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An Equilibrium-Econometric Analysis of Rental Housing Markets with Indivisibilities*

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Abstract

We develop a theory of an equilibrium-econometric analysis of rental housing markets with indivisibilities. It provides a bridge between a (competitive) market equilibrium theory and a statistical/econometric analysis. The listing service of apartments provides the information to both economic agents and an econometric analyzer: each economic agent uses a small part of the data from the service for his economic behavior, and the analyzer uses them to estimate the market structure. It is argued that the latter may be done by assuming that the economic agents take the standard price-taking behavior. We apply our theory to the data in the rental housing markets in the Tokyo area, and examine the law of diminishing marginal utility for household. It holds strictly with respect to the consumption, less with commuting time-distance, and much less with the sizes of apartments

Key-Words: Rental housing market, Indivisibilities, Competitive equilibrium, Discrepancy measure, Law of diminishing marginal utility, *Ex post* rationalization

JEL Classification: C10, D45, R20

1. Introduction

1.1. General idea

We develop a theory of an equilibrium-econometric analysis of rental housing markets, and test it with some data from the Tokyo area. Our theory has the following salient features:

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(i): An econometric method is developed through an (market) equilibrium theory.

(ii): Both *economic agents* and an *econometric analyzer* are facing statistical components in the economy. We show that an equilibrium theory without statistical components is regarded as an idealization, and that its structure is estimated by our equilibrium-econometric analysis.

(iii): We define the measure of discrepancy between the prediction by our theory and the best statistical estimator, and show that the prediction is quite satisfactory in the example of the Tokyo area.

Feature (ii) tells why and how we can use an equilibrium theory for an econometric analysis of (i). Feature (iii) is a requirement from the econometric point of view. Here, focusing on these features, we discuss our motivations and backgrounds.

One fundamental question arises in an application of an equilibrium theory to real economic problems with an econometric method: what is the source for errors in the econometric analysis? This may be answered in the same way as classical statistics: the source is attributed to partial observations. In many economic problems, this answer is applied not only to the economic analyzer but also to economic agents. Both face non-unique (perturbed) rents of goods. Error terms represent the effects of variables not included in available information to either economic agents or the econometric analyzer.

We look at a rental housing market in Tokyo. In the Tokyo area, the rental housing market is held, day by day, in a highly decentralized manner, i.e., many households (demanders) and many landlords (suppliers) look for better opportunities[1]. Various weekly magazines, daily newspapers, and internet services for listing apartments for rental prices (rents) are available as media for information transmission of supplied units together with rents from suppliers to demanders[2]. With the help of those media, rental housing markets function well, even though rents are not uniform over the "same" category of apartment units. We will call these media *housing magazines*.

Housing magazines give *concise and coarse* date about each listed apartment unit, following a fixed number of criteria, rents, size, location, age, geography, etc. This information is far from the description of its full characteristics. This is because the number of weekly listed units is large; e.g., $100-1,200$ listed around one railway station, and an weekly issue may exceed 500 pages.

The data of rents show that they are heterogeneous over the "same" category of apartment units. The market can still be regarded as "perfectly competitive" in that each has many competitors. These may appear contradictory, but can be reconciled;

[1] In the city of Tokyo (about 12 millions of residents), the percentage of households renting apartments is about 55% in 2005, and in the entire Japan, the percentage is about 37%.

[2] There are many decentralized real estate agents. In our analysis, we do not explicitly count real-estate companies. But we should remember that behind the market description, many real-estate companies are included.

households and landlords look at summary statistics, and behave as if they are facing uniform rents. Taking this interpretation into account, the econometric analyzer may make estimation of a structure of the market. These are the two faces of our theory.

We call the attributes listed in the magazine as *systematic* components and the others as *non-systematic* factors. The systematic ones are described as a market model $\mathbb{E}$, which is assumed to be an equilibrium theory without perturbations, and the non-systematic factors are summarized by error terms ϵ. The listed rents in housing magazines are given as $p(\mathbb{E}) + \epsilon$, where the rent vector $p(\mathbb{E})$ is determined by $\mathbb{E}$. Both economic agents and econometric analyzer observe the rents $p(\mathbb{E}) + \epsilon$, but they have different purposes. The economic agents use them for their behavioral choices, while the econometric analyzer does for the estimation of the systematic components of $\mathbb{E}$. Those structures are depicted in Fig.1. In the left box, the rents and behavioral choices are simultaneously determined as an equilibrium. The determined, yet perturbed, rents are used by the analyzer. We study each of those, and then synthesize them.

$$\boxed{p(\mathbb{E}) + \epsilon \ \rightleftarrows \ \text{behavioral choices}} \quad \Longrightarrow \quad \boxed{\text{estimation of } \mathbb{E}}$$

Fig.1; $\mathbb{E}(\boldsymbol{\epsilon}) = (\mathbb{E}; \boldsymbol{\epsilon})$

We adopt the theory of *assignment markets* for the systematic part $\mathbb{E}$, which was initiated by Böhm-Bawerk [17] and developed by von Neumann-Morgenstern [21] and Shapley-Shubik [16]. In this theory, housings are treated as indivisible commodities, which significantly differs from the urban economics literature of bid-rent theory from Alonso [1][3]. In particular, we adopt a theoretical model given by Kaneko [7] in which income effects are allowed. In the model, apartments units are classified into a finite number, T, of *categories,* and are traded for rents measured by the composite commodity other than housing services.

1.2. Specific developments

Let us discuss specific developments of our theory. First, the *systematic* part of the housing market is summarized as:

$$\mathbb{E} = (M, u, I; N, C), \tag{1.1}$$

where M is the set of households, u their utility functions, I the income distribution for households, and N the set of landlords, C the cost functions for landlords. The details of (1.1) and the market equilibrium theory are given in Section 2.

[3] See van der Laan *et al.* [18] and its references for recent papers for the literature of assignment markets, and see Arnott [2] for a recent survey on the urban economics literature from Alonso [1].

In the systematic part, the rents are uniform over each category of apartments. However, the rents listed in housing magazines are not uniform over each category. Those non-uniform rents are resulted by *non-systematic factors* other than the components listed in (1.1). The effects of non-systematic factors are summarized by one random variable $\boldsymbol{\epsilon}_k$ for each category k. That is, the apartment rent for a unit d in category k is determined by $p_k + \boldsymbol{\epsilon}_{kd}$, where p_k is the competitive rent for category k and $\boldsymbol{\epsilon}_{kd}$ is an independent random variable identical to $\boldsymbol{\epsilon}_k$. This ϵ_{kd} represents properties of unit d such as its specific location in addition to the systematic components in $\mathbb{E}$. The rent p_k is *latent* in that only $p_k + \boldsymbol{\epsilon}_{kd}$ is observed in housing magazines. The market model with housing magazines is denoted by $\mathbb{E}(\boldsymbol{\epsilon}) = (\mathbb{E}; \boldsymbol{\epsilon})$.

As described above, the housing market model $\mathbb{E}(\boldsymbol{\epsilon})$ has two faces: it is purely the trading place with media for information transmissions; and it is a target of an econometric study. In both faces, housing magazines serve information about rents to households/landlords and to the econometric analyzer. Here, we emphasize that these two faces are asymmetric.

An economic agent pursues his utility or profit in the market, rather than to understand the market structure. If he looks at the average of the rents of randomly taken 10 apartment units from one category, its variance becomes 1/10 of the original distribution. Thus, the uniform rent assumption for each category seems to be an approximation. This interpretation will be expressed by the *convergence* theorem (see Theorem 3.2). Once this is obtained, we can use a housing market model $\mathbb{E}$ without errors as representing a market structure.

The econometric analyzer estimates the components in $\mathbb{E}$. Let Γ be some class of market models $\mathbb{E}$ so that each $\mathbb{E}$ in Γ has a *competitive rent vector* $p(\mathbb{E})$. He minimizes the total sum of square residuals $T_R(P_D, p(\mathbb{E}))$ from the observed data P_D to $p(\mathbb{E})$ by choosing $\mathbb{E}$ in Γ. This will be formulated in Section 4.

Here, we consider two specific choice problems:

A: a measure η of discrepancy between the data and predicted rent vector;

B: a candidate set of market models Γ.

For A, the discrepancy measure η is defined in terms of $T_R(P_D, p(\mathbb{E}))$ in Section 4, to describe how much the estimated result deviates from the optimal estimates. In our application to the data in Tokyo, the value of the measure will be shown to be $1.025 \sim 1.032$, i.e., $2.5\% \sim 3.2\%$ of the optimal estimates, by specifying certain classes of market models with homogeneous utility functions.

As an application, we examine the law of diminishing marginal utility for the household. It holds strictly with respect to, particularly, the consumption other than the housing services.

For B, we consider two classes of market models. We show the *Ex Post Rationalization Theorem* in Section 6 that we make the value of the discrepancy measure exactly 1 by choosing a certain set Γ of market models. However, this has no prediction power in that only after observations, we adjust a model to fit to the data, because this candidate set Γ has enough freedom. This means that a too general candidate set is meaningless for an econometric analysis. As a kind of opposite, we consider the standard linear regression in our equilibrium-econometric analysis. When the households have the common linear utility functions with respect to attributes of housing and consumption, our econometric analysis becomes linear regression, which is "too specific" in that income effects cannot be taken into account. The choice of an appropriate candidate set is subtle.

This chapter is organized as follows: In Section 2, the market equilibrium theory of Kaneko [7] is described together with the example from the Tokyo area. In Section 3, a market equilibrium theory with perturbed rents is discussed. In Section 4, statistical/econometric treatments are developed as well as a definition of the measure for discrepancy is defined. In Section 5, we apply those concepts to a data set from the Tokyo metropolitan area. In Section 6, we consider two classes of utility functions. Section 7 gives conclusions and concluding remarks.

2. Equilibrium Theory of Rental Housing Markets

In Section 2.1, we describe the market structure $\mathbb{E}$ of (1.1), and state the existence results of a competitive equilibrium in $\mathbb{E}$ due to Kaneko [7]. In Section 2.2, we describe a rental housing market in the Tokyo area.

2.1. Basic theory: the assignment market

The target situation is summarized as $\mathbb{E} = (M, u, I; N, C)$, where

M1: $M = \{1, ..., m\}$ - the set of *households*, and each $i \in M$ has a utility function u_i and an income $I_i > 0$ measured by the composite commodity other than housing services;

M2: $N = \{1, ..., T\}$ - the set of *landlords* and each $k \in N$ has a cost function C_k.

Each $i \in M$ looks for (at most) one unit of an apartment, and each $k \in N$ supplies some units of apartments to the market. The apartments are classified into *categories* $1, ..., T$. These categories of apartments are interpreted as potentially supplied. Multiple units in one category of apartments may be at the market. When no confusion is expected, we use the term "apartment" for either one unit or a category of apartments.

Each household $i \in M$ chooses a consumption bundle from the consumption set $X := \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\} \times \mathbf{R}_+$, where $\mathbf{e}^k$ is the unit T-vector with its k-th component 1 for $k = 1, ..., T$ and $\mathbf{R}_+$ is the set of nonnegative real numbers. We may write $\mathbf{e}^0$ for $\mathbf{0}$, meaning that he decides to rent no apartment. A typical element $(\mathbf{e}^k, m_i)$ means that household i rents one unit from the k-th category and enjoys the consumption $m_i = I_i - p_k$ after paying the rent p_k for $\mathbf{e}^k$ from his *income* $I_i > 0$.

The *initial endowment* of each household $i \in M$ is given as $(\mathbf{0}, I_i)$ with $I_i > 0$. His *utility function* $u_i : X \to \mathbf{R}$ is assumed to satisfy:

Assumption A (Continuity and Monotonicity): For each $x_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$, $u_i(x_i, m_i)$ is a continuous and strictly monotone function of m_i and $u_i(\mathbf{0}, I_i) > u_i(\mathbf{e}^k, 0)$ for $k = 1, ..., T$.

The last inequality, $u_i(\mathbf{0}, I_i) > u_i(\mathbf{e}^k, 0)$, means that going out of the market is preferred to renting an apartment by paying all his income.

Remark 1. *The emphasis of the model $\mathbb{E}$ is on the households and their behavior, rather than on the landlords. We simplify the descriptions of landlords: As long as competitive equilibrium is concerned, we can* assume without loss of generality *that only one landlord k provides all the apartments of category k (cf., Sai [15]). Still, he is a price-taker.*

By this remark, we assume that the set of landlords is given as $N = \{1, ..., T\}$, where only one landlord k provides the apartments of category k $(k = 1, ..., T)$. Each landlord k has a cost function $C_k(y_k) : \mathbf{Z}^*_+ \to \mathbf{R}_+$ with $C_k(0) = 0 < C_k(1)$, where $\mathbf{Z}^*_+ = \{0, 1, ..., z^*\}$ and z^* is an integer greater than the number of households m. The cost of providing y_k units is $C_k(y_k)$. No fixed costs are required when no units are provided to the market[4]. The finiteness of $\mathbf{Z}^*_+$ will be used only in Theorem 3.2.

We impose the following on the cost functions:

Assumption B (Convexity): For each landlord $k \in N$,

$$C_k(y_k + 1) - C_k(y_k) \leq C_k(y_k + 2) - C_k(y_k + 1) \text{ for all } y_k \in \mathbf{Z}^*_+ \text{ with } y_k \leq z^* - 2.$$

This means that the marginal cost of providing an additional unit is increasing.

We write the set of all economic models $\mathbb{E} = (M, u, I; N, C)$ satisfying Assumptions A and B by Γ_0.

Now, we define the concept of a competitive equilibrium in $\mathbb{E} = (M, u, I; N, C)$. Let (p, x, y) be a triple of $p \in \mathbf{R}^T_+$, $x \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}^m$ and $y \in (\mathbf{Z}^*_+)^T$. We say that (p, x, y) is a *competitive equilibrium* in $\mathbb{E}$ iff

UM(Utility Maximization Under the Budget Constraint): for all $i \in M$,
$I_i - px_i \geq 0$; and $u_i(x_i, I_i - px_i) \geq u_i(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ with $I_i - px'_i \geq 0$;

[4] The cost functions here should not be interpreted as measuring costs for building new apartments. In our rental housing market, the apartment units are already built and fixed. Therefore, $C_j(y_j)$ is the valuation of apartment units y_j below which he is not willing to rent y_j unit for the contract period. This will be clearer in the numerical example in Section 2.2.

PM(Profit Maximization): for all $k \in N$,

$p_k y_k - C_k(y_k) \geq p_k y'_k - C_k(y'_k)$ for all $y'_k \in \mathbf{Z}^*_+$;

BDS(Balance of the Total Demand and Supply): $\sum_{i \in M} x_i = \sum_{k=1}^{T} y_k \mathbf{e}^k$.

Note $px_i := \sum_{k=1}^{T} p_k x_{ik}$. These conditions constitute the standard notion of competitive equilibrium. Here, each agent maximizes his utility (or profits) as if he can observe all rents $p_1, ..., p_T$, and then the total demand and supply balance.

The above housing market model is a special case of Kaneko [7], where the existence of a competitive equilibrium is proved.

Theorem 2.1. ***(Existence)*** *In each $\mathbb{E} = (M, u, I;\ N, C)$ in Γ_0, there is a competitive equilibrium (p, x, y).*

A competitive equilibrium may not be unique, but we choose a particular competitive rent vector. We say that p is a *competitive rent vector* iff (p, x, y) is a competitive equilibrium for some x and y, and that $p = (p_1, ..., p_T)$ is a *maximum competitive rent vector* iff $p \geq p'$ for any competitive rent vector p'. By definition, a maximum competitive rent vector would be unique if it ever exists. We have the existence of a maximum competitive rent vector in $\mathbb{E} = (M, u, I; N, C)$. This fact has been known in slightly different models since the pioneering work of Shapley-Shubik [16] and Gale-Shapley [4]. Also, see Miyake [13].

Theorem 2.2. ***(Existence of a maximum competitive rent vector)*** *There is a maximum competitive rent vector in each $\mathbb{E} = (M, u, I; N, C)$ in Γ_0.*

We can define also a minimal competitive rent vector, but here we focus on the maximum one.

2.2. Application to a rental housing market in Tokyo (1)

Consider the JR (Japan Railway) Chuo line from Tokyo station in the west direction along which residential areas are spread out. See Fig.1. The line has 30 stations from Tokyo to Takao station, which is almost on the west boundary of the Tokyo great metropolitan area. Here, we consider only a submarket: we take six stations and three types of sizes for apartments. We explain how we formulate this market as a market model $\mathbb{E} = (M, u, I; N, C)$.

Look at Table 1. The first column shows the time distance from Tokyo to each station, i.e., 18, 23, 31, 52, 64, and 70 min. It is assumed that people commute to Tokyo station (office area) from their apartments. The first raw designates the sizes of apartments, and the three intervals are represented by the medians, 15, 35, and 55 m^2. Thus, the apartments are classified into $T = 6 \times 3 = 18$ categories.

Figure 2.1: Chuo Line

We assume that the households have the common *base utility function* $U^0(t, s, m_i)$, from which the utility function $u_i(x_i, m_i)$ in the previous sense follows: it is given as

$$U^0(t, s, m_i) = -2.2t + 4.0s + 100\sqrt{m_i}, \tag{2.1}$$

where t is the time distance, 18, 23, 31, 52, 64, or 70 (minutes), s is the size 15, 35, or 55 (m^2), and m_i is the consumption after paying the rent. A pair (t, s) determines a category. By calculating the first part $-2.2t + 4.0s$ of $U^0(t, s, c)$, we obtain h_k for the corresponding cell of Table 1. These h_k's give the ordering over the 18 categories: For example, $-2.2t + 4.0s$ takes the largest value at $(t, s) = (18, 55)$; we label $k = 1$ to the category of $(t, s) = (18, 55)$. Similarly, it takes the 7-th value at $(t, s) = (64, 55)$, and thus $k = 7$. We have the correspondence $\lambda_0(t, s) = k$ from (t, s)'s to k's. We call λ_0 the *category function.*

Now, we define the utility function $u : X = \{\mathbf{e}^0, \mathbf{e}^1, ..., \mathbf{e}^{18}\} \times \mathbf{R}_+ \to \mathbf{R}$ by

$$u(\mathbf{e}^k, m_i) = h_k + 100\sqrt{m_i}, \tag{2.2}$$

where $\lambda_0(t, s) = k$ and $h_k = -2.2t{+}4.0s$ for $k \geq 1$ and h_0 is chosen so that $h_0{+}100\sqrt{I_m} > h_1$. The derived utility function in (2.2) satisfies Assumption A. The concavity of $100\sqrt{m_i}$ expresses the law of diminishing marginal utility of consumption.

The third entry w_k of category k in Table 1 is the number of units listed for sale in housing magazines; particularly, *the Yahoo Real Estate* (15, June 2005). The largest number of supplied units is $w_{11} = 1176$ for the smallest apartments in the Nakano area, and the smallest number is $w_9 = 102$ for the largest apartments in the Takao area. The total number of apartment units on the market is $\sum_{k=1}^{18} w_k = 8957$. These large numbers will be important for statistical treatments in subsequent sections.

Table 1: Basic data for the rental housing market

k	h_k	w_k

time (min) \ **size** (m^2)	< 25			$25 - 45$			$45 - 65$		
18:Nakano	11	20.4	1176	5	100.4	761	1	180.4	269
23:Ogikubo	12	9.4	1153	6	89.4	739	2	169.4	367
31:Mitaka	14	-8.2	716	8	71.8	571	3	151.8	267
52:Tachikawa	16	-54.4	460	10	25.6	283	4	105.6	260
64:Hachio-ji	17	-80.8	1095	13	-0.8	346	7	79.2	184
70:Takao	18	-94.0	103	15	-14.0	105	9	66.0	102

We *assume* that the same number, $m = 8957$, of households come to the market to look for apartments and they rent all the units.

To determine a competitive equilibrium, we separate between the cost functions for $k = 1, ..., T-1$ and $k = T$. For $k = 1, ..., T-1$, we define the cost function $C_k(y_k)$ as:

$$C_k(y_k) = \begin{cases} c_k y_k & \text{if } y_k \leq w_k \\ \text{``large''} & \text{if } y_k > w_k, \end{cases} \tag{2.3}$$

where $c_k > 0$ for $k = 1, ..., T-1$ and "large" is a number greater than I_1. Thus, only the supplied units are in the scope of cost functions. For $k = T$, we assume that more units are waiting for the market. Let w_T^0 be an integer with $w_T^0 > w_T$. We define $C_T(y_T)$ by (2.3) with $c_T > 0$ and substitution of w_T^0 for w_k. Hence, the market rent for an apartment in category T must be c_T. This satisfies Assumption B.

For calculation of the maximum competitive rent vector, we take $c_{18} = 48.0$ and $c_1, ..., c_{17}$ are "small" in the sense that all the w_k units are supplied at the competitive rents for $k = 1, ..., 17$. The cost 48,000 yen is about the average rents of the smallest category in Takao around in 2005.

Finally, we assume that the (monthly) income distribution $I = (I_1, ..., I_{8957})$ over $M = \{1, ..., 8957\}$ is uniform from 100,000 yen to 850,000 yen. Hence, $I_{8957} = 100,000$ and $I_1 = 850,000$. In fact, this uniform distribution is just for the purpose of calculation, and can be changed into other distributions[5].

[5] At this stage, the result is not sensitive with the uniform distribution assumption, i.e., if we change it to a truncated normal distribution, the calculated rents are not much changed. However, in the later calculation in Section 5, a change of this assumption seems to affect the result.

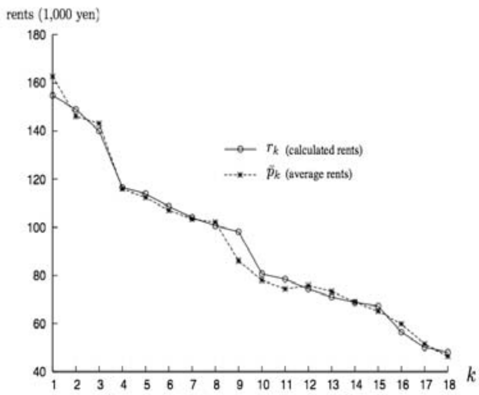

Figure 2.2: Calculated and average prices

Under the above specification of $\mathbb{E} = (M, u, I; N, C)$, we can calculate the maximum competitive rent vector $p = (p_1, ..., p_{18})$, which is given in Table 2. The average rents $\overline{p} = (\overline{p}_1, ..., \overline{p}_{18})$ as well as the standard deviations $(s_1, ..., s_T)$ from *the Yahoo Real Estate* are given. Fig.2 depicts the average rents $\overline{p} = (\overline{p}_1, ..., \overline{p}_{18})$ from the data of as well as $p = (p_1, ..., p_{18})$.

In Section 4.1, we will define the discrepancy measure in order to consider how much the calculated rent vector $p = (p_1, ..., p_T)$ fits the data from housing magazines. For the present data, the value is about 1.032, i.e., the discrepancy is 3.2%.

Table 2: Calculated and average rents

k	p_k	$\bar{p}_k$	s_k	(1,000yen)

time (min) \ **size** (m^2)	< 25				$25 - 45$				$45 - 65$			
18:Nakano	11	78.5	74.4	12.7	5	113.9	112.5	23.8	1	154.8	162.7	26.7
23:Ogikubo	12	74.3	75.8	13.6	6	108.6	107.0	23.1	2	149.0	146.2	20.9
31:Mitaka	14	68.7	68.9	9.8	8	110.6	102.1	21.2	3	140.0	143.1	21.6
52:Tachikawa	16	56.4	59.8	11.0	10	80.7	78.1	12.5	4	116.6	116.0	16.5
64:Hachio-ji	17	50.0	51.5	7.5	13	71.0	73.3	11.3	7	104.0	103.5	17.9
70:Takao	18	48.0	46.4	5.9	15	67.2	65.1	9.6	9	98.1	86.1	11.3

3. Rental Housing Markets with Housing Magazines

In a competitive equilibrium in $\mathbb{E} = (M, u, I; N, C)$, all the apartment units in each category are uniformly priced, but in reality, rents are not uniform. This non-uniformity represents the effects of non-systematic factors. Here, we modify a housing market model by taking non-systematic factors into account. We show that the market model $\mathbb{E}$ can still be used as an analytic tool for the markets with non-systematic factors.

3.1. Time structure of the rental housing market

Since our approach is a static equilibrium theory, we do not need time indices. However, it would be easier first to describe the economy with the time structure for the consideration of decision making with housing magazines. We use time indices only for this explanation.

The market is recurrent and is described using the "week" due to Hicks [5] in Fig.4. In week t, market $\mathbb{E}^t(\boldsymbol{\epsilon}^t) = (\mathbb{E}^t; \boldsymbol{\epsilon}^t)$ has, in addition to the systematic part $\mathbb{E}^t = (M^t, u^t, I^t; N^t, C^t)$, a perturbation term $\boldsymbol{\epsilon}^t = (\boldsymbol{\epsilon}_1^t, ..., \boldsymbol{\epsilon}_T^t)$ as the summary of non-systematic factors.

$$\cdots \longrightarrow \boxed{\begin{array}{c}\mathbb{E}^{t-1}(\boldsymbol{\epsilon}^{t-1})\\ \text{week } t-1\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbb{E}^{t}(\boldsymbol{\epsilon}^{t})\\ \text{week } t\end{array}} \longrightarrow \cdots$$

Diagram 2: weekly markets

Interactions between information from housing magazines and decision making by households/landlords have a complex temporal structure. For logical clarity, here we simplify the story in the following manner. Before going to the market of week t, households M^t look at housing magazines of week $t-1$ and decide which category they go to. Landlords N^t decide to a supply quantity, also looking at the same housing magazine (recall Remark 1). Decision making by each household is only about category choice; and decision making by each landlord is only about supply quantity choice. Then, households M^t and landlords N^t go to the market of week t, and there they trade apartment units (no explicit decision making is considered here in our theory) and disappear from the market.

In $\mathbb{E}^t(\boldsymbol{\epsilon}^t)$, the rental prices are realized with error term $\boldsymbol{\epsilon}^t$. This $\boldsymbol{\epsilon}^t$ is a T-vector of independent random variables which perturb the market rents p_k^t for apartments in category $k = 1, ..., T$ to $p_k^t + \boldsymbol{\epsilon}_k^t$. However, when apartment unit d in category k is provided, the error term applied to the unit d is $\boldsymbol{\epsilon}_{kd}^t$. Here, it is assumed that $\boldsymbol{\epsilon}_{kd}^t$ is independently and identically distributed as $\boldsymbol{\epsilon}_k^t$ over those units in category k.

To distinguish between random variables and their realizations, we prepare the underlying probability space $(\Omega, \mathcal{F}, \mu)$ which all the random variables in this paper follow.

In week $t-1$, apartment units of categories $1, ..., T$ were brought to the housing market. Let $D_1^{t-1}, ..., D_T^{t-1}$ are the (finite nonempty) sets of those units. Each unit d in D_k^{t-1} is listed in housing magazines with its realized rent $p_k^{t-1} + \boldsymbol{\epsilon}_{kd}^{t-1}(\omega_o^{t-1})$, where ω_o^{t-1} is the realized value of the state of nature. The entire *housing magazine of week* $t-1$ is described as

$$\left\{p_1^{t-1} + \boldsymbol{\epsilon}_{1d}^{t-1}(\omega_o^{t-1}) : d \in D_1^{t-1}\right\} \quad \cdots \quad \left\{p_T^{t-1} + \boldsymbol{\epsilon}_{Td}^{t-1}(\omega_o^{t-1}) : d \in D_T^{t-1}\right\}. \tag{3.1}$$

Each household $i \in M^t$ looks at the housing magazines (3.1) of week $t-1$, and then forms an estimate of the rent distribution:

$$P_k^{i,t} = p_k^{t-1} + \boldsymbol{\epsilon}_k^{i,t} \text{ for each } k = 1, ..., T. \tag{3.2}$$

In general, $P_k^{i,t}(\omega^t) = p_k^{t-1} + \boldsymbol{\epsilon}_k^{i,t}(\omega^t)$ is a random variable for each k; possibly, it may be degenerated such as $P_k^{i,t}(\omega^t) = p_k^{t-1} + \boldsymbol{\epsilon}_{kd}^{t-1}(\omega_o^{t-1})$ given by the observation of a particular unit d. Household i makes a choice of a category by looking at his rent estimator in (3.2). That is, he maximizes the expected utility (subject to the budget constraint) relative to this rent expectation.

Each landlord k $(k = 1, ..., T)$ decides the supply quantity of apartment units in category k based on his estimate $p_k^{t-1} + \boldsymbol{\epsilon}_k^{k,t}$ of the rents of apartments in category k.

3.2. Equilibrium with subjective estimates

Assuming that the market is stationary and $P_k^{i,t}$, as a random variable, is independent of week t, we drop the superscript t from $\mathbb{E}^t(\boldsymbol{\epsilon}^t)$ and $P_k^{i,t}$. The economy where the households and landlords have their estimators are denoted by $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M\cup N}) = (\mathbb{E}(\boldsymbol{\epsilon}); \boldsymbol{\epsilon}^{M\cup N})$, where $\boldsymbol{\epsilon}^{M\cup N} = (\{\boldsymbol{\epsilon}^i\}_{i\in M}, \{\boldsymbol{\epsilon}^k\}_{k\in N})$. Thus, each household $i \in M$ has his own subjective estimate $P_k^i = p_k + \boldsymbol{\epsilon}_k^i$ in each $k = 1, ..., T$, and each landlord $k \in N$ has the rent estimate $P_k^k = p_k + \boldsymbol{\epsilon}_k^k$. We assume that these rent estimates do not take negative values:

$$P_k^i(\omega) \geq 0 \text{ and } P_k^k(\omega) \geq 0 \text{ for all } \omega \in \Omega. \tag{3.3}$$

Since the realization ω_o^{t-1} of the previous week differs typically from the realization ω_o^t of the present week, we should distinguish between the realization of the previous week and the present one. We still use the symbol ω_o^{t-1} for the previous week, and the symbol ω without the time index for the present week.

We give two examples for such subjective rent estimates.

Example 3.1. ***(Average rents)*** *Looking at the housing magazine (3.1), household i (landlord k) takes some samples of rents from category k. Let L_i be the samples taken. Then, if he uses the average of the observed rents, he has a single-value for estimate:*

$$P_k^i(\omega) = \sum_{d\in L_k} (p_k + \boldsymbol{\epsilon}_{kd}(\omega_o^{t-1})) / |L_k|, \tag{3.4}$$

which is independent of ω. On the other hand, if he is more careful and take some uncertainty about rents into account, he could have the random average P_k^i :

$$P_k^i(\omega) = \sum_{d \in L_k} (p_k + \boldsymbol{\epsilon}_{kd}(\omega)) / |L_k| . \tag{3.5}$$

This exact form must be very rare. The point of this example is: The number of samples $|L_k|$ is typically small such as $5 \sim 25$. In the second case, since $\boldsymbol{\epsilon}_{kd}$ are independent random variables identical to $\boldsymbol{\epsilon}_k$ for $d \in D_k$, the expected value is $E(P_k^i) = p_k + E(\boldsymbol{\epsilon}_k)$ and its variance is $E(P_k^i - E(P_k^i))^2 = E(\boldsymbol{\epsilon}_k - E(\boldsymbol{\epsilon}_k))^2 / |L_k|$. Thus, the variance is reciprocal to the number of samples.

The above examples suggest that the economic agents may take rents with smaller variances than the actual variances of $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1, ..., \boldsymbol{\epsilon}_T)$. If household i (landlord k) very carefully scrutinizes the housing magazines by drawing a histogram. Since the number of units listed in the magazine is quite large, it is close to the true $P_k^i = p_k + \boldsymbol{\epsilon}_k^i = p_k + \boldsymbol{\epsilon}_k$. Since, however, the magazine is quite large and not well-organized, it is costly to extract the distribution $p_k + \boldsymbol{\epsilon}_k(\cdot)$. Instead, often, the information publicly used is the average rent of samples; the examples of (3.4) and (3.5) are better fitting to reality.

In the model $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M \cup N})$, the concept of a competitive equilibrium is adjusted by incorporating each agent's rent estimation. We, first, take this estimation into account in utility maximization for each household, and then we formulate a landlord's profit maximization.

To capture the budget constraint for household i with estimation $P^i = (P_1^i, ..., P_T^i)$, we define the following utility function: for $x_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ and $\omega \in \Omega$,

$$U_i(x_i, I_i - P^i(\omega) \cdot x_i) = \begin{cases} u_i(x_i, I_i - P^i(\omega) \cdot x_i) & \text{if } 0 \leq I_i - P^i(\omega) \cdot x_i \\ \\ u_i(\mathbf{0}, I_i) & \text{otherwise.} \end{cases} \tag{3.6}$$

In the second case, his budget is violated; so, no trade occurs. In general, this utility function $U_i(x_i, I_i - P^i(\cdot) \cdot x_i)$ is a random variable. We define the expected utility before going to a category:

$$EU_i(x_i, I_i - P^i \cdot x_i) = \int_{\omega \in \Omega} U_i(x_i, I_i - P^i(\omega) \cdot x_i) d\mu(\omega). \tag{3.7}$$

He chooses a category by maximizing this expected utility function over $\{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$.

We assume that each landlord k has a *risk-neutral utility function.* Then his expected utility is calculated as the expected payoff:

$$E(y_k P_k^k - C_k(y_k)) = y_k E(P_k^k) - C_k(y_k). \tag{3.8}$$

If $E(\boldsymbol{\epsilon}_k^k) = 0$, i.e., $E(P_k^k) = p_k$, (3.8) becomes simply the profit function $y_k p_k - C_k(y_k)$. However, we treat landlords in the same way as households in that he may construct his rent estimate P_k^k without assuming $E(\boldsymbol{\epsilon}_k^k) = 0$.

In the housing market $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M\cup N})$, a competitive equilibrium is simply defined by substituting the objective functions (3.7) and (3.8) for the utility functions and profit functions in UM and PM. Nevertheless, we need to take two approximations: a γ-competitive equilibrium and a convergent sequence of rent estimates.

Let γ be a nonnegative real number. We call (p, x, y) is a *γ-competitive equilibrium* $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M\cup N})$ when the following two conditions and the BDS condition, $\sum_{i\in M} x_i = \sum_{k=1}^{T} y_k \mathbf{e}^k$, hold:

γ-Expected Utility Maximization: for all household $i \in M$,

$$EU_i(x_i, I_i - P^i \cdot x_i) + \gamma \geq EU_i(x_i', I - P^i \cdot x_i') \text{ for all } x_i' \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}.$$

γ-Expected Profit Maximization: for all landlord $k = 1, ..., T$,

$$E(P_k^k y_k - C_k(y_k)) + \gamma \geq E(P_k^k y_k' - C_k(y_k')) \text{ for all } y_k' \in \mathbf{Z}_+^*.$$

The other notion is that $\boldsymbol{\epsilon}^{M\cup N}$ is "small perturbations". To describe this, we introduce the convergence of the vectors of estimators $\boldsymbol{\epsilon}^{M\cup N}$. We say that an error sequence $\{\boldsymbol{\epsilon}^{M\cup N,\nu} : \nu = 1, ...\} = \{(\{\boldsymbol{\epsilon}^{i,\nu}\}_{i\in M}, \{\boldsymbol{\epsilon}^{k,\nu}\}_{k\in N}) : \nu = 1, ...\}$ is *convergent to 0 in probability* iff for any $\delta > 0$,

$$\mu(\{\omega : \max_{j\in M\cup N} \left\| \boldsymbol{\epsilon}^{j,\nu}(\omega) \right\| < \delta\}) \to 1 \text{ as } \nu \to +\infty, \tag{3.9}$$

where $\|\cdot\|$ is the max-norm $\|(y_1, ..., y_T)\| = \max_{1\leq t\leq T} |y_t|$. This mean that when ν is large enough, the estimation $\boldsymbol{\epsilon}^{j,\nu}(\omega)$ is distributed closely to $\mathbf{0}$.

We have the following theorem. The proof will be given in Section 8.

Theorem 3.2. ***(Convergence to $\mathbb{E}$)*** *Suppose that the sequence of estimation errors $\{\boldsymbol{\epsilon}^{M\cup N,\nu} : \nu = 1, ...\}$ is convergent to 0 in probability.*

(1): *If (p, x, y) be a competitive equilibrium in $\mathbb{E}$, then for any $\gamma > 0$, there is a ν_o such that for any $\nu \geq \nu_o$, (p, x, y) is a γ-competitive equilibrium in $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M\cup N,\nu})$.*

(2): *Suppose that a triple (p, x, y) satisfies $px^i < I_i$ for all $i \in M$. Then, the converse of (1) holds.*

When the rent expectation for landlord $k \in N$ satisfies $E(\boldsymbol{\epsilon}^k) = 0$, his expected profit is simply given as the profit function, and so we need to consider neither the convergent sequence nor the γ-modification for landlord k. Also, if a competitive equilibrium (p, x, y)

is strict in the sense that a household (landlord) maximizes his utility at a unique choice, then we do not need the γ-modification for the household (landlord).

The convergence condition is interpreted as meaning that the subjective rent expectation of each household (landlord) has a small variance. Theorem 3.2 states that when each household i (landlord j) has his rent expectation $\boldsymbol{\epsilon}^i$ (or $\boldsymbol{\epsilon}^j$) with a small variance, his utility maximization (or profit maximization) in the idealized market $\mathbb{E}$ is preserved approximately in the market $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M \cup N, \nu})$ for large ν, and *vice versa*. Thus, the competitive equilibrium in $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M \cup N, \nu})$ can well be represented by one in $\mathbb{E}$.

4. Statistical Analysis of Rental Housing Markets

We turn our attention to estimation of the structures of the rental housing market from the data given in housing magazines. In Section 4.1 we develop various concepts to connect the data with possible market models and to evaluate such a connection. In Section 4.2 we specify a class of market models for estimation.

4.1. Estimation of the market structure

We denote, by $\mathbb{E}^o(\boldsymbol{\epsilon}^o) = (\mathbb{E}^o; \boldsymbol{\epsilon}^o)$, the true market, to distinguish between $\mathbb{E}^o(\boldsymbol{\epsilon}^o)$ and an estimated $\mathbb{E}$. We call $\mathbb{E}^o = (M^o, u^o, I^o; N^o, C^o)$ the *latent* true market structure. We assume that this $\mathbb{E}^o$ satisfies Assumptions A and B of Section 2, i.e., $\mathbb{E}^o \in \Gamma_0$. The maximum competitive rent vector $p^o = (p_1^o, ..., p_T^o)$ of $\mathbb{E}^o$ is called the *latent market rent vector*. Let D_k^o be a nonempty set of apartment units listed in category $k = 1, ..., T$. Once the perturbation term $\boldsymbol{\epsilon}^o$ is realized at $\omega \in \Omega$ for each $d \in D_k^o, k = 1, ..., T$, we have the housing magazines $\{P_{1d}^o(\omega) : d \in D_1^o\}, ..., \{P_{Td}^o(\omega) : d \in D_T^o\}$. Here, ω is not fixed to be a specific ω_o. The listed rent for each unit $d \in D_k^o$, $k = 1, ..., T$ is given as: $P_{kd}^o(\omega) = p_k^o + \boldsymbol{\epsilon}_{kd}^o(\omega)$. We estimate components of $\mathbb{E}^o$ from the housing magazines; in this paper, specifically, we estimate the utility functions of households.

Let $p = (p_1, ..., p_T)$ be a rent vector in $\mathbb{R}^T$, which is intended to be an estimated one. Then, the *total sum of square residuals* $T_R(P_D^o(\omega), p)$ is given as

$$T_R(P_D^o(\omega), p) = \sum_{k=1}^{T} \sum_{d \in D_k^o} (P_{kd}^o(\omega) - p_k)^2. \tag{4.1}$$

This is the distance between the data and estimated rent vector.

Let Γ be a subset of Γ_0 where T is fixed and $M = \{1, ..., m\}$ is determined by $m = \sum_{k=1}^{T} |D_k^o|$. Our problem is to choose $\mathbb{E} = (M, u, I; N, C)$ to minimize $T_R(P_D^o(\omega), p(\mathbb{E}))$ in Γ, where $p(\mathbb{E})$ is the maximum competitive rent vector in $\mathbb{E}$. We write our problem explicitly:

Definition 4.1. *(*Γ*-**MSE**) We choose a model* $\mathbb{E}$ *from* Γ *to minimize* $T_R(P_D^o(\omega),\, p(\mathbb{E}))$ *subject to the condition:*

$(*)$: $(p(\mathbb{E}), x, y)$ *is a maximum competitive equilibrium in* $\mathbb{E}$ *for some* (x, y) *with* $y_k = |D_k^o|$ *for* $k = 1, ..., T$.

The additional condition $y_k = |D_k^o|$ for $k = 1, ..., T$ requires $(p(\mathbb{E}), x, y)$ to be compatible with the number of apartment units listed in the housing magazines.

If the latent true structure $\mathbb{E}^o$ belongs to Γ, it is a candidate for the solution of the Γ-MSE. However, we do not know whether or not $\mathbb{E}^o$ belongs to Γ. A simple idea is to choose a large class for Γ to guarantee that $\mathbb{E}^o$ could be in Γ. In fact, this idea does not work well: in Section 6.1, we discuss the negative result for this; we should somehow look at a narrower class for Γ.

As the benchmark, we consider the *average rent estimator*: given the housing magazines $P_D^o = \{P_{kd}^o : d \in D_k^o \text{ and } k = 1, ..., T\}$, we define $\overline{P}^o = (\overline{P}_1^o, ..., \overline{P}_T^o)$ by

$$\overline{P}_k^o(\omega) = \frac{\sum_{d \in D_k^o} P_{kd}^o(\omega)}{|D_k^o|} \text{ for each } \omega \in \Omega \text{ and } k = 1, ..., T. \tag{4.2}$$

This is the best estimator of the latent market rents $p^o = (p_1^o, ..., p_T^o)$. Each realization $\overline{P}^o(\omega)$ $(\omega \in \Omega)$ is the unique minimizer of $T_R(P_D^o(\omega), p)$ with no constraints. When $E(\boldsymbol{\epsilon}_k^o) = 0$, $\overline{P}_k^o$ is an unbiased estimator of p_k^o.

Lemma 4.2. ***(1)*** *For each* $\omega \in \Omega$, $T_R(P_D^o(\omega), \overline{P}^o(\omega)) \leq T_R(P_D^o(\omega), p)$ *for any* $p = (p_1, ..., p_T) \in \mathbf{R}^T$.

(2) *When* $E(\boldsymbol{\epsilon}_k^o) = 0$, $\overline{P}_k^o$ *is an unbiased estimator of* p_k^o*, i.e.,* $E(\overline{P}_k^o) = p_k^o$.

Proof.(1) *Let* $\omega \in \Omega$ *be fixed. Since* $T_R(P_D^o(\omega), p)$ *is a strictly convex function of* $p = (p_1, ..., p_T) \in \mathbf{R}^T$*, the necessary and sufficient condition for* p *to be the minimizer of* $T_R(P_D^o(\omega), p)$ *is given as* $\partial T_E(P_D(\omega), p)/\partial p_k = 0$ *for all* $k = 1, ..., T$*. Only the average* $\overline{P}^o(\omega) = (\overline{P}_1^o(\omega), ..., \overline{P}_T^o(\omega))$ *satisfies this condition.*

(2) *Since* $\boldsymbol{\epsilon}_{kd}^o$ *is identical to* $\boldsymbol{\epsilon}_k^o$ *for all* $d \in D_k^o$ *and* $E(\boldsymbol{\epsilon}_k^o) = 0$*, we have* $E(\boldsymbol{\epsilon}_{kd}^o) = 0$ *for all* $d \in D_k^o$*. Hence* $E(\overline{P}_k^o) = \sum_{d \in D_k^o} E(P_{kd}^o)/\,|D_k^o| = \sum_{d \in D_k^o}(p_k^o + E(\boldsymbol{\epsilon}_{kd}^o))/\,|D_k^o| = p_k^o$. ■

The estimator $\overline{P}^o$ enjoys various desired properties such as consistency (i.e., convergence to the latent market rent vector p^o in probability as $\min_k |D_k|$ tends to infinity) and efficiency in the sense of Cramer-Rao. For these, see van der Vaart [20].

We have the decomposition of the total sum of square residuals, which corresponds to the well-known decomposition property in the regression model (cf., Wooldridge [19]). This will be a base for our further analysis.

Lemma 4.3. ***(Decomposition)*** *For each $\omega \in \Omega$,*

$$T_R(P_D^o(\omega), p) = T_R(P_D^o(\omega), \overline{P}^o(\omega)) + \sum_{k=1}^{T} |D_k^o| \, (\overline{P}_k^o(\omega) - p_k)^2. \tag{4.3}$$

Proof *The term $\sum_{d \in D_k^o} (P_{kd}^o(\omega) - p_k)^2$ of $T_R(P_D^o(\omega), p)$ for each k is transformed to:*

$$\sum_{d \in D_k^o} (P_{kd}^o(\omega) - \overline{P}_k^o(\omega) + \overline{P}_k^o(\omega) - p_k)^2 = \sum_{d \in D_k^o} (P_{kd}^o(\omega) - \overline{P}_k^o(\omega))^2$$
$$+ \sum_{d \in D_k^o} 2(P_{kd}^o(\omega) - \overline{P}_k^o(\omega)) \cdot (\overline{P}_k^o(\omega) - p_k) + \sum_{d \in D_k^o} (\overline{P}_k^o(\omega) - p_k)^2.$$

The second term of the last expression vanishes by (4.2). The third is written as $|D_k^o| \, (\overline{P}_k^o(\omega) - p_k)^2$. We have (4.3) by summing these over $k = 1, ..., T$. ■

The first term of (4.3) is the residual between the data and the averages (optimal estimates) of rents. The second is the total sum of the differences between the average $\overline{P}^o(\omega)$ and p, and this is newly generated by the estimates $p = (p_1, ..., p_T)$. We call the ratio

$$\eta(p)(\omega) = \frac{T_R(P_D^o(\omega), p)}{T_R(P_D^o(\omega), \overline{P}^o(\omega))} = 1 + \frac{\sum_{k=1}^{T} |D_k^o| \, (\overline{P}_k^o(\omega) - p_k)^2}{T_R(P_D^o(\omega), \overline{P}^o(\omega))} \tag{4.4}$$

the *discrepancy measure* of p from of $\overline{P}^o(\omega)$. The second is the *theoretical discrepancy*, relative to the smallest total sum of residuals from $\overline{P}^o(\omega)$. In the example of Section 2.2, $\eta \doteqdot 1.032$ (denoted by η^0), i.e., the theoretical discrepancy is only 3.2%[6].

4.2. Subclass Γ_{sep} of Γ_0

We estimate the utility functions $u^o = (u_1^o, ..., u_m^o)$ in $\mathbb{E}^o = (M^o, u^o, I^o; N^o, C^o)$ from the rents listed in the housing magazines, assuming that the other components in $\mathbb{E}^o = (M^o, u^o, I^o; N^o, C^o)$ are given from the other information included in the housing magazines. For example, the set of households M^o is given as $\{1, ..., m\}$, where m is the cardinality of the data set $D^o = \cup_{k=1}^{T} D_k^o$.

The set of market models Γ_{sep} consists of $\mathbb{E} = (M, u, I; N, C)$ satisfying the following three conditions:

S1: The incomes of households are ordered as $I_1 \geq ... \geq I_m > 0$.

[6] Incidentally, in the present context, the *coefficient of determination* is defined as $\sum_{k=1}^{T} |D_k^o| \, (\overline{P}_k^o(\omega) - \overline{\overline{P}}^o(\omega))^2 / T_V(P_D^o(\omega), \overline{\overline{P}}^o(\omega))$, where $\overline{\overline{P}}^o(\omega)$ is the entire average of P_D^o. It indicates how much the systematic factors explain the observed rental prices. In the above example, the coefficient is approximately 0.757.

S2: Every household in M has the same utility function $u_1 = ... = u_m$ expressed as

$$u(\mathbf{e}^k, m_i) = h_k + g(m_i) \text{ for all } (\mathbf{e}^k, m_i) \in X, \tag{4.5}$$

where $h_0, h_1, ..., h_T$ are given real numbers with $h_k > h_0$ for $k = 1, ..., T$ and $g : \mathbf{R}_+ \to \mathbf{R}$ is an increasing and continuous concave function with $g(m_i) \to +\infty$ as $m_i \to +\infty$ and $h_0 + g(I_i) > h_k + g(0)$ for $k = 1, ..., T$.

S3: Each landlord $k = 1, ..., T$ has a cost function of the form (2.3).

In S1, the households are ordered by their incomes. Condition S2 has two parts: Every household has the same utility function; and the utility function is expressed in the separable form. The former part is interpreted as requiring the households to have the same location of their offices. The latter still allows the law of diminishing marginal utility over consumption, i.e., $g(m_i)$ may be strictly concave. Condition S3 is for simplification: Our theory emphasizes on the households' side.

The set Γ_{sep} may be regarded as very narrow from the viewpoint of mathematical economics in that the households have the same utility functions and the landlords' cost functions are also very specific. However, we will show in Section 6.1 that the class Γ_{sep} is still too large in that the estimated model has no prediction power. Thus, we will consider a narrower class for Γ.

A method of calculating a maximum competitive equilibrium (p, x, y) in $\mathbb{E} = (M, u, I; N, C)$ was given in Kaneko [8] and Kaneko *et al.* [10]. This method is used to implement our econometrics. Here, we describe this method without a proof.

Consider a rent vector $p = (p_1, ..., p_T)$ with $p_1 \geq ... \geq p_T > 0$. This is obtained by renaming $1, ..., T$. Then, we regard the units in category 1 as the best, and will suppose that the richest households $1, ..., |D_1^o|$ rent them. Similarly, the units in category 2 are the second best and the second richest households $|D_1^o| + 1, ..., |D_1^o| + |D_2^o|$ rent them. In general, defining

$$G(k) = \sum_{t=1}^{k} |D_t^o| \text{ for all } k = 1, ..., T, \tag{4.6}$$

we suppose the households $G(k-1)+1, ..., G(k)$ rent units in category k. We focus on the boundary households $G(1), G(2), ..., G(T-1)$ and their incomes $I_{G(1)}, I_{G(2)}, ..., I_{G(T-1)}$.

We have the following lemma due to Kaneko [8] and Kaneko, *et al.* [10]. Our econometric calculation is based on this lemma.

Lemma 4.4. ***(Rent Equations)*** *Consider a vector* $(p_1, ..., p_T)$ *with* $p_1 \geq ... \geq p_T > 0$. *Let* $\mathbb{E} = (M, u, I; N, C) \in \Gamma_{sep}$ *satisfying*

(1)*:* $p_k \leq I_{G(k)}$ *for all* $k = 1, ..., T - 1$*;*

(2)*:* $c_k \leq p_k$ *and* $w_k = |D_k^o|$ *for all* $k = 1, ..., T$*,* *where c_k is the marginal cost given in (2.3). Suppose also that $(p_1, ..., p_T)$ satisfies*

$h_{T-1} + g(I_{G(T-1)} - p_{T-1}) = h_T + g(I_{G(T-1)} - p_T)$
$h_{T-2} + g(I_{G(T-2)} - p_{T-2}) = h_{T-1} + g(I_{G(T-2)} - p_{T-1})$
$\cdot \quad \cdot \quad \cdot$
$h_1 + g(I_{G(1)} - p_1) = h_2 + g(I_{G(1)} - p_2).$

(4.7)

Then, there is an allocation (x, y) such that (p, x, y) is a maximum competitive equilibrium in $\mathbb{E}$ with $y_k = |D_k^o|$ for all $k = 1, ..., T$.

In (4.7), the boundary household $G(T-1)$ compares his utility $h_{T-1} + g(I_{G(T-1)} - p_{T-1})$ from staying in a unit in category $T-1$ with the utility $h_T + g(I_{G(T-1)} - p_T)$ obtained by switching to category T. Also, the household $G(T-2)$ makes a parallel comparison between $h_{T-2} + g(I_{G(T-2)} - p_{T-2})$ and $h_{T-1} + g(I_{G(T-2)} - p_{T-1})$, and so on. The logic of this argument is essentially the same as Ricardo's [14] differential rents. The rent $p_T = c_T$ in the worst category T is regarded as the land rent-cost of farm lands, which corresponds to Ricardo's absolute rent.

5. Application to the Market in Tokyo (2)

Here, we apply our equilibrium-econometric analysis to the rental housing market in Tokyo described in Section 2.2. First, we give a simple heuristic discussion on our application, and then give a more systematic study of it.

5.1. Heuristic discussion

For a study of a specific target, we consider a more concrete class for Γ than the class Γ_{sep} given in Section 4.2. In Section 2.2, we used a specific form of the base utility function $U^0(t, s, m_i) = -2.2t + 4.0s + 100\sqrt{m_i}$ and obtained the resulting value of the discrepancy measure, $\eta^0 = 1.032$. Perhaps, we should explain how we have found it and how good it is relative to others.

Let us compare several other base utility functions with (2.1):

$U^1(t, s, m_i) = -t + s + 100\sqrt{m_i}$	$\eta^1 = 3.259$
$U^2(t, s, m_i) = -2t + 255\sqrt{s + 1000} + 100\sqrt{m_i}$	$\eta^2 = 1.036$
$U^3(t, s, m_i) = -74t + 165s + 100m_i$	$\eta^3 = 1.124.$

(5.1)

With U^1, the discrepancy measure η takes large value 3.259. Thus, the total sum of square residuals from the estimated rents is more than the three-times of that from the average rents. With U^2, the value of η is already almost as small as 1.032 given by (2.1). With U^3, it is larger than this value, but U^3 is entirely linear. The law of diminishing marginal utility does not hold.

The case of U^1 tells that if coefficients are arbitrarily chosen, the discrepancy value could be large. On the other hand, U^0 is chosen by minimizing the discrepancy measure η by changing the coefficients of t and s in the class of base utility functions:

$$\mathcal{U}(1,1,\tfrac{1}{2}) := \{U(t,s,m_i) = -\alpha_1 t + \alpha_2 s + 100\sqrt{m_i} : \alpha_1, \alpha_2 \in R\}, \tag{5.2}$$

where $1,1$ and $\frac{1}{2}$ are the exponents of t, s and m_i. The coefficient 100 of the third term is chosen to make the values of α_1, α_2 clearly visible. Both U^0 and U^1 belong to this class. Then, $U^0(t,s,m_i)$ is obtained by minimizing the discrepancy measure η in this class. This is not the exact solution but is calculated using a method of grid-search by a computer.

Consider our computation procedure more concretely. Suppose that $U \in \mathcal{U}(1,1,\frac{1}{2})$ is given. For each $(t,s) \in \{18,23,31,52,64,70\} \times \{15,35,55\}$, we have the value $-\alpha_1 t + \alpha_2 s$, which gives the ranking, $1, ..., 18$ over $\{18,23,31,52,64,70\} \times \{15,35,55\}$. Recall that this is described by the category function λ_0. The k-th category has $h_k = \alpha_1 t + \alpha_2 s$ and $\lambda_0(k) = (t,s)$. This method is the same as in Section 2.2. Hence, U determines

$$u(\mathbf{e}^k, m_i) = h_k + 100\sqrt{m_i} \text{ for } k = 0, 1, ..., T. \tag{5.3}$$

Thus, each $U \in \mathcal{U}(1,1,\frac{1}{2})$ determines $\mathbb{E} \in \Gamma_{sep}$.

Now, we consider the subclass $\Gamma(1,1,\frac{1}{2})$ of Γ_{sep} defined by

$$\{(M,u,I;N,C) \in \Gamma_{sep} : u \text{ is determined by some } U \in \mathcal{U}(1,1,\tfrac{1}{2})\}. \tag{5.4}$$

Then, we apply the $\Gamma(1,1,\frac{1}{2})$-MSE problem to the data in Section 2.2, and find an approximate solution (α_1, α_2) for it.

An approximate solution will be obtained by the following process.

Step 1: we assume that each of α_1 and α_2 takes a (integer) value from some intervals, say, $[1, 100]$. Then, we have $100^2 = 10^4$ combinations of (α_1, α_2).

Step 2: for each combination (α_1, α_2), we find a maximum competitive rent vector p compatible with the data set $P^o_D(\omega)$ and we have the value η of discrepancy measure. The algorithm to find a maximum competitive rent vector given by Lemma 4.4 is used to find the rent vector.

Step 3: we find a combination (α_1, α_2) with the minimum value of η among 10^4 combinations of (α_1, α_2).

If a solution is on the boundary, we calibrate the intervals, and if not, we repeat these steps by choosing a smaller intervals with finer grids. Hence, the computation to obtain the minimum value of η is not exact: it may be a local optimum as well as an approximation.

By the above simulation method, we have found the utility function $U^0(t,s,m_i)$ of (2.1) in the class $\Gamma(1,1,\frac{1}{2})$ with $\eta^0 = 1.032$.

The base utility function U^2 is obtained by minimization in the class $\mathcal{U}(1,\frac{1}{2}\circledast\beta_2,\frac{1}{2})$:

$$\{U(t,s,m_i) = -\alpha_1 t + \alpha_2\sqrt{s+\beta_2} + 100\sqrt{m_i} : \alpha_1,\alpha_2,\beta_2 \in R\}. \tag{5.5}$$

In fact, when β_2 is increased, the optimal value of η is decreasing (we calculated η up to $\beta_2 = 400,000$), but it does not reach $\eta^0 = 1.032$. Since β_2 is getting large, the second term is getting closer to the linear function. Therefore, we interpret this result as meaning that the base utility function $U^0(t,s,m_i) = -2.2t + 4.0s + 100\sqrt{m_i}$ of (2.1) would be the limit function.

The utility function $U^3(t,s,m_i)$ is obtained by minimizing the value η in the class $\mathcal{U}(1,1,1)$:

$$\{U(t,s,m_i) = -\alpha_1 t + \alpha_2 s + 100 m_i : \alpha_1,\alpha_2 \in R\}. \tag{5.6}$$

That is, the utility functions are entirely linear. The estimation in this class is only interested in seeing the relationship between our Γ-MSE problem and the standard linear regression. This will be discussed in Section 6.2.

5.2. Law of diminishing marginal utility

In the above classes of base utility functions, $U^0(t,s,m_i)$ gave the best value to the discrepancy measure. The law of diminishing marginal utility holds strictly only for the consumption term m_i, but not for the other variables, the commuting time-distance t and size of an apartment s. One possible test of this observation is to broaden the class of base utility functions. Here, we will give this test.

Consider the following class $\mathcal{U}(\pi_1\circledast\beta_1,\pi_2\circledast\beta_2,\pi_3\circledast\beta_3)$:

$$U(t,s,m_i) = \alpha_1(\beta_1 - t)^{\pi_1} + \alpha_2(s+\beta_2)^{\pi_2} + 100(m_i+\beta_3)^{\pi_3}, \tag{5.7}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\pi_1,\pi_2,\pi_3$ are all real numbers. The introduction of β_1 is natural, since the commuting time-distance t has a limit. The parameters β_2 and β_3 will be interpreted after stating the calculation result. The parameters π_1,π_2,π_3 are related to the law of diminishing marginal utility. When they are close to 1, the law is regarded as not holding, and when they are far away from 1, the law holds.

The computation result is given as

$$U^{MU}(t,s,m_i) = 3.53(140-t)^{0.75} + 2.68(s+200)^{0.91} + 100(m_i-25)^{0.40}, \tag{5.8}$$

and the discrepancy value is $\eta^{MU} = 1.025$. Assuming the incomes are uniformly distributed, we adjusted parameters of the lowest income I_{8957}, the highest income I_1 and the lowest rent p_{18}, and obtained the estimated minimum income as $I_{8957} = 94$ ($\times 1000$yen) and the highest as $I_1 = 1120$[7].

First, the law of diminishing marginal utility holds for each variable. However, the degree is quite different: the degree of diminishing marginal utility is the largest with consumption, the second with the commuting time-distance, and is the least with the size of an apartment unit.

The fact that it is the least with the size may be caused by our restriction on apartments up to $65m^2$. In Tokyo, we may find a quite small number of apartments larger than $85m^2$, and omitted these "large" apartments, since the number of supply is much smaller than the smaller types. This may be the reason for almost constant marginal utility.

Second, the degree for the commuting time-distance is higher than that for the apartment size. This suggests, perhaps, that the time-distance 70 minutes to Takao station is already quite large. Our computation result is sensitive with $\beta_1 = 140$, i.e., if we change $\beta_1 = 140$ slightly either up or down, the value of η changed. Thus, this upper limit has a specific meaning; it may be an upper limit for commuting.

Finally, the degree of diminishing marginal utility for consumption is quite large. This means that the choice by a household renting an apartment crucially depends upon its income level. The dependence of willingness-to-pay for an apartment upon income is quite strong: a poor people do not (or cannot) want to pay for a rent for a good apartment, but if they become rich, they would change their attitudes.

Nevertheless, the discrepancy value $\eta^{MU} = 1.025$ for U^{MU} is not very different from $\eta^0 = 1.032$ for U^0; despite of the fact that the latter has 2 parameters controlled and the former has 8. This means that more precision after U^0 does not give much differences. It is more important to see the difference between the discrepancy values for U^0 and U^3 ($\eta^3 = 1.124$) in (5.1). After all, we conclude that the law of diminishing marginal utility surely holds for consumption, but less for other variables.

This conclusion differs from the estimation result of a utility function in Kanemoto-Nakamura [11] in the hedonic approach (cf., Epple [3]). It is stated in [11], p. 227, that the degree of diminishing marginal utility is very low, for example, consumption term is $x^{0.978}$. The approach itself is totally different from ours. One difference is: all variables take continuous values in the hedonic approach. This approach requires a very large variety of attributes of apartment units. In contrast, the number T of apartment categories should not be so large, because the choice of description criteria is restricted, as discussed in Section 1.1.

[7] One possible amendment of our estimation is to change the assumption on the income distribution. We have assumed that the incomes are distributed from the lowest I_{8957} to the highest I_1. The above computation result seems to be quite sensitive by changing these lowest and highest income levels. Hence, it could give a better result if we replace the assumption of a uniform distribution by the data available from the other source. This is an open problem.

6. Two Classes of Market Models

Here, first we argue that Γ_{sep} is too large as a candidate set of models for estimation. Second, we consider the other extreme, i.e., the class of linear utility functions, and show that the Γ-MSE problem is equivalent to linear regression.

6.1. Γ_{sep}-market structure estimation: *ex post* rationalization

From the viewpoint of mathematical economics, the class Γ_{sep} of market models is quite restrictive. However, the following theorem implies that it is too large to have meaningful estimation. A proof will be given in the end of this section.

Theorem 6.1. **(Ex post *rationalization*)** *Suppose that each D_k^o is nonempty and the average rents $\overline{P}^o(\omega) = (\overline{P}_1^o(\omega), ..., \overline{P}_T^o(\omega))$ are positive. Then, there exists a market model $\mathbb{E} = (M, u, I; N, C)$ in the class Γ_{sep} such that for some (x, y), $(\overline{P}^o(\omega), x, y)$ is a maximum competitive equilibrium in $\mathbb{E}$ with $y_k = |D_k^o| > 0$ for $k = 1, ..., T$. This existence assertion holds for any fixed $g : R_+ \to R$ in Condition S2 of Section 4.2.*

Within the class Γ_{sep}, we can "fully explain" any data set from housing magazines in the sense that the estimate coincides with the average rents $\overline{P}^o(\omega)$ and the discrepancy measure η takes the exact value 1. The key fact for this is that the number of dependent variables $\overline{P}^o(\omega) = (\overline{P}_1^o(\omega), ..., \overline{P}_T^o(\omega))$ is the same as that of independent variables $(h_1, ..., h_T)$ in utility function $u(e^k, m_i) = h_k + g(m_i)$. For a different observed $\overline{P}(\omega) = (\overline{P}_1(\omega), ..., \overline{P}_T(\omega))$, the theorem gives different $(h_1, ..., h_T)$. The $\Gamma(1, 1, \frac{1}{2})$-MSE problem in Section 5.1 exhibits a clear-cut contrast: 18 average rents are explained by the choice of parameters by changing essentially 2 parameter values, and $\eta^o = 1.032$.

Should we be pleased by finding a class to guarantee to "fully explain" each data set? Or should we interpret this theorem as meaning that the true market $\mathbb{E}^0$ is included in the class Γ_{sep}?

Contrary to these interpretations, we regard the above theorem as a negative result. The estimated economic model critically depends upon the observed average rents $\overline{P}^o(\omega) = (\overline{P}_1^o(\omega), ..., \overline{P}_T^o(\omega))$. If a different ω' happens and the realized rents $\overline{P}^o(\omega')$ are different, the estimated model $\mathbb{E}'$ differs, too. This estimation explains the observed rents only after observations; it cannot make any meaningful forecast. In particular, since the assertion is done with an arbitrary given function g, it is totally incapable in talking about the law of diminishing marginal utility[8].

[8]The reader may recall the Debreu-Mandel-Sonnenshein Theorem in general equilibrium theory (see Mas-Colell, *et al.* [12]) stating that any demand function with a certain required condition is derived from some economic model. It describes the equivalence between the set of demand curves and the set of economic models. In this sense, it gives an important implication to the theory of general equilibrium theory.

Perhaps, this is related to the fact that the degree of diminishing marginal utility is very low in the hedonic price approach mentioned in Section 5.2. It allows a great variety of attributes, which is contrary to the above negative interpretation of Theorem 6.1.

Proof of Theorem 6.1 We denote $(\overline{P}_1^o(\omega), ..., \overline{P}_T^o(\omega))$ by $(p_1, ..., p_T)$, and let $G(k) = \sum_{t=1}^k |D_t^o|$ for $k = 1, ..., T$. We assume without loss of generality that $p_1 \geq ... \geq p_T > 0$. First, we let $g : \mathbf{R}_+ \to \mathbf{R}$ be any monotone, strictly concave and continuous function with $\lim_{m_i \to +\infty} g(m_i) = +\infty$.

Let $h_0 = 0$. We choose $I_m, I_{G(T-1)}, ..., I_{G(1)}$ and define $h_T, h_{T-1}, ..., h_1$ inductively as follows: the base case is as follows:

(T-0): choose an income level I_m so that $I_m > p_T > 0$, and then define $h_T > h_0 + g(I_m) - g(I_m - p_T)$.

The choices of I_m and h_T are possible by the monotonicity of g. Here, $h_T + g(I_m - p_T) > h_0 + g(I_m)$.

Let k be an arbitrary number with $1 \leq k < T$. The inductive hypothesis is that $I_{G(k)}$ and h_k are already defined. First, we choose $I_{G(k-1)}$ so that

(k-1) : $I_{G(k-1)} > p_{k-1}$ and $I_{G(k-1)} > I_{G(k)}$.

This choice is simply possible. Then we define h_{k-1} by

(k-2) : $h_{k-1} = h_k + g(I_{G(k-1)} - p_k) - g(I_{G(k-1)} - p_{k-1})$.

Since $g(I_{G(k-1)} - p_{k-1}) \leq g(I_{G(k-1)} - p_k)$, we have $h_{k-1} \geq h_k$.

By the above inductive definition, we have $I_m, I_{G(T-1)}, ..., I_{G(1)}$ and $h_T, h_{T-1}, ..., h_1$. We also choose other I_i's ($i \neq m$ and $i \neq G(k)$ for $k = 1, ..., T$) so that $I_m \leq I_{m-1} \leq ... \leq I_1$.

Thus, we have the utility function $u(\mathbf{e}^k, m_i) = h_k + g(m_i)$ for $(\mathbf{e}^k, m_i) \in X$. By the above inductive definition, $(p_1, ..., p_T)$ satisfies the recursive equation (4.7).

Let us define the cost function $C_k(\cdot)$ for landlord k. We assume $0 < c_k \leq p_k$ for all $k = 1, ..., T-1$ and $c_T = p_T$. Then each $C_k(\cdot)$ is defined by (2.3) for $k = 1, ..., T$. Then, by Lemma 4.4, $(p_1, ..., p_T)$ is the maximum competitive rent vector of $\mathbb{E}$ with $y_k = |D_k^o|$ for $k = 1, ..., T$.$\square$

6.2. Linear utility functions and linear regression

Here, we compare our approach with the linear utility assumption to linear regression. We assume that there are L attributes for the base utility function U for each household, and the domain of U is expressed as $Y = \mathbf{R}_+^L \times \mathbf{R}_+$. In the example of Section 2.2, there are only two attributes, the commuting time t and the apartment size s. A linear base utility function over Y is expressed as

$$U(a_1, ..., a_L, m_i) = \sum_{l=1}^{L} \alpha_l a_l + m_i \text{ for all } (a_1, ..., a_L, m_i) \in Y. \tag{6.1}$$

Here, a_l represents the magnitude of the l-th attribute of an apartment and $\alpha_l \in \mathbf{R}$ is its coefficient for $l = 1, ..., L$. We denote the set of all base utility function of the form (6.1) by $\mathcal{U}_{lin}$. We choose 1 for the coefficient of m_i for a direct comparison to the linear regression analysis, while it was 100 in the previous examples.

An *attribute vector* $\tau^k = (\tau_1^k, ..., \tau_L^k) \in \mathbf{R}_+^L$ is given for each $k = 0, 1, ..., T$. That is, the choice $\mathbf{e}^k$ gives the attribute vector τ^k, which means that an apartment in category k has the magnitudes $\tau_1^k, ..., \tau_L^k$ of attributes $1, ..., L$. For $k = 0$, τ^0 is interpreted as the attributes of the outside option. In the example of Section 2.2, category $k = 5$ (Nakano, size: 25-45) has the attribute vector $\tau^5 = (18\,\text{min}, 35m^2)$. Then, each U in $\mathcal{U}_{lin}$ determines

$$u(\mathbf{e}^k, m_i) = U(\tau^k, m_i) = \sum_{l=1}^{L} \alpha_l \tau_l^k + m_i \text{ for all } k = 0, 1, ..., T. \tag{6.2}$$

We define the class $\Gamma_{lin} := \{\mathbb{E} \in \Gamma_{sep} : u \text{ is determined by } U \in \mathcal{U}_{lin} \text{ and } \tau^0, ..., \tau^T\}$. The boundary condition "$u(0, I_i) > u(\mathbf{e}^k, 0)$ for all $k = 1, ..., T$" holds $\mathbb{E} \in \Gamma_{lin}$, because $\mathbb{E} \in \Gamma_{sep}$. Once this class is defined, we have the Γ_{lin}-MSE problem.

The next lemma states that the competitive rents in $\mathbb{E} \in \Gamma_{lin}$ are simply described by the utility from the attributes of an apartment and some constant.

Lemma 6.2. *Let $\mathbb{E} \in \Gamma_{lin}$. If $p = (p_1, ..., p_T)$ is a maximum competitive rent vector in $\mathbb{E} \in \Gamma_{lin}$, then there is some β such that*

$$p_k = \sum_l \alpha_l \tau_l^k + \beta > 0 \text{ for } k = 1, ..., T \text{ and } \beta < -\sum_{l=1}^{L} \alpha_l \tau_l^0. \tag{6.3}$$

Proof. Let (p, x, y) be any competitive equilibrium in $\mathbb{E} = (M, u, I; N, C)$ in Γ_{lin} with $|D_k^o| = y_k > 0$ for all $k = 1, ..., T$. Without loss of generality, we assume that $p_k \geq p_T$ for $k = 1, ..., T - 1$. First, we show

$$p_k - p_T = \sum_l \alpha_l \tau_l^k - \sum_l \alpha_l \tau_l^T \text{ for all } k = 1, ..., T. \tag{6.4}$$

Suppose that this is shown. Let $\beta = p_T - \sum_l \alpha_l \tau_l^T$. We have, by (6.4), $p_k = \sum_l \alpha_l \tau_l^k + \beta$ for $k = 1, ..., T$. For each k, since $|D_k^o| = y_k > 0$ and $c_k > 0$, we have $p_k \geq c_k$. Hence $p_k > 0$ for $k = 1, ..., T$, which is the first half of (6.3). Since any household i in

D^o_T chooses the T-th apartment rather than $(0, I_i)$, i.e., $U(e^T, I_i - p_T) = \sum_l \alpha_l \tau^T_l + I_i - (\sum_l \alpha_l \tau^T_l + \beta) = I_i - \beta > u(0, I_i) = h_0 + I_i = \sum_l \alpha_l \tau^0_l + I_i$, which implies $\beta < -\sum_l \alpha_l \tau^0_l$.

Now let us prove (6.4). Consider any $k = 1, ..., T$. Since $|D^o_k| > 1$, we take a household i with $x_i = e^k$. Since he chooses $x_i = e^k$ by utility maximization under $p = (p_1, ..., p_T)$, we have $\sum_l \alpha_l \tau_l(k) + I_i - p_k \geq \sum_l \alpha_l \tau_l(T) + I_i - p_T$. By the same argument for a household i' with $x_{i'} = e^T$, we have $\sum_l \alpha_l \tau_l(t) + I_{i'} - p_k \leq \sum_l \alpha_l \tau_l(T) + I_{i'} - p_T$. Equation (6.4) follows from these two inequalities. ■

Now let us turn our consideration to linear regression: the rent of an apartment in category k is assumed to be a linear combination of the magnitudes of attributes and some constant. Mathematically, it is exactly the same as (6.3) subject to some error, that is[9],

$$P_k = \sum_{l=1}^{L} \alpha_l \tau^k_l + \beta + \boldsymbol{\epsilon}_k \quad \text{for } k = 1, ..., T. \tag{6.5}$$

The attribute vectors $\tau^0, ..., \tau^T$ are fixed. Given the housing magazine $P^o_D(\omega)$ as data, we estimate $\alpha = (\alpha_1, ..., \alpha_L)$ and β by minimizing the sum of square residuals, i.e., *the method of least squares*. It is formulated by the following minimization problem:

$$\min_{\alpha,\beta} \sum_{k=1}^{T} \sum_{d \in D^o_k} (P^o_{kd}(\omega) - p_k)^2 = \min_{\alpha,\beta} \sum_k \sum_d \left(P^o_{kd}(\omega) - (\sum_{l=1}^{L} \alpha_l \tau^k_l + \beta) \right)^2. \tag{6.6}$$

This is a no-constraint minimization problem and has a solution $(\widehat{\alpha}, \widehat{\beta})$.

The above linear regression problem is very close to the Γ_{lin}-MSE problem. In linear regression, however, neither utility maximization nor profit maximization is included. It would be worth considering the exact relationship.

The minimization (6.6) is applied to any data set $P^o_D(\omega)$, even if $P^o_D(\omega)$ contains negative elements. On the other hand, the Γ_{lin}-MSE problem may not be if it contains negative elements: if the estimated rent for category k is negative, landlord k provides no apartments, i.e., condition $y_k = |D^o_k|$ is violated. We need a certain condition to avoid such a case. For this, the following condition is enough, though it is not directly on $P^o_D(\omega)$:

$$\sum_{l=1}^{L} \alpha_l \tau^k_l + \beta > 0 \text{ for all } k = 1, ..., T \text{ and } \beta < -\sum_{l=1}^{L} \alpha_l \tau^0_l. \tag{6.7}$$

Again, this corresponds to (6.3) in Lemma 6.2. Using this condition, we can state the equivalence between the Γ_{lin}-MSE problem and the linear regression problem.

Theorem 6.3. ***(Linear Regression)*** *Let $(\widehat{\alpha}, \widehat{\beta}) \in \mathbf{R}^L \times \mathbf{R}$. Then, $(\widehat{\alpha}, \widehat{\beta})$ is a solution of the minimization (6.6) and satisfies (6.7) if and only if there is a solution model $\widehat{\mathbb{E}}$*

[9] This is regarded as a linear hedonic price model.

in the Γ_{lin}-MSE problem such that $\widehat{u}$ of $\widehat{\mathbb{E}}$ is determined by U of (6.2) with $\widehat{\alpha}$ and the maximum competitive rent vector $p(\widehat{\mathbb{E}}) = \widehat{p}$ is given as[10]

$$\widehat{p}_k = \sum_{l=1}^{L} \widehat{\alpha}\tau_l^k + \widehat{\beta} \ \ \textit{for all } k = 1, ..., T. \tag{6.8}$$

In the example of Chuo line in Section 2.2, the base utility function and rents are estimated as follows:

$$U(t, s, m_i) = -0.74t + 1.65s + m_i \ \text{ and } \ p_{\lambda_0(t,s)} = -0.74t + 1.65s + 41.3, \tag{6.9}$$

where $\lambda_0(t, s)$ is the category function. This U is the same as U^3 of (5.1) as a utility function in that $U^3 = U/100$. The discrepancy value $\eta = \eta^3 = 1.124$ is larger than the corresponding values given in Section 5 except U^1.

The next lemma states that the rent vector given in (6.7) is sustained as a competitive vector by some $\mathbb{E}$ in Γ_{lin}.

Lemma 6.4. ***(Sustainability)*** *Let (6.7) hold for $\alpha = (\alpha_1, ..., \alpha_L)$ and β, and let $p_k = \sum_l \alpha_l \tau_l^k + \beta > 0$ for $k = 1, ..., T$. Then, there is a model $\mathbb{E}$ in Γ_{lin} such that $p = p(\mathbb{E})$.*

Proof: First, we define the base utility function by $U(a_1, ..., a_L, m_i) = \sum_l \alpha_l a_l + m_i$. Let $I_1, ..., I_m$ be incomes with $I_1 > ... > I_m > p_1$. We define cost functions $C_1, ..., C_{T-1}$ by (2.3) with $w_k = |D_k^o|$ and $c_k < p_k$ for $k = 1, ..., T-1$. Define C_T by (2.3) with $w_T^0 > |D_T^o|$ and $c_T = p_T$. In this case, for each $k = 1, ..., T$, $y_k = |D_k^o|$ maximizes landlord k's profits.

The rents $p_k = \sum_l \alpha_l \tau_l^k + \beta$ satisfies the rent equation (4.7). Also, since $\beta < -\sum_l \alpha_l \tau_l^0$, each household i has the utility, $u(e^k, I_i - p_k) = I_i - \beta > I_i + \sum_l \alpha_l \tau_l^0 = u(\mathbf{0}, I_i)$. Hence, his choice of an apartment is better than choosing no apartments. ■

Proof of Theorem 6.3 (Only-If) Let (α, β) be any vector satisfying (6.7) and let $p_k = \sum_l \alpha_l \tau_l^k + \beta > 0$ for $k = 1, ..., T$. By Lemma 6.4, $p = (p_1, ..., p_T)$ is the maximum competitive rent vector of some $\mathbb{E} \in \Gamma_{lin}$. Hence, if $(\widehat{\alpha}, \widehat{\beta})$ minimizes the total sum of total square errors in (6.6), then it also minimizes $T_R(P_D^o(\omega), p(\mathbb{E}))$ over Γ_{lin} with $y_k = |D_k^o|$ for $k = 1, ..., T$.

(If) Suppose that $\widehat{\mathbb{E}}$ is a solution of the Γ_{lin}-MSE problem, and that its maximum competitive rent vector $\widehat{p} = (\widehat{p}_1, ..., \widehat{p}_T)$ is expressed by (6.8). Let $\widehat{\alpha}$ be the coefficients of the utility function in $\widehat{\mathbb{E}}$ and let $\widehat{\beta}$ be the constant given in (6.8). For each $k = 1, ..., T$, it holds that $\sum_{l=1}^L \widehat{\alpha}_l \tau_l^k + \widehat{\beta} = \widehat{p}_k \geq \widehat{c}_k > 0$, since some unit in category k is supplied in $\widehat{\mathbb{E}}$. Then, $\widehat{\beta} < -\sum_l \widehat{\alpha}_l \tau_l^0$ by the boundary condition in $\widehat{\mathbb{E}}$.

[10] In fact, "maximum" can be dropped in here in the sense that each $\mathbb{E}$ has a unique competitive rent vector.

Suppose that $\widehat{p} = (\widehat{p}_1, ..., \widehat{p}_T)$ is not a solution of (6.6). Then, some other $p' = (p'_1, ..., p'_T)$ with α' and β' gives the smallest total sum of square errors in (6.6). Consider the convex combination $\alpha(\pi) = \pi\alpha' + (1-\pi)\widehat{\alpha}$ and $\beta(\pi) = \pi\beta' + (1-\pi)\widehat{\beta}$ with $0 \leq \pi \leq 1$. The, $(\alpha(1), \beta(1))$ gives the smaller total sum of square errors than any other $(\alpha(\pi), \beta(\pi))$. Since the total sum is a convex function of α and β, $(\alpha(0), \beta(0))$ gives a larger value than $(\alpha(\pi), \beta(\pi))$ for any π $(0 < \pi < 1)$. We can take a small $\pi > 0$ so that $\beta(\pi) < -\sum_l \alpha_l(\pi)\tau_l^0$ and $\sum_{l=1}^L \alpha_l(\pi)\tau_l^k + \beta(\pi) > 0$. By Lemma 6.4, there is an economy $\mathbb{E}$ in Γ_{lin} such that $p(\mathbb{E}) = \sum_{l=1}^L \alpha_l(\pi)\tau_l^k + \beta(\pi)$. This is a contradiction to the supposition that $\widehat{\mathbb{E}}$ is a solution of the Γ_{lin}-MSE problem. Hence, $(\widehat{\alpha}, \widehat{\beta})$ is a solution of (6.6). ■.

7. Conclusions

We developed the equilibrium-econometric analysis of rental housing markets. Our analysis provides a bridge between a market equilibrium theory and an econometric analysis. This is built by focusing on housing magazines as serving information about apartment units to economic agents (households, landlords) as well as to the econometric analyzer. We modified the equilibrium theory by incorporating the former aspect, but at the same time, we showed that we can ignore the error terms, which is the convergence theorem (Theorem 3.2) for equilibrium theory.

Then, we introduced the discrepancy measure as the ratio of the total sum of square residuals from the predicted rents over that from the average rents. In the best estimation we obtained in Section 5, the measure takes about the value 1.025. This result has strong implications on the law of diminishing marginal utility. It holds strictly for consumption, less for the commuting time-distance to the office area, and much less for the sizes of apartment units.

We have many untouched problems, which are divided into three classes: We end this paper by mentioning some problems in each class.

(1): *Subjective estimation*: we simply assumed that each economic agent forms an estimate of a rent distribution from housing magazines. Theorem 3.2 is a study of this subjective estimation. However, a more study is of great interests also from the viewpoint of inductive game theory (Kaneko-Kline [9]): the question is whether an agent with a limited analytical ability can derive a meaningful estimation. This should be studied not only theoretically but also empirically.

(2): *Applications to housing markets along different railway lines and in different cities*: we discussed only a submarket along the JR Chuo railway line in Tokyo. The authors have been applying the theory to some other railway lines, but those are not more than pilot studies. A more systematic study of rental housing markets in different places and in different time is an important future problem. Then, for example, the law of diminishing marginal utility can be tested in different areas.

Although there are almost no clear-cut segregations, in the Tokyo area (also in Japan), with different income groups and/or ethnic groups, such segregations are common phenomena in the world. The theory of assignment markets has not been developed to treat such problems. To treat it, we need to develop a more general procedure to calculate a competitive equilibrium than that used in this paper. An application to such cases will make our theory more fruitful.

(3): *Applications to panel data*: this is related to (2). Each housing magazine is issued daily or weekly. Accumulating these housing magazines, we have panel data, and can study the temporal changes of the housing market. One problem is to check the comparative statics results obtained in Kaneko *et al.* [10] and Ito [6] with those railway lines. In doing so, we may have better understanding of the structure of the housing market.

8. Proof of Theorem 3.2

Since the condition BDS in $\mathbb{E}$ is preserved to $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M\cup N,\nu})$, we show that the γ-UM and γ-PM hold for $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M\cup N,\nu})$ for all $\nu \geq$ some ν_0, but show it only for a household $i \in M$. It is similar to prove it for $j \in N$; the assumption that the domain of the profit function is finite is used for it.

Now, let γ be an arbitrary positive number, and $P^{i,\nu} = p + \boldsymbol{\epsilon}^{i,\nu}$ for $\nu = 1, \ldots$. Consider any $i \in M$. Let $z^i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ with $I_i - pz^i \geq 0$. Then, by UM,

$$u_i(x^i, I_i - px^i) \geq u_i(z^i, I_i - pz^i). \tag{8.1}$$

We should consider two cases: $x^i = \mathbf{e}^t$ $(t \neq 0)$ and $x^i = \mathbf{0}$, but now we consider the case of $x^i = \mathbf{e}^t$.

As $\delta \to 0$, the utility value $u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t))$ converges to $u_i(x^i, I_i - px^i) = u_i(\mathbf{e}^t, I_i - p_t)$ by continuity of u_i in Assumption A. Since $\{\boldsymbol{\epsilon}^{i,\nu}\}$ converges to $\mathbf{0}$ in probability, for any $\delta > 0$, there is a $\nu(\delta)$ such that for any $\nu \geq \nu(\delta)$,

$$\mu(\{\omega : \left\|\boldsymbol{\epsilon}^{i,\nu}(\omega)\right\| < \delta\}) < 1 - \frac{\delta}{2}. \tag{8.2}$$

Since u_i is increasing in consumption by Assumption A, it holds that for all $\nu \geq \nu(\delta)$,

$$EU_i(\mathbf{e}^t, I_i - P^{i,\nu} \cdot \mathbf{e}^t) \geq (1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i). \tag{8.3}$$

Since the right-hand side converges to $u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t)$ as $\delta \to 0$, there is some δ_1 such that for all $\delta \geq \delta_1$,

$$(1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i) \geq u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) - \frac{\gamma}{2}. \tag{8.4}$$

Since δ in (8.3) is arbitrary, we can take the above δ_1 for δ. From (8.3) for δ_1 and (8.4), for any $\nu \geq \nu(\delta_1)$, we have

$$EU_i(\mathbf{e}^t, I_i - P^{i,\nu}\mathbf{e}^t) \geq u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) - \frac{\gamma}{2}. \tag{8.5}$$

Now, let z^i be in $\{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$. Since u_i is increasing in consumption, we have, using (8.2) and (8.1), for all $\nu \geq \nu(\delta)$,

$$(1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_{t'} - \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i) \geq EU_i(\mathbf{e}^t, I_i - P^{i,\nu}\mathbf{e}^t) \geq EU_i(z^i, I_i - P^{i,\nu}z^{it}) \tag{8.6}$$

The first term converge to $u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t)$ as $\delta \to 0$. Hence, there is some δ_2 such that for any $\delta \geq \delta_2$,

$$u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) + \frac{\gamma}{2} \geq (1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i). \tag{8.7}$$

Hence, from (8.6) and (8.7), it holds that for any $\nu \geq \nu(\delta_2)$,

$$u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) + \frac{\gamma}{2} \geq EU_i(z^i, I_i - P^{i,\nu}z^i) \tag{8.8}$$

Let $\delta_3 = \min(\delta_1, \delta_2)$. Then, it follows from (8.5) and (8.8) that for all $\nu \geq \delta_3$,

$$EU_i(\mathbf{e}^t, I_i - P^{i,\nu}e^t) + \frac{\gamma}{2} \geq u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) \geq EU_i(z^i, I_i - P^{i,\nu}z^i) - \frac{\gamma}{2}.$$

Connecting the first term with the last term, we have the final target: $EU_i(\mathbf{e}^t, I_i - P^{i,\nu}\mathbf{e}^t) + \gamma \geq EU_i(z^i, I_i - P^{i,\nu}z^i)$.

In the case $x^i = \mathbf{0}$, the first half of the above proof should be modified.

(2): Suppose the *if clause* of the assertion. Now, let $\{\gamma_\beta\}$ a positive decreasing and converging sequence to 0. For each γ_β, we find a ν_β such that for all $\nu \geq \nu_\beta$, (p, x, y) is a γ_β-competitive equilibrium in $\mathbb{E}(\boldsymbol{\epsilon}; \boldsymbol{\epsilon}^{M \cup N, \nu})$. We show that the utility maximization and profit maximization hold under rent vector p.

Consider utility maximization for x_i. We have, for all β,

$$EU_i(x_i, I_i - P^{i,\nu_\beta}x_i) + \gamma_\beta \geq EU_i(z^i, I_i - P^{i,\nu_\beta}z_i) \text{ for all } z_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}. \tag{8.9}$$

Let $z_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ be fixed. Suppose $I_i - pz_i > 0$. Then, both $EU_i(x_i, I_i - P^{i,\nu_\beta}x_i)$ and $EU_i(z^i, I_i - P^{i,\nu_\beta}z_i)$ converge to $u_i(x_i, I_i - px_i)$ and $u_i(z_i, I_i - pz_i)$; by (8.9), we have $u_i(x_i, I_i - px_i) \geq u_i(z^i, I_i - pz_i)$.

Now, suppose $I_i - pz_i = 0$. Since $u_i(z_i, 0) < u_i(0, I_i)$ by Assumption A, there is a β_0 such that for all $\beta \geq \beta_0$, $EU_i(z^i, I_i - P^{i,\nu_\beta}z_i) > u_i(z_i, 0)$. Hence, by (8.9), we have $u_i(x_i, I_i - px_i) \geq u_i(z_i, 0) = u_i(z^i, I_i - pz_i)$.

The profit maximization for y_j can be proved even in a simpler manner.□

References

[1] Alonso, W., (1964), *Location and Land Use*, Harvard University Press, Cambridge.

[2] Arnott, R., (1991), Economic Theory and Housing, *Handbook of Regional and Urban Economics*, Vo.2, E. S. Mills, North-Holland, Amsterdam.

[3] Epple, D. (1987), Hedonic Prices and Implicit Markets: Estimating Demand and Supply Functions for Differentiated Products, *Journal of Political Economy* 95, 58-80.

[4] Gale, D., and L. Shapley, (1962), College Admissions and the Stability of Marriage, *American Mathematical Monthly* 69, 9-15.

[5] Hicks, J., (1939), *Value and Capital*, Carendon Press, Oxford.

[6] Ito, T., (2007), Effects of Quality Changes in Rental Housing Markets with Indivisibilities, *Regional Science and Urban Economics* 37, 602-617.

[7] Kaneko, M., (1982), The Central Assignment Game and the Assignment Markets, *Journal of Mathematical Economics* 11, 205–232.

[8] Kaneko, M., (1983), Housing market with indivisibilities, *Journal of Urban Economics* 13, 22–50.

[9] Kaneko, M., and J. J. Kline (2008), Inductive Game Theory: A Basic Scenario, *Journal of Mathematical Economics* 44, 1332–1363

[10] Kaneko, M. T. Ito and Y.-I. Osawa, (2006), Duality in comparative statics in rental housing markets with indivisibilities, *Journal of Urban Economics* 59, 142-170.

[11] Kanemoto, Y., and R. Nakamura, (1986), A New Approach to the Estimation of Structural Equations in Hedonic Models, *Journal of Urban Economics* 19, 218-233.

[12] Mas-Colell, A., M. D. Whinston, and J. R. Green, (1995), *Microeconomic Theory, Oxford University Press*, New York.

[13] Miyake, M., (1994), Comparative Statics of Assignment Markets with General Utilities, *Journal of Mathematical Economics* 23, 519–531.

[14] Ricardo, D., (1965, 1817: original), *The Principles of Political Economy and Taxation*, J. M. Dent and Sons, London.

[15] Sai, S., (2014), The Structure of Competitive Equilibria in an Assignment Market, *Journal of Mathematical Economics* 51, 42-49.

[16] Shapley, L., and M. Shubik, (1972), Assignment Game I: the Core, *International Journal of Game Theory* 1, 111–130.

[17] von Böhm-Bawerk, E., (1921), *Positive Theory of Capital*, translated by W. Smart, (Original publication in 1891) , Books for Libraries, New York.

[18] van der Laan, G., D. Talman, and Z. Yang, (2002), Existence and Welfare Properties of Equilibrium in an Exchange Economies with Multiple Divisible and Indivisible Commodities and Linear Production, *Journal of Economic Theory* 103, 411-428.

[19] Wooldridge, J. M., (2000), *Introductory Econometrics: Modern Approach*, South-Western College Publishing, United States.

[20] van der Vaart, A. W., (1998), *Asymptotic Statistics*, Cambridge University Press, Cambridge.

[21] von Neumann, J., and O. Morgenstern, (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton.