epistemic logic KD^n with the $\wedge$-Barcan axiom (conjunctive analogue of the Barcan axiom $\forall x(\Box A(x)) \supset \Box(\forall x A(x))$ in the first order modal logic). Infinitary aspects are included in L4, $\wedge$-rule, and $\wedge$-Barcan, while the other axioms and inference rules do not directly operate on infinitary structures. The definition of $\boldsymbol{L}_\alpha$ guarantees the well-definedness of L4, $\wedge$-rule, and $\wedge$-Barcan. Indeed, an instance $\wedge\Phi \supset C$ for L4 is in $\boldsymbol{L}_\alpha$ for all $C \in \Phi$ by Lemma 2.1 and Iα1. The sequence $\langle A \supset C : C \in \Phi\rangle$ of the upper formulae in $\wedge$-rule is α-permissible by Iα2.(i). Since $\mathbf{B}_i(\wedge\Phi) \in \boldsymbol{L}_\alpha$ by Iα1 and $\wedge\mathbf{B}_i(\Phi) \in \boldsymbol{L}_\alpha$ by Iα2.(ii), the formula $\wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$ of the $\wedge$-Barcan axiom is in $\boldsymbol{L}_\alpha$. An equivalent form of Axiom D is $\mathbf{B}_i(\neg A) \supset \neg\mathbf{B}_i(A)$, which is used in (20) in Section 5.

A *proof* $P = \langle X, <; f\rangle$ in GL($\boldsymbol{L}_\alpha$) consists of a countable tree $\langle X, <\rangle$ and a function $f : X \to \boldsymbol{L}_\alpha$ with the following requirements:

(o): $\langle X, <\rangle$ has no infinite path from its root;

(i): for each node x in $\langle X, <\rangle$, $f(x)$ is a formula attached to x;

(ii): for each leaf x in $\langle X, <\rangle$, $f(x)$ is an instance of the axiom schemata;

(iii): for each non-leaf x in $\langle X, <\rangle$,

$$\frac{\{f(y) : y \text{ is an immediate successor of } x\}}{f(x)}$$

is an instance of the inference rules, MP, $\wedge$-rule, and Nec.

Infinite branching is possible in (iii) to allow inferences with $\wedge$-rule. Thus, the width of $(X, <)$ can be countably infinite and also the supremum of the depths can be infinite.

When A is attached to the root node of $P = \langle X, <; f\rangle$, we call P a *proof of* A. We say that A is *provable* in GL($\boldsymbol{L}_\alpha$), denoted by $\vdash A$, iff there is a proof of A in GL($\boldsymbol{L}_\alpha$).

Lemma 3.1 states basic properties of the provability relation $\vdash$ in GL($\boldsymbol{L}_\alpha$). Since we adopt a particular axiomatization of classical logic, these should be proved. Since the fragment determined by $\supset$ and $\neg$ with L1-L3, MP is a standard formulation of classical proposition logic, a proof of (1) is found in a textbook (e.g., Mendelson [24]). Since our system additionally includes the connective $\wedge$, (2) is crucial; a proof is given in Kaneko [18], Lemma 11.1. (3) is the converse of $\wedge$-Barcan, which is proved for any permissible or finite Φ: indeed, since $\vdash \wedge\Phi \supset A$ for $A \in \Phi$

by L4, we have $\vdash \mathbf{B}_i(\wedge\Phi) \supset \mathbf{B}_i(A)$ by Nec and K. Since this holds for all $A \in \Phi$, we have, by $\wedge$-rule, $\vdash \mathbf{B}_i(\wedge\Phi) \supset \wedge\mathbf{B}_i(\Phi)$. Incidentally, when Φ is a *finite* set, the $\wedge$-Barcan axiom is unnecessary, i.e., $\wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$ is derived without using $\wedge$-Barcan.

Lemma 3.1. *For any $A, B, C, \wedge\Phi \in \boldsymbol{L}_\alpha$, and $i \in N$,*
(1): $\vdash A \supset B$ *and* $\vdash B \supset C$ *imply* $\vdash A \supset C$;
(2): $\vdash [A \wedge B \supset C] \equiv A \supset (B \supset C)$;
(3): $\vdash \mathbf{B}_i(\wedge\Phi) \supset \wedge\mathbf{B}_i(\Phi)$.

Remark 3.1.(1): We can take the standard de Morgan method to define the disjunction formula as $\vee\Phi := \neg \wedge \{\neg A : A \in \Phi\}$ for a finite set of formulae Φ. For an α-permissible sequence Φ, this could work when we extend Iα2 to include $\langle \neg A : A \in \Phi \rangle$ for any $\wedge\Phi \in \boldsymbol{L}_\alpha$, which is not included in this paper.
(2): In GL($\boldsymbol{L}_\alpha$), the substitution-rule is stated as follows: for any $A[p]$ and B in $\boldsymbol{L}_\alpha$,

$$\text{if} \vdash A[p] \text{ and } A[B] \in \boldsymbol{L}_\alpha, \text{ then } \vdash A[B], \tag{8}$$

where $A[p]$ is a formula in $\boldsymbol{L}_\alpha$ and $A[B]$ is the formula obtained from $A[p]$ by substituting B for all occurrences of p. This fact will be used in Lemma 4.2.

3.2 Kripke completeness

A *Kripke frame* $\mathbb{K} = \langle W; R_1, ..., R_n \rangle$ is an $(n+1)$-tuple of a set of possible worlds and n accessibility relations over W, where W is an arbitrary nonempty set and R_i is a *serial* binary relation over W for each $i \in N$, i.e., for any $w \in W$, $(w, u) \in R_i$ for some $u \in W$. A *truth assignment* τ is a function from $W \times \mathcal{P}_0$ to $\{\top, \bot\}$. A pair $(\mathbb{K}, \tau)$ is a *Kripke model*.

Let $\mathcal{G}$ be a fixed countable set of germinal forms. The valuation $(\mathbb{K}, \tau, w) \models$ for $w \in W$ is inductively defined over $\boldsymbol{L}_\alpha = \boldsymbol{L}_\alpha(\mathcal{G})$ as follows: for any A, C, $\wedge\Phi \in \boldsymbol{L}_\alpha = \boldsymbol{L}_\alpha(\mathcal{G})$, and any $w \in W$,

V0: for any $p \in \mathcal{P}_0$, $(\mathbb{K}, w, \tau) \models p \iff \tau(w, p) = \top$;
V1: $(\mathbb{K}, \tau, w) \models \neg A \iff (\mathbb{K}, \tau, w) \not\models A$;
V2: $(\mathbb{K}, \tau, w) \models A \supset C \iff (\mathbb{K}, \tau, w) \not\models A$ or $(\mathbb{K}, \tau, w) \models C$;
V3: $(\mathbb{K}, \tau, w) \models \wedge\Phi \iff (\mathbb{K}, \tau, w) \models A$ for all $A \in \Phi$;
V4: $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(A) \iff (\mathbb{K}, \tau, v) \models A$ for all v with wR_iv.

Since $\boldsymbol{L}_\alpha \subseteq \boldsymbol{L}_\omega$ $(\alpha \leq \omega)$, the valuation $(\mathbb{K}, \tau, w) \models$ is uniform over $\boldsymbol{L}_\alpha$ for all $\alpha \leq \omega$; that is, it is defined over $\boldsymbol{L}_\omega$ and it can be restricted to $\boldsymbol{L}_\alpha$. For any $A \in \boldsymbol{L}_\alpha$, we write $\models A$ iff $(\mathbb{K}, \tau, w) \models A$ for all $\mathbb{K}$, $w \in W$ and τ.

We have the following soundness-completeness theorem; the proof of soundness is standard and mentioned below, and completeness will be proved in Section 6. In the theorem, let $\mathcal{G}$ be a fixed (at most countable) set of germinal forms.

Theorem 3.1. ***(Soundness and completeness for GL($\boldsymbol{L}_\alpha$))*** *Let α be an ordinal with $0 \leq \alpha \leq \omega$. For any $A \in \boldsymbol{L}_\alpha$, GL($\boldsymbol{L}_\alpha$) $\vdash A$ if and only if $\models A$.*

Soundness (the only-if part) implies the contradiction-freeness of logic GL($\boldsymbol{L}_\alpha$), which will be used in the proof of completeness. Also, by soundness, we can see consistency of the conjunctions of both germinal forms in (7) by the following Kripke model; both are true in the middle world.

$$\begin{array}{ccccc} \bigcirc\ p_1, \neg p_2 & \longleftarrow_1 & \bigcirc\ p_1, p_2 & \longrightarrow_2 & \bigcirc\ \neg p_1, p_2 \\ \circlearrowleft_{1,2} & & \circlearrowleft_{1,2} & & \circlearrowleft_{1,2} \end{array}$$

Figure 3.1

Soundness is proved as follows: Let $P = \langle X, <; f\rangle$ be a proof of A in GL($\boldsymbol{L}_\alpha$). We prove by induction on the tree structure of P from its leaves that $\models C$ for each formula $C = f(x)$ attached to a node x in P. Each step is verified in the following lemma.

Lemma 3.2. *(1): Let A be an instance of L1-L4 in $\boldsymbol{L}_\alpha$. Then $\models A$.*

(2): Let A be an instance of Axioms K, D, $\wedge$-Barcan in $\boldsymbol{L}_\alpha$. Then $\models A$.

(3): $\models$ satisfies inference rules MP, $\wedge$-rule, and Necessitation.

Proof. We see only the truthfulness of $\wedge$-Barcan. Let $(\mathbb{K}, \tau, w) \models \wedge\mathbf{B}_i(\Phi)$. Then, $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(C)$ for any $C \in \Phi$. Then, for any accessible $v \in W$ from w by R_i, it holds that for any $C \in \Phi$, $(\mathbb{K}, \tau, v) \models C$, equivalently, $(\mathbb{K}, \tau, v) \models C$; thus, $(\mathbb{K}, \tau, v) \models \wedge\Phi$ holds for any accessible $v \in W$ by R_i. This implies $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(\wedge\Phi)$. Thus, $(\mathbb{K}, \tau, w) \models \wedge\mathbf{B}_i(\Phi) \supset \mathbf{B}_i(\wedge\Phi)$ is true.■

For completeness, a difficulty is to show the existence of a maximal consistent set. For this aim, Karp [19] assumes Axiom of Choice within her axiomatic system. We do not choose this method; instead, we adopt the Q-filter method due to Rasiowa-Sikorski [29] and the multi-modal extension given by Tanaka-Ono [33]. Here, a Q-filter plays the role of a maximal consistent set. A sketch of a proof of our proof will be given in Section 6.1.

The above completeness result holds when we add Axioms T, 4, and 5 (or drop D), either in combination or in isolation, and add the corresponding conditions, reflexivity, transitivity, and euclidean (or drop seriality) on accessibility relation R_i $(i \in N)$. Required modifications of the proof will be stated in Remark 6.1. On the contrary, in our framework, we can evaluate these axioms by studying *explicit definability* of each axiom, which will be undertaken in Section 4.

3.3 Conservativity and four meta-lemmas

We have the conservativity result between two logics with orders over α's and $\mathcal{G}$'s.

Theorem 3.2. *(Conservativity) Let $\alpha \le \beta \le \omega$ and $\mathcal{G}, \mathcal{G}'$ two sets of germinal forms with $\mathcal{G} \subseteq \mathcal{G}'$. Then, for any $A \in \boldsymbol{L}_\alpha(\mathcal{G})$, $GL(\boldsymbol{L}_\alpha(\mathcal{G})) \vdash A$ if and only if $GL(\boldsymbol{L}_\beta(\mathcal{G}')) \vdash A$.*

Proof. The *if* part is essential. Let GL($\boldsymbol{L}_\beta(\mathcal{G}')$) $\vdash A$. Let $(\mathbb{K}, \tau)$ be any serial Kripke model, and w any world in $\mathbb{K}$. By Theorem 3.1, we have $(\mathbb{K}, \tau, w) \models A$. Because of subformula-closedness (Lemma 2.1) and the definition V0-V4 for $(\mathbb{K}, \tau, w) \models$, the statement $(\mathbb{K}, \tau, w) \models A$ is determined in $\boldsymbol{L}_\alpha(\mathcal{G})$. Since this holds for any $\mathbb{K}, \tau, w \in W$, we have GL($\boldsymbol{L}_\alpha(\mathcal{G})$) $\vdash A$ by Theorem 3.1.■

By Theorem 3.2, our infinitary logics form the hierarchy with the conservative extension relation $\rightrightarrows$, described as in Table 3.1: each row is a series of logics with the same $\mathcal{G}$, corresponding to (1), and each column is a series with the same α with $\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{G}''$. The weakest logic is

GL($\boldsymbol{L}_0$) = KD^n and the strongest is GL($\boldsymbol{L}_\omega(\mathcal{G})$) in the row with the same $\mathcal{G}$. It holds that for each fixed $A \in \boldsymbol{L}_\omega(\mathcal{G})$, we can find the smallest $\alpha_A < \omega$ and $\mathcal{G}_A \subseteq \mathcal{G}$ such that $A \in \boldsymbol{L}_{\alpha_A}(\mathcal{G}_A)$; then Theorem 3.2 implies that $\mathrm{GL}(\boldsymbol{L}_{\alpha_A}(\mathcal{G}_A)) \vdash A \Longleftrightarrow \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G})) \vdash A$.

Table 3.1 Hierarchy of infinitary epistemic logics

$$\begin{array}{ccccccc}
\mathrm{GL}(\boldsymbol{L}_0) & \rightrightarrows & \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) & \rightrightarrows & \cdots & \rightrightarrows & \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G})) \\
 & & \downdownarrows & & & & \downdownarrows \\
 & & \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}')) & & & & \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G}')) \\
 & & \downdownarrows & & & & \downdownarrows \\
 & & \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}'')) & \rightrightarrows & \cdots & \rightrightarrows & \mathrm{GL}(\boldsymbol{L}_\omega(\mathcal{G}''))
\end{array}$$

In terms of languages, the arrows $\rightrightarrows$ and $\downdownarrows$ are strict; $\boldsymbol{L}_\alpha(\mathcal{G})$ is a proper subset of $\boldsymbol{L}_\beta(\mathcal{G}')$ whenever $\alpha < \beta$ or $\mathcal{G} \subsetneq \mathcal{G}'$. In terms of provability, it is more subtle. Consider positive introspection (Example 2.1.(1)) and let $\mathcal{G} = \{\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle\}$, and $\mathcal{G}' = \{\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle, \langle \mathbf{B}_i^\nu(p \vee \neg p) : \nu \geq 0\rangle\}$. Then the vertical relation $\downdownarrows$ between GL($\boldsymbol{L}_1(\mathcal{G})$) and GL($\boldsymbol{L}_1(\mathcal{G}')$) collapses in the sense that for any $A' \in \boldsymbol{L}_1(\mathcal{G}')$, there is a formula $A \in \boldsymbol{L}_1(\mathcal{G})$ such that $\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}')) \vdash A' \equiv A$.[4] When $\mathcal{G}^o = \{\langle \mathbf{B}_i^\nu(p \vee \neg p) : \nu \geq 0\rangle\}$, we have the entire collapse result from GL($\boldsymbol{L}_\alpha(\mathcal{G}^o)$) to GL($\boldsymbol{L}_0$) = KD^n for any $\alpha \geq 0$, though $\boldsymbol{L}_\alpha(\mathcal{G}^o)$ contains infinite formulae.

Conversely, both arrows can be strict. Here, we give only two examples. The strictness holds between GL($\boldsymbol{L}_0(\mathcal{G})$) = KD^n and GL($\boldsymbol{L}_1(\mathcal{G})$); we show by Lemma 3.3, given below, that for any $A \in \boldsymbol{L}_0(\mathcal{G}) = \boldsymbol{L}_0$,

$$\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) \vdash A \supset \mathbf{B}_i^\omega(p) \Longrightarrow \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) \vdash \neg A. \tag{9}$$

Thus, there is no formula $A \in \boldsymbol{L}_0(\mathcal{G})$ such that $\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) \vdash A \equiv \mathbf{B}_i^\omega(p)$. Now, let $\mathcal{G}' = \{\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle, \langle \mathbf{B}_j^\nu(p) : \nu \geq 0\rangle\}$ $(i \neq j)$. It holds that for any $A \in \boldsymbol{L}_1(\mathcal{G})$,

$$\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}')) \vdash A \supset \mathbf{B}_j^\omega(p) \Longrightarrow \mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) \vdash \neg A. \tag{10}$$

A proof is given in the Appendix. Then, there is no formula $A \in \boldsymbol{L}_1(\mathcal{G})$ such that $\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}')) \vdash A \equiv \mathbf{B}_j^\omega(p)$. However, a general study of the hierarchy in Table 2.1 is beyond the scope of the current paper.

Here, we give four meta-results; two are known in a finitary KD^n (cf., Kaneko-Suzuki [22]) and the other two are new. First, the *depth lemma* for GL($\boldsymbol{L}_0$) = KD^n is converted to GL($\boldsymbol{L}_\alpha$) by Theorem 3.2. Recall the depth measure δ given in Section 2.[5]

Lemma 3.3. *(Depth lemma) Let A and C be two formulae in $\boldsymbol{L}_0$. Let $(i_1, ..., i_k)$ be a sequence of agents in N and $\delta(A) < k$. In GL($\boldsymbol{L}_\alpha$), if $\vdash A \supset \mathbf{B}_{i_1}...\mathbf{B}_{i_k}(C)$, then $\vdash \neg A$ or $\vdash C$.*

Assertion (9) is proved by this lemma. Let $\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) \vdash A \supset \mathbf{B}_i^\omega(p)$ and $k > \delta(A)$. Then, $\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G})) \vdash A \supset \mathbf{B}_i^k(p)$, which implies $\vdash \neg A$ by Lemma 3.3.

The second result is an extension of the *epistemic disjunction lemma* for KD^n. The following lemma is stated in GL($\boldsymbol{L}_\alpha$), but can be proved in the same manner as in [22], i.e., by constructing a counter-model based upon Theorem 3.1. Recall Remark 3.1 about disjunction $\vee$.

Lemma 3.4. *(Epistemic Disjunction lemma) Let $A, C \in \boldsymbol{L}_\alpha$. In GL($\boldsymbol{L}_\alpha$), $\vdash \mathbf{B}_i(A) \vee \mathbf{B}_i(C)$ if and only if $\vdash \mathbf{B}_i(A)$ or $\vdash \mathbf{B}_i(C)$.*

[4] A referee gave a similar example to show the collapse of $\downdownarrows$.

[5] In [22], the epistemic depth to count only the nested occurrences of $\mathbf{B}_i, i \in N$ is used for this lemma.

The third result enables us to move forward/backward from the beliefs and their contents. This will be used in Section 5.

Lemma 3.5. *(Scope Lemma) Let $A, C \in \boldsymbol{L}_\alpha$. In GL($\boldsymbol{L}_\alpha$), $\vdash \mathbf{B}_i(A) \supset \mathbf{B}_i(C)$ if and only if $\vdash A \supset C$.*

Proof. The *if* part is straightforward. We show the contrapositive of the *only-if* part. Suppose $\nvdash A \supset C$. By Theorem 3.1, there is a model $(\mathbb{K}, \tau)$ such that $(\mathbb{K}, \tau, w) \models A$ but $(\mathbb{K}, \tau, w) \not\models C$ for some world $w \in W$. Now, we add a new world w^* to W so that $W^* = W \cup \{w^*\}$, $R_i^* = R_i \cup \{(w^*, w)\}$ and $R_j^* = R_i \cup \{(w^*, w^*)\}$ for all $j \neq i$. We extend τ to $\tau^* : W^* \times \mathcal{P}_0 \to \{\top, \bot\}$ so that $\tau^*(u, p) = \tau(u, p)$ for all $(u, p) \in W \times \mathcal{P}_0$ and $\tau^*(w^*, p)$ is arbitrary for all $p \in \mathcal{P}_0$. We have a new model $(\mathbb{K}^*, \tau^*)$. In this new model, all valuations are preserved from $(\mathbb{K}, \tau)$. Since agent i refers only to w at w^*, we have $(\mathbb{K}, \tau, w^*) \models \mathbf{B}_i(A)$ but $(\mathbb{K}, \tau, w^*) \not\models \mathbf{B}_i(C)$. Hence, $(\mathbb{K}, \tau, w^*) \not\models \mathbf{B}_i(A) \supset \mathbf{B}_i(C)$. By Theorem 3.1, $\nvdash \mathbf{B}_i(A) \supset \mathbf{B}_i(C)$.■

Using this lemma and Theorem 3.1, we can prove that in GL($\boldsymbol{L}_\alpha$), $\nvdash \mathbf{B}_i(p) \supset \mathbf{B}_i\mathbf{B}_i(p)$ and $\nvdash \mathbf{B}_i\mathbf{B}_i(p) \supset \mathbf{B}_i(p)$. Thus, Axioms 4 and T are not provable in our logic. Nevertheless, $\vdash \mathbf{B}_i^\omega(p) \supset \mathbf{B}_i\mathbf{B}_i^\omega(p)$ but $\nvdash \mathbf{B}_i\mathbf{B}_i^\omega(p) \supset \mathbf{B}_i^\omega(p)$ in GL($\boldsymbol{L}_\alpha$) with $\alpha \geq 1$. This unprovability is shown by the counter-model:

$$\bigcirc\; p \;\longrightarrow_i\; \bigcirc\; \neg p \;\longrightarrow_i\; \underset{\circlearrowleft_i}{\bigcirc\; p}$$

This is a counter-model also for $\mathbf{B}_i^\omega\mathbf{B}_i^\omega(p) \supset \mathbf{B}_i^\omega(p)$ in GL($\boldsymbol{L}_2(\mathcal{G})$).

The next lemma will be used in Section 5.

Lemma 3.6. *(Infinitary conjunctions) Let $A, \wedge\langle C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. In GL($\boldsymbol{L}_\alpha$), if $\vdash A \supset \neg C^\nu$ for some $\nu \geq 0$, then $\vdash A \supset \neg \wedge \langle C^\nu : \nu \geq 0\rangle$.*

Proof. Let $\vdash A \supset \neg C^\nu$ for some $\nu \geq 0$. Let $(\mathbb{K}, \tau)$ be any model and w any world in W with $(\mathbb{K}, \tau, w) \models A$. By Theorem 3.1, $(\mathbb{K}, \tau, w) \models \neg C^\nu$, i.e., $(\mathbb{K}, \tau, w) \not\models C^\nu$. Thus, $(\mathbb{K}, \tau, w) \not\models \wedge\langle C^\nu : \nu \geq 0\rangle$, equivalently, $(\mathbb{K}, \tau, w) \models \neg\wedge\langle C^\nu : \nu \geq 0\rangle$. Thus, $(\mathbb{K}, \tau, w) \models A \supset \neg\wedge\langle C^\nu : \nu \geq 0\rangle$. Since $(\mathbb{K}, \tau)$ and w are arbitrary, we have, by Theorem 3.1, $\vdash A \supset \neg \wedge \langle C^\nu : \nu \geq 0\rangle$.■

4 Application 1: Evaluations of Various Epistemic Concepts

From the viewpoint of epistemic logics, the choices of Axioms T, 4, and 5 are of great importance. Completeness is one criterion but is neutral in the sense that our logics accommodate all these axioms, as stated after Theorem 3.1. Axioms 4 and 5 include infinitary aspects, though they are formulated in a finitary logic. Here, we ask whether each can be explicitly defined in our infinitary logics. The answers differ for T, 4, and 5. Then, we consider the possibility of embedding a logic with such an axiom to GL($\boldsymbol{L}_\alpha$). A similar consideration is given to the concept of common knowledge. In the end of Section 4.3, we give a small summary of differences between our approach and modal μ-calculus.

4.1 Explicit definabilities of Axioms T, 4, and 5 in $\mathrm{GL}(\boldsymbol{L}_\alpha)$

We fix one agent i throughout Sections 4.1 and 4.2. Also, a set of germinal forms $\mathcal{G}$ is fixed here. We begin with the following requirements for a target formula $F_i(p)$ in $\boldsymbol{L}_\alpha$: for any $A, C \in \boldsymbol{L}_\alpha$,

$$\begin{array}{lcl} \mathrm{F0}_i & : & F_i(A) \in \boldsymbol{L}_\alpha; \\ \mathrm{FE}_i & : & \vdash F_i(A) \supset \mathbf{B}_i(A); \\ \mathrm{FK}_i & : & \vdash F_i(A \supset C) \supset (F_i(A) \supset F_i(C)); \\ \mathrm{FN}_i & : & \vdash A \text{ implies } \vdash F_i(A), \end{array} \tag{11}$$

where $F_i(p)$ contains only propositional variable p and $\vdash$ is the provability relation in $\mathrm{GL}(\boldsymbol{L}_\alpha)$. $\mathrm{F0}_i$ means that $F_i(\cdot)$ is applicable to any $A \in \boldsymbol{L}_\alpha$, and FE_i that $F_i(\cdot)$ is an extension of the belief operator $\mathbf{B}_i(\cdot)$. FK_i and FN_i correspond to Axiom K and Nec. The corresponding requirement to Axiom D, $\vdash \neg F_i(A \wedge \neg A)$, is implied by the contrapositive of $\mathrm{FE}_i \vdash \neg \mathbf{B}_i(A \wedge \neg A) \supset \neg F_i(A \wedge \neg A)$ and Axiom D for $\mathbf{B}_i(\cdot)$.

The above requirements are conditions not only for $F_i(p)$ but also for $\boldsymbol{L}_\alpha$, since formulae A, C vary in $\boldsymbol{L}_\alpha$. Lemma 4.1 states that when $F_i(p) \in \boldsymbol{L}_\alpha$ satisfies $\mathrm{F0}_i$, $F_i(p)$ is finitary or $\alpha = \omega$.

Lemma 4.1. *If $F0_i$ holds for $F_i(p) \in \boldsymbol{L}_\alpha$, then $\delta(F_i(p)) < \omega$ or $\alpha = \omega$.*

Proof. Let $\delta(F_i(p)) \geq \omega$. Then, some infinitary conjunction $\wedge\Phi$ with $\delta(\wedge\Phi) \geq \omega$ is included in $F_i(p)$. Since $F_i(p)$ contains only propositional variable p, so does $\wedge\Phi$. Since $F_i(F_i(p)) \in \boldsymbol{L}_\alpha$ by $\mathrm{F0}_i$ and $\wedge\Phi(F_i(p))$ is a subformula of $F_i(F_i(p))$, it holds by Lemma 2.1 that $\wedge\Phi(F_i(p)) \in \boldsymbol{L}_\alpha$. But $\delta(\wedge\Phi(F_i(p))) \geq \omega + \omega$. This implies $\delta(F_i(F_i(p))) \geq \omega \cdot 2$. In general, we can prove by induction on $\beta \geq 1$ that $\delta(F_i^\beta(p)) > \omega \cdot \beta$ for all $\beta < \omega$. Using $\mathrm{F0}_i$, $F_i^\beta(p) \in \boldsymbol{L}_\alpha$ for any $\beta < \omega$. Thus, $\omega^2 \leq \sup_{\beta<\omega} \delta(F_i^\beta(p)) \leq \delta(\boldsymbol{L}_\alpha)$. By Theorem 2.2, we have $\alpha = \omega$.■

Another lemma is about the consistency of $F_i(p)$. We say that a *formula A is consistent* in $\mathrm{GL}(\boldsymbol{L}_\alpha)$ iff $\nvdash A \supset \neg p \wedge p$ in $\mathrm{GL}(\boldsymbol{L}_\alpha)$. A formula A is not consistent if and only if $\vdash \neg A$.

Lemma 4.2. *Let $0 \leq \alpha \leq \omega$. Any $F_i(p)$ satisfying FN_i is consistent in $GL(\boldsymbol{L}_\alpha)$.*

Proof. Suppose that $F_i(p)$ is not consistent in $\mathrm{GL}(\boldsymbol{L}_\alpha)$, i.e., $\vdash \neg F_i(p)$. By the substitution-rule mentioned in Remark 3.1.(2), it holds that $\vdash \neg F_i(p \supset p)$. On the other hand, by FN_i, $\vdash F_i(p \supset p)$. This is impossible because $\mathrm{GL}(\boldsymbol{L}_\alpha)$ is contradiction-free, as remarked just after Theorem 3.1.■

The conditions corresponding to Axioms T, 4, and 5 are as follows: for any $A \in \boldsymbol{L}_\alpha$,

$$\begin{array}{lcl} \mathrm{FT}_i & : & \vdash F_i(A) \supset A; \\ \mathrm{F4}_i & : & \vdash F_i(A) \supset F_i(F_i(A)); \\ \mathrm{F5}_i & : & \vdash \neg F_i(A) \supset F_i(\neg F_i(A)). \end{array} \tag{12}$$

We look for a formula $F_i(p)$ satisfying each of these in addition to $\mathrm{F0}_i$ to FN_i. Whether or not such an $F_i(p)$ exists is explicit definability of Axioms T, 4, and/or 5 in $\mathrm{GL}(\boldsymbol{L}_\alpha)$.

In the case of Axiom T, we observe that $\mathbf{B}_i(p) \wedge p$ satisfies $\mathrm{F0}_i$, FE_i, and FT_i, and it is also the deductively weakest among such formulae; we say that $F_i(p)$ is the *deductively weakest* among the formulae satisfying given conditions iff it satisfies them and for any $F_i'(p)$ among those formulae, $\vdash F_i'(A) \supset F_i(A)$ for any $A \in \boldsymbol{L}_\alpha$.

Theorem 4.1. *(Explicit definability for Axiom T) Let $0 \leq \alpha \leq \omega$. In GL($\boldsymbol{L}_\alpha$), $\mathbf{B}_i(p) \wedge p$ is the deductively weakest among the formulae satisfying F0$_i$, FE$_i$, and FT$_i$.*

Proof. We can verify that $\mathbf{B}_i(p) \wedge p$ satisfies F0$_i$, FE$_i$, and FT$_i$ in GL($\boldsymbol{L}_\alpha$). Let $F_i'(p)$ satisfy F0$_i$, FE$_i$, and FT$_i$. By FE$_i$ and FT$_i$, $\vdash F_i'(A) \supset \mathbf{B}_i(A)$ and $\vdash F_i'(A) \supset A$. By $\wedge$-rule, $\vdash F_i'(A) \supset \mathbf{B}_i(A) \wedge A$, which holds for any $A \in \boldsymbol{L}_\alpha$. Thus, $\mathbf{B}_i(p) \wedge p$ is deductively weakest among $F_i(p)$ satisfying F0$_i$, FE$_i$, and FT$_i$.■

This theorem holds for every α ($0 \leq \alpha \leq \omega$). Also, we can include FK$_i$ and FN$_i$ as required conditions in Theorem 4.1. Note that $\mathcal{G}$ is arbitrary up to this theorem.

Now, we go to the evaluation of Axiom 4. We assume that $\mathcal{G}$ contains $\mathbf{B}_i^\omega(p) = \wedge\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle$ with $\mathbf{B}_i^0(p) = \mathbf{B}_i(p)$.

Theorem 4.2. *(Explicit definability for Axiom 4) (1): Let $F_i(p) \in \boldsymbol{L}_\alpha$ satisfy F0$_i$, FE$_i$, FK$_i$, FN$_i$, and F4$_i$. Then $\alpha = \omega$ and $\vdash F_i(p) \supset \mathbf{B}_i^\omega(p)$ in GL($\boldsymbol{L}_\omega$).*

(2): $\mathbf{B}_i^\omega(p)$ is the deductively weakest among the formulae $F_i(p)$ satisfying F0$_i$, FE$_i$, FK$_i$, FN$_i$, and F4$_i$ in GL($\boldsymbol{L}_\omega$).

Proof. **(1)**: We prove GL($\boldsymbol{L}_\alpha$) $\vdash F_i(p) \supset \mathbf{B}_i^\nu(p)$ for all $\nu < \omega$ by induction over $\nu \geq 0$. For $\nu = 0$, the claim is FE$_i$. Suppose the induction hypothesis that $\vdash F_i(p) \supset \mathbf{B}_i^\nu(p)$. Then, by F0$_i$, FN$_i$, and FK$_i$, we have $\vdash F_i(F_i(p)) \supset F_i(\mathbf{B}_i^\nu(p))$. By this and $\vdash F_i(p) \supset F_i(F_i(p))$ by F4$_i$, we have $\vdash F_i(p) \supset F_i(\mathbf{B}_i^\nu(p))$. Since $\vdash F_i(\mathbf{B}_i^\nu(p)) \supset \mathbf{B}_i\mathbf{B}_i^\nu(p)$ by FE$_i$, we have $\vdash F_i(A) \supset \mathbf{B}_i^{\nu+1}(A)$.

Let $\delta(F_i(p)) < \omega$. Take a $\nu > \delta(F_i(p))$. By Lemma 3.3, we have $\vdash \neg F_i(p)$ or $\vdash p$ in GL($\boldsymbol{L}_0$). The first is impossible since $F_i(p)$ is consistent in GL($\boldsymbol{L}_\alpha$) by Lemma 4.2. The second is also impossible. Hence, $\delta(F_i(p)) \geq \omega$. By Lemma 4.1, $\alpha = \omega$. Using F0$_i$, FE$_i$, FK$_i$, FN$_i$, F4$_i$, we have GL($\boldsymbol{L}_\omega$) $\vdash F_i(p) \supset \mathbf{B}_i^\nu(p)$ for all $\nu < \omega$. Thus, GL($\boldsymbol{L}_\omega$) $\vdash F_i(p) \supset \mathbf{B}_i^\omega(p)$ by $\wedge$-rule.

(2): We can verify that F0$_i$, FE$_i$, FK$_i$, FN$_i$, F4$_i$ hold for $\mathbf{B}_i^\omega(p)$ in GL($\boldsymbol{L}_\omega$). By (1) of this theorem, it is deductively weakest among $F_i(p)$ satisfying these requirements.■

In contrast to Theorem 4.1, Theorem 4.2 states that Axiom 4 is explicitly definable only in GL($\boldsymbol{L}_\omega$). It has the implication that $\vdash \mathbf{B}_i^{\omega k}(p) \supset \mathbf{B}_i^\omega(\mathbf{B}_i^{\omega k}(p))$ for any $k < \omega$ in GL($\boldsymbol{L}_\omega$), though $\nvdash \mathbf{B}_i^\nu(p) \supset \mathbf{B}_i\mathbf{B}_i^\nu(p)$ for $\nu < \omega$; i.e., after ω, further introspection carries no additional information. F4$_i$ with the closure property F0$_i$ directly brings us to infinity.

We showed that both Axiom T and 4 can be explicitly defined in our system, though the depth requirements differs. For Axiom 5, the answer is entirely negative, independent of the choices of α and $\mathcal{G}$.

Theorem 4.3. *(Explicit indefinability of Axiom 5) There is no consistent formula $F_i(p)$ in GL($\boldsymbol{L}_\alpha$) ($0 \leq \alpha \leq \omega$) such that it satisfies FE$_i$ and F5$_i$.*

Proof. Suppose that there is some consistent formula $F_i(p)$ in GL($\boldsymbol{L}_\alpha$) satisfying FE$_i$ and F5$_i$. Then, F5$_i$ is equivalent to $\vdash F_i(p) \vee F_i(\neg F_i(p))$, which further implies, by FE$_i$, $\vdash \mathbf{B}_i(p) \vee \mathbf{B}_i(\neg F_i(p))$. By Lemma 3.4, we have $\vdash \mathbf{B}_i(p)$ or $\vdash \mathbf{B}_i(\neg F_i(p))$. By Lemma 3.5, we have $\vdash p$ or $\vdash \neg F_i(p)$. The former is impossible; and so is the latter because $F_i(p)$ is consistent in GL($\boldsymbol{L}_\alpha$).■

Thus, Axiom 5 cannot be defined explicitly by a formula in GL($\boldsymbol{L}_\alpha$). However, it can still be treated as a logical axiom keeping completeness, as remarked in Section 3.2.

4.2 Faithful embedding

The explicit definability results for Axioms T and 4 may imply that an extension GL($\boldsymbol{L}_\alpha$) with Axiom T or 4 is faithfully embedded into GL($\boldsymbol{L}_\alpha$). For Axiom T, the embedding result is available from $\boldsymbol{L}_\alpha$ to $\boldsymbol{L}_\alpha$ for any α in terms of language, but for Axiom 4, it can be only from $\boldsymbol{L}_0$ to $\boldsymbol{L}_\omega$. We have no embedding result for Axiom 5.[6] Here, we give a full embedding argument in the case of Axiom 4, and a sketch for the embedding result in the case of Axiom T.

Consider the case of Axiom 4 and recall $F_i^4(p) = \mathbf{B}_i^\omega(p)$. Let $\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle \in \mathcal{G}$. We define the F_i^4-*translator* $\psi^4 : \boldsymbol{L}_0 \to \boldsymbol{L}_\omega = L_\omega(\mathcal{G})$ inductively as follows: for all $A, C \in L_0$ and $\wedge\Phi \in L_0$,

E0: $\psi^4(p) = p$ if $p \in \mathcal{P}_0$;

E1$_0$: $\psi^4(\neg A) = \neg\psi^4(A)$;

E2$_0$: $\psi^4(A \supset C) = \psi^4(A) \supset \psi^4(C)$;

E3$_0$: $\psi^4(\wedge\Phi) = \wedge\psi^4(\Phi)$;

E4$_0$: $\psi^4(\mathbf{B}_i(A)) = F_i^4(\psi^4(A))$ and $\psi^4(\mathbf{B}_j(A)) = \mathbf{B}_j(\psi^4(A))$ for $j \neq i$.

The following theorem states that KD^n+ 4_i is faithfully embedded to GL($\boldsymbol{L}_\omega$). The depth of the embedded fragment $\psi^4(\boldsymbol{L}_0)$ is $\delta(\psi^4(\boldsymbol{L}_0)) = \sup_{\nu<\omega} \psi^4(\mathbf{B}_i^\nu(A)) = \sup_{\nu<\omega}(\omega \cdot \nu) = \omega^2 = \delta(\boldsymbol{L}_\omega)$.

Theorem 4.4. *(Faithful embedding of KD4 to GL($\boldsymbol{L}_\omega$)) (1): For any $A \in \boldsymbol{L}_0$, KD$^n + 4_i \vdash A$ in if and only if GL($\boldsymbol{L}_\omega$) $\vdash \psi^4(A)$.*

(2): For any $A \in \boldsymbol{L}_0$, there exists an $\alpha < \omega$ such that KD$^n + 4_i \vdash A$ if and only if GL($\boldsymbol{L}_\alpha$) $\vdash$ $\psi^4(A)$.

Proof. **(1)**: Take an arbitrary Kripke model $(\mathbb{K}, \tau)$ for KD^n, which is also a model for GL($\boldsymbol{L}_\omega$). We replace the accessibility relation R_i in $(\mathbb{K}, \tau)$ by its transitive closure R_i^{tr}, and we denote the resulting Kripke model by $(\mathbb{K}^{tr}, \tau)$. Then, KD^n + 4_i is Kripke complete with respect to those models $(\mathbb{K}^{tr}, \tau)$. Then, we prove by induction on the length of $A \in \boldsymbol{L}_0$ that for any world $w \in W$, $(\mathbb{K}^{tr}, \tau, w) \models A$ if and only if $(\mathbb{K}, \tau, w) \models \psi^4(A)$. We consider only case of $A = \mathbf{B}_i(C)$. Let $(\mathbb{K}^{tr}, \tau, w) \models \mathbf{B}_i(C)$. Then, $(\mathbb{K}^{tr}, \tau, v) \models C$ for any $v \in R_i^{tr}(w)$. By the induction hypothesis, $(\mathbb{K}, \tau, v) \models \psi^4(C)$ for any $v \in R_i^{tr}(w)$. Since R_i^{tr} is the transitive closure of R_i, it is equivalent to that $(\mathbb{K}, \tau, v) \models \psi^4(C)$ for any v reachable from w by R_i. This means $(\mathbb{K}, \tau, w) \models \mathbf{B}_i^\nu(\psi^4(C))$ for any $\nu \geq 0$, i.e., $(\mathbb{K}, \tau, w) \models \mathbf{B}_i^\omega(\psi^4(C))$, implying $(\mathbb{K}, \tau, w) \models \psi^4(\mathbf{B}_i(C))$. Tracing this argument back, we have a proof of the converse. For the cases of other connectives, the argument is similar.

(2): For a given $A \in \boldsymbol{L}_0$, we find the maximal iterations, α, of $\mathbf{B}_i(\cdot)$ inside A; then, by Theorem 3.2 (conservativity), KD^n+ $4_i \vdash A \Longleftrightarrow$ GL($\boldsymbol{L}_\alpha$) $\vdash \psi^4(A)$.■

Now, consider the embedding of Axiom T to GL($\boldsymbol{L}_\alpha$). Now, we do not need $\langle \mathbf{B}_i^\nu(p) : \nu \geq 0\rangle \in \mathcal{G}$. In this case, we use the translator ψ^T based on $F_i^T(p) = \mathbf{B}_i(p) \wedge p$. Then, the formal definition of $\psi^T : \boldsymbol{L}_0 \longrightarrow \boldsymbol{L}_0$ is obtained by the same rules E0, E1$_0$-E3$_0$, but E4$_0$ with $F_i^T(p) = \mathbf{B}_i(p) \wedge p$ instead of $F_i^4(p)$. This translator ψ^T is also uniquely defined. Then, we have

$$\mathrm{KD}^n + \mathrm{T}_i \vdash A \Longleftrightarrow \mathrm{KD}^n \vdash \psi^T(A). \tag{13}$$

This embedding result is essentially the same as the result given in Kaneko [18], Section 5.

[6] Halpern *et al.* [11] consider two modalities, one called belief (KD45) and the other called knowledge (S5), and discuss whether the latter can be reduced to the former via various notions of definability. In contrast, our embedding results are about reducing one logic system (e.g., KD4) to GL($\boldsymbol{L}_\omega$).

However, the result (13) holds, under a minor additional condition, from GL($\boldsymbol{L}_\alpha$)+T$_i$ to GL($\boldsymbol{L}_\alpha$) for all α ($0 \leq \alpha \leq \omega$). When $\alpha \geq 1$, the definition ψ^T over $\boldsymbol{L}_\alpha$ needs one requirement on the set of germinal forms $\mathcal{G}$ to be closed under the translation ψ^T:

$$\langle C^\nu : \nu \geq 0\rangle \in \mathcal{G} \Longrightarrow \langle \psi^T(C^\nu) : \nu \geq 0\rangle \in \mathcal{G}. \tag{14}$$

This implies that $\mathcal{G}$ is countably infinite.

We have the following lemma. Proofs of this lemma and the next theorem are given in the Appendix.

Lemma 4.3. $\psi^T : \boldsymbol{L}_\omega \to \boldsymbol{L}_\omega$ *is uniquely defined by E0, E1$_\alpha$ to E4$_\alpha$* ($\alpha \leq \omega$).

Now, we have the following theorem, where GL($\boldsymbol{L}_\alpha$)+T$_i$ denotes the logic GL($\boldsymbol{L}_\alpha$) plus Axiom T for $\mathbf{B}_i(\cdot)$. Then, the logic GL($\boldsymbol{L}_\alpha$)+T$_i$ is faithfully embedded into GL($\boldsymbol{L}_\alpha$) with the translator ψ^T. Let $0 \leq \alpha \leq \omega$.

Theorem 4.5. *For any* $A \in \boldsymbol{L}_\alpha$, *GL*($\boldsymbol{L}_\alpha$)*+T$_i$* $\vdash A$ *if and only if GL*($\boldsymbol{L}_\alpha$) $\vdash \psi^T(A)$.

Theorem 4.5 compares logic GL($\boldsymbol{L}_\alpha$)+T$_i$ with the fragment ψ^T(GL($\boldsymbol{L}_\alpha$)) obtained by the translator ψ^T. It is the main difference from Theorem 4.4 that the translator ψ^T does not change the layer, i.e., it embeds $\boldsymbol{L}_\alpha$ to $\boldsymbol{L}_\alpha$ for each α, while ψ^4 embeds $\boldsymbol{L}_0$ to $\boldsymbol{L}_\omega$. We remark here that Iα2.(*iii*) is used in proving Lemma 4.3 and Theorem 4.5, but otherwise, it is not needed for any other results in the present paper.

4.3 Evaluation of common knowledge in GL($\boldsymbol{L}_\alpha$)

The concept of common knowledge can be formulated in a fixed-point extension of a finitary epistemic logic, often S5-type, (Halpern, *et al.* [8], Meyer-van der Hoek [25]). Here, we consider its KD^n variant, and show that this fixed-point logic is embedded to GL($\boldsymbol{L}_\alpha$).

The finitary language $\boldsymbol{L}_0$ is extended by adding the unary operator symbol $\mathbf{C}_N(\cdot)$ to the basic symbols listed in Section 2.1, and use $\boldsymbol{L}^{\mathbf{C}_N}$ to denote the extended language. A formula $\mathbf{C}_N(A)$ means the common knowledge of A among the group of agents N. The *common knowledge logic* CK($\boldsymbol{L}^{\mathbf{C}_N}$) is defined to be the extension of KD^n with the language $\boldsymbol{L}^{\mathbf{C}_N}$ by adding the following axiom scheme and an inference rule: for any A,$D \in L^{\mathbf{C}_N}$,

Axiom CKA: $\mathbf{C}_N(A) \supset [A \wedge \wedge_{i\in N}\mathbf{B}_i\mathbf{C}_N(A)]$;

Rule CKI: $\dfrac{D \supset [A \wedge \wedge_{i\in N}\mathbf{B}_i(D)]}{D \supset \mathbf{C}_N(A)}$.

A (finite) proof is defined in the same way as in Section 3.1. In this logic, it is shown by repeated use of CKA that $\vdash \mathbf{C}_N(A) \supset \mathbf{B}_N^\nu(A)$ for all $\nu \geq 0$, where $\mathbf{B}_N^\nu(A)$ is defined in (2). Thus, $\mathbf{C}_N(A)$ contains the common knowledge of A. Rule CKI means that if any D has the property described by CKA, then D contains $\mathbf{C}_N(A)$, i.e., $\mathbf{C}_N(A)$ is the deductively weakest among the formulae having the property.

In CK($\boldsymbol{L}^{\mathbf{C}_N}$), the formula $\mathbf{C}_N(A)$ is not explicitly expressed in terms of $\mathbf{B}_1(\cdot), ..., \mathbf{B}_n(\cdot)$ in CK($\boldsymbol{L}^{\mathbf{C}_N}$), but $\mathbf{C}_N(A)$ is *implicitly definable.* To see this, we add another operator symbol $\mathbf{C}'_N(\cdot)$ to the language $\boldsymbol{L}^{\mathbf{C}_N}$ and assume CKA, CKI for $\mathbf{C}'_N(\cdot)$. By CKA for $\mathbf{C}'_N(A)$ and CKI with $D = \mathbf{C}'_N(A)$, we have $\vdash \mathbf{C}'_N(A) \supset \mathbf{C}_N(A)$. We have the converse by a parallel argument. Thus, $\vdash \mathbf{C}'_N(A) \equiv \mathbf{C}_N(A)$.

In contrast, our infinitary logic GL($\boldsymbol{L}_\alpha$) allows us to express the concept of common knowledge explicitly, i.e., $\mathbf{B}^\omega_N(A) = \wedge\langle \mathbf{B}^\nu_N(A) : \nu \geq 0\rangle$, assuming $\langle \mathbf{B}^\nu_N(p) : \nu \geq 0\rangle \in \mathcal{G}$. In a similar manner to Section 4.1, we look for a formula $F(p) \in \boldsymbol{L}_\alpha$ in GL($\boldsymbol{L}_\alpha$) having the following properties: for $A \in \boldsymbol{L}_\alpha$ and $D \in \boldsymbol{L}_\alpha$, F0 with the replacement of $F_i(p)$ by $F(p)$ and

$$\begin{aligned} \mathrm{FCA}_\alpha &: \quad \vdash F(A) \supset A \wedge [\wedge_{i\in N}\mathbf{B}_i(F(A))]; \\ \mathrm{FCI}_\alpha &: \quad \text{if } \vdash D \supset A \wedge [\wedge_{i\in N}\mathbf{B}_i(D)], \text{ then} \vdash D \supset F(A). \end{aligned}$$

These require $F(p)$ satisfy the properties corresponding to CKA and CKI in CK($\boldsymbol{L}^{\mathbf{C}_N}$).

The following theorem states that the common knowledge is explicitly definable in GL($\boldsymbol{L}_\alpha$). Since it follows from FCA_α and Nec, K for $\mathbf{B}_i(\cdot)$'s that $F(A)$ is an infinitary formula, Lemma 4.1 is applied to $F(p)$, the explicit definability holds only for $\alpha = \omega$.

Theorem 4.6. *(Explicit definability of common knowledge). In GL($\boldsymbol{L}_\omega$), the common knowledge $F(p) = \wedge\langle \mathbf{B}^\nu_N(p) : \nu \geq 0\rangle$ is a unique, up to the deductive equivalence, formula satisfying FCA_ω and FCI_ω.*

Now, we look at the relation between CK($\boldsymbol{L}^{\mathbf{C}_N}$) and GL($\boldsymbol{L}_\alpha$). The Kripke semantics for CK($\boldsymbol{L}^{\mathbf{C}_N}$) is the same as that for GL($\boldsymbol{L}_\alpha$). Here, $M = ((W; R_1, ..., R_n), \tau)$ is a serial model as in Section 3.2 and the valuation of $\mathbf{C}_N(A)$ is defined in the same way except the following:

$(M, w) \models \mathbf{C}_N(A)$ iff $(M, v) \models A$ for all $\mathbf{C}_N$-reachable v from w,

where v is *$\mathbf{C}_N$-reachable from w* iff there is a finite sequence $\langle w_0, ..., w_m\rangle$ $(m \geq 0)$ in W such that $w_0 = w$, $w_m = v$, and for all $k = 0, ..., m-1$, $(w_k, w_{k+1}) \in R_i$ for some $i \in N$.

We have the completeness/soundness result for CK($\boldsymbol{L}^{\mathbf{C}_N}$), which is a variant of the well-known result (cf., Fagin *et al.* [8]); for any $A \in L^{\mathbf{C}_N}$, A is valid if and only if CK($\boldsymbol{L}^{\mathbf{C}_N}$) $\vdash A$.

Now we show that CK($\boldsymbol{L}^{\mathbf{C}_N}$) can be faithfully embedded into GL($\boldsymbol{L}_\omega$) with the translator $\psi^{C_N} : \boldsymbol{L}^{\mathbf{C}_N} \to \boldsymbol{L}_\omega$ by E0 and $\mathrm{E1}_0$ - $\mathrm{E3}_0$, and

$$\begin{aligned} \mathrm{E4}_0 &: \quad \psi^{\mathbf{C}_N}(\mathbf{B}_i(A)) = \mathbf{B}_i(\psi^{\mathbf{C}_N}(A)) \text{ for all } i \in N; \\ \mathrm{EC} &: \quad \psi^{\mathbf{C}_N}(\mathbf{C}_N(A)) = \mathbf{B}^\omega_N(\psi^{\mathbf{C}_N}(A)). \end{aligned}$$

Then, we have the following theorem.

Theorem 4.7. *(Faithful embedding of CK($\boldsymbol{L}^{\mathbf{C}_N}$) to GL($\boldsymbol{L}_\omega$))* ***(1):*** *For any $A \in \boldsymbol{L}^{\mathbf{C}_N}$, CK($\boldsymbol{L}^{\mathbf{C}_N}$) $\vdash A$ if and only if GL($\boldsymbol{L}_\omega$) $\vdash \psi^{\mathbf{C}_N}(A)$.*

(2): *For any $A \in \boldsymbol{L}^{\mathbf{C}_N}$, there exists an $\alpha_A < \omega$ such that CK($\boldsymbol{L}^{\mathbf{C}_N}$) $\vdash A$ if and only if GL($\boldsymbol{L}_\alpha$) $\vdash \psi^{\mathbf{C}_N}(A)$.*

Proof. (1) can be proved by observing that with the translation ψ^{C_N}, the Kripke semantics for CK($\boldsymbol{L}^{C_N}$) and for GL($\boldsymbol{L}_\omega$) are the same. For (2), we take the maximum nested depth α of $\mathbf{C}_N(\cdot)$ in $A \in \boldsymbol{L}^{\mathbf{C}_N}$. By Theorem 3.2, we have GL($\boldsymbol{L}_\alpha$) $\vdash \psi^{\mathbf{C}_N}(A) \Longleftrightarrow$ GL($\boldsymbol{L}_\omega$) $\vdash \psi^{\mathbf{C}_N}(A)$. By part (1) and this, we have (2).■

This theorem is similar to Theorem 4.4 with respect to the depths required, that is, the finitary logics are faithfully embedded to GL($\boldsymbol{L}_\omega$).

It may be relevant to see the *rank function* given by Alberucci *et al.* [2] in this context; this concept is defined in modal μ-calculus, but Alberucci [1] shows that CK($\boldsymbol{L}^{\mathbf{C}_N}$) (based on K-type) can be regarded as a fragment of modal μ-calculus. In our context with $\mathcal{G} = \{\langle \mathbf{B}_N^{\nu}(p) : \nu \geq 0\rangle\}$, their problem is to find a function f over $\boldsymbol{L}^{\mathbf{C}_N}$ assigning an ordinal to each formula in $\boldsymbol{L}^{\mathbf{C}_N}$ having the following two properties: for all $A \in \boldsymbol{L}^{\mathbf{C}_N}$,

(a): if B is a proper subformula of A, then $f(B) < f(A)$;

(b): $f(\mathbf{C}_N(A)) > f(\mathbf{B}_N^{\nu}(A))$ for all $\nu < \omega$.

The second is motivated by the fact that $\vdash \mathbf{C}_N(A) \supset \mathbf{B}_N^{\nu}(A)$ for all $\nu \geq 0$. In the present context, their rank function f is defined by the inductive definition of our depth function δ by replacing the second part of **d3** by: $f(\mathbf{C}_N(A)) = f(A) + \omega$ for all $A \in \boldsymbol{L}^{\mathbf{C}_N}$. This function f satisfies the requirements (a) and (b). Furthermore, we have:

$$f(A) = \delta(\psi^{\mathbf{C}_N}(A)) \text{ for all } A \in \boldsymbol{L}^{\mathbf{C}_N}. \tag{15}$$

Thus, their rank function for $\boldsymbol{L}^{\mathbf{C}_N}$ corresponds to our depth function δ for $\boldsymbol{L}_{\omega}$. In the same manner as Theorem 4.7.(2), we can evaluate the depth for each $A \in \boldsymbol{L}^{\mathbf{C}_N}$. Since each $A \in \boldsymbol{L}^{\mathbf{C}_N}$ has the maximum nested depth $\alpha < \omega$ of $\mathbf{C}_N(\cdot)$, it follows from (15) and Theorem 2.2 that for each $A \in \boldsymbol{L}^{\mathbf{C}_N}$, there is an $\alpha_A < \omega$ such that $\omega\alpha_A \leq f(A) = \delta(\psi^{\mathbf{C}_N}(A)) < \omega(\alpha_A + 1)$.

We remark that Theorem 4.7 does not hold for *generic common knowledge* (Sato [30], Artemov [4], Antonakos [3]). In one version of such logics, the language $\boldsymbol{L}^{\mathbf{J}}$ is obtained from $\boldsymbol{L}^{\mathbf{C}_N}$ by adding $\mathbf{J}(\cdot)$. Here, we consider the extension JL($\boldsymbol{L}^{\mathbf{J}}$) of CK($\boldsymbol{L}^{\mathbf{C}_N}$) in which the belief operators $\mathbf{B}_i(\cdot)$ obey KD^n and $\mathbf{J}(\cdot)$ obeys S4 axioms (including Nec), and

Interaction axiom (IA): $\mathbf{J}(A) \supset \wedge_{i \in N}\mathbf{B}_i(A)$ for all $A \in \boldsymbol{L}^{\mathbf{J}}$.

The expression $\mathbf{J}(A)$ is interpreted as meaning that A is "obvious fact" in that it is known to all agents. Interaction Axiom connects $\mathbf{J}(A)$ to $\wedge_{i \in N}\mathbf{B}_i(A)$, but the converse is not guaranteed. Also, JL($\boldsymbol{L}^{\mathbf{J}}$) $\vdash \mathbf{J}(A) \supset \mathbf{C}_N(A)$; since $\vdash \mathbf{JJ}(A) \supset \wedge_{i \in N}\mathbf{B}_i(\mathbf{J}(A))$ by plugging $\mathbf{J}(A)$ to A in IA and $\vdash \mathbf{J}(A) \equiv \mathbf{JJ}(A)$ by the S4 axioms for $\mathbf{J}(\cdot)$, we have $\vdash \mathbf{J}(A) \supset \wedge_{i \in N}\mathbf{B}_i(\mathbf{J}(A))$, and since this is the upper formula of CKI, we have $\vdash \mathbf{J}(A) \supset \mathbf{C}_N(A)$.

In JL($\boldsymbol{L}^{\mathbf{J}}$), the operator $\mathbf{J}(A)$ is not explicitly defined in terms of $\mathbf{B}_1(\cdot), ..., \mathbf{B}_n(\cdot)$ and $\mathbf{C}_N(\cdot)$. Contrary to this, in GL($\boldsymbol{L}_{\omega}$), there are multiple formulae satisfying the corresponding properties to the axioms for $\mathbf{J}(\cdot)$. The formula $F(p) = \mathbf{B}_N^{\omega}(p)$ enjoys the S4 properties and IA, but for another propositional variable $q \neq p$, the formula $F'(p) = \mathbf{B}_N^{\omega}(p) \wedge \mathbf{B}_N^{\omega}(q)$ also enjoys all of these properties, but is deductively stronger than $F(p)$.

A more general development in the fixed-point logic literature is given in the study of modal μ-calculus (cf., Enqvist, *et al.* [7]). Our approach looks similar in that germinal forms can be based on iterated substitutions. However, the two approaches also have significant difference, as summarized below.

(i) The definition of germinal forms in Section 2 allows non-constructive germinal forms, and even when germinal forms are constructive in terms of iterated substitutions, they may include negative occurrences of propositional variables for substitution. See Example 2.1.(3). In contrast, the positivity assumption that the μ-operator (and ν-operator) is applied only to a formula is crucial. See Enqvist, *et al.* [7], Section 3, and Fountaine [9] for related problems.

(ii) The required depth for the language of GL($\boldsymbol{L}_{\alpha}$) is $\omega(\alpha+1)$ $(0 \leq \alpha < \omega)$ and that of GL($\boldsymbol{L}_{\omega}$) is ω^2. On the other hand, Alberrucci *et al.* [2] showed that their notion of ordinal ranks to evaluate the depths of formulae in modal μ-calculus and it goes up to ω^{ω}. Our germinal forms are sequences in $\mathrm{GL}(\boldsymbol{L}_0)$ and are assumed to be uniform in generating the series $\mathrm{GL}(\boldsymbol{L}_0)$, $\mathrm{GL}(\boldsymbol{L}_1), ...$ In modal μ-calculus, this is regarded as corresponding to $A(\mu x.A(x))$, $A^2(\mu x.A(x)), ...$, and the μ-operator is also applied to formulae already including the μ-operator, that is, $\mu y(\mu x.A(x,y))$ as long as the variable condition is satisfied. The difference in the required depths is caused by these facts.

5 Application 2: Rationalizability in Game Theory

We apply our framework to the study of decision making in game theory, called the theory of *rationalizability* (cf., Bernheim [5], Pearce [28], and Osborne-Rubinstein [27]). This application has two purposes. First, we show that our framework enables us to formalize each agent's decision-making process in terms of agents' logical inference. Second, it gives a concrete example of a discourse requiring $\mathrm{GL}(\boldsymbol{L}_\alpha)$ exactly with $\alpha = 2$, which differs from the infinitary concepts discussed in Section 4. Also, the theory requires more complex germinal forms involving disjunctions, and we will use the sound/completeness theorem (Theorem 3.1) to prove one step (Lemma 5.4) of the main theorem (Theorem 5.2). We remark that Axiom D is used for (20) in this section.

A 2-person game is given as $G = (\{1,2\}, S_1, S_2, g_1, g_2)$, where 1 and 2 are agents, S_i is a finite nonempty set of *available actions*, and $g_i : S_1 \times S_2 \to \mathbb{R}$ (reals) is the *payoff function* of agent $i = 1, 2$. Before the actual play of the game, each agent chooses his action to be played without knowing the other's choice. The focus is on this *ex ante* decision making.

A crucial component for rationalizability is the best response property: an action $s_i \in S_i$ for agent i is a *best response* to an action $t_j \in S_j$ for agent j iff $g_i(s_i; t_j) \geq g_i(s'_i; t_j)$ for all $s'_i \in S_i$, where we often write $g_i(s_1, s_2)$ as $g_i(s_i; s_j)$. We stipulate that when agent i is focused, the other agent is denoted by j. We say that an action $s_i \in S_i$ for agent i is *rationalizable* iff s_i is a best response to *some* action $s_j^1 \in S_j$ for j, and s_j^1 is a best response to some s_i^2, and s_i^2 is a best response to some s_j^3, and so on *ad infinitum.*[7] The referred action s_i^{t+1} for t is interpreted as a prediction inferred in the interpersonal beliefs of depth t in the mind of agent i. Here, this interpretation is informal; to make it explicit, we go to our formal system.

To express the above game theoretical concepts, we add the following atomic propositions as propositional variables to the basic symbols listed in the beginning of Section 2: for $i = 1, 2$,

preference symbols $\mathrm{Pr}_i(s_1, s_2 : t_1, t_2)$ for $(s_1, s_2), (t_1, t_2) \in S_1 \times S_2$;

decision symbols $\mathrm{I}_i(s_i)$ for $s_i \in S_i$.

The atomic proposition $\mathrm{Pr}_i(s_1, s_2 : t_1, t_2)$ intends to mean that "agent i weakly prefers (s_1, s_2) to (t_1, t_2)", which is also written as $\mathrm{Pr}_i(s_i; s_j : t_i; t_j)$ with $\{i, j\} = \{1, 2\}$. The expression $\mathrm{I}_i(s_i)$ means that "s_i is a *possible* final decision for agent i". The finitary language $\boldsymbol{L}_0$ is now defined by Iα0 and Iα1 with $\alpha = 0$ based on these additional symbols and the list of primitive symbols in Section 2. In $\boldsymbol{L}_0$, the best response property is described as a formula: for $s_i \in S_i$ and $t_j \in S_j$,

$$\mathrm{Bst}_i(s_i; t_j) := \wedge\{\mathrm{Pr}_i(s_i; t_j : s'_i; t_j) : s'_i \in S_i\}. \tag{16}$$

[7] In the literature, this is called *point-rationalizability,* which is the degenerate version of "rationalizability" allowing mixed strategies with the interpretation that they express probabilistic beliefs about the other's choices (Bernheim [5], Peace [28]). In the recent game-theory literature, rationalizability is studied in a state space with probabilistic (common) beliefs (cf., Tan and Werlang [31] and Hu [13]). However, this approach does not explicitly formulate logical inferences as in proof theory, since it does not have a formal language.

For rationalizability, we use two types of germinal forms. The first is the germinal forms for epistemic infinite regresses $\langle \mathbf{Ir}_i^{\nu}[p_1,p_2] : \nu \geq 0\rangle$ in Example 2.1.(2). We denote $\mathcal{G}^{IR} = \{\langle \mathbf{Ir}_i^{\nu}[p_1,p_2] : \nu \geq 0\rangle : i = 1,2\}$. The other will be introduced after giving the decision making criterion.

Consider the following criterion for decision making by agent i :

$\boldsymbol{D}_i^R : \wedge_{s_i\in S_i}(\mathrm{I}_i(s_i) \supset \vee_{t_j\in S_j}\langle \mathbf{B}_j(\mathrm{I}_j(t_j))\wedge \mathrm{Bst}_i(s_i;t_j)\rangle)$.

This is used in his mind, i.e., $\boldsymbol{D}_i^R$ occurs in the scope $\mathbf{B}_i(\cdot)$. It states that agent i makes some prediction about the other's decision t_j and his decision s_i is a best response to the prediction t_j. The disjunction $\vee_{t_j\in S_j}$ is specific to the rationalizability theory and to capture the idea of *rationalization.*

The criterion D_i^R is self-insufficient in that it lacks the description of how agent j infers t_j in agent i's mind; that is, agent i needs to have a certain criterion for it. We assume that agent i has the same (symmetric) criterion, D_j^R, to predict a possible t_j for the imaginary agent j in agent i's mind. This is formally expressed as $\mathbf{B}_i\mathbf{B}_j(D_j^R)$. However, this formula includes $\mathbf{B}_i(\mathrm{I}_i(t_i))$ in the innermost D_j^R, and by the parallel argument to the above, $\mathbf{B}_i\mathbf{B}_j\mathbf{B}_i(D_i^R)$ is required. Unless we force this argument to stop at some finite level, this leads to an infinite regress:

$$\mathbf{B}_i(D_i^R) \rightarrow \mathbf{B}_i\mathbf{B}_j(D_j^R) \rightarrow \mathbf{B}_i\mathbf{B}_j\mathbf{B}_i(D_i^R) \rightarrow ... \tag{17}$$

The conjunction of this sequence is exactly the infinite regress formula $\mathbf{Ir}_i[\boldsymbol{D}^R] = \mathbf{Ir}_i[D_1^R, D_2^R]$.

We regard the infinite regress $\mathbf{Ir}_i[\boldsymbol{D}^R]$ as a system of equations with unknowns $\mathrm{I}_1(s_1)$ and $\mathrm{I}_2(s_2)$; agent i may find some formulae so that they could be regarded as solutions for $\mathbf{Ir}_i[\boldsymbol{D}^R]$. To discuss solutions for $\mathbf{Ir}_i[\boldsymbol{D}^R]$, we introduce the germinal forms to express the *rationalizability property.*

First we choose subsets of propositional variables $\{p_i(t_1;t_2) : (t_1,t_2) \in S_1 \times S_2\}$ for $i = 1,2$ from $\{\mathbf{p}_0, \mathbf{p}_1, ...\}$, where $p_i(t_i;t_j)$'s are all distinct. We define two sets of sequences $\{\langle \mathrm{rat}_i^{\nu}(s_i) : \nu \geq 0\rangle : s_i \in S_i\}$, $i = 1,2$, interactively as follows: for $i = 1,2$,

$$\begin{aligned} \mathrm{rat}_i^0(s_i) &= \vee_{t_j\in S_j} p_i(s_i;t_j); \\ \mathrm{rat}_i^{\nu}(s_i) &= \vee_{t_j\in S_j}\langle \mathbf{B}_j(\mathrm{rat}_j^{\nu-1}(t_j)) \wedge p_i(s_i;t_j)\rangle \text{ for } \nu \geq 1. \end{aligned} \tag{18}$$

Recall $\vee\Phi = \neg\wedge\{\neg A : A \in \Phi\}$ for a finite nonempty set Φ in $\boldsymbol{L}_\alpha$. Let $\mathcal{G}^R = \{\langle \mathrm{rat}_i^{\nu}(s_i) : \nu \geq 0\rangle : s_i \in S_i, i = 1,2\}$. Hence, $\mathcal{G}^R$ consists of $|S_1| + |S_2|$ germinal forms, and each $\langle \mathrm{rat}_i^{\nu}(s_i) : \nu \geq 0\rangle$ contains $2\times|S_1 \times S_2|$ propositional variables for substitution. set of these germinal forms by $\mathcal{G}^R$. We adopt the set of germinal forms $\mathcal{G}^{IR+R} := \mathcal{G}^{IR} \cup \mathcal{G}^R$. The series of languages $\{\boldsymbol{L}_\alpha : \alpha \geq 0\}$ is defined based on $\mathcal{G}^{IR+R}$.

Let $s_i \in S_i$ and $i = 1,2$. For each $\nu \geq 0$, let $\mathrm{Rat}_i^{\nu}(s_i)$ be the formula obtained from $\mathrm{rat}_i^{\nu}(s_i)$ by substituting $\mathrm{Bst}_i(t_i;t_j)$ for all occurrences of each $p_i(t_i;t_j)$ in $\mathrm{rat}_i^{\nu}(s_i)$, which is still in $\boldsymbol{L}_0$. The *rationalizability formula* is given as $\mathrm{Rat}_i(s_i) := \wedge\langle \mathrm{Rat}_i^{\nu}(s_i) : \nu \geq 0\rangle$, which is in $\boldsymbol{L}_1$. Again, we note that $\mathrm{Rat}_i(s_i)$ occurs in the scope of $\mathbf{B}_i(\cdot)$.

The formula $\mathrm{Rat}_i(s_i)$ is intended to be a solution of the inference process (17), i.e., $\mathbf{Ir}_i[\boldsymbol{D}^R]$. However, the directions of predictions are opposite to (17); in (17), predictions go to deeper layers along $\nu = 0,1,...$, but $\mathrm{Rat}_i^{\nu}(s_i) = \vee_{t_j\in S_j}(\mathbf{B}_j(\mathrm{Rat}_j^{\nu-1}(t_j)) \wedge \mathrm{Bst}_i(s_i;t_j))$ has a prediction $\mathbf{B}_j(\mathrm{Rat}_j^{\nu-1}(t_j))$, and $\mathrm{Rat}_j^{\nu-1}(t_j)$ has a prediction $\mathbf{B}_i(\mathrm{Rat}_i^{\nu-1}(t_i))$, and so on to $\nu = 0$. In the latter, we require s_i to satisfy this backward argument for all $\nu \geq 0$. For this reason, it holds that $\mathbf{Ir}_i[\boldsymbol{D}^R] = \mathbf{Ir}_i[D_1^R, D_2^R]$ with some additional axiom determines $\mathrm{I}_i(s_i)$ to be equivalent to $\mathrm{Rat}_i(s_i)$. The one direction is given by the following theorem, which will be proved later in this section.

Theorem 5.1. ***(Necessity)*** *Let $s_i \in S_i, s_j \in S_j$ and $\{i, j\} = \{1, 2\}$. Then,*

(1): $\vdash \mathbf{Ir}_i[\boldsymbol{D}^R] \supset [\mathbf{B}_i(I_i(s_i)) \supset \mathbf{B}_i(Rat_i(s_i))]$ *in GL($\boldsymbol{L}_1$);*

(2): $\vdash \mathbf{Ir}_i[\boldsymbol{D}^R] \supset \mathbf{Ir}_i[I_i(s_i) \supset Rat_i(s_i); I_j(s_j) \supset Rat_j(s_j)]$ *in GL($\boldsymbol{L}_2$).*

In (1), $\mathbf{Ir}_i[\boldsymbol{D}^R]$ implies that if agent i believes that s_i is a final decision, then he believes the rationalizability property for s_i. In (2), the conclusions for both agents in (1) form an infinite regress. The epistemic logic GL($\boldsymbol{L}_1$) is sufficient for (1), but GL($\boldsymbol{L}_2$) is required for (2) since the infinitary formulae $\{\mathrm{Rat}_i(s_i)\}_{s_i \in S_i}, i = 1, 2$ occur in the germinal form $\mathbf{Ir}_i[\cdot;\cdot]$ of infinite regress.

Consider the converse of the conclusions of Theorem 5.1. If we plug $\{\mathrm{Rat}_i(s_i)\}_{s_i \in S_i}, i = 1, 2$ to $\{\mathrm{I}_i(s_i)\}_{s_i \in S_i}, i = 1, 2$ in $\mathbf{Ir}_i[\boldsymbol{D}^R]$, they could be regarded as a solution for $\boldsymbol{D}^R$. Formally, we substitute each $\mathrm{Rat}_i(s_i)$ for the corresponding $\mathrm{I}_i(s_i)$ in $\boldsymbol{D}^R$ for $i = 1, 2$, and we denote the resulting formulae by $\boldsymbol{D}^R(\mathrm{Rat}) = [D_1^R(\mathrm{Rat}), D_2^R(\mathrm{Rat})]$. If $D_i^R(\mathrm{Rat})$ is provable, then each $\mathrm{Rat}_i(s_i)$ would be a candidate for $\mathrm{I}_i(s_i)$. This argument is formulated as follows:

$\mathbf{V}_i^R$: $D_i^R(\mathrm{Rat}) \supset \wedge_{t_i \in S_i}(\mathrm{Rat}_i(t_i) \supset \mathrm{I}_i(t_i))$.

We write $\mathbf{V}^R = (V_1^R, V_2^R)$.[8] In fact, we need the infinite regress $\mathbf{Ir}_i[\mathbf{V}^R]$ of $\mathbf{V}^R = (V_1^R, V_2^R)$ in order to have the converse of the conclusions of Theorem 5.1. We have the following theorem, which will be proved below.

Theorem 5.2. ***(Full Characterization)*** *Let $(s_1, s_2) \in S_1 \times S_2$ and $i = 1, 2$. Then, both hold in GL($\boldsymbol{L}_2$) :*

(1): $\vdash \mathbf{Ir}_i[\mathbf{V}^R] \supset \mathbf{Ir}_i[Rat_1(s_1) \supset I_1(s_1), Rat_2(s_2) \supset I_2(s_2)]$;

(2): $\vdash \mathbf{Ir}_i[\mathbf{D}^R] \wedge \mathbf{Ir}_i[\mathbf{V}^R] \supset \mathbf{Ir}_i[Rat_1(s_1) \equiv I_1(s_1), Rat_2(s_2) \equiv I_2(s_2)]$.

The first is the converse of Theorem 5.1.(2). Combining this and Theorem 5.1.(2), we obtain the second assertion, the full characterization of $\mathrm{I}_1(s_1)$ and $\mathrm{I}_2(s_2)$, which is done in GL($\boldsymbol{L}_\alpha$) with $\alpha = 2$. The infinitary logic GL($\boldsymbol{L}_2$) is required and is sufficient to have these results.

Theorems 5.1 and 5.2 study the logical inferences required for decision making and possible final decisions. These are not about an actual play of a recommended action. The next stage for agent i is the play of such an action. For this, the agent needs the detailed information about the payoff functions g_1 and g_2 of the game $G = (\{1, 2\}, S_1, S_2, g_1, g_2)$. The payoff function g_i $(i = 1, 2)$ is formulated in terms of atomic propositions as follows:

$$\{\mathrm{Pr}_i(s_1, s_2 : t_1, t_2) : g_i(s_1, s_2) \geq g_i(t_1, t_2)\} \cup \{\neg \mathrm{Pr}_i(s_1, s_2 : t_1, t_2) : g_i(s_1, s_2) < g_i(t_1, t_2)\}, \quad (19)$$

which is denoted by Γ_i. We assume the infinite regress of these preferences, i.e., $\mathbf{Ir}_i[\boldsymbol{\Gamma}] = \mathbf{Ir}_i[\wedge\Gamma_1, \wedge\Gamma_2]$.

We denote the set of rationalizable actions by R_i in the sense of the non-formalized game theory. Incidentally, Bernheim [5] proved that $R_i \neq \emptyset$ for $i = 1, 2$ in any finite game G.

[8] In the infinitary logic, we can formulate the choice of weakest formulae enjoying the property $\boldsymbol{D}^R$ as infinitary formulae without using inference rules

Returning to our logical framework, it holds that

$$\mathrm{GL}(\boldsymbol{L}_1) \vdash \mathbf{Ir}_i[\boldsymbol{\Gamma}] \supset [\underset{s_i \in R_i}{\wedge} \mathbf{B}_i(\mathrm{Rat}_i(s_i))] \wedge [\underset{s_i \in S_i - R_i}{\wedge} \mathbf{B}_i(\neg \mathrm{Rat}_i(s_i))], \tag{20}$$

which will be proved in the end of this section. Thus, under the infinite regress of preferences $\mathbf{Ir}_i[\boldsymbol{\Gamma}]$, agent i can decide whether a given action s_i is rationalizable or not. To relate this to a description of agent i's decision, we combine (20) with Theorem 5.2.(2), and we have the following theorem; under the infinite regresses $\mathbf{Ir}_i[\mathbf{D}^R] \wedge \mathbf{Ir}_i[\mathbf{V}^R] \wedge \mathbf{Ir}_i[\boldsymbol{\Gamma}]$, agent i can tell whether a given s_i is a decision for him or not. Mathematically, Theorem 5.3 is a corollary of Theorem 5.2.(2) and (20).

Theorem 5.3. ***(Playability)*** *Let $s_i \in S_i$ and $i = 1, 2$. We have, in GL($\boldsymbol{L}_2$),*

$$\vdash \mathbf{Ir}_i[\mathbf{D}^R] \wedge \mathbf{Ir}_i[\mathbf{V}^R] \wedge \mathbf{Ir}_i[\boldsymbol{\Gamma}] \supset [\underset{s_i \in R_i}{\wedge} \mathbf{B}_i(I_i(s_i))] \wedge [\underset{s_i \in S_i - R_i}{\wedge} \mathbf{B}_i(\neg I_i(s_i))].$$

Note that the conclusions of (20) and Theorem 5.3 can be formulated in the form of infinite regress including predictions.

The above discourse starts with the decision/prediction criterion and goes to the consideration of a play of the game. The main engine is logical inferences by agent i and the imaginary agents in his mind. The discourse of decision making is done within the infinitary logic GL($\boldsymbol{L}_2$). In the game theory literature, decision making and existence of a resulting outcome have been discussed a lot, but these are not explicitly connected by agents' logical inferences. The above discourse is the very first attempt in this respect.

From the viewpoint of logic, the above discourse is based upon complex germinal forms, $\mathcal{G}^{IR}$ and $\mathcal{G}^{R}$, though they are still obtained by iterations of substitution. The germinal forms $\mathcal{G}^{IR}$ for infinite epistemic regress are conceptually not specific to the theory of rationalizability, but the germinal forms $\mathcal{G}_i^R$ are specific to the theory of rationalizability. In fact, infinite epistemic regress can be captured in terms of a fixed-point logic, similar to the common knowledge logic. However, so far, we do not know whether the rationalizability property is captured in terms of a fixed-point logic, though we conjecture an affirmative answer.

We remark that when "some prediction" in $\boldsymbol{D}_i^R$ is replaced by "all predictions", the theory becomes the decision making following the line of Nash's [26] theory; specifically, $\boldsymbol{D}_i^R$ is changed into $\wedge_{s_i \in S_i}(\mathrm{I}_i(s_i) \supset \wedge_{t_j \in S_j}[\mathbf{B}_j(\mathrm{I}_j(t_j)) \supset \mathrm{Bst}_i(s_i; t_j)]\rangle$. Then, we can develop the theory in a parallel manner, with the use of only germinal forms of infinite regress, to the discourse in this section, but this theory depends more upon the payoff structure and is more complex as a whole (see Hu-Kaneko [14] within the framework of a fixed-point logic).

Finally, we prove the above theorems and (20). All steps, except for Lemma 5.4, are done in proof-theoretic ways in GL($\boldsymbol{L}_1$) and GL($\boldsymbol{L}_2$). Lemma 5.4 is proved using the Kripke semantics. The proof of (20) is partially semantic since Lemma 3.6 is used.

Lemma 5.1 states various properties of infinitary regress formulae $\mathbf{Ir}_i[\boldsymbol{A}]$. GL($\boldsymbol{L}_2$) is required for (3), but GL($\boldsymbol{L}_1$) is enough for the others as long as content formulae are in $\boldsymbol{L}_1$. We define the *epistemic content* of $\mathbf{Ir}_i^o[\boldsymbol{A}]$ by $\mathbf{Ir}_i^o[\boldsymbol{A}] := A_i \wedge \mathbf{Ir}_j[\boldsymbol{A}]$.

Lemma 5.1. ***(1):*** $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \equiv \mathbf{B}_i(\mathbf{Ir}_i^o[\boldsymbol{A}])$;

(2): *if* $\vdash A_k$ *for* $k = 1, 2$, *then* $\vdash \mathbf{Ir}_i[\boldsymbol{A}]$;

(3): $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i[\mathbf{Ir}_i^o[\boldsymbol{A}]; \mathbf{Ir}_j^o[\boldsymbol{A}]]$;

(4): $\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \wedge \mathbf{Ir}_i[A_1, A_2] \supset \mathbf{Ir}_i[C_1, C_2]$;

(5): $\vdash \mathbf{Ir}_i[A_1, A_2] \wedge \mathbf{Ir}_i[C_1, C_2] \equiv \mathbf{Ir}_i[A_1 \wedge C_1, A_2 \wedge C_2]$.

Proof. We prove (1), (3), and (4).
(1): Recall $\mathbf{Ir}_i[\boldsymbol{A}] = \wedge\langle \mathbf{Ir}_i^{\nu}[\boldsymbol{A}] : \nu \geq 0\rangle$, where $\mathbf{Ir}_i^0[\boldsymbol{A}] = \mathbf{B}_i(A_i)$ and $\mathbf{Ir}_i^{\nu+1}[\boldsymbol{A}] = \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j^{\nu}[\boldsymbol{A}])$ for all $\nu \geq 0$. Hence, $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j^{\nu}[\boldsymbol{A}])$ for all $\nu \geq 0$; so $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \wedge\langle \mathbf{B}_i(\mathbf{Ir}_j^{\nu}[\boldsymbol{A}]) : \nu \geq 0\rangle$. By $\wedge$-Barcan, we have $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(\wedge\langle \mathbf{Ir}_j^{\nu}[\boldsymbol{A}] : \nu \geq 0\rangle)$. Thus, $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j[\boldsymbol{A}])$. The converse is similar.
(3): By (1), $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(\mathbf{Ir}_i^o[\boldsymbol{A}])$ for $i = 1, 2$. Suppose that $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i^{\nu}[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_2^o[\boldsymbol{A}]]$ for $i = 1, 2$. Since $\vdash \mathbf{B}_i(\mathbf{Ir}_j[\boldsymbol{A}]) \supset \mathbf{B}_i(\mathbf{Ir}_j^{\nu}[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_2^o[\boldsymbol{A}]])$ by Nec and K, and since $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{B}_i(A_i \wedge \mathbf{Ir}_j[\boldsymbol{A}])$ by (1), we have $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i^{\nu+1}[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_2^o[\boldsymbol{A}]]$. By $\wedge$-rule, $\vdash \mathbf{Ir}_i[\boldsymbol{A}] \supset \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{A}], \mathbf{Ir}_2^o[\boldsymbol{A}]]$.
(4): It suffices to show that $\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \wedge \mathbf{Ir}_i(A_1, A_2) \supset \mathbf{Ir}_i[C_1, C_2]$. It is proved by induction over ν that $\vdash \mathbf{Ir}_i[A_1 \supset C_1, A_2 \supset C_2] \wedge \mathbf{Ir}_i[A_1, A_2] \supset \mathbf{Ir}_i^{\nu}[C_1, C_2]$ for all $\nu \geq 0$. By $\wedge$-rule, we have the result.■

Lemma 5.2. *$GL(\boldsymbol{L}_1) \vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [I_i(s_i) \supset Rat_i^{\nu}(s_i)]$ for all $\nu \geq 0$, $s_i \in S_i$, $i = 1, 2$.*

Proof. We show this by induction on ν. Since $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset D_i^R$ and $\vdash D_i^R \supset [\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j}\mathrm{Bst}_i(s_i; t_j)]$, we have the assertion for $\nu = 0$. Suppose the assertion for ν. Then, $\vdash \mathbf{Ir}_j[\boldsymbol{D}^R] \supset [\mathbf{B}_j(\mathrm{I}_j(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j) \supset \mathbf{B}_j(\mathrm{Rat}_j^{\nu}(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j)]$. Hence, we have $\vdash \mathbf{Ir}_j[\boldsymbol{D}^R] \supset [\vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle \supset \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle]$. Since $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset \mathbf{Ir}_j[\boldsymbol{D}^R]$, we have $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle \supset \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle]$. Also, since $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{I}_j(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle]$, we have $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\mathrm{I}_i(s_i) \supset \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle]$. Thus, $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset [\mathrm{I}_i(s_i) \supset \mathrm{Rat}_i^{\nu+1}(s_i)]$. Hence, we have the assertion for $\nu + 1$.■

Proof of Theorem 5.1.(1): This is obtained by Lemma 5.2.
(2): Lemma 5.2 implies $\vdash \mathbf{Ir}_i^o[\boldsymbol{D}^R] \supset (\mathrm{I}_i(s_i) \supset \mathrm{Rat}_i(s_i))$ for $i = 1, 2$. By Lemma 5.1.(2), we have $\vdash \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{D}^R] \supset (\mathrm{I}_1(s_1) \supset \mathrm{Rat}_1(s_1)), \mathbf{Ir}_2^o[\boldsymbol{D}^R] \supset (\mathrm{I}_2(s_2) \supset \mathrm{Rat}_2(s_1))]$. Using Lemma 5.1.(4), we have $\vdash \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{D}^R], \mathbf{Ir}_2^o[\boldsymbol{D}^R]] \supset \mathbf{Ir}_i[\mathrm{I}_1(s_1) \supset \mathrm{Rat}_1(s_1), \mathrm{I}_2(s_2) \supset \mathrm{Rat}_2(s_1)]$. Since $\vdash \mathbf{Ir}_i[\boldsymbol{D}^R] \supset \mathbf{Ir}_i[\mathbf{Ir}_1^o[\boldsymbol{D}^R], \mathbf{Ir}_2^o[\boldsymbol{D}^R]]$ by Lemma 5.1.(3), we have the assertion.■

To prove Theorem 5.2, we will show that $\vdash D_i^R(\mathrm{Rat})$ for $i = 1, 2$. Then, we have $\vdash \mathbf{Ir}_i[D_1^R(\mathrm{Rat}), D_2^R(\mathrm{Rat})]$ by Lemma 5.1.(2). By Lemma 5.1.(4), we have $\vdash \mathbf{Ir}_i[\mathbf{V}^R] \supset \mathbf{Ir}_i[\mathrm{Rat}_1(s_1) \supset \mathrm{I}_1(s_1), \mathrm{Rat}_1(s_1) \supset \mathrm{I}_2(s_1)]$. This is Theorem 5.2.(1). Combining this with Theorem 5.1.(2) by Lemma 5.1.(5), we have Theorem 5.2.(2).

The first step for $\vdash D_i^R(\mathrm{Rat})$ for $i = 1, 2$ is the following lemma.

Lemma 5.3. *(Monotonicity): $GL(L_0) \vdash Rat_i^{\nu+1}(s_i) \supset Rat_i^{\nu}(s_i)$ for all $\nu \geq 0$, $s_i \in S_i$, $i = 1, 2$.*

Proof. We prove the assertion by induction over $\nu \geq 0$. Recall $\mathrm{Rat}_i^0(s_i) = \vee_{t_j \in S_j}\mathrm{Bst}_i(s_i; t_j)$. Since $\mathrm{Rat}_i^1(s_i) = \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{Rat}_j^0(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle$, we have $\vdash \mathrm{Rat}_i^1(s_i) \supset \vee_{t_j \in S_j}\mathrm{Best}(s_i; t_j)$, i.e., $\vdash \mathrm{Rat}_i^1(s_i) \supset \mathrm{Rat}_i^0(s_i)$. Suppose that $\vdash \mathrm{Rat}_i^{\nu+1}(s_i) \supset \mathrm{Rat}_i^{\nu}(s_i)$ for $i = 1, 2$. This implies $\vdash \mathbf{B}_j(\mathrm{Rat}_j^{\nu+1}(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j) \supset \mathbf{B}_j(\mathrm{Rat}_j^{\nu}(s_j)) \wedge \mathrm{Bst}_i(s_i; s_j)$, and then $\vdash \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{Rat}_j^{\nu+1}(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle \supset \vee_{t_j \in S_j}\langle \mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_j)) \wedge \mathrm{Bst}_i(s_i; t_j)\rangle$, i.e., $\vdash \mathrm{Rat}_i^{\nu+2}(s_i) \supset \mathrm{Rat}_i^{\nu+1}(s_i)$.■

Now, we prove $\vdash D_i^R(\text{Rat})$ for $i = 1, 2$. The proof of part (1) is based on the soundness/completeness (Theorem 3.1); the finiteness of S_i and Lemma 5.3 are used. In the following lemma, we use the abbreviation $\wedge_\nu A_\nu$ of $\wedge\langle A_\nu : \nu \geq 0\rangle$.

Lemma 5.4. $GL(L_1) \vdash Rat_i(s_i) \supset \vee_{t_j \in S_j}[\mathbf{B}_j(Rat_j(t_j)) \wedge Bst_i(s_i; t_j)]$.

Proof. First, we recall $\wedge_\nu \text{Rat}_i^\nu(s_i) = \wedge_\nu \vee_{t_j \in S_j}[\mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \wedge \text{Bst}_i(s_i; t_j)]$. We prove $\vdash \wedge_\nu \text{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\wedge_\nu \langle \mathbf{B}_j(\text{Rat}_j^\nu(t_j))\rangle \wedge \text{Bst}(s_i; t_j)]$. By rule I$\alpha$2.$(ii)$ with $\alpha = 1$, $\wedge_\nu \langle \mathbf{B}_j(\text{Rat}_j^\nu(t_j))\rangle$ is a permissible conjunction. Since $\vdash \wedge_\nu \langle \mathbf{B}_j(\text{Rat}_j^\nu(t_j))\rangle \equiv \mathbf{B}_j(\wedge_\nu \text{Rat}_j^\nu(t_j))$, it follows that $\vdash \wedge_\nu \text{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\mathbf{B}_j(\wedge_\nu \text{Rat}_j^\nu(t_j)) \wedge \text{Bst}(s_i; t_j)]$, which is the assertion of the lemma.

Let $\mathcal{M} = (\mathcal{F}, \tau)$ be a serial Kripke model, and w any possible world in W. Suppose $(\mathcal{M}, w) \models \wedge_\nu \vee_{t_j \in S_j} \langle \mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \wedge \text{Bst}_i(s_i; t_j)\rangle$. Then, $(\mathcal{M}, w) \models \vee_{t_j \in S_j} \langle \mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \wedge \text{Bst}_i(s_i; t_j)\rangle$ for any $\nu \geq 0$. Let

$$T_j^\nu = \{t_j \in S_j : (\mathcal{M}, w) \models \mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \wedge \text{Bst}_i(s_i; t_j)\} \text{ for } \nu \geq 0.$$

Since $(\mathcal{M}, w) \models \vee_{t_j \in S_j} \langle \mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \wedge \text{Bst}_i(s_i; t_j)\rangle$, we have $T_j^\nu \neq \emptyset$ for all ν. Since $(\mathcal{M}, w) \models \text{Rat}_j^{\nu+1}(s_j) \supset \text{Rat}_j^\nu(s_j)$ by Lemma 5.3 and Soundness, we have $T_j^\nu \supseteq T_j^{\nu+1}$ for all $\nu \geq 0$. Since S_j is a finite set, there is some ν_0 such that T_j^ν is constant for all $\nu \geq \nu_0$. Hence, we find an $s_j \in \cap_\nu T_j^\nu$, which implies $(\mathcal{M}, w) \models \langle \wedge_\nu \mathbf{B}_j(\text{Rat}_j^\nu(s_j))\rangle \wedge \text{Bst}_i(s_i; s_j)$. Thus, $(\mathcal{M}, w) \models \vee_{t_j \in S_j}[\langle \wedge_\nu \mathbf{B}_j(\text{Rat}_j^\nu(t_j))\rangle \wedge \text{Bst}_i(s_i; t_j)]$. Thus, $(\mathcal{M}, w) \models \wedge_\nu \text{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\langle \wedge_\nu \mathbf{B}_j(\text{Rat}_j^\nu(t_j))\rangle \wedge \text{Bst}_i(s_i; t_j)]$. Since $\mathcal{F}, \tau, w \in W$ are all arbitrary, we have $\vdash \wedge_\nu \text{Rat}_i^\nu(s_i) \supset \vee_{t_j \in S_j}[\langle \wedge_\nu \mathbf{B}_j(\text{Rat}_j^\nu(t_j))\rangle \wedge \text{Bst}_i(s_i; t_j)]$ by completeness.■

Proof of (20): We use Lemma 3.6, which allows us to infer (20) from assertions about $\text{Rat}_j^\nu(s_j)$ for finite ν's. Now, we work with GL($\boldsymbol{L}_0$). In fact, the main argument uses the technique that eliminates the belief operators $\mathbf{B}_1(\cdot)$ and $\mathbf{B}_2(\cdot)$ from KD^n and hence we can work with finitary classical logic, whose provability relation is denoted by $\vdash_0$. Correspondingly, we denote, by $\text{Nat}_j^\nu(s_j)$, the formula obtained from $\text{Rat}_j^\nu(s_j)$ eliminating all $\mathbf{B}_1(\cdot)$ and $\mathbf{B}_2(\cdot)$. The set $\wedge(\Gamma_1 \cup \Gamma_2)$ is complete by (19) with respect to atomic preference propositions; for a finitary nonepistemic formula A containing only atomic preference propositions,

$$\vdash_0 \ \wedge (\Gamma_1 \cup \Gamma_2) \supset A \text{ or } \vdash_0 \ \wedge (\Gamma_1 \cup \Gamma_2) \supset \neg A. \tag{21}$$

This is applied to $\text{Nat}_j^\nu(s_j)$ for all $i = 1, 2$, $s_i \in S_i$, and $\nu \geq 0$. Also, when A contains only atomic preference propositions for agent i, the premise in (21) can be $\wedge\Gamma_i$.

We prove, by induction over ν, that for $i = 1, 2$, $s_i \in S_i$, and $\nu \geq 0$,

$$\text{if } \vdash_0 \ \wedge (\Gamma_1 \cup \Gamma_2) \supset \text{Nat}_i^\nu(s_i), \text{ then} \vdash \mathbf{Ir}_i[\mathbf{\Gamma}] \supset \mathbf{B}_i(\text{Rat}_i^\nu(s_i)); \tag{22}$$

$$\text{if } \vdash_0 \ \wedge (\Gamma_1 \cup \Gamma_2) \supset \neg\text{Nat}_i^\nu(s_i), \text{ then } \vdash \mathbf{Ir}_i[\mathbf{\Gamma}] \supset \mathbf{B}_i(\neg\text{Rat}_i^\nu(s_i)). \tag{23}$$

For $\nu = 0$, $\text{Nat}_i^0(s_i) = \text{Rat}_i^0(s_i) = \vee_{t_j \in S_j} \text{Bst}_i(s_i; t_j)$. Since $\mathbf{Ir}_i^o[\mathbf{\Gamma}] = (\wedge\Gamma_i) \wedge \mathbf{Ir}_j[\mathbf{\Gamma}]$, we obtain (22) and (23) for $\nu = 0$ by applying Nec and K. Suppose that (22) and (23) hold for ν. By (21), $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \text{Nat}_i^{\nu+1}(s_i)$ or $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \neg\text{Nat}_i^{\nu+1}(s_i)$. First, let $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \text{Nat}_i^{\nu+1}(s_i)$; by definition, $\text{Nat}_i^{\nu+1}(s_i) = \vee_{t_j \in S_j}(\text{Nat}_j^\nu(t_j) \wedge \text{Bst}_i(s_i; t_j))$, and hence, by (21) again, $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset (\text{Nat}_j^\nu(t_j) \wedge \text{Bst}_i(s_i; t_j))$ for some $t_j \in S_j$. For this t_j, it holds that $\vdash \wedge\mathbf{\Gamma}_i \supset \text{Bst}_i(s_i; t_j)$ and $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \supset \mathbf{B}_j(\text{Rat}_j^\nu(t_j))$ by (22) for ν. Combining these, we have $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \wedge (\wedge\mathbf{\Gamma}_i) \supset \mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \wedge \text{Bst}_i(s_i; t_j)$. Hence, $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \wedge (\wedge\mathbf{\Gamma}_i) \supset \vee_{t_j \in S_j}[\mathbf{B}_j(\text{Rat}_j^\nu(t_i)) \wedge \text{Bst}_i(s_i; t_j)]$. Thus, $\vdash \mathbf{Ir}_i^o[\mathbf{\Gamma}] \supset \text{Rat}_i^{\nu+1}(s_i)$, so, $\vdash \mathbf{Ir}_i[\mathbf{\Gamma}] \supset \mathbf{B}_i(\text{Rat}_i^{\nu+1}(s_i))$ by Nec and K.

Second, let $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \neg\mathrm{Nat}_i^{\nu+1}(s_i)$. Again, by definition and (21), $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \neg(\mathrm{Nat}_j^{\nu}(t_j) \wedge \mathrm{Bst}_i(s_i;t_j))$ for all $t_j \in S_j$. Let $t_j \in S_j$. Then, $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \neg\mathrm{Nat}_j^{\nu}(t_j)$ or $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \neg\mathrm{Bst}_i(s_i;t_j)$. Then, by (23) for ν, we have $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \supset \mathbf{B}_j(\neg\mathrm{Rat}_j^{\nu}(t_i))$ or $\vdash \wedge\Gamma_i \supset \neg\mathrm{Bst}_i(s_i;t_j)$. Combining these, we have $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \wedge (\wedge\Gamma_i) \supset \mathbf{B}_j(\neg\mathrm{Rat}_j^{\nu}(t_i)) \vee (\neg\mathrm{Bst}_i(s_i;t_j))$. This and Axiom D for $\mathbf{B}_j(\cdot)$ imply $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \wedge (\wedge\Gamma_i) \supset \neg\mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_i)) \vee(\neg\mathrm{Bst}_i(s_i;t_j))$, i.e., $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \wedge (\wedge\Gamma_i) \supset \neg(\mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_j)) \wedge \mathrm{Bst}_i(s_i;t_j))$. Since t_j is arbitrary, we have $\vdash \mathbf{Ir}_j[\mathbf{\Gamma}] \wedge (\wedge\mathbf{\Gamma}_i) \supset \neg \vee_{t_j \in S_j} (\mathbf{B}_j(\mathrm{Rat}_j^{\nu}(t_i)) \wedge \mathrm{Bst}_i(s_i;t_j))$. That is, $\vdash \mathbf{Ir}_i^o[\mathbf{\Gamma}] \supset \neg\mathrm{Rat}_i^{\nu+1}(s_i)$, which, by Nec and K, implies (23) for $\nu + 1$.

Now, take any $s_i \in S_i$. Then, let s_i be rationalizable action. Then, $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \mathrm{Nat}_i^{\nu}(s_i)$ for all $\nu \geq 0$. In this case, by (22), $\vdash \mathbf{Ir}_i^o[\mathbf{\Gamma}] \supset \mathrm{Rat}_i^{\nu}(s_i)$ for all $\nu \geq 0$. Thus, $\vdash \mathbf{Ir}_i^o[\mathbf{\Gamma}] \supset \mathrm{Rat}_i(s_i)$ by $\wedge$-rule. Hence, $\vdash \mathbf{Ir}_i[\mathbf{\Gamma}] \supset \mathbf{B}_i(\mathrm{Rat}_i(s_i))$.

Let s_i be a non-rationalizable action. Then, $\vdash_0 \wedge(\Gamma_1 \cup \Gamma_2) \supset \neg\mathrm{Nat}_i^{\nu}(s_i)$ for some $\nu \geq 0$. In this case, by (23), $\vdash \mathbf{Ir}_i^o[\mathbf{\Gamma}] \supset \neg\mathrm{Rat}_i^{\nu}(s_j)$ for some $\nu \geq 0$. By Lemma 3.6, we have $\vdash \mathbf{Ir}_i^o[\mathbf{\Gamma}] \supset \neg\langle\mathrm{Rat}_j^{\nu}(s_j) : \nu \geq 0\rangle$. Hence, $\vdash \mathbf{Ir}_i[\mathbf{\Gamma}] \supset \mathbf{B}_i(\neg\mathrm{Rat}_j(s_j))$. These imply (20).$\blacksquare$

6 Proof of the Completeness of GL($\boldsymbol{L}_\alpha$) by Q-filters

We adopt the $\mathcal{Q}$-filter method to prove completeness of GL($\boldsymbol{L}_\alpha$). First, we give a sketch of the proof, a summary of the concepts to be used, and then go to the main body of the proof.

6.1 Sketch of the proof

As usual, we show that if a formula $A \in \boldsymbol{L}_\alpha$ is not provable, we find a Kripke model so that A is not true in some world. It is standard in the literature to construct maximal consistent sets as those possible worlds via the Henkin method (cf. Hughes and Cresswell [15]). This may appear to be applicable to our logics because the set of formulae $\boldsymbol{L}_\alpha$ ($0 \leq \alpha \leq \omega$) is kept countable. But this does not work in our case for two reasons. Since GL($\boldsymbol{L}_\alpha$) allows infinite conjunctions, the Henkin method to extend a consistent set does not fit our purpose; the infinitary approach from Karp [19] avoids this difficulty by requiring Axiom of Choice in the axiomatic system (cf. Heifeitz [12] in the epistemic logic context). Instead, we adopt the Q-filter method, due to Rasiowa-Sikorski [29] for algebraic semantics and Tanaka-Ono [33] for Kripke semantics. A Q-filter is a strengthened version of a prime filter to deal with infinitary conjunctions. This method has been developed as an alternative to prove completeness for a first-order logic as well as for infinitary modal logics (cf., Tanaka [32]). We note that the countability of the language $\boldsymbol{L}_\alpha$ is crucial in applications of these lemmas.

The Q-filter method relies upon various concepts in Boolean algebra, though we deal with Kripke semantics rather than algebraic semantics. Utilizing the Q-filter method, we construct a counter-model. This is not the canonical model; instead, we start with the Lindenbaum algebra $\boldsymbol{L}_\alpha/\equiv$, where $\equiv$ is the equivalence relation of provability in GL($\boldsymbol{L}_\alpha$). Then, a Q-filter is a subset of $\boldsymbol{L}_\alpha/\equiv$ and is a possible world for the counter-model. A Q-filter is required to satisfy certain closure properties in addition to the prime filter condition. These closure properties are guaranteed by the formula construction steps, Iα2.(i) and (ii), for the definition of $\boldsymbol{L}_\alpha$. Once the set of possible worlds is defined, accessibility relations $R_i, i \in N$ are defined in a similar manner as in the standard proof based on maximal consistent sets.

In Section 6.2, we provide a small summary of $\mathcal{Q}$-filters in a Boolean algebra. In Section 6.3, we define the Lindenbaum algebra based on GL($\boldsymbol{L}_\alpha$), and prepare for applications of the Rasiowa-Sikorski and Tanaka-Ono lemmas. In Section 6.4, we construct a counter-model. A key step is the truth lemma that a formula A is true in a world w if and only if $[\![A]\!] \in w$, where $[\![A]\!]$ is the equivalence class including A. This step requires the Tanaka-Ono Lemma to deal with $\mathbf{B}_i(\cdot)$. Finally, we show that if $\nvdash A$, there is a $\mathcal{Q}$-filter w such that $[\![A]\!] \notin w$; the existence of such a $\mathcal{Q}$-filter w is guaranteed by the Rasiowa-Sikorski Lemma.

6.2 Boolean algebra and Q-filters

We give basic definitions and relevant properties of a Boolean algebra (cf., Halmos [10] and Mendelson [23]). Consider a Boolean algebra $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1})$. We define $a \leq b$ iff $a \sqcup b = b$. Then $\leq$ is a lattice ordering on $\mathbf{B}$ (i.e., $a \sqcap b$ and $a \sqcup b$ are the *greatest lower bound* and *least upper bound* of a, b with respect to $\leq$). We say that a nonempty subset F of $\mathbf{B}$ is a *filter* iff **F1**(upward closed): $a \leq b$ and $a \in F \Longrightarrow b \in F$; and **F2**($\sqcap$-closed): $a, b \in F \Longrightarrow a \sqcap b \in F$. Also, we say that a filter F is *prime* iff **P1**(Non-triviality): $F \neq \mathbf{B}$; and **P2**($\sqcup$-property): $a \sqcup b \in F \Longrightarrow a \in F$ or $b \in F$. We have the following fact on a prime filter F:

$$a \in F \;\Leftrightarrow\; (-a) \notin F. \tag{24}$$

In the following, we write $a \rightarrow b$ for $-a \sqcup b = (-a) \sqcup b$. When F is a prime filter, $a \rightarrow b \in F$ if and only if $a \notin F$ or $b \in F$, since $(-a) \sqcup a = \mathbf{1} \in F$.

For any subset S of $\mathbf{B}$, the *greatest lower bound* of S in $(\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1})$ is denoted by $\sqcap S$, and the *least upper bound* of S is denoted by $\sqcup S$. Note that $\sqcap S$ and $\sqcup S$ may not exist, but if either exists, it is unique. Let $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ be a pair of countable sets of nonempty subsets of $\mathbf{B}$ so that

(($\sqcap, \sqcup$)**-closed**): $\sqcap Q_1$ and $\sqcup Q_2$ exist for all $Q_1 \in \mathcal{Q}_1$ and $Q_2 \in \mathcal{Q}_2$.

We say that a prime filter F is a $\mathcal{Q}$-*filter* iff

Q1: for any $Q_1 \in \mathcal{Q}_1$, $Q_1 \subseteq F \Longrightarrow \sqcap Q_1 \in F$;

Q2: for any $Q_2 \in \mathcal{Q}_2$, $\sqcup Q_2 \in F \Longrightarrow a \in Q_2$ for some $a \in F$.

These correspond to the conditions F2 and P2. The following is Rasiowa-Sikorski lemma (see also Tanaka-Ono [**33**]).

Lemma 6.1. ***(Rasiowa-Sikorski [29])*** *Let $\mathbb{B}$ be a Boolean algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a pair of countable sets of nonempty subsets of $\mathbf{B}$ with $(\sqcap, \sqcup)$-closedness. For any $a, b \in \mathbf{B}$, if $a \nleq b$, then there is a $\mathcal{Q}$-filter F such that $a \in F$ and $b \notin F$.*

For a given $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$, we denote the set of all $\mathcal{Q}$-filters of $\mathbb{B}$ by $\mathbb{F}_{\mathcal{Q}}(\mathbb{B})$. The nonemptiness of $\mathbb{F}_{\mathcal{Q}}(\mathbb{B})$ follows from Lemma 6.1 if $\mathbf{0} \neq \mathbf{1}$. The set $\mathbb{F}_{\mathcal{Q}}(\mathbb{B})$ will be adopted for the set of all possible worlds in our construction of a Kripke model.

Since the logic GL($\boldsymbol{L}_\alpha$) has belief operators, Rasiowa-Sikorski lemma is not enough: We extend it, which is Tanaka-Ono lemma. We say that $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ is a *multi-modal algebra* iff

ma1: $(\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1})$ is a Boolean algebra;

ma2: for $i \in N$, $\Box_i$ is a unary operator on $\mathbf{B}$ satisfying the property that $\Box_i \mathbf{1} = \mathbf{1}$

and $\Box_i(a \sqcap b) = \Box_i a \sqcap \Box_i b$ for all $a, b \in \mathbf{B}$.

We define $\Box_i^{-1}F = \{x \in \mathbf{B} : \Box_i x \in F\}$ for any $F \subseteq \mathbf{B}$.

Let $\mathbb{B}$ be a multi-modal algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a fixed pair of countable sets of nonempty subsets of $\mathbf{B}$ satisfying $(\sqcap, \sqcup)$-closedness. The following three conditions are crucial for the Tanaka-Ono Lemma: for all $i \in N$,

q0: for all $Q_1 \in \mathcal{Q}_1$, $\sqcap(\Box_i Q_1) := \sqcap\{\Box_i a : a \in Q_1\}$ exists and $\sqcap(\Box_i Q_1) = \Box_i(\sqcap Q_1)$;

q1: $\{\Box_i(a \to b) : b \in Q_1\} \in \mathcal{Q}_1$ for all $a \in \mathbf{B}$ and all $Q_1 \in \mathcal{Q}_1$;

q2: $\{\Box_i(b \to a) : b \in Q_2\} \in \mathcal{Q}_1$ for all $a \in \mathbf{B}$ and all $Q_2 \in \mathcal{Q}_2$.

Lemma 6.2. ***(Tanaka-Ono [33])*** *Let $\mathbb{B} = (\mathbf{B}, \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1}, \Box_1, ..., \Box_n)$ be a multi-modal algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a fixed pair of countable sets of nonempty subsets of $\mathbf{B}$. Suppose that $\mathcal{Q}$ satisfies $(\sqcap, \sqcup)$-closedness, and the conditions $q0$, $q1$, and $q2$ for $i \in N$. Then, for any $i \in N$, $b \in \mathbf{B}$, and $F \in \mathbb{F}_{\mathcal{Q}}(\mathbb{B})$, if $\Box_i b \notin F$, there exists a $G \in \mathbb{F}_{\mathcal{Q}}(\mathbb{B})$ such that $\Box_i^{-1}F \subseteq G$ and $b \notin G$.*

6.3 Lindenbaum algebra

Recall that for any $A, B \in \boldsymbol{L}_\alpha$, $A \equiv B$ iff $\vdash (A \supset B) \wedge (B \supset A)$ in GL($\boldsymbol{L}_\alpha$). We take the quotient set $\boldsymbol{L}_\alpha / \equiv$. For any $A \in \boldsymbol{L}_\alpha$, we denote, by $[\![A]\!]$, the equivalence class in $\boldsymbol{L}_\alpha / \equiv$ including A. In $\mathbf{B} := \boldsymbol{L}_\alpha / \equiv$, we define elements $\mathbf{0}, \mathbf{1}$ and operations $\sqcap, \sqcup, -$, and $\Box_1, ..., \Box_n$ by

ℓ**1**: $\mathbf{0} = [\![\neg \mathbf{p}_0 \wedge \mathbf{p}_0]\!]$ and $\mathbf{1} = [\![\mathbf{p}_0 \supset \mathbf{p}_0]\!]$;

ℓ**2**: for any $A, B \in \boldsymbol{L}_\alpha$, $[\![A]\!] \sqcap [\![B]\!] = [\![A \wedge B]\!]$, $[\![A]\!] \sqcup [\![B]\!] = [\![\neg(\neg A \wedge \neg B)]\!]$, $-[\![A]\!] = [\![\neg A]\!]$;

ℓ**3**: for any $A \in \boldsymbol{L}_\alpha$, $\Box_i[\![A]\!] = [\![\mathbf{B}_i(A)]\!]$ for $i \in N$.

Using these, we have, for any $A, B \in \boldsymbol{L}_\alpha$,

$$[\![A]\!] \to [\![B]\!] = (-[\![A]\!]) \sqcup [\![B]\!] = [\![\neg A]\!] \sqcup [\![B]\!] = [\![\neg(\neg\neg A \wedge \neg B)]\!] = [\![A \supset B]\!]. \tag{25}$$

It follows from this that $\Box_i([\![A]\!] \to [\![B]\!]) = \Box_i([\![A \supset B]\!]) = [\![\mathbf{B}_i(A \supset B)]\!]$.

Lemma 6.3. *$\mathbb{L} = (\mathbf{B}, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, -, \Box_1, ..., \Box_n)$ with $\mathbf{B} = (\boldsymbol{L}_\alpha / \equiv)$ is a multi-modal algebra.*

Proof. We can show in the standard manner that $(\mathbf{B}, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, -)$ with $\mathbf{B} = (\boldsymbol{L}_\alpha / \equiv)$ is a Boolean algebra. It remains to show condition ma2. Let $i \in N$. Since $\vdash [(A \supset A) \supset \mathbf{B}_i(A \supset A)] \wedge [\mathbf{B}_i(A \supset A) \supset (A \supset A)]$, we have $\Box_i \mathbf{1} = \mathbf{1}$. Since $\vdash [\mathbf{B}_i(A \wedge C) \supset \mathbf{B}_i(A) \wedge \mathbf{B}_i(C)] \wedge [\mathbf{B}_i(A) \wedge \mathbf{B}_i(C) \supset \mathbf{B}_i(A \wedge C)]$, we have $\Box_i([\![A]\!] \sqcap [\![C]\!]) = \Box_i[\![A]\!] \sqcap \Box_i[\![C]\!]$.$\blacksquare$

In the following, we call $\mathbb{L}$ in Lemma 6.3 the *Lindenbaum algebra*. We prove the following lemma, which guarantees we can use Lemmas 6.1 and 6.2 in the proof of completeness.

Lemma 6.4. *For any $\wedge\Phi \in \boldsymbol{L}_\alpha$ and $i \in N$,*

***(a)**:* $\sqcap\{[\![C]\!] : C \in \Phi\} = [\![\wedge\Phi]\!]$;

***(b)**:* $\sqcap\{\Box_i[\![C]\!] : C \in \Phi\} = [\![\mathbf{B}_i(\wedge\Phi)]\!]$.

Proof. **(a)**: First, let us see that $[\![\wedge\Phi]\!]$ is a lower bound of $\{[\![C]\!] : C \in \Phi\}$. Since $\vdash \wedge\Phi \supset C$ for all $C \in \Phi$ by L4, we have $(-[\![\wedge\Phi]\!]) \sqcup [\![C]\!] = \mathbf{1}$. Let $C \in \Phi$. Then, we have

$$\begin{aligned} [\![\wedge\Phi]\!] &= [\![\wedge\Phi]\!] \sqcap \mathbf{1} = [\![\wedge\Phi]\!] \sqcap \langle(-[\![\wedge\Phi]\!]) \sqcup [\![C]\!]\rangle \\ &= \langle[\![\wedge\Phi]\!] \sqcap (-[\![\wedge\Phi]\!])\rangle \sqcup \langle[\![\wedge\Phi]\!] \sqcap [\![C]\!]\rangle = \mathbf{0} \sqcup \langle[\![\wedge\Phi]\!] \sqcap [\![C]\!]\rangle = [\![\wedge\Phi]\!] \sqcap [\![C]\!]. \end{aligned}$$

Hence, $[\![\wedge\Phi]\!] \leq [\![C]\!]$. Since C is arbitrary in Φ, $[\![\wedge\Phi]\!]$ is a lower bound of $\{[\![C]\!] : C \in \Phi\}$.

It remains to show that $[\![\wedge\Phi]\!]$ is the greatest lower bound of $\{[\![C]\!] : C \in \Phi\}$. Now, let $[\![D]\!]$ be a lower bound of $\{[\![C]\!] : C \in \Phi\}$. This means $[\![D]\!] \leq [\![C]\!]$, i.e., $[\![D]\!] \sqcup [\![C]\!] = [\![C]\!]$, for any $C \in \Phi$. Let $C \in \Phi$. Then $(-[\![D]\!]) \sqcup [\![C]\!] = (-[\![D]\!]) \sqcup ([\![D]\!] \sqcup [\![C]\!]) = (-[\![D]\!] \sqcup [\![D]\!]) \sqcup [\![C]\!] = \mathbf{1} \sqcup [\![C]\!] = \mathbf{1}$. This implies $\vdash D \supset C$. Since C is arbitrary in Φ, we have, by $\wedge$-rule, we have $\vdash D \supset \wedge\Phi$. This means that $[\![\wedge\Phi]\!]$ is greater than or equal to $[\![D]\!]$ in $\mathbb{L}$. Thus, $[\![\wedge\Phi]\!]$ is the greatest lower bound of $\{[\![C]\!] : C \in \Phi\}$.

(b): Since $\vdash \mathbf{B}_i(\wedge\Phi) \supset \mathbf{B}_i(A)$ for all $A \in \Phi$, and since $\{\Box_i[\![A]\!] : A \in \Phi\} = \{[\![\mathbf{B}_i(A)]\!] : A \in \Phi\}$, $[\![\mathbf{B}_i(\wedge\Phi)]\!]$ is a lower bound of $\{\Box_i[\![A]\!] : A \in \Phi\}$. Now, let $[\![D]\!]$ be a lower bound of $\{\Box_i[\![A]\!] : A \in \Phi\}$. Using the same argument as in (a), we have $\vdash D \supset \mathbf{B}_i(A)$ for all $A \in \Phi$. Thus, $\vdash D \supset \wedge\mathbf{B}_i(\Phi)$ by $\wedge$-rule. By $\wedge$-Barcan, we have $\vdash D \supset \mathbf{B}_i(\wedge\Phi)$. This means that $[\![\mathbf{B}_i(\wedge\Phi)]\!]$ is the greatest lower bound of $\{\Box_i[\![A]\!] : A \in \Phi\}$.■

Now we define a pair $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ as follows:

$$\mathcal{Q}_1 = \{\{[\![A]\!] : A \in \Phi\} : \wedge\Phi \in \boldsymbol{L}_\alpha\} \text{ and } \mathcal{Q}_2 = \emptyset. \tag{26}$$

Then, $\mathcal{Q}_1$ is a countable. Then, the following lemma holds.

Lemma 6.5. *(1): $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies $(\sqcap, \sqcup)$-closedness.*

(2): $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies the conditions $q0$, $q1$, $q2$.

Proof. Since $\mathcal{Q}_2 = \emptyset$, the $(\sqcap, \sqcup)$-closedness for $\sqcup$ and $q2$ are vacuous.

(1): Let $Q \in \mathcal{Q}_1$. This Q is written as $\{[\![A]\!] : A \in \Phi\}$ for some $\wedge\Phi \in \boldsymbol{L}_\alpha$. Since $\sqcap Q = [\![\wedge\Phi]\!]$ by Lemma 6.4.(a), $\sqcap Q$ belongs to $\mathbf{B} = \boldsymbol{L}_\alpha/\equiv$.

(2)($q0$): We show that for any $Q \in \mathcal{Q}_1$, $\sqcap(\Box_i Q) := \sqcap\{\Box_i a : a \in Q\}$ exists and $\sqcap(\Box_i Q) = \Box_i(\sqcap Q)$. Since $Q \in \mathcal{Q}_1$, $\{\Box_i a : a \in Q\}$ is expressed as $\{[\![\mathbf{B}_i(A)]\!] : A \in \Phi\}$ for some $\wedge\Phi \in \boldsymbol{L}_\alpha$. By I$\beta$1-I$\beta$2, $\wedge\Phi \in \boldsymbol{L}_\alpha$ implies $\wedge\mathbf{B}_i(\Phi) \in \boldsymbol{L}_\alpha$. Then, by Lemma 6.4.(b) and $\vdash \wedge\mathbf{B}_i(\Phi) \equiv \mathbf{B}_i(\wedge\Phi)$, it holds that $\sqcap(\Box_i Q) = [\![\wedge\langle\mathbf{B}_i(A) : A \in \Phi\rangle]\!] = [\![\mathbf{B}_i(\wedge\Phi)]\!] = \Box_i[\![\wedge\Phi]\!] = \Box_i \sqcap Q$.

($q1$) : Let $Q \in \mathcal{Q}_1$ and $a \in \mathbf{B}$. We show $\{\Box_i(a \to b) : b \in Q\} \in \mathcal{Q}_1$. Since $a = [\![A]\!]$ for some $A \in \boldsymbol{L}_\alpha$ and Q is also expressed as $\{[\![B]\!] : B \in \Phi\}$ for some $\wedge\Phi \in \boldsymbol{L}_\alpha$, we have, by (25),

$$\{\Box_i(a \to b) : b \in Q\} = \{[\![\mathbf{B}_i(A \supset B)]\!] : B \in \Phi\}. \tag{27}$$

Since $\wedge\langle A \supset B : B \in \Phi\rangle \in \boldsymbol{L}_\alpha$ by Iα2.(i), we have $\wedge\langle\mathbf{B}_i(A \supset B) : B \in \Phi\rangle \in \boldsymbol{L}_\alpha$ by Iα2.(ii). Let $\Phi' = \langle\mathbf{B}_i(A \supset B) : B \in \Phi\rangle$. Then, since $\wedge\Phi' \in \boldsymbol{L}_\alpha$, we have, by (27), $\{\Box_i(a \to b) : b \in Q\} = \{[\![\mathbf{B}_i(A \supset B)]\!] : B \in \Phi\} \in \mathcal{Q}_1$.■

6.4 Construction of a counter-model

Recall that $\mathbb{L} = (\mathbf{B}, \mathbf{0}, \mathbf{1}, \sqcap, \sqcup, -, \Box_1, ..., \Box_n)$ with $\mathbf{B} = \boldsymbol{L}_\alpha/\equiv$ is the Lindenbaum algebra given in Lemma 6.3. Also, let $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ be given by (26). Now, we define a Kripke frame $\mathbb{K} = (W; R_1, ..., R_n)$ and an assignment τ as follows:

- **(i)**: $W = \mathbb{F}_\mathcal{Q}(\mathbb{L})$, where $\mathbb{F}_\mathcal{Q}(\mathbb{L})$ is the set of all $\mathcal{Q}$-filters for $\mathbb{L}$;
- **(ii)**: for all $i \in N$, wR_iu if and only if $\Box_i^{-1}w \subseteq u$;
- **(iii)**: for any $w \in W$ and any propositional variable p, $\tau(w, p) = \top$ if and only if $[\![p]\!] \in w$.

The nonemptiness of $\mathbb{F}_\mathcal{Q}(\mathbb{L})$ follows from Lemma 6.1 and the contradiction-freeness of GL($\boldsymbol{L}_\alpha$) noted after Theorem 3.1. Then, $\mathbb{M} = (\mathbb{K}, \tau) = (W; R_1, ..., R_n, \tau)$ is a Kripke model.

Lemma 6.6. *R_i is serial for each $i \in N$.*

Proof. Let $w \in W$. Consider $\Box_i \mathbf{0} = \Box_i [\![\neg \mathbf{p}_0 \wedge \mathbf{p}_0]\!]$ Then, $\Box_i \mathbf{0} = [\![\mathbf{B}_i(\neg \mathbf{p}_0 \wedge \mathbf{p}_0)]\!] = [\![\neg \mathbf{p}_0 \wedge \mathbf{p}_0]\!] = \mathbf{0}$ by $\ell 1$ and by Axiom D. Since w is a prime filter, we have $\Box_i \mathbf{0} = \mathbf{0} \notin w$. By Lemma 6.2 (Tanaka-Ono Lemma), we have $u \in \mathbb{F}_{\mathcal{Q}}(\mathbb{L})$ such that $\Box_i^{-1} w \subseteq u$, i.e., wR_iu, and $\mathbf{0} \notin u$.■

The following lemma is central to the completeness theorem.

Lemma 6.7. *(Truth lemma) For any $A \in \boldsymbol{L}_\alpha$ and $w \in W$, $(\mathbb{K}, \tau, w) \models A$ if and only if $[\![A]\!] \in w$.*

Proof. We prove the assertion by induction along the definition Iβ0-Iβ2 ($\beta \leq \alpha$) of formulae. Consider a propositional variable p. Then $(\mathbb{K}, \tau, w) \models p \Leftrightarrow \tau(w, p) = \top \Leftrightarrow [\![p]\!] \in w$.

Now, consider a non-propositional formula A in $\boldsymbol{L}_\beta$. Suppose that A is generated by Iβ1. Here, the induction hypothesis (abbreviated as IH), is simply that the assertion holds for any proper subformulae of A. The case $\wedge$ is applied to an infinitary conjunctive formula.

$(\supset)$: Let $(\mathbb{K}, \tau, w) \models A \supset B$. Then $(\mathbb{K}, \tau, w) \nvDash A$ or $(\mathbb{K}, \tau, w) \vDash B$. By the induction hypothesis, we have $[\![A]\!] \notin w$ or $[\![B]\!] \in w$. Since $[\![\neg A]\!] \in w$ or $[\![B]\!] \in w$, and since $[\![\neg A]\!] \leq [\![A \supset B]\!]$ and $[\![B]\!] \leq [\![A \supset B]\!]$, we have $[\![A \supset B]\!] \in w$.

Let $[\![A \supset B]\!] \in w$. Then $[\![\neg A \vee B]\!] = [\![\neg A]\!] \sqcup [\![B]\!] \in w$. Since w is a prime filter, we have $[\![\neg A]\!] \in w$ or $[\![B]\!] \in w$. Hence $[\![A]\!] \notin w$ or $[\![B]\!] \in w$. By IH, we have $(\mathbb{K}, \tau, w) \nvDash A$ or $(\mathbb{K}, \tau, w) \vDash B$. Thus, $(\mathbb{K}, \tau, w) \vDash A \supset B$.

$(\neg)$: The proof is similar.

$(\mathbf{B}_i)$: Let $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(A)$. Then $(\mathbb{K}, \tau, u) \models A$ for any u with $(w, u) \in R_i$. By IH, $[\![A]\!] \in u$ for any u with $(w, u) \in R_i$. Now, on the contrary, suppose that $\Box_i [\![A]\!] \notin w$. Then, by Lemma 6.2 (Tanaka-Ono Lemma), there is a $u \in \mathbb{F}_{\mathcal{Q}}(\mathbb{L})$ such that $\Box_i^{-1} w \subseteq u$ and $[\![A]\!] \notin u$. This is a contradiction. Hence, $[\![\mathbf{B}_i(A)]\!] = \Box_i [\![A]\!] \in w$.

Let $[\![\mathbf{B}_i(A)]\!] = \Box_i [\![A]\!] \in w$. Then $[\![A]\!] \in u$ for all u with $\Box_i^{-1} w \subseteq u$. By IH, we have $(\mathbb{K}, \tau, u) \models A$ for all u with $(w, u) \in R_i$. Hence, $(\mathbb{K}, \tau, u) \models \mathbf{B}_i(A)$.

$(\wedge)$: Let $\wedge \Phi$ be a finite conjunctive formula generated by Iβ1, or an infinite conjunctive formula given from a germinal form. In the latter case, any $A \in \Phi$ belongs to $\cup_{\gamma < \beta} \boldsymbol{L}_\gamma$. In either case, IH is that the assertion holds for any $A \in \Phi$. In these cases, we have the following proof.

Let $(\mathbb{K}, \tau, w) \models \wedge \Phi$. Then $(\mathbb{K}, \tau, w) \models A$ for all $A \in \Phi$. By IH, $[\![A]\!] \in w$ for all $A \in \Phi$. Then $\sqcap \{[\![A]\!] : A \in \Phi\}$ exists by Lemma 6.5.(1), and it belongs to w by Q1. Hence, $[\![\wedge \Phi]\!] = \sqcap \{[\![A]\!] : A \in \Phi\} \in w$.

Let $[\![\wedge \Phi]\!] \in w$. Then $[\![\wedge \Phi]\!] \leq [\![A]\!]$ for all $A \in \Phi$. Since w is a filter, we have $[\![A]\!] \in w$ for all $A \in \Phi$ by F1. Hence $(\mathbb{K}, \tau, w) \models A$ for all $A \in \Phi$ by IH, which implies $(\mathbb{K}, \tau, w) \models \wedge \Phi$.

Now, consider the cases of Iβ2.(i), Iβ2.(ii), and Iβ2.(iii). Suppose that $\wedge \Phi = \wedge \langle D \supset C_\nu : \nu \geq 0 \rangle$, $\wedge \Phi = \wedge \langle \mathbf{B}_i(C_\nu) : \nu \geq 0 \rangle$, or $\wedge \Phi = \wedge \langle C_\nu \wedge D_\nu : \nu \geq 0 \rangle$ be generated by Iβ2.(i), Iβ2.(ii), or Iβ2.(iii) from D, $\wedge \langle C_\nu : \nu \geq 0 \rangle$, and $\wedge \langle D_\nu : \nu \geq 0 \rangle$. Here, IH is that the assertion holds form D, $\wedge \langle C_\nu : \nu \geq 0 \rangle$, and $\wedge \langle D_\nu : \nu \geq 0 \rangle$.

Let $(\mathbb{K}, \tau, w) \models \wedge \langle D \supset C_\nu : \nu \geq 0 \rangle$. Then $(\mathbb{K}, \tau, w) \models D \supset C_\nu$, i.e., $(\mathbb{K}, \tau, w) \nvDash D$ or $(\mathbb{K}, \tau, w) \models C_\nu$, for all $\nu \geq 0$. The latter part implies $(\mathbb{K}, \tau, w) \models \wedge \langle C_\nu : \nu \geq 0 \rangle$. By IH, we have $[\![D]\!] \notin w$ or $[\![\wedge \langle C_\nu : \nu \geq 0 \rangle]\!] \in w$. Since w is a prime filter, we have $[\![D]\!] \rightarrow [\![\wedge \langle C_\nu : \nu \geq 0 \rangle]\!] \in w$, which implies $[\![D \supset \wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$ by (25). Since $\vdash (D \supset \wedge\langle C_\nu : \nu \geq 0\rangle) \equiv \wedge\langle D \supset C_\nu : \nu \geq 0\rangle$, we have $[\![\wedge\langle D \supset C_\nu : \nu \geq 0\rangle]\!] \in w$. The converse can be obtained by tracing back this argument.

Let $(\mathbb{K}, \tau, w) \models \wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle$. This implies $(\mathbb{K}, \tau, w) \models \mathbf{B}_i(\wedge\langle C_\nu : \nu \geq 0\rangle)$. Let u be any world with $(w, u) \in R_i$. Then, $(\mathbb{K}, \tau, u) \models \wedge\langle C_\nu : \nu \geq 0\rangle$. By IH, we have $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in u$. Now, on the contrary, suppose that $\Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \notin w$. Then, by Lemma 6.2 (Tanaka-Ono Lemma), there is a $u^o \in \mathbb{F}_{\mathcal{Q}}(\mathbb{L})$ such that $\Box_i^{-1}w \subseteq u^o$ and $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \notin u^o$. Since $(w, u^o) \in R_i$ by the definition of R_i, this is a contradiction. Hence, $\Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$. Thus, $[\![\wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle)]\!] = [\![\mathbf{B}_i(\wedge\langle C_\nu : \nu \geq 0\rangle)]\!] = \Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$, using $\wedge$-Barcan.

Conversely, let $[\![\wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle)]\!] \in w$. Then, $\Box_i[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] = [\![\mathbf{B}_i(\wedge\langle C_\nu : \nu \geq 0\rangle)]\!] = [\![\wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle]\!] \in w$ using $\wedge$-Barcan. Let $u \in W$ be an arbitrary world with $\Box_i^{-1}w \subseteq u$. Then, $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in u$. By IH, we have $(\mathbb{K}, \tau, u) \models \wedge\langle C_\nu : \nu \geq 0\rangle$. Since u is arbitrary with $(w, u) \in R_i$, we have $(\mathbb{K}, \tau, u) \models \mathbf{B}_i(C_\nu)$ for all $\nu \geq 0$. Hence, $(\mathbb{K}, \tau, u) \models \wedge\langle \mathbf{B}_i(C_\nu) : \nu \geq 0\rangle$.

Let $(\mathbb{K}, \tau, w) \models \wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle$. Then $(\mathbb{K}, \tau, w) \models C_\nu \wedge D_\nu$, i.e., $(\mathbb{K}, \tau, w) \models C_\nu$ and $(\mathbb{K}, \tau, w) \models D_\nu$ for all $\nu \geq 0$. This implies $(\mathbb{K}, \tau, w) \models \wedge\langle C_\nu : \nu \geq 0\rangle$ and $(\mathbb{K}, \tau, w) \models \wedge\langle D_\nu : \nu \geq 0\rangle$. By IH, $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \in w$ and $[\![\wedge\langle D_\nu : \nu \geq 0\rangle]\!] \in w$. Since w is a filter, we have $[\![\wedge\langle C_\nu : \nu \geq 0\rangle]\!] \sqcap [\![\wedge\langle D_\nu : \nu \geq 0\rangle]\!] \in w_\nu$. Since $\vdash [\langle C_\nu : \nu \geq 0\rangle \wedge \langle D_\nu : \nu \geq 0\rangle] \equiv \wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle$, we have $[\![\wedge\langle C_\nu \wedge D_\nu : \nu \geq 0\rangle]\!] \in w$. The converse can be obtained by tracing back this argument.■

The final step of completeness is to show that for any $A \in \boldsymbol{L}_\alpha$, if $\nvdash A$, then $(\mathbb{K}, \tau, w) \not\models A$ for some world $w \in W$. Suppose $\nvdash A$. This means $[\![A]\!] \neq \mathbf{1}$; hence, $[\![A]\!] \not\geq \mathbf{1}$. Applying Lemma 6.1 (Rasiowa-Sikorski lemma) to $[\![A]\!]$ and $\mathbf{1}$, there is a $\mathcal{Q}$-filter F such that $\mathbf{1} \in F$ and $[\![A]\!] \notin F$. Denote F by w. Then, by Lemma 6.7, we have $(\mathbb{K}, \tau, w) \not\models A$.■

Remark 6.1. Lemma 6.6 can be extended to other epistemic axioms, T, 4, or 5, and the corresponding conditions, reflexivity, transitivity, or euclidean for R_i. Transitivity is derived from Axiom 4: Let $\Box_i^{-1}w \subseteq u$ and $\Box_i^{-1}u \subseteq v$ and $[\![A]\!] \in \Box_i^{-1}w$. Then, $[\![\mathbf{B}_i(A)]\!] \in w$. Since $\vdash \mathbf{B}_i(A) \supset \mathbf{B}_i\mathbf{B}_i(A)$, we have $[\![\mathbf{B}_i(A)]\!] \leq [\![\mathbf{B}_i\mathbf{B}_i(A)]\!]$. Since w is a filter, we have $[\![\mathbf{B}_i\mathbf{B}_i(A)]\!] \in w$; so $[\![\mathbf{B}_i(A)]\!] \in \Box_i^{-1}w$. Hence, $[\![\mathbf{B}_i(A)]\!] \in u$, i.e., $[\![A]\!] \in \Box_i^{-1}u$. Repeating this argument, we have $[\![A]\!] \in \Box_i^{-1}v$. Also, euclidean: wR_iu and $wR_iv \Longrightarrow uR_iv$ is derived from Axiom 5. Let $\Box_i^{-1}w \subseteq u$ and $\Box_i^{-1}w \subseteq v$. Suppose that for some A, $[\![A]\!] \in \Box_i^{-1}u$ but $[\![A]\!] \notin v$. Since $\Box_i^{-1}w \subseteq v$, we have $[\![\mathbf{B}_i(A)]\!] \notin w$. Thus, $[\![\neg\mathbf{B}_i(A)]\!] \in w$. By Axiom 5, $[\![\mathbf{B}_i(\neg\mathbf{B}_i(A))]\!] \in w$. Thus, $[\![\neg\mathbf{B}_i(A))]\!] \in \Box_i^{-1}w \subseteq u$, which is a contradiction to $[\![A]\!] \in \Box_i^{-1}u$. Hence, $\Box_i^{-1}u \subseteq v$.

7 Conclusions

We developed a series of small infinitary epistemic logics. This series inherits useful features from both infinitary logic approach and fixed-point logic approach. Similar to the infinitary logic approach, our framework allows for explicit and unified formulations of infinitary concepts such as common knowledge; and it allows for the direct evaluation of depths of such infinitary concepts. Similar to the fixed-point approach, we can control infinitary expressions by imposing specific germinal forms. Moreover, we have shown that our completeness result holds for each layer, and our logics in different layers are connected by the conservative extension relation.

We provided two applications. The first is about explicit definabilities of epistemic axioms T, 4, and 5. Specifically, we showed that Axiom T can be captured in GL($\boldsymbol{L}_\alpha$) for any α ($0 \leq \alpha \leq \omega$), Axiom 4 can be done in GL($\boldsymbol{L}_\omega$), since it needs infinite iterations of the belief operator. Axiom 5 is not explicitly definable for any α ($0 \leq \alpha \leq \omega$). These results differentiate the three axioms. The second is for game theory: we considered an agent's decision-making in a game, based on the idea of rationalizability. We gave a full epistemic characterization, which was done within GL($\boldsymbol{L}_2$), a shallow part in the series in (1), and, based on this characterization, we obtained the playability result for an agent in a game.

Our approach gives rise to new open problems. As already stated, a full study of Table 3.1 is an open problem of great importance. As seen in Section 4.3, we showed that some known fixed-point logics such as common knowledge logic can be faithfully embedded into our system (Theorem 4.7). In recent years, the fixed-point approach has been extensively developed in modal μ-calculus, and a natural question is whether such embedding results can be extended to (some specific fragments of) those logics, and what relationship exists between our system and modal μ-calculus. A full answer to this question remains open, though we gave a summary of differences in our approach and modal μ-calculus in the end of Section 4.3.

There are open problems related to explicit definability and embedding. We studied explicit definability and embedding for each of the three epistemic axioms and common knowledge. However, a general criterion for an infinitary (and/or finitary) concept to be explicitly definable in some GL($\boldsymbol{L}_\alpha$) remains open.[9] A related problem is to have a general understanding of when a fixed-point logic can be embedded into our system.

Our framework adopts the Hilbert-style proof theory. One alternative would be to formulate it in the Gentzen style sequence calculus. In particular, if cut-elimination is available, then one can discuss the sizes of proofs. For this purpose, there are two possibilities from the literature. One is to adopt Kaneko-Nagashima [21]'s formulation in the context of an infinitary logic, which is close to the original Gentzen formulation. Cut-elimination is available, while $\wedge$-Barcan prevents it from implying the full subformula property. Another is in the modal μ-calculus, for which Brünnler-Studer [6] provided a different Gentzen style formulation, focusing on some shallow fragments for cut-elimination. A full study of these systems remains open.

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Appendix

Proof of (10): Let $N = \{1,2\}$, $i = 1, j = 2$, and $\mathcal{G} = \{\langle \mathbf{B}_1^\nu(p) : \nu \geq 0\rangle\}$, $\mathcal{G}' = \{\langle \mathbf{B}_1^\nu(p) : \nu \geq 0\rangle$, $\langle \mathbf{B}_2^\nu(p) : \nu \geq 0\rangle\}$; this can be extended to a general case without difficulty. The contrapositive of the claim is proved by constructing a counter-model. Let $A \in \boldsymbol{L}_1(\mathcal{G})$ be any formula with $\mathrm{GL}(\boldsymbol{L}_1(\mathcal{G}')) \nvdash \neg A$. By Theorem 3.1, there is a Kripke model $(\mathbb{K}, \tau) = ((W, R_1, R_2), \tau)$ such that $(\mathbb{K}, \tau, w_0) \models A$ for some $w_0 \in W$. If there is a world $w \in W$ such that w is (sequentially) accessible from w_o by R_2 and $\tau(w,p) = \bot$, then $(\mathbb{K}, \tau, w_0) \nvDash A \supset \mathbf{B}_2^\nu(p)$ for some $\nu < \omega$; so, $(\mathbb{K}, \tau, w_0) \nvDash A \supset \mathbf{B}_2^\omega(p)$. In the following, we assume that $\tau(w,p) = \top$ for any sequentially accessible w from w_0 by R_2. We extend $(\mathbb{K}, \tau)$ to $(\mathbb{K}', \tau')$ so that $(\mathbb{K}', \tau', w_0) \models A$ but $\tau'(w,p) = \bot$ for some sequentially accessible w by R_2'.

Since $A \in \boldsymbol{L}_1(\mathcal{G})$, the nested depth of $\mathbf{B}_2(\cdot)$ is finite, and let m_o be the maximum of the nested depths of $\mathbf{B}_2(\cdot)$ in A. We denote the set of all subformulae of A by $Sub(A)$, and the maximum nested depth of $\mathbf{B}_2(\cdot)$ in $C \in Sub(A)$ by $\eta_2(C)$. There are two cases to be considered:

(A): there is a sequence $\{w_0, w_1, ..., w_{\ell_1}\}$ such that $w_0, w_1, ..., w_{\ell_1}$ are all distinct, $w_t R_2 w_{t+1}$ for all $t = 0, ..., \ell_1 - 1$, but $w_{\ell_1} R_2 w_{\ell_0}$ for some $w_{\ell_0} \in \{w_0, w_1, ..., w_{\ell_1}\}$;

(B): for any infinite sequence $\{w_0, w_1, ...\}$ with $w_t R_2 w_{t+1}$ for all $t \geq 0$ and for any $w_\ell \in \{w_0, w_1, ...\}$, there is no element $w_{\ell_0} \in \{w_0, w_1, ..., w_\ell\}$ with $w_\ell R_2 w_{\ell_0}$.

These two cases are mutually exclusive and exhaustive because of seriality for R_2.

Case (A): We choose a sequence $\{w_0, w_1, ..., w_{\ell_1}\}$ with the smallest ℓ_1. Then, we choose an ℓ_0 given in (A). Incidentally, $\ell_0 = \ell_1$ could be possible. The idea of extending $(\mathbb{K}, \tau)$ to $(\mathbb{K}', \tau')$

is as follows: a sequence $\{v_1, ..., v_{m_o+2}\}$ of new symbols (for new possible worlds) is sequentially connected to w_{ℓ_0} by the new R_2' but not R_1'. The additional accessibilities in R_2' are the connections from w_{ℓ_o} to v_1, from v_1 to $v_2, ...,$ and from v_{m_o+1} to v_{m_o+2}, but the out-going references from v_t $(1 \leq t \leq v_{m_o+1})$, in addition to the connection to v_{t+1}, are the same as $w_{\ell_0+t'}$ where $t = k(\ell_1 - \ell_0 + 1) + t'$ and $t' \leq \ell_1 - \ell_0$. The last v_{m_o+2} is the dead-end. In addition to $\{v_1, ..., v_{m_o+2}\}$, we define the sequence $\langle w_{\ell_0+t} : t = 1, ..., m_o + 2\rangle$ in W so that $w_{\ell_0+t} = w_{\ell_0+t'}$ when $t = k(\ell_1 - \ell_0 + 1) + t'$ for some $k \geq 0$ and $t' \leq \ell_1 - \ell_0$. That is, this is the sequence counting $w_{\ell_0+1}, ..., w_{\ell_1}$ and then counting along the same cycle and it stops at $w_{\ell_0+m_0+2}$.

$$
\begin{array}{|lcr|c|c|c|l}
\hline
w_0 & \to_2 \cdots \to_2 & w_{\ell_0} \to_2 & v_1 \to_2 & v_2 \to_2 \cdots & v_{m_o+1} \to_2 & v_{m_o+2} \circlearrowleft_{1,2} \\
\downarrow_1 \searrow_2 & & \downarrow_1 \searrow_2 & \downarrow_1 \searrow_2 & \downarrow_1 \searrow_2 & \downarrow_1 \searrow_2 & \\
\cline{4-6}
 & & W & W & W & W & \\
\hline
\end{array}
$$

In the above figure, we consider the special case where $\ell_0 = \ell_1$. We replicate the connections from w_{ℓ_0} and attach them to each v_t up to v_{m_o+1}. We can extend τ to τ' so that $\tau'(v_t, \cdot)$ takes the same value as $\tau(w_{\ell_0}, \cdot)$, but $\tau'(v_{m_o+2}, \cdot)$ takes the negative value $\perp$ for all $q \in \mathcal{P}_0$. Then, we prove $(\mathbb{K}', \tau', w_0) \models A$ but $(\mathbb{K}', \tau', w_0) \not\models \mathbf{B}_2^{\ell_0+m_o+2}(p)$.

Now, we prove the assertion more rigorously. We extend $(\mathbb{K}, \tau)$ to $(\mathbb{K}', \tau')$ as follows:

$$W' = W \cup \{v_1, ..., v_{m_o+2}\}; \tag{28}$$

$$\begin{aligned}
R_1' &= R_1 \cup \{(v_t, w) : (w_{\ell_0+t}, w) \in R_1 \text{ and } t = 1, ..., m_o + 1\} \cup \{(v_{m_o+2}, v_{m_o+2})\} \qquad (29)\\
R_2' &= R_2 \cup \{(w_{\ell_0}, v_1)\} \cup \{(v_t, v_{t+1}) : t = 1, ..., m_o + 1\} \cup \{(v_{m_o+2}, v_{m_o+2})\} \cup \\
&\quad \{(v_t, w) : (w_{\ell_0+t}, w) \in R_2 \text{ and } t = 1, ..., m_o + 1\};
\end{aligned}$$

and for any $q \in \mathcal{P}_0$,

$$\tau'(v, q) = \begin{cases} \tau(w, q) & \text{if } v \in W \\ \tau(w_{\ell_0+t}, q) & \text{if } v = v_t \ (1 \leq t \leq m_0 + 1) \\ \perp & \text{if } v = v_{m_o+2}. \end{cases} \tag{30}$$

In (29), only v_1 is directly connected only to w_{ℓ_0}, and there may be references from v_t to some worlds in W in the same way from w_{ℓ_0+t}.

Now, we show by induction on $k = 0, ..., m_o$ that for any $C \in Sub(A)$ with $\eta_2(C) \leq k$ and for any $t = 1, ..., m_o - k$,

$$(\mathbb{K}, \tau, w_{\ell_0+t}) \models C \Longleftrightarrow (\mathbb{K}', \tau', v_t) \models C. \tag{31}$$

We show this claim by double induction over $k = 0, ..., m_o - 1$ and the length of formula $C \in Sub(A)$ with $\eta_2(C) = k$. We call the induction over k the *main induction*, and the induction for the length of formula the *sub-induction*.

Let $k = 0$, which is the induction base of the main induction. We prove the assertion for any formula C with $\eta_2(C) = 0$. By (30), the truth assignment $\tau'(v_t, \cdot)$ is the same as $\tau(w_{\ell_0+t}, \cdot)$ for t $(1 \leq t \leq m_o)$. Also, C has no occurrences of $\mathbf{B}_2(\cdot)$. For the valuation of C, R_2' is not used at v_t and w_{ℓ_0+t}. When the outermost connective of C is $\mathbf{B}_1(\cdot)$, R_1' does not connect v_t with v_{t+1} and the out-going references from v_t are the same from w_{ℓ_0+t}. Hence, (31) for C. The cases of the other logical connectives, including infinitary formulae, are similar. Hence, for all $t = 1, ..., m_o$, we have (31) for $C \in Sub(A)$ with $\eta_2(C) = 0$.

Suppose (31) for $k < m_o$. Then, it is sufficient to consider the case $\eta_2(C) = k+1$. The main induction hypothesis is that (31) holds for any $D \in Sub(A)$ with $\eta_2(D) = k$. The next step is induction by the length of C. The first case is that $C = \mathbf{B}_2(D)$ for some D. Then $\eta_2(D) = k$; we have (31) for D by the main induction hypothesis. By the choice of $w_{\ell_0+1}, ..., w_{\ell_1}$ and (29), each of v_{t+1} and w_{ℓ_0+t+1} is accessible, respectively, from each of v_t and w_{ℓ_0+t} by R_2'; thus, the out-going references by R_2' from v_t are the same as w_{ℓ_0+t} for all $t \leq m_o - k$, we have (31) for $C = \mathbf{B}_2(D)$.

Then, we should consider the other connectives, $\neg, \supset, \wedge$ and $\mathbf{B}_1(\cdot)$. Consider the case where $C = \mathbf{B}_1(D)$ for some D. Then $\eta_2(D) = k+1$. By the sub-induction hypothesis, we have (31) for D. Since the out-going references by R_1' from v_t are the same as w_{ℓ_0+t} for all $t \leq m_o - k$, we have (31) for $C = \mathbf{B}_1(D)$. The cases of the other connectives are similar.

The equivalence (31) for $k = m_o$ implies that for all $C \in Sub(A)$,

$$(\mathbb{K}, \tau, w_{\ell_0+1}) \models C \Longleftrightarrow (\mathbb{K}', \tau', v_1) \models C.$$

Since v_1 is connected to w_{ℓ_0} by R_2' and not by R_1', we have, for all $C \in Sub(A)$, $(\mathbb{K}, \tau, w_{\ell_0}) \models C \Longleftrightarrow (\mathbb{K}', \tau', w_{\ell_0}) \models C$. We do not change the out-going references with respect to R_1' and R_2' from $w_0, ..., w_{\ell_0-1}$. Also, the truth assignments τ and τ' are the same over W. Hence, for $k = 0, ..., \ell_0$ and for all $C \in Sub(A)$, $(\mathbb{K}, \tau, w_k) \models C \Longleftrightarrow (\mathbb{K}', \tau', w_k) \models C$. Hence, $(\mathbb{K}', \tau', w_0) \models A$.

In sum, we have $(\mathbb{K}', \tau', w_0) \models A$ and $(\mathbb{K}', \tau', v_{m_0+2}) \models \neg p$. Thus, $(\mathbb{K}', \tau', w_0) \not\models A \supset \mathbf{B}_2^{\omega}(p)$.

Case (B): In this case, for any infinite sequence $\{w_0, w_1, ...\}$ with $w_t R_2 w_{t+1}$ for all $t \geq 0$ and for any $w_\ell \in \{w_0, w_1, ...\}$, there is no element $w_\ell R_2 w_{\ell_0}$ for some $w_{\ell_0} \in \{w_0, w_1, ..., w_\ell\}$. In this case, we can take any w_ℓ and plug $v_1, ..., v_{m_o+2}$ to w_ℓ. Then, we construct $(\mathbb{K}', \tau')$ in the same way as above. Only we take $w_{\ell+1}, ..., w_{\ell+m_o+2}$ from $\{w_0, w_1, ...\}$; here, the above cyclical argument is unnecessary. The remaining part is the same as above, and we have $(\mathbb{K}', \tau', w_0) \models A$ and $(\mathbb{K}', \tau', v_{m_0+2}) \models \neg p$. Thus, $(\mathbb{K}', \tau', w_0) \not\models A \supset \mathbf{B}_2^{\omega}(p)$.$\blacksquare$

Proof of Lemma 4.3: We show by induction over $\alpha \leq \omega$ that $\psi^T : \boldsymbol{L}_\alpha \to \boldsymbol{L}_\alpha$ is uniquely extended by E0, E1$_\alpha$-E4$_\alpha$. This holds for $\alpha = 0$. Suppose the induction hypothesis that it holds for all $\beta < \alpha$. Then, we show that E1$_\alpha$-E4$_\alpha$ uniquely define $\psi^T : \boldsymbol{L}_\alpha \to \boldsymbol{L}_\alpha$.

First, we prove that for any formula $G(p_1, ..., p_m)$ in L_0 and $G'(p_1, ..., p_m) = \psi^T(G(p_1, ..., p_m))$,

$$\psi^T G(A_1, ..., A_m) = G'(\psi^T A_1, ..., \psi^T A_m) \text{ for any } A \in \mathcal{P}_\alpha. \tag{32}$$

This is proved by induction on the length of formula G. We consider only the case of $m = 1$, from which a general case is simply obtained. If $G(p) = p$, then $\psi^T(G(A)) = \psi^T(A) = G'(\psi^T(A))$. We consider the induction step only for the case where the outermost connective of $G(p) \in L_0$ is $\mathbf{B}_i(\cdot)$, i.e., $G(p) = \mathbf{B}_i(D(p))$ for some $D(p)$, supposing (32) for $D(p)$. Then, $G'(p) = \psi^T \mathbf{B}_i(D(p))$ $= \mathbf{B}_i(\psi^T D(p)) \wedge \psi^T D(p)$ by E4$_\alpha$, and let $D'(p) = \psi^T D(p)$. Since $\psi^T D(A) = D'(\psi^T(A))$ by (32) for $D(p)$, we have $G'(p) = \mathbf{B}_i(D'(\psi^T A)) \wedge D'(\psi^T A)$, which is $G'(\psi^T A)$.

Now, we show that ψ_T is well-defined over $\mathcal{F}_\alpha$. Suppose that $\wedge\langle C^\nu(A_1, ..., A_m) : \nu \geq 0\rangle$ is generated by a germinal form $\wedge\langle C^\nu(p_1, ..., p_m) : \nu \geq 0\rangle$ and germs $A_1, ..., A_m \in \mathcal{P}_\alpha$. By (14), $C^{\nu\prime}(p_1, ..., p_m) = \psi^T C^\nu(p_1, ..., p_m) \in \mathcal{G}$, and let $A_k' = \psi^T(A_k) \in \mathcal{P}_\alpha$ for $k = 1, ..., m$. Then, abbreviating $C^\nu(A_1, ..., A_m)$ as C^ν, we have $\psi^T(\wedge\langle C^\nu : \nu \geq 0\rangle) = \wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$ by E3$_\alpha$. Now, by (32), $C'^\nu = \psi^T C^\nu = \psi^T(C^\nu(A_1, ..., A_m)) = C^{\nu\prime}(\psi^T A_1, ..., \psi^T A_1) \in \mathcal{P}_\alpha$ for all $\nu \geq 0$. Hence, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$ is generated by germinal form $\wedge\langle C^{\nu\prime}(p_1, ..., p_m) : \nu \geq 0\rangle$ and germs $A_1', ..., A_m' \in \mathcal{P}_\alpha$. Thus, $\psi^T(\wedge\langle C^\nu(A_1, ..., A_\nu) : \nu \geq 0\rangle) \in \mathcal{F}_\alpha$.

We extend ψ_T from $\mathcal{P}_\alpha \cup \mathcal{F}_\alpha$ to the entire $\boldsymbol{L}_\alpha$ along Iα1 - Iα2. This is also by induction. The steps in Iα1 are standard. Consider Iα2. Let A, $\wedge\langle C^\nu : \nu \geq 0\rangle$, $\wedge\langle D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$, and assume the induction hypothesis IH$_\alpha$: $\psi^T A$, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$, $\wedge\langle \psi^T D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. We prove only I$\alpha$2.$(ii)$ and Iα2.(iii).

Consider Iα2.(ii). By IH$_\alpha$, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle \in L_\alpha$, and by I$\alpha$2.$(ii)$, $\wedge\langle \mathbf{B}_i(\psi^T C^\nu) : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By I$\alpha$2.$(iii)$, we have $\wedge\langle \mathbf{B}_i(\psi^T C^\nu) \wedge \psi^T C^\nu : \nu \geq 0\rangle \in L_\alpha$. This is written as $\wedge\langle \psi^T \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle \in L_\alpha$ by E4$_\alpha$. Thus, $\psi^T(\wedge\langle \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle) = \wedge\langle \psi^T \mathbf{B}_i(C^\nu) : \nu \geq 0\rangle = \wedge\langle \mathbf{B}_i(\psi^T C^\nu) \wedge \psi^T C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. In the case of $j \neq i$, this proof becomes simpler.

Consider Iα2.(iii). By In$_\alpha$, $\wedge\langle \psi^T C^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$ and $\wedge\langle \psi^T D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By I$\alpha$2.$(iii)$, we have $\wedge\langle \psi^T C^\nu \wedge \psi^T D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$. By E3$_\alpha$, $\psi^T(\wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle) = \wedge\langle \psi^T(C^\nu \wedge D^\nu) : \nu \geq 0\rangle = \wedge\langle \psi^T C^\nu \wedge \psi^T D^\nu : \nu \geq 0\rangle \in \boldsymbol{L}_\alpha$.■

Proof of Theorem 4.5. Take an arbitrary serial model $(\mathbb{K}, \tau)$ and let $\mathbb{K}^{ref}$ be the reflexive closure of $\mathbb{K}$ with respect to R_i. We show by induction on the length of A that

$$\text{for all } w \in W, \quad (\mathbb{K}^{ref}, \tau, w) \models A \Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T(A). \tag{33}$$

Thus, $\mathbb{K}^{ref} \models A \Longleftrightarrow \mathbb{K} \models \psi^T(A)$. Since $\mathbb{K}$ is an arbitrary serial model, this equivalence implies, by the completeness theorems for GL($\boldsymbol{L}_\alpha$) and GL($\boldsymbol{L}_\alpha$)+ T$_i$ (Theorem 3.1 and its variant) that (GL($\boldsymbol{L}_\alpha$)+T$_i$) $\vdash A$ if and only if GL($\boldsymbol{L}_\alpha$) $\vdash \psi^T(A)$.

We prove (33) by induction on the length of $A \in \boldsymbol{L}_\alpha$. We consider only the two cases: Case $A = \wedge\langle C^\nu : \nu \geq 0\rangle \in \mathcal{F}_\alpha$ and Case $A = \wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle$ generated by Iα2.(iii).

Consider Case $A = \wedge\langle C^\nu : \nu \geq 0\rangle \in \mathcal{F}_\alpha$. Then, $(\mathbb{K}^{ref}, \tau, w) \models \wedge\langle C^\nu : \nu \geq 0\rangle \Longleftrightarrow (\mathbb{K}^{ref}, \tau, w) \models C^\nu$ for all $v \geq 0 \Longleftrightarrow (\mathbb{K}, \tau, v) \models \psi^T C^\nu$ for all $v \geq 0$ by the induction hypothesis. This is further equivalent to $(\mathbb{K}, \tau, w) \models \wedge\langle \psi^T C^\nu : \nu \geq 0\rangle$, which is equivalent to $\Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T(\wedge\langle C^\nu : \nu \geq 0\rangle)$.

Consider Case $A = \wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle$. The induction hypothesis is: (33) holds for C^ν and D^ν for all $\nu \geq 0$. Now, $(\mathbb{K}^{ref}, \tau, w) \models \wedge\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle \Longleftrightarrow (\mathbb{K}, \tau, w) \models C^\nu \wedge D^\nu$ for all $\nu \geq 0 \Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T(C^\nu \wedge D^\nu)$ for all $\nu \geq 0$, which is equivalent to $(\mathbb{K}, \tau, w) \models \langle \psi^T(C^\nu \wedge D^\nu) : \nu \geq 0\rangle \Longleftrightarrow (\mathbb{K}, \tau, w) \models \psi^T\langle C^\nu \wedge D^\nu : \nu \geq 0\rangle$.■