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Hedonic Games with Externalities:

Myopic, Farsighted and Individual Stability Concepts

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Abstract

Our objective is to study the stability of coalition structures in symmetric majority games and apex games. In this paper, we define hedonic games with externalities, in which each player's preferences are defined over all possible partitions of the player set. We consider the coalition formation problems of symmetric majority games and apex games to be subclasses of hedonic games with externalities. Our stability concepts are classified into three types: myopic, farsighted, and individual. We demonstrate which coalition structure satisfies which stability concept.

Keywords: Externalities; Farsightedness; Hedonic games; Myopia
JEL Classification: C71

1 Introduction

In this paper, we attempt to answer the following question: Which coalition structure is "stable" in majority voting? To clarify our question, we assume that there are three identical players (namely, voters). Call them i, j and k. Each player can form a coalition with some of the other players, such as a political party. In three-player majority voting, a coalition consisting of two or three players wins. All possible coalition structures are $\{\{i,j,k\}\}$, $\{\{i,j\},\{k\}\}$ and $\{\{i\},\{j\},\{k\}\}$. Now, which coalition structure can we consider a "stable" coalition structure? To formally analyze this problem, we need to revisit each player's *preferences* and the definitions of *stability* concepts.

Hart and Kurz (1984)'s work is a leading attempt to solve this problem. They use a *coalition structure value* (CS-value) to represent each player's preferences over coalition structures.[*1] A CS-value is a function that assigns a real number to every player in a given coalition structure. In their analysis,

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[*1] Hart and Kurz (1983) focus on the theoretical aspect of the CS-value, whereas Hart and Kurz (1984) focus on its applications.

they use the Owen power index as a CS-value to evaluate each player's power in a coalition structure, which is an extension of the Shapley-Shubik power index to games with (a priori) coalition structures. Their objective is to study which coalition structure satisfies the α-, β-, γ- and δ-stability concepts in symmetric majority games and apex games. They show that some coalition structures satisfy some of the four stability concepts. We compare our results with theirs in Section 5.

Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002) introduce the concept of *hedonic games*, which are simple models consisting only of players and their preferences. Although hedonic games allow us to analyze many problems of coalition formation, they are not straightforwardly applicable to the majority voting that Hart and Kurz (1984) formulated. This is because of *externalities among coalitions* (occasionally refereed to as *spillovers* across coalitions) in majority voting. Original hedonic games are models without externalities among coalitions, in which each player's preferences (representing his power, payoff or utilities) depend only on the members of his coalition, regardless of the structure of the other players outside his coalition. In other words, each player's preferences are defined over all coalitions that contain him as a member. This model is, however, not suitable to depict majority voting. To see this, we consider, for example, four-player majority voting and call the players i, j, k and h. We consider the power of a two-player coalition $\{i, j\}$. The power of the coalition $\{i, j\}$ can vary depending on the structure of the other players k, h: the power of $\{i, j\}$ in $\{\{i, j\}, \{k\}, \{h\}\}$ is actually different from that in $\{\{i, j\}, \{k, h\}\}$.

In this paper, we define *hedonic games with externalities.* Formally, each player's preferences are defined over all possible partitions (*i.e.*, coalition structures) of the player set. We consider the coalition formation problems based on majority voting (more precisely, symmetric majority games and apex games, as described below) to be subclasses of hedonic games with externalities.

We define several stability concepts in our framework. These concepts are categorized into three types: myopic, farsighted, and individual. We define the *projective*, *pessimistic*, and *optimistic* cores as the myopic stability concepts. As for the farsighted stability concept, we extend the notion of the *farsighted vNM stable set*, which is introduced by Diamantoudi and Xue (2003), to our framework. The individual stability notions include the extended versions of the *Nash stability* and the *individual stability*, which are intensively studied by Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002) in hedonic games without externalities.

We show which coalition structure satisfies which stability concept. All of our main results are summarized in Table 4.

The rest of this paper is organized as follows. In Section 2, we offer the basic definitions and some general results. We analyze symmetric majority games in Section 3. Apex games are studied in Section 4. In section 5, we offer a summary of our results and comparisons with related works.

2 Preliminaries

Let N be a set of players. A coalition of players is a subset of the player set: $S \subseteq N$. We denote the cardinality of S by $|S|$. Let n be $|N|$. Throughout this paper, we assume $n \geq 3$. We typically use $\mathcal{P}$ to denote a partition of N. We denote the set of all partitions of N by $\Pi(N)$. For every player $i \in N$, let

$\succsim_i$ denote i's preferences over $\Pi(N)$. A *hedonic game with externalities* is a pair $(N, \{\succsim_i\}_{i\in N})$.

For any $S \subseteq N$, we define $\mathcal{P}' \succ_S \mathcal{P}$ as $\mathcal{P}' \succ_j \mathcal{P}$ for all $j \in S$.[*2] For any partition $\mathcal{P}$ and any coalition $S \subseteq N$, let $\mathcal{P}|_S$ be a *projection* of $\mathcal{P}$ onto S, which is given by $\mathcal{P}|_S = \{S \cap C \mid C \in \mathcal{P},\ S \cap C \neq \emptyset\}$. Note that $\mathcal{P}|_S$ is a partition of S.

Example 2.1. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$. If $S = \{1, 2\}$, then $\mathcal{P}|_S = \{\{1, 2\}\}$. If $S = \{2, 3\}$, then $\mathcal{P}|_S = \{\{2\}, \{3\}\}$. If $S = \{1, 4\}$, then $\mathcal{P}|_S = \{\{1\}, \{4\}\}$.

Example 2.2. Let $N = \{1, 2, 3\}$. The following game is an example of hedonic games with externalities:

$$\begin{aligned}
&\succsim_1 : \{\{1,2\},\{3\}\} \sim_1 \{\{1,3\},\{2\}\} \succ_1 \{\{1,2,3\}\} \sim_1 \{\{1\},\{2\},\{3\}\} \succ_1 \{\{2,3\},\{1\}\},\\
&\succsim_2 : \{\{1,2\},\{3\}\} \sim_2 \{\{2,3\},\{1\}\} \succ_2 \{\{1,2,3\}\} \sim_2 \{\{1\},\{2\},\{3\}\} \succ_2 \{\{1,3\},\{2\}\},\\
&\succsim_3 : \{\{2,3\},\{1\}\} \sim_3 \{\{1,3\},\{2\}\} \succ_3 \{\{1,2,3\}\} \sim_3 \{\{1\},\{2\},\{3\}\} \succ_3 \{\{1,2\},\{3\}\}.
\end{aligned}$$

2.1 Myopic Stability

We define the three myopic stability concepts in hedonic games with externalities.

Definition 2.3. A partition $\mathcal{P}$ is in the *projective core*, C^{pro}, if there is no $S \subseteq N$ such that $S \notin \mathcal{P}$ and $\mathcal{P}' \succ_S \mathcal{P}$ where $\mathcal{P}' = \{S\} \cup (\mathcal{P}|_{N\setminus S})$.

Definition 2.4. A partition $\mathcal{P}$ is in the *pessimistic core*, C^{pes}, if there is no $S \subseteq N$ such that $S \notin \mathcal{P}$ and for all $\mathcal{P}' \in \Pi(N)$ with $S \in \mathcal{P}'$, $\mathcal{P}' \succ_S \mathcal{P}$.

Definition 2.5. A partition $\mathcal{P}$ is in the *optimistic core*, C^{opt}, if there is no $S \subseteq N$ such that $S \notin \mathcal{P}$ and for some $\mathcal{P}' \in \Pi(N)$ with $S \in \mathcal{P}'$, $\mathcal{P}' \succ_S \mathcal{P}$.

Note that we have

$$C^{\text{opt}} \subseteq C^{\text{pro}} \subseteq C^{\text{pes}}. \tag{2.1}$$

By a "*deviating coalition* S from $\mathcal{P}$ to $\mathcal{P}'$," we generally mean a coalition $S \subseteq N$ such that $\mathcal{P}' \succ_S \mathcal{P}$, where we exclude coalitions $S \in \mathcal{P}$ from the definition of the deviating coalition. For example, for a partition $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$, coalitions $\{1, 2\}$ and $\{3, 4\}$ are not called a deviating coalition form $\mathcal{P}$ even if there exists a partition $\mathcal{P}'$ such that $\mathcal{P}' \succ_S \mathcal{P}$ and $S \in \mathcal{P}'$. All the other coalitions can be deviating coalitions.[*3]

These three types of cores are distinguished by the *reaction* of the players who are not contained in the deviating coalition. The projective core is the set of partitions from which no coalition deviates with keeping the structure of the other players unchanged. In other words, the players outside the deviating coalition do not react to the deviation at all. We call this type of deviation *projective deviation.* In contrast, the pessimistic (optimistic) core exhibits the deviating players' pessimistic (optimistic) expectation for the reaction of the other players. In the pessimistic (optimistic) view, every deviating player

[*2] Similarly, for any $S \subseteq N$, we define $\mathcal{P}' \succsim_S \mathcal{P}$ as $\mathcal{P}' \succsim_j \mathcal{P}$ for all $j \in S$

[*3] To be more mathematically precise, we do not have to exclude $S \in \mathcal{P}$ from the definition of deviation for the projective core and the pessimistic core.

anticipates that the other players regroup their coalition structure to minimize (maximize) the deviating coalition. We call this type of deviation *pessimistic* (*optimistic*) *deviation*. Myopia is reflected in the fact that the deviating coalition considers no reaction or, at most, a one-step reaction from other players.

Example 2.6. Consider Example 2.2. We have

$$C^{\text{opt}} = C^{\text{pro}} = C^{\text{pes}} = \{\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{2,3\},\{1\}\}\}.$$

2.2 Farsighted Stability

Diamantoudi and Xue (2003) introduce the farsighted stability concepts to hedonic games (without externalities). We extend the concepts to hedonic games with externalities.

Definition 2.7. A partition $\mathcal{P}'$ *indirectly dominates* $\mathcal{P}$ if there exists a sequence of partitions $\mathcal{P}^1, ..., \mathcal{P}^k$ with $\mathcal{P}^1 = \mathcal{P}$ and $\mathcal{P}^k = \mathcal{P}'$ and a sequence of coalitions $S^1, ..., S^{k-1}$ such that, for every $j = 1, ..., k-1$,

i. $\mathcal{P}^{j+1} = \{S^j\} \cup (\mathcal{P}^j|_{N\setminus S^j})$, and
ii. $\mathcal{P}' \succ_{S^j} \mathcal{P}^j$.

The indirect domination can be seen as a sequence of projective deviations. However, in each step j, the deviating coalition S^j compares the current partition $\mathcal{P}^j$ with the final destination $\mathcal{P}'$. This is the difference between the indirect domination and an ordinal sequence of projective deviations. Note that the *direct domination*, namely, only one-step indirect domination, is the same as the projective deviation. It follows from Definition 2.7 that if $\mathcal{P}'$ directly dominates $\mathcal{P}$, then $\mathcal{P}'$ indirectly dominates $\mathcal{P}$. We next define the vNM stable set in hedonic games with externalities.

Definition 2.8. The *vNM stable set* V is the set of partitions satisfying the following two conditions:

i. for any $\mathcal{P}$ and $\mathcal{P}' \in V$, $\mathcal{P}'$ does not indirectly dominate $\mathcal{P}$, and
ii. for any $\mathcal{P} \in \Pi(N) \setminus V$, there exists $\mathcal{P}' \in V$ such that $\mathcal{P}'$ indirectly dominates $\mathcal{P}$.

The first condition exhibits *internal stability*, and the second exhibits *external stability*.

Example 2.9. Consider the game of Example 2.2. We have

$$V = \{\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{2,3\},\{1\}\}\}.$$

2.3 Individual Stability

Our individual stability concepts include Nash stability and individual stability, which are intensively studied in the leading works of hedonic games, such as Banerjee *et al.* (2001) and Bogomolnaia and Jackson (2002). Diamantoudi and Xue (2003) call these concepts "non-cooperative" stability concepts and extend them to farsighted notions. We extend these (original) notions to hedonic games with externalities.

Definition 2.10. A partition $\mathcal{P}$ is *Nash stable* if there are no $i \in N$ and $S \in \mathcal{P} \cup \{\emptyset\}$ such that

- $\mathcal{P}_{i \to S} \succ_i \mathcal{P}$,

where $\mathcal{P}_{i \to S} = \{S \cup \{i\}\} \cup (\mathcal{P}|_{N \setminus (S \cup \{i\})})$. We denote the set of Nash stable partitions by $\text{Nash}(N, \{\succsim_i\}_{i \in N})$.

Definition 2.11. A partition $\mathcal{P}$ is *individually stable* if there are no $i \in N$ and $S \in \mathcal{P} \cup \{\emptyset\}$ such that

- $\mathcal{P}_{i \to S} \succ_i \mathcal{P}$, and
- for every $j \in S$, $\mathcal{P}_{i \to S} \succsim_j \mathcal{P}$.

We denote the set of individually stable partitions by $\text{IS}(N, \{\succsim_i\}_{i \in N})$.

Note that for any game $(N, \{\succsim_i\}_{i \in N})$, we have $\text{Nash}(N, \{\succsim_i\}_{i \in N}) \subseteq \text{IS}(N, \{\succsim_i\}_{i \in N})$.

Example 2.12. Consider the game of Example 2.2. We have

$$\text{Nash} = \text{IS} = \{\{1, 2, 3\}\}.$$

2.4 A General Condition for the Nonemptiness of the Projective Core

Before moving on to symmetric majority games and apex games, we offer some additional notations and a sufficient condition for the nonemptiness of the projective core, which holds for general hedonic games with externalities.

For any $i \in N$ and any $\mathcal{P} \in \Pi(N)$, let $\mathcal{D}_i(\mathcal{P})$ denote a set of partitions that player i strictly prefers to $\mathcal{P}$, namely, $\mathcal{D}_i(\mathcal{P}) = \{\mathcal{P}' \in \Pi(N) | \mathcal{P}' \succ_i \mathcal{P}\}$. A partition $\mathcal{P}'$ is *unprojectable* from $\mathcal{P}$ if for any $S \in \mathcal{P}'$, we have $\mathcal{P}'|_{N \setminus S} \neq \mathcal{P}|_{N \setminus S}$.

Example 2.13. Let $N = \{1, 2, 3, 4\}$.

- $\mathcal{P}' = \{\{1\}, \{2\}, \{3, 4\}\}$ is unprojectable from $\mathcal{P} = \{\{1, 2\}, \{3\}, \{4\}\}$, because
  for $S = \{1\}$, $\{\{2\}, \{3, 4\}\} \neq \{\{2\}, \{3\}, \{4\}\}$;
  for $S = \{2\}$, $\{\{1\}, \{3, 4\}\} \neq \{\{1\}, \{3\}, \{4\}\}$ and
  for $S = \{3, 4\}$, $\{\{1\}, \{2\}\} \neq \{\{1, 2\}\}$.
- $\mathcal{P}' = \{\{1, 2\}, \{3, 4\}\}$ is unprojectable from $\mathcal{P} = \{\{1\}, \{2, 3\}, \{4\}\}$.
- $\mathcal{P}' = \{\{1\}, \{2, 3\}, \{4\}\}$ is projectable (not unprojectable) from $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$ via $\{2, 3\}$.

We denote by $\mu(\mathcal{P})$ the set of partitions that are unprojectable from $\mathcal{P}$. Formally, $\mu(\mathcal{P}) = \{\mathcal{P}' \in \Pi(N) | \forall S \in \mathcal{P}', \mathcal{P}'|_{N \setminus S} \neq \mathcal{P}|_{N \setminus S}\}$. Moreover, for any nonempty $T \subseteq N$, we define $\mu^T(\mathcal{P}) = \{\mathcal{P}' \in \Pi(N) | \forall S \in \mathcal{P}' : S \subseteq T, \mathcal{P}'|_{N \setminus S} \neq \mathcal{P}|_{N \setminus S}\}$. Note that $\mu(\mathcal{P}) = \mu^N(\mathcal{P}) \subseteq \mu^T(\mathcal{P})$.

Proposition 2.14. Let $(N, \{\succsim_i\}_{i \in N})$ be a hedonic game with externalities. The projective core is nonempty if there exist a nonempty coalition $S \subseteq N$ and a partition $\mathcal{P} \in \Pi(N)$ with $S \in \mathcal{P}$ such that

i. $\mathcal{D}_i(\mathcal{P}) \subseteq \mu(\mathcal{P})$ for any $i \in S$, and
ii. $\mathcal{D}_i(\mathcal{P}) \subseteq \mu^{N \setminus S}(\mathcal{P})$ for any $i \in N \setminus S$.

Proof. See the Appendix. □

Every symmetric majority game and apex game satisfies this condition, as discussed below.

3 Symmetric Majority Games

3.1 Definitions and Examples

The objective of this section is to study which coalition structure (*i.e.*, partition) is myopically, farsightedly or individually stable in symmetric majority games. An n-person symmetric majority *cooperative* game is given by

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where k is the "majority," namely, the minimal number of players needed to win. In this section, we define k as follows:

$$k = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2}+1 & \text{if } n \text{ is even.} \end{cases} \tag{3.2}$$

The word "majority" usually covers not only exact k but also any natural number $k^+ \in [k, n]$. Therefore, to avoid confusion, we specifically use "exact majority" to refer to k.

Now we compute the power of each player in every partition. We consider a coalition to be a player and compute the Shapley-Shubik power index of the corresponding weighted majority game. Below, we demonstrate the computation of each player's power in partition, for instance, $\{\{12\}, \{3\}, \{4\}\}$. First, we consider a weighted majority game $[3; 2, 1, 1]$, where 3 is the quota of this game (*i.e.*, the exact majority) and $(2, 1, 1)$ is the weight based on the number of players of each coalition (*i.e.*, the coalition $\{1, 2\}$ has two votes). The S-S power index ψ of this weighted majority game is $\psi([3; 2, 1, 1]) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. Namely, the coalition $\{1, 2\}$ has power $\psi_{\{1,2\}} = \frac{2}{3}$. We equally divide the S-S index ψ by the number of players of each coalition and obtain the power index ϕ with respect to each individual player: $\phi(\{\{12\}, \{3\}, \{4\}\}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$. Formally, for any $\mathcal{P} \in \Pi(N)$ and any $i \in N$, the power of i in $\mathcal{P}$ is given by

$$\phi_i(\mathcal{P}) = \frac{1}{|S_i|}\psi_{S_i}(\mathcal{P}), \tag{3.3}$$

where S_i is the coalition in $\mathcal{P}$ which contains i as its member. For any $j \in S \in \mathcal{P}$, $\phi_S(\mathcal{P}) := \sum_{j \in S} \phi_j(\mathcal{P}) = |S|\phi_j(\mathcal{P})$ $(= \psi_S(\mathcal{P}))$.

In this section, the power index given by (3.3) is the same as the Owen value, *i.e.*, Owen (1977)'s extension of the Shapley value to games with a priori coalition structures. In general, the Owen value is given, for any cooperative game v and any partition $\mathcal{P}$ and any player i, by

$$\phi_i(v, \mathcal{P}) = E[v(\rho \cup \{i\}) - v(\rho)],$$

where $E[\cdot]$ is the expected value taken over all random orders which are consistent with the partition $\mathcal{P}$.[*4] Moreover, ρ is the set of predecessors of player i in the random order. To obtain the Owen *power*

[*4] For example, if $N = \{1, 2, 3\}$ and $\mathcal{P} = \{\{12\}, \{3\}\}$, then the all consistent orders are $[12, 3]$, $[21, 3]$, $[3, 12]$, $[3, 21]$.

index, we focus on v, given by (3.1), and k, given by (3.2). Then, for any partition $\mathcal{P} \in \Pi(N)$ and any player $i \in N$, we obtain the Owen power index of i in $\mathcal{P}$, $\phi_i(\mathcal{P})$.

For any given N, computing $\phi_i(\mathcal{P})$ with respect to every $\mathcal{P} \in \Pi(N)$ generates a list of power index profiles as Tables 1 and 2. In the tables, we use, for example, 12|3 to denote partition $\{\{1,2\},\{3\}\}$ for notational simplicity. Symbol "+" shows that the partition satisfies the stability concept (to be discussed in the following propositions). Moreover, we offer the examples of $n = 5, 6, 7$ in the Appendix. Note that all players are symmetric in the tables of symmetric majority games. In this paper, we see a list of power index profiles as a coalition formation game of symmetric majority games. We call it, simply, a game or a symmetric majority game, hereafter. Each symmetric majority game can be represented by a hedonic game with externalities. For example, Example 2.2 represents Table 1.

Table 1 The three-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
123	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$					+	+
12|3	$\frac{1}{2}$	$\frac{1}{2}$	0	+	+	+	+		+
1|2|3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$						

Table 2 The four-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
1234	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$					+	+
123|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	+	+	+	+		+
12|34	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$						
12|3|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$						
1|2|3|4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$						

Below, we show our results in symmetric majority games. Tables 1, 2 in this section and Tables 8,9,10 in the Appendix could be helpful to examine our results.

3.2 Myopic Stability

Let K be a coalition whose size is exactly k, *i.e.*, $|K| = k$, and, for any nonempty coalition $T \subseteq N$, $[T]$ be a partition of coalition T into singletons.

Proposition 3.1. Any symmetric majority game satisfies the conditions of Proposition 2.14. Moreover, the partitions $\{K\} \cup [N \setminus K]$ are always in the projective core.

Proof. See the Appendix. □

Proposition 3.2. For any symmetric majority game, a partition $\mathcal{P}$ is in the pessimistic core if and only if $\mathcal{P}$ contains an exact majority coalition K.

Proof. See the Appendix. □

Remark 3.3. For $n = 3, 4, 5, 6, 8, 10$, $C^{\text{pro}} = C^{\text{pes}}$. For $n = 7, 9, 11, ...$, $C^{\text{pro}} \subsetneq C^{\text{pes}}$.

An example of the gap between the projective core and the pessimistic core is found in Table 10 in the Appendix. This gap always appears in the coalition structure such that the partition of $N \setminus K$ is coarse (for example, $\{K, N \setminus K\}$ has the coarsest partition of $N \setminus K$). We conjecture that this gap expands as n grows.

Proposition 3.4. For any symmetric majority game with $n \geq 5$, the optimistic core is empty. For $n = 3, 4$, we have $C^{\text{opt}} = C^{\text{pro}} = C^{\text{pes}}$

Proof. From Proposition 3.2 and $C^{\text{opt}} \subseteq C^{\text{pes}}$, it follows that if a partition $\mathcal{P} \in C^{\text{opt}}$, then $\mathcal{P} \in C^{\text{pes}}$: $\mathcal{P}$ contains an exact majority coalition K. In $\mathcal{P}$, players in $N \setminus K$ get zero. For $n \geq 5$, in the optimistic view, any single player $i \in N \setminus K$ can expect to obtain a positive value by deviating alone. If $\mathcal{P} = \{K\} \cup [N \setminus K]$, then the singleton players in $[N \setminus K]$ form coalition $N \setminus K$ and obtain a positive value. For $n = 3$ or 4, $N \setminus K$ consists of just one player. He has an incentive to deviate but cannot implement this deviation. □

3.3 Farsighted Stability

Proposition 3.5. For any symmetric majority game, $V = C^{\text{pes}}$.

Proof. See the Appendix. □

Proposition 3.5 states that the farsighted vNM stable set coincides with the pessimistic core, which is the largest myopically stable set, in symmetric majority games.

Moreover, we briefly mention two more concepts related to the farsighted vNM stable set. The first concept is the *sequentially stable coalition structure* (SSCS). The SSCS is the partition that *sequentially dominates* all the other partitions by itself.[*5] In symmetric majority games, the SSCS, however, is empty for any n. Even for $n = 3$, $\{\{1, 2\}, \{3\}\}$ does not sequentially dominate, for example, $\{\{1, 3\}, \{2\}\}$.

Second, we slightly change the farsighted vNM and consider the *myopic* vNM, *i.e.*, internal and external stabilities are defined by the direct (one-step) domination. In symmetric majority games, the myopic vNM, V^{my}, coincides with farsighted vNM, V, for any odd n. For any even n, $V^{\text{my}} = V \cup \{\mathcal{P} | \hat{K}_{\mathcal{P}} \in \mathcal{P} \text{ and } |\hat{K}_{\mathcal{P}}| = \frac{n}{2}\}$, where $\hat{K}_{\mathcal{P}} = \{i \in N | \phi_i(\mathcal{P}) \geq \frac{1}{k}\}$. For example, $\mathcal{P} = \{\{1, 2\}, \{3\}, \{4\}\}$ with $\phi(\mathcal{P}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$ satisfies this condition, because we have $\frac{1}{k} = \frac{1}{3}$ and $\hat{K}_{\mathcal{P}} = \{1, 2\} \in \mathcal{P}$.

3.4 Individual Stability

Proposition 3.6. For every symmetric majority game, $\{N\}$ is Nash stable.

Proof. For any $i \in N$, we have $\phi_i(\{N\}) = \frac{1}{n}$ and $\phi_i(\{N \setminus \{i\}, \{i\}\}) = 0$. Hence, $\{N\} \succ_i \{N \setminus \{i\}, \{i\}\}$. □

[*5] For details, see Funaki and Yamato (2014).

Note that $\{N\}$ is not the unique Nash stable coalition structure for some n. The first example is $n = 9$. Consider the partition $\mathcal{P} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. For any player i, we have $\phi_i(\mathcal{P}) = \frac{1}{9}$. Without loss of generality, let 9 be the player who tries to move. Let $\mathcal{P}' = \{\{1, 2, 3, 9\}, \{4, 5, 6\}, \{7, 8\}\}$ and $\mathcal{P}'' = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}, \{9\}\}$. We have $\phi_9(\mathcal{P}') = \frac{1}{12} < \frac{1}{9}$ and $\phi_9(\mathcal{P}'') = 0 < \frac{1}{9}$.

Proposition 3.7. For every symmetric majority game, the partitions $\{K\} \cup [N \setminus K]$ are individually stable. Moreover, any partition that contains a winning coalition whose size is strictly greater than k is individually stable.

Proof. See the Appendix. □

Similar to the Nash stability, the partitions mentioned in Proposition 3.7 are not the unique individually stable coalition structures for some n. For example, in Table 10, we can find the partition 12|34|56|7, which is individually stable but not supported by Proposition 3.7.

4 Apex Games

4.1 Definitions and Examples

The purpose of this section is to analyze the stability of coalition structures in apex games. Let player 1 be a major player (the "apex player") and players in $N \setminus \{1\}$ be minor players. An apex game is given by

$$v(S) = \begin{cases} 1 & \text{if } 1 \in S \text{ and } S \setminus \{1\} \neq \emptyset, \text{ or } S = N \setminus \{1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We similarly evaluate each player's power in every partition by ϕ_i given by (3.3).[*6] Note that each apex game can be represented as a weighted majority game as follows: $[n - 1; n - 2, 1, ..., 1]$.

If (and only if) $n = 3$, the apex game is equal to the three-player symmetric majority game. Therefore, in this section, we assume $n \geq 4$. The apex game of $n = 4$ is shown in Table 3. Note that the minor players 2, 3 and 4 are symmetric.

If a partition has a winning coalition, we can easily obtain each player's power in the partition. However, if a partition does not have any winning coalition, then it is not straightforward to compute it. The following remark helps us compute the players' power when the apex player forms his singleton coalition.

Remark 4.1. Let $\mathcal{P}$ be a partition such that $\{1\} \in \mathcal{P}$. Let $\mathcal{Q}_r$ denote a partition of $N \setminus \{1\}$ consisting of r coalitions. We have

$$\phi_1(\mathcal{P}) = \frac{r-1}{r+1},$$
$$\phi_S(\mathcal{P}) = \sum_{j \in S} \phi_j(\mathcal{P}) = \frac{2}{r(r+1)} \text{ for any } S \in \mathcal{P} \setminus \{1\}.$$

[*6] In apex games, because of asymmetry among players, the power index ϕ given by (3.3) is slightly different from the Owen value. To be precise, although each coalition S's power ϕ_S is equal to $\sum_{j \in S} \text{Owen}_j$ for any $S \in \mathcal{P}$, ϕ_j is not necessarily equal to Owen_j for $j \in S$.

Table 3 The four-player apex game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
1234	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$					+	+
123|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0						
12|34	$\frac{1}{2}$	$\frac{1}{2}$	0	0	+	+		+		+
12|3|4	$\frac{1}{2}$	$\frac{1}{2}$	0	0	+	+		+		+
1|234	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$						
1|23|4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$						
1|2|3|4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$				+		

Therefore, for any $S \in \mathcal{P} \setminus \{1\}$ and any $i \in S$, we have $\phi_i(\mathcal{P}) = \frac{2}{r(r+1)|S|}$.

Below, we show our results in apex games. Because of the asymmetry of apex games, each table with $n \geq 5$ becomes too large to appear in this paper. However, by using Remark 4.1, we can obtain a table with any size of n.

4.2 Myopic Stability

Lemma 4.2. Any apex game satisfies the conditions of Proposition 2.14. Moreover, for any minor player $i^* \in N \setminus \{1\}$, $\mathcal{P} = \{\{1, i^*\}\} \cup \bar{\mathcal{Q}}_t$ is in the projective core if $t = 1, 2$, where $\bar{\mathcal{Q}}_t$ is a partition of $N \setminus \{1, i^*\}$ consisting of t coalitions. For $t \geq 3$, it is not.

Proof. See the Appendix. □

Proposition 4.3. For any apex game, a partition $\mathcal{P}$ is in the pessimistic core if and only if $\mathcal{P}$ satisfies either (i) or (ii):

i. $\mathcal{P}$ contains coalition $\{1, i^*\}$ ($i^* \in N \setminus \{1\}$),
ii. $\mathcal{P} = \{1\} \cup \mathcal{Q}_r$, where $\mathcal{Q}_r$ is a partition of $N \setminus \{1\}$ consisting of r coalitions, satisfies
  - $r \geq 3$, and
  - there exists $S \in \mathcal{Q}_r$ such that $|S| \leq \frac{2(n-1)}{r(r+1)}$.

Proof. See the Appendix. □

Proposition 4.3 shows that the pessimistic core consists of two types of partitions. In the first type of partitions (the partitions satisfying (i)), the apex player is paired with one minor player. Note that some of these partitions also belong to the projective core. On the other hand, in the second type of partitions, the apex player is isolated. This is unique to the pessimistic core. We can find such partitions for $n \geq 7$. For example, for $n = 7$, consider $\mathcal{P} = \{\{1\}, \{2\}, \{3, 4\}, \{5, 6, 7\}\}$. We have $\phi(\mathcal{P}) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{18}, \frac{1}{18}, \frac{1}{18})$. Any two-player deviation that contains the apex player 1 is refused by the apex player himself, as the apex player obtains $\frac{1}{2}$ even after the deviation. The deviation by all six minor players to form $\{2, ..., 7\}$ is rejected by the minor player 2, who already obtains $\frac{1}{6}$ in $\mathcal{P}$. In the pessimistic view, minor players do not deviate to form a coalition smaller than $n-2$ because, after their deviation (call them S), the apex player 1 and the other minor players $N \setminus (S \cup \{1\})$ form the (winning) coalition $N \setminus S$, and then, in the pessimistic view, the players in S lose and obtain 0.

Lemma 4.4. Partitions satisfying (ii) of Proposition 4.3 are not in the projective core.

Proof. See the Appendix. □

Combining Lemmas 4.2 and 4.4 together with Proposition 4.3, we obtain the following necessary and sufficient condition for the projective core.

Proposition 4.5. For any apex game, a partition $\mathcal{P}$ is in the projective core if and only if $\mathcal{P}$ satisfies the condition of Lemma 4.2: $\mathcal{P} = \{1, i^*\} \cup \mathcal{Q}_t$ $(t = 1, 2)$.

Proof. Proposition 4.3 and (2.1) imply that the partitions satisfying (ii) of Proposition 4.3 may be in the projective core. Lemma 4.4, however, shows that each partition satisfying (ii) of Proposition 4.3 is not in the projective core. Thus, the projective core is limited to the set of partitions shown in Lemma 4.2. □

Proposition 4.6. For any apex game, the optimistic core is empty.

Proof. See the Appendix. □

4.3 Farsighted Stability

We offer the result of $n = 4$. We mention the cases of $n \geq 5$ in Remark 4.8.

Proposition 4.7. For $n = 4$, $V = \mathscr{B} \cup \{\mathcal{P}^*\}$, where

$$\mathscr{B} = \{\mathcal{P} \in \Pi(N) \mid \{1, i^*\} \in \mathcal{P}, (i^* \in N \setminus \{1\})\},$$
$$\mathcal{P}^* = [N].$$

Proof. See the Appendix. □

Remark 4.8. In general, the extension of Proposition 4.7 does not hold for $n \geq 5$ because of the expansion of partitions satisfying Case C in the proof of Proposition 4.7, that is, partitions in which the apex player is isolated (for example, see Table 7). To be more precise, for $n \geq 5$, if we let $V = \mathscr{B}$, then V satisfies internal stability but not external stability for partitions in Case C. If we let $V = \mathscr{B} \cup \{\text{all partitions in C}\}$, then V satisfies external stability but not internal stability for the partitions in C. Moreover, even if we let $V = C^{\text{pes}}$ (see Proposition 4.3), then V violates internal stability.

Similar to symmetric majority games, we briefly mention the SSCS and the myopic vNM. In apex games, the SSCS is empty for any n. The myopic vNM coincides with the farsighted vNM for $n = 4$. For $n \geq 5$, the same difficulties in Remark 4.8 occur.

4.4 Individual Stability

Proposition 4.9. For any apex game, $\{N\}$ is the unique Nash stable partition.

Proof. See the Appendix. □

In contrast to symmetric majority games, we have the uniqueness of $\{N\}$ in apex games. If we change the concept of Nash stability to *farsighted* Nash stability, then no partitions satisfy the farsighted Nash stability. To define farsighted Nash stability, we consider a repetition of Definition 2.10, where each player in each step compares the current partition and the final destination he wants to reach. In this farsighted setting, every player is easier to move than the original Nash stability, and, therefore, the farsighted Nash stable set is a refinement of the original Nash stable set. As mentioned above, $\{N\}$ does not satisfy the farsighted Nash stability, as the apex player 1 can deviate from N to be the sole player and one of the minor players, i^*, moves from $N \setminus \{1\}$ to form $\{1, i^*\}$.

Proposition 4.10. For any apex game with $n \geq 5$, the partitions $\{N\}$, $[N]$ and all partitions in the projective core are individually stable. The other partitions are not individually stable. If $n = 4$, the partition $\{N\}$ and all partitions in the projective core are individually stable. The other partitions are not individually stable (namely, $[N]$ is not individually stable if $n = 4$).

Proof. See the Appendix. □

Proposition 4.10 shows that we can identify the individually stable coalition structures in apex games. The partition $[N]$ features the individual stability. In general, $[N]$ is individually stable but neither Nash stable nor in the projective core.

5 Concluding Remarks

In this paper, we define hedonic games with externalities and the stability of coalition structures. We see symmetric majority games and apex games as two specific subclasses of hedonic games with externalities. Our results are summarized in Table 4. In the following subsections, we offer the comparisons between our results and two related works.

5.1 Comparisons with Hart and Kurz (1984) and Bloch (1996) in Symmetric Majority Games

We compare our results of symmetric majority games with two eminent works: Hart and Kurz (1984) and Bloch (1996). The following two paragraphs summarize their approaches.

- The approach of Hart and Kurz (1984) is basically the same as that in our paper. However, there are three differences. The first difference is the stability concept. They use the α-, β-, γ- and δ-stability concepts. Second, they analyze stable coalition structures for some specific numbers of n. Our results hold for general n. Third, they analyze not only exact majority k but also $k^+ \in [k, n]$ for some n.

Table 4 Summary of results

	Myopic			Farsighted	Individual	
	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
Symmetric majority games	Prop. 3.1	Prop. 3.2 Rmk. 3.3	Prop. 3.4	Prop. 3.5	Prop. 3.6	Prop. 3.7
Apex games	Lma. 4.2 Lma. 4.4 Prop. 4.5	Prop. 4.3	Prop. 4.6	Prop. 4.7 Rmk. 4.8	Prop. 4.9	Prop. 4.10

- In contrast, Bloch (1996)'s approach is different from ours. He models the process of coalition formation as a noncooperative game and studies "stable" coalition structures of symmetric majority games as an application of his model. His "stability" is described as the *equilibrium coalition stricture*, which is an outcome of his noncooperative game (more specifically, a coalition structure as a result of a subgame perfect equilibrium of his noncooperative game). The results for general n are not offered in his analysis. He also analyzes not only exact majority k but also $k^+ \in [k, n]$ for some n.

We compare our results with theirs for the case of $n = 5$. Hart and Kurz (1984) show that the partitions 123|4|5 and 123|45 satisfy all the α-, β-, γ- and δ-stabilities, while other partitions satisfy none of them. Moreover, Bloch (1996) also shows that exactly the same partitions are the equilibrium coalition strictures. In our analysis, according to Table 8, the pessimistic core, the projective core and the farsighted vNM stable set coincide with their results. We have the same correspondence for $n = 6$. Although this coincidence does not necessarily hold for general n, we conjecture that Bloch's equilibrium coalition structures generally coincide with the pessimistic core and the farsighted vNM stable set in symmetric majority games from the fact that they always coincide for $3 \leq n \leq 10$.

5.2 A Comparison with Hart and Kurz (1984) in Apex Games

We compare our results for apex games with those of Hart and Kurz (1984). In apex games, these authors focus on the γ- and δ-stabilities. Their result is as follows: for any $n \geq 5$, $\{\{1\}, \{2, ..., n\}\}$ is the unique γ-stable partition, and there are no δ-stable partitions. For $n = 4$, $\{\{1\}, \{2, 3, 4\}\}$ is γ-stable (but not δ-stable), and the partitions $\{\{1, 2\}, \{3\}, \{4\}\}$ (the minor players $2, 3, 4$ are symmetric) are both γ- and δ-stable.

For $n = 4$, according to Table 3, the γ- and δ-stable partitions $\{\{1, 2\}, \{3\}, \{4\}\}$ satisfy all the stability concepts except for the optimistic core and the Nash stability. For $n \geq 4$, the γ-stable partition $\{\{1\}, \{2, ..., n\}\}$, in which all minor players form their coalition, is not stable in any sense of stability concepts we defined. This is mainly because we focus on the incentive that each minor player has to deviate from the group of minor players to merge with the apex player.

Appendix

Proof of Proposition 2.14

Proof. Assume that there exist a partition $\mathcal{P}^*$ and a coalition $S^* \in \mathcal{P}^*$ satisfying the conditions (i) and (ii) of Proposition 2.14. First, for any $S \subseteq N$ with $S \cap S^* \neq \emptyset$, consider a partition $\{S\} \cup (\mathcal{P}^*|_{N\setminus S})$. We have

$$\{S\} \cup (\mathcal{P}^*|_{N\setminus S}) \notin \mu(\mathcal{P}^*), \tag{A.1}$$

because $\{S\}\cup(\mathcal{P}^*|_{N\setminus S})$ is projectable (not unprojectable) from $\mathcal{P}^*$ via S, namely, $(\{S\}\cup(\mathcal{P}^*|_{N\setminus S}))|_{N\setminus S} = \mathcal{P}^*|_{N\setminus S}$. Hence, (i) and (A.1) imply that for any $i \in S^* \cap S$, $\{S\} \cup (\mathcal{P}^*|_{N\setminus S}) \notin \mathcal{D}_i(\mathcal{P}^*)$. Equivalently, $\mathcal{P}^* \succsim_i \{S\} \cup (\mathcal{P}^*|_{N\setminus S})$ for any $i \in S^* \cap S$.

Next, for any $S \subseteq N$ with $S \cap S^* = \emptyset$ (namely, $S \subseteq N \setminus S^*$), we have

$$\{S\} \cup (\mathcal{P}^*|_{N\setminus S}) \notin \mu^{N\setminus S}(\mathcal{P}^*). \tag{A.2}$$

Hence, (ii) and (A.2) imply that for any $i \in S$, $\{S\} \cup (\mathcal{P}^*|_{N\setminus S}) \notin \mathcal{D}_i(\mathcal{P}^*)$. It follows that $\mathcal{P}^* \succsim_i \{S\} \cup (\mathcal{P}^*|_{N\setminus S})$ for any $i \in S$. Thus, there is no S such that $\{S\} \cup (\mathcal{P}^*|_{N\setminus S}) \succ_S \mathcal{P}^*$. □

Proof of Proposition 3.1

Proof. For any given partition $\mathcal{P}^* = \{K\} \cup [N \setminus K]$, we show that $\mathcal{P}^*$ satisfies the conditions of Proposition 2.14 and it is in the projective core. We consider the following two cases: odd n and even n.

Odd, Condition (i): We show that for any $S \subseteq N$ such that $S \cap K \neq \emptyset$ and $|S| \leq k-1$, we have

$$\mathcal{P}^* = \{K\} \cup [N \setminus K] \succsim_i \{K \setminus S\} \cup \{S\} \cup [(N \setminus K) \setminus S] =: \mathcal{P}$$

for any player $i \in S \cap K$.*7 To compute $\phi_S(\mathcal{P})$, let $l = |S \cap K|$ and $r = |(N \setminus K) \setminus S|$. Note that

$$1 \leq |S| \leq k-1, \tag{A.3}$$
$$1 \leq l \leq k-1,$$
$$0 \leq r \leq n-k. \tag{A.4}$$

The number of orders that S becomes a pivot as a coalition in $\mathcal{P}$ (formally, a pivot in the corresponding weighted majority game) is

$$\begin{aligned}\sum_{j=0}^{l-1}\left[(1+j)!(r-j)!\begin{pmatrix} r \\ j \end{pmatrix} \times 2\right] &= 2\sum_{j=0}^{l-1}[(j+1)r!] \\ &= 2r!\sum_{j=1}^{l} j \\ &= r!l(1+l).\end{aligned} \tag{A.5}$$

*7 If $|S| \geq k$, then the coalition S should be a winning coalition and gets 1 in $\mathcal{P}$. Hence, $\phi_i(\mathcal{P}) \leq \frac{1}{k}$ for any $i \in S$. Since $\phi_i(\mathcal{P}^*) = \frac{1}{k}$ for any $i \in K$, we have $\mathcal{P}^* \succsim_i \mathcal{P}$ for any $i \in K \cap S$.

The number of all orders (of coalitions in $\mathcal{P}$) is $(r+2)!$. Hence, by (A.5), we have

$$\phi_S(\mathcal{P}) = \frac{r!l(1+l)}{(r+2)!} = \frac{l(1+l)}{(r+2)(r+1)}.$$

Since the number of players in S is equal to $(l+n-k-r)$, we have, for any $i \in S$,

$$\phi_i(\mathcal{P}) = \frac{l(1+l)}{(r+2)(r+1)} \cdot \frac{1}{l+n-k-r}. \tag{A.6}$$

This is increasing with respect to l. Since $|S| = l+n-k-r$, the right hand side of (A.3) implies $l \leq 2k+r-1-n$. Moreover, because of $k = \frac{n+1}{2}$, we have $l \leq r$. Hence, with respect to l, (A.6) attains its maximum at $l = r$. Assuming $l = r$, (A.6) is

$$\frac{r(1+r)}{(r+2)(r+1)} \cdot \frac{1}{r+n-k-r} = \frac{r}{r+2} \cdot \frac{1}{n-k}. \tag{A.7}$$

This is increasing with respect to r. By (A.4), substituting $r = n-k$ for (A.7), we have

$$\frac{1}{n-k+2} = \frac{2}{n+3} < \frac{2}{n+1} = \frac{1}{k}. \tag{A.8}$$

Thus, (A.6), (A.7) and (A.8) imply that for any $i \in S \cap K$, $\phi_i(\mathcal{P}) \leq \frac{1}{n-k+2} < \frac{1}{k} = \phi_i(\mathcal{P}^*)$.

Odd, Condition (ii): In the partition $\mathcal{P}^*$, the exact majority coalition K is winning. As long as K forms, the players in $N \setminus K$ get zero. In other words, it is impossible for the players in $N \setminus K$ to get a positive value by regrouping their partition within $N \setminus K$.

Even, Condition (i): The approach is the same as the case of odd n. We have (A.3) to (A.4). We change the computation of (A.5) as follows:

$$\begin{aligned}\sum_{j=0}^{l-2}\left[(1+j)!(r-j)!\begin{pmatrix} r \\ j \end{pmatrix} \times 2\right] + l!(r-l+1)!\begin{pmatrix} r \\ l-1 \end{pmatrix} &= 2\sum_{j=0}^{l-2}[(j+1)r!] + l \cdot r! \\ &= r! \cdot l^2. \end{aligned} \tag{A.9}$$

The number of all orders is $(r+2)!$. Hence, we obtain

$$\phi_S(\mathcal{P}) = \frac{r! \cdot l^2.}{(r+2)!} = \frac{l^2}{(r+2)(r+1)}.$$

Since $|S| = (l+n-k-r)$, for any $i \in S$,

$$\phi_i(\mathcal{P}) = \frac{l^2}{(r+2)(r+1)} \cdot \frac{1}{l+n-k-r}. \tag{A.10}$$

This is increasing with respect to l. From $|S| = l+n-k-r$ and (A.3), we have $l \leq 2k+r-1-n$. Moreover, because of $k = \frac{n}{2}+1$, we obtain $l \leq r+1$. Hence, with respect to l, (A.10) attains its maximum at $l = r+1$. Assuming $l = r+1$, (A.10) is

$$\frac{(r+1)^2}{(r+2)(r+1)} \cdot \frac{1}{r+1+n-k-r} = \frac{r+1}{r+2} \cdot \frac{1}{n-k+1}. \tag{A.11}$$

This is increasing with respect to r. By (A.4), substituting $r = n-k$ for (A.11), we have

$$\frac{1}{n-k+2} = \frac{2}{n+2} < \frac{2}{n+1} = \frac{1}{k}. \tag{A.12}$$

Thus, from (A.10), (A.11) and (A.12), we have the same conclusion as the case of odd n.

Even, Condition (ii): This part is exactly the same as the case of odd n. □

Proof of Proposition 3.2

Proof. For any partition $\mathcal{P}$, consider the following three cases:

A. $\mathcal{P}$ contains a majority coalition whose size is strictly greater than k;

B. $\mathcal{P}$ contains an exact majority coalition;

C. For every coalition in $\mathcal{P}$, its size is strictly less than k.

We show that the partitions in B are in the pessimistic core and the partitions in A or C are not. Let K be an exact majority coalition.

Case A. Let $\mathcal{P}$ be in Case A. Let $K^+ \in \mathcal{P}$ be a coalition such that $k^+ = |K^+| > k$. Let $\mathcal{P}^*$ be in Case B such that $K \in \mathcal{P}^*$ and $K \subseteq K^+$. Since for any $i \in K \cap K^+ = K$, $\phi_i(\mathcal{P}^*) = \frac{1}{k} > \frac{1}{k^*} = \phi_i(\mathcal{P})$, the players in $K \subsetneq K^+$ have an incentive to deviate to $\mathcal{P}^*$. Whatever partition the other players, $N \setminus K$, form, the players in K win and get $\frac{1}{k}$. Thus, each partition in Case A is not in the pessimistic core.

Case B. Let $\mathcal{P}$ be in Case B and $K \in \mathcal{P}$. First, every coalition S with $|S| \geq k$ does not deviate from $\mathcal{P}$, because there exists at least one player $i \in K \cap S$ who prefers the original partition $\mathcal{P}$ to any new partition $\mathcal{P}^*$. Next, consider a coalition S with $|S| < k$. We focus on the case of $S \cap K \neq \emptyset$, because if $S \cap K = \emptyset$, then $S \subseteq N \setminus K$ and any player i in S gets zero both before and after their deviation: $\phi_i(\mathcal{P}) = 0 = \phi_i(\mathcal{P}^*)$.

In the case of odd n, $|N \setminus S| \geq k$. Hence, in the pessimistic view, $\phi_S(\mathcal{P}^*) = 0$, because the players in $N \setminus S$ form their coalition $\{N \setminus S\}$ and win in $\mathcal{P}^*$. In other words, any S with $|S| < k$ has no incentive to deviate from $\mathcal{P}$.

In the case of even n, if $|S| \leq k - 2$, then it is the same as the case of odd n. If $|S| = k - 1 = \frac{n}{2}$, then the power of coalition S decreases as the number of coalitions except S increases.[*8] Therefore, in the pessimistic view, the players in coalition S expect that the players in $N \setminus S$ form their coalition $(N \setminus S)$, where $|S| = |N \setminus S| = \frac{n}{2} = k - 1$. Hence, each player in coalition S expects their minimum $\frac{1}{2} \cdot \frac{1}{|S|}$ per capita after their deviation. We have, for any $i \in K \cap S$,

$$\phi_i(\mathcal{P}) - \phi_i(\mathcal{P}^*) = \frac{1}{k} - \frac{1}{2}\frac{1}{|S|} = \frac{2}{n+2} - \frac{1}{n} = \frac{n-2}{n(n+2)} \geq 0,$$

where the last inequality holds as $n \geq 3$.

Case C. Let $\mathcal{P}$ be in Case C. We consider the players who obtain at least $\frac{1}{k}$ in $\mathcal{P}$. Let $\hat{K} = \{j \in N | \phi_j(\mathcal{P}) \geq \frac{1}{k}\}$ and $\hat{k} = |\hat{K}|$. We must have $\hat{k} \leq k$ because of $\sum_{j \in N} \phi_j(\mathcal{P}) = 1$. Moreover, we claim that $\hat{k} \neq k$ as follows.

Claim 1 $\hat{k} \leq k - 1$.

Proof. Assume that $\hat{k} = k$. Let $\mathcal{Z}$ be a partition of $\hat{K}$, namely, $\mathcal{Z}$ is a subpartition of $\mathcal{P}$. Now, $\mathcal{Z}$ must consist of at least two coalitions, because $\mathcal{Z}$ is a partition of exactly k players and for every coalition in $\mathcal{P}$ its size is strictly less than k. (and $\mathcal{Z}$ is a subpartition of $\mathcal{P}$). We consider the following order of coalitions in $\mathcal{P}$. First, we arrange the coalitions in $\mathcal{Z}$ in order of their size (the biggest coalition in $\mathcal{Z}$ is

[*8] In general, for any S with $|S| = \frac{n}{2}$, we have $\phi_S(\mathcal{P}) = \frac{r-1}{r+1}$, where r is the number of coalitions in $\mathcal{P} \setminus S$.

located at the top) and remove (one of) the smallest coalition(s) in $\mathcal{Z}$ from the order. At least $\frac{k}{2}$ players are now lined up. Moreover, we additionally arrange the coalitions which are not in $\mathcal{Z}$. The number of the players who are not in $\mathcal{Z}$ (which is the partition of $\hat{K}$) is $n-\hat{k}=n-k=\frac{n}{2}-1$. There must exists a coalition which becomes a pivot and is not in $\mathcal{Z}$. In other words, the power index, ϕ, assigns a positive value to this coalition. This, however, contradicts $\hat{k}=k$, because $\hat{k}=k$ implies that the players not in $\hat{K}$ get zero. □

We assume $\hat{k} \leq k-1$ hereafter.

In the case of odd n, the fact $\hat{k} \leq k-1$ implies that $n-\hat{k} \geq n-(k-1)$: in $\mathcal{P}$, the number of players who get strictly less than $\frac{1}{k}$ is greater than $n-(k-1)=n-(\frac{n+1}{2}-1)=k$. Hence, these players have an incentive to form their coalition S such that $|S|=k$ and deviate from $\mathcal{P}$.

In the case of even n, if $\hat{k} \leq k-2$, then it is the same as the case of odd n. If $\hat{k}=k-1=\frac{n}{2}$, the other $\frac{n}{2}$ players (namely, $N \setminus \hat{K}$) obtain strictly less than $\frac{1}{k}$. In the pessimistic view, these players in $N \setminus \hat{K}$ have an incentive to form a deviating coalition S only when $S=N \setminus \hat{K}$ with $|S|=n-\hat{k}=\frac{n}{2}$, because for any S with $|S|<n-\hat{k}=\frac{n}{2}$, $|N \setminus S|>k$ and S gets zero after the deviation. If $S=N \setminus \hat{K}$ and $|S|=\frac{n}{2}$, then the deviating players in S expect to get $\frac{1}{2}\frac{1}{|S|}=\frac{1}{n}$ per capita after their deviation. Therefore, it suffices to show that $\phi_i(\mathcal{P})<\frac{1}{n}$ for any $i \in N \setminus \hat{K}$.

Claim 2 If $\hat{k}=k-1=\frac{n}{2}$, then $\phi_i(\mathcal{P})<\frac{1}{n}$ for any $i \in N \setminus \hat{K}$.

Proof. We denote by t the number of players $i \in N \setminus \hat{K}$ such that $\phi_i(\mathcal{P}) \geq \frac{1}{n}$. We have

$$\hat{k}\cdot\frac{1}{k}+t\cdot\frac{1}{n}+0 \leq \sum_{j\in\hat{K}}\phi_j(\mathcal{P})+\sum_{j\in T}\phi_j(\mathcal{P})+\sum_{j\notin(\hat{K}\cup T)}\phi_j(\mathcal{P})=1, \text{ and}$$
$$\hat{k}\cdot\frac{1}{k}+t\cdot\frac{1}{n}=\frac{n}{n+2}+\frac{t}{n}.$$

Equivalently, we must have $t \leq \frac{2n}{n+2}$. As $n \geq 3$, $1 \leq \frac{2n}{n+2}<2$. Hence, t should be 0 or 1. We assume $t=1$ and call the player t. The player t forms his singleton coalition in $\mathcal{P}$, because if he is in a coalition consisting of more than two players, *symmetry within a coalition* implies that his partners also obtain at least $\frac{1}{n}$, which contradicts $t=1$.[*9] Moreover, similarly, *symmetry across coalitions* demands that the player t's singleton coalition is the only singleton coalition in $\mathcal{P}$.[*10] In other words, the other coalitions consist of at least two players. We take a coalition $S' \in \mathcal{P}$ such that $|S'| \geq 2$, $S' \subseteq N \setminus \hat{K}$. Each player in S' gets strictly less than $\frac{1}{k}$ in $\mathcal{P}$ because they are not members of $\hat{K}$. By *local monotonicity* and inequality $|S'|>|\{t\}|=1$, we obtain $\phi_{S'}(\mathcal{P}) \geq \phi_t(\mathcal{P})=\frac{1}{n}$.[*11] Hence, we have

$$\hat{k}\cdot\frac{1}{k}+\frac{1}{n}+\frac{1}{n} \leq \sum_{j\in\hat{K}}\phi_j(\mathcal{P})+\phi_t(\mathcal{P})+\phi_{S'}(\mathcal{P})+\sum_{j\notin(\hat{K}\cup\{t\}\cup S')}\phi_j(\mathcal{P})=1, \text{ and}$$
$$\hat{k}\cdot\frac{1}{k}+\frac{1}{n}+\frac{1}{n}=\frac{n}{n+2}+\frac{2}{n}=\frac{n^2+2n+4}{n^2+2n}>1.$$

[*9] This property and the two properties below are ascribed to the Owen value. For any partition $\mathcal{P}$, any coalition $S \in \mathcal{P}$ ($|S| \geq 2$) and any $i,j \in S$, we have $\phi_i(\mathcal{P})=\phi_j(\mathcal{P})$. See Hart and Kurz (1984).

[*10] For any partition $\mathcal{P}$ and any coalition $S,T \in \mathcal{P}$, if $|S|=|T|$ then $\sum_{j\in S}\phi_j(\mathcal{P})=\sum_{j\in T}\phi_j(\mathcal{P})$. See Hart and Kurz (1984).

[*11] For any partition $\mathcal{P}$ and any coalition $S,T \in \mathcal{P}$, if $|S| \geq |T|$ then $\sum_{j\in S}\phi_j(\mathcal{P}) \geq \sum_{j\in T}\phi_j(\mathcal{P})$. Note that this implies symmetry across coalitions. See Alonso-Meijide et al. (2009).

This is a contradiction. Hence, we have $t = 0$, *i.e.*, every player in $N \setminus \hat{K}$ gets strictly less than $\frac{1}{n}$ in $\mathcal{P}$. $\square$

Thus, in $\mathcal{P}$, the players in $N \setminus \hat{K}$ have an incentive to from their coalition $N \setminus \hat{K}$ and deviate to get $\frac{1}{n}$ even in the pessimistic view. $\square$

Proof of Proposition 3.5

Proof. For any partition $\mathcal{P}$, consider the following three cases:

A. $\mathcal{P}$ contains a majority coalition whose size is strictly greater than k;
B. $\mathcal{P}$ contains an exact majority coalition;
C. For every coalition in $\mathcal{P}$, its size is strictly less than k.

Let V be the set of partitions satisfying B (namely, the pessimistic core; see Proposition 3.2).
Internal stability. Let $\mathcal{P}, \mathcal{P}'$ be any two different partitions in V. Let $K \in \mathcal{P}$ and $K' \in \mathcal{P}'$ ($K \neq K'$ and $k = k'$). Since K and K' are exact majority coalitions, we have $K \cap K' \neq \emptyset$. Now, assume that there exists a path from $\mathcal{P}$ to $\mathcal{P}'$ satisfying the both conditions of Definition 2.7. In the path, there is at least one coalition S^j and $\mathcal{P}^j$ (at step j) such that $K \cap K' \subseteq S^j$ and $\mathcal{P}' \succ_{S^j} \mathcal{P}^j$; otherwise, $K = K'$. However, we have $\phi_i(\mathcal{P}^j) = \frac{1}{k} = \frac{1}{k'} = \phi_i(\mathcal{P}')$ for any $i \in K \cap K'$. This is a contradiction.

For any K and any two different partitions $\mathcal{P}, \mathcal{P}'$ such that $K \in \mathcal{P}$ and $K \in \mathcal{P}'$, we have $\phi_i(\mathcal{P}) = \phi_i(\mathcal{P}')$ for any player $i \in N$. Thus, there is no indirect dominance within V.

External stability. We show that for any partition $\mathcal{P}$ satisfying A or C, there exists a partition $\mathcal{P}^* \in V$ such that $\mathcal{P}^*$ indirectly dominates $\mathcal{P}$.
Case A. For any $\mathcal{P}$ in Case A, let $K^+ \in \mathcal{P}$ denote a majority coalition whose size k^+ is strictly greater than k. There exists a partition $\mathcal{P}^* \in V$ and an exact majority coalition $K \in \mathcal{P}$ such that $K \subset K^+$ and for any $i \in K \subsetneq K^+$, $\phi_i(\mathcal{P}^*) = \frac{1}{k} > \frac{1}{k^*} = \phi_i(\mathcal{P})$. Thus, $\mathcal{P}^*$ (in)directly dominates $\mathcal{P}$.
Case C. Let $\mathcal{P}$ be a partition in Case C. We consider the set of players who obtain at least $\frac{1}{k}$ in $\mathcal{P}$. Let $\hat{K} = \{j \in N | \phi_j(\mathcal{P}) \geq \frac{1}{k}\}$ and $\hat{k} = |\hat{K}|$. As Claim 1 in the proof of Proposition 3.2, we have $\hat{k} \leq k - 1$.

In the case of odd n, $\hat{k} \leq k - 1$ means that at least k players get strictly less than $\frac{1}{k}$ in $\mathcal{P}$. They form their coalition S consisting of k players and directly deviate to partition $\{S\} \cup (\mathcal{P}|_{N \setminus S}) \in V$.

In the case of even n, if $\hat{k} \leq k - 2$, then it is the same as the case of odd n. If $\hat{k} = k - 1 = \frac{n}{2}$, then the other $\frac{n}{2}$ players (namely, $N \setminus \hat{K}$) obtain strictly less than $\frac{1}{k}$. Let $\mathcal{Q}$ be a partition of $N \setminus \hat{K}$, namely $\mathcal{Q}$ is a subpartition of $\mathcal{P}$. We first show the following claim.
Claim 3 $|\mathcal{Q}| \geq 2$.

Proof. Assume that $|\mathcal{Q}| = 1$ (*i.e.*, $\mathcal{Q} = \{N \setminus \hat{K}\}$, which contains $k - 1$ players). We have

$$\phi_{N \setminus \hat{K}}(\mathcal{P}) = 1 - (|\mathcal{P}| - 1)\frac{(|\mathcal{P}| - 2)!}{|\mathcal{P}|!} = 1 - (|\mathcal{P}| - 1)\frac{1}{|\mathcal{P}|(|\mathcal{P}| - 1)} = \frac{|\mathcal{P}| - 1}{|\mathcal{P}|}, \tag{A.13}$$

for $2 \leq |\mathcal{P}| \leq k$.[*12] This attains its minimum at $|\mathcal{P}| = 2$. Hence, $\phi_{N\setminus\hat{K}}(\mathcal{P}) \geq \frac{1}{2}$. However, since $\phi_i(\mathcal{P}) \geq \frac{1}{k}$ for any $i \in \hat{K}$, we must have $\phi_{N\setminus\hat{K}}(\mathcal{P}) = 1 - \phi_{\hat{K}}(\mathcal{P}) \leq 1 - (\frac{1}{k} \cdot \hat{k}) = 1 - (\frac{k-1}{k}) = \frac{1}{k}$, which contradicts that $\phi_{N\setminus\hat{K}}(\mathcal{P}) \geq \frac{1}{2}$ as $n \geq 4$ ($k \geq 3$ when $n \geq 4$, and 4 is the smallest even n). □

Next, we construct a path from $\mathcal{P}$ to $\mathcal{P}^* \in V$ via a partition $\bar{\mathcal{P}}$ given by

$$\bar{\mathcal{P}} = (\mathcal{P}|_{\hat{K}}) \cup \{N \setminus \hat{K}\},$$
$$\mathcal{P}^* = (\mathcal{P}|_{\hat{K}\setminus\{i^*\}}) \cup \{(N \setminus \hat{K}) \cup \{i^*\}\},$$

where i^* is any player in $\hat{K}$. Note that $\mathcal{P}^* \in V$ because coalition $(N \setminus \hat{K}) \cup \{i^*\}$ is an exact majority.

The first deviating coalition (from $\mathcal{P}$ to $\bar{\mathcal{P}}$) is $N \setminus \hat{K}$. For every player $i \in N \setminus \hat{K}$, $\mathcal{P}^* \succ_i \mathcal{P}$, because $\phi_i(\mathcal{P}) < \frac{1}{k}$ and $\phi_i(\mathcal{P}^*) = \frac{1}{k}$. Moreover, by $|\mathcal{Q}| \geq 2$, the all players in $\mathcal{Q}$ (which is a partition of $N \setminus \hat{K}$) can form one coalition $N \setminus \hat{K}$ and move to $\bar{\mathcal{P}}$.

The second deviating coalition (from $\bar{\mathcal{P}}$ to $\mathcal{P}^*$) is $(N \setminus \hat{K}) \cup \{i^*\}$. We show that for any $i \in N \setminus \hat{K}$, $\phi_i(\bar{\mathcal{P}}) < \frac{1}{k}$ (for i^*, see Claim 5).

Claim 4 In $\bar{\mathcal{P}}$, for any $i \in N \setminus \hat{K}$, $\phi_i(\bar{\mathcal{P}}) < \frac{1}{k} = \phi_i(\mathcal{P}^*)$.

Proof. We first show that $\phi_i(\bar{\mathcal{P}}) \leq \frac{1}{k}$. We have

$$\frac{1}{k} - \phi_i(\bar{\mathcal{P}}) = \frac{1}{k} - \frac{|\bar{\mathcal{P}}| - 1}{|\bar{\mathcal{P}}|} \cdot \frac{1}{k-1} = \frac{1}{k(k-1)} \left(\frac{k}{|\bar{\mathcal{P}}|} - 1 \right) \geq 0,$$

where the first equality holds in the same manner with (A.13) and the final inequality follows from $k \geq |\bar{\mathcal{P}}|$. Hence, if $k = |\bar{\mathcal{P}}|$, then $\phi_i(\bar{\mathcal{P}}) = \frac{1}{k}$. Below, we show that this case does not occur. Assume that $|\bar{\mathcal{P}}| = k$. Note that $\hat{k} = k - 1$. From $\bar{\mathcal{P}} = (\mathcal{P}|_{\hat{K}}) \cup \{N \setminus \hat{K}\}$, it follows that $\mathcal{P}|_{\hat{K}}$ must be a partition of $\hat{K}$ into $\hat{k}$ singletons, which means that in the first partition $\mathcal{P}$, the players in $\hat{K}$ are singletons. Local monotonicity of the Owen power index demands that for any $i \in \hat{K}$ and any $j \in N \setminus \hat{K}$, $\phi_i(\mathcal{P}) \leq \phi_j(\mathcal{P})$, because $i \in \hat{K}$ forms his singleton coalition in $\mathcal{P}$ and $j \in N \setminus \hat{K}$ belongs to a coalition consisting of at least one player in $\mathcal{P}$. This contradicts that for $i \in \hat{K}$, $\phi_i(\mathcal{P}) \geq \frac{1}{k}$ and for $j \in N \setminus \hat{K}$, $\phi_j(\mathcal{P}) < \frac{1}{k}$. □

We next show that in $\bar{\mathcal{P}}$, there exists $i^* \in \hat{K}$ who agrees with the second deviation.

Claim 5 In $\bar{\mathcal{P}}$, there exists $i^* \in \hat{K}$ such that $\phi_{i^*}(\bar{\mathcal{P}}) < \frac{1}{k} = \phi_{i^*}(\mathcal{P}^*)$.

Proof. Assume that for every $i \in \hat{K}$, $\phi_i(\bar{\mathcal{P}}) \geq \frac{1}{k}$. We have $\phi_{N\setminus\hat{K}}(\bar{\mathcal{P}}) = \frac{|\bar{\mathcal{P}}|-1}{|\bar{\mathcal{P}}|}$ ($2 \leq |\bar{\mathcal{P}}| \leq k$) in the same manner with (A.13). This attains its minimum $\frac{1}{2}$ at $|\bar{\mathcal{P}}| = 2$ (namely, $\phi_{N\setminus\hat{K}}(\bar{\mathcal{P}}) = \sum_{j\in N\setminus\hat{K}} \phi_j(\bar{\mathcal{P}}) \geq \frac{1}{2}$). Hence,

$$\sum_{j\in N} \phi_j(\bar{\mathcal{P}}) = \sum_{j\in\hat{K}} \phi_j(\bar{\mathcal{P}}) + \sum_{j\in N\setminus\hat{K}} \phi_j(\bar{\mathcal{P}}) \geq (k-1)\frac{1}{k} + \frac{1}{2} = \frac{3}{2} - \frac{1}{k}.$$

This, however, contradicts $\sum_{j\in N} \phi_j(\bar{\mathcal{P}}) = 1$ as $k \geq 3$ (*i.e.*, $n \geq 4$). □

Thus, for every player $i \in (N \setminus \hat{K}) \cup \{i^*\}$, $\mathcal{P}^* \succ_i \bar{\mathcal{P}}$. □

[*12] In general, for any partition $\mathcal{P}$ and any coalition $S \in \mathcal{P}$ with $|S| = k - 1$, we have $\phi_S(\mathcal{P}) = \frac{|\mathcal{P}|-1}{|\mathcal{P}|}$.

Proof of Proposition 3.7

Proof. We first consider $\{K\} \cup [N \setminus K]$. Although every singleton player has an incentive to join coalition K, all players in K refuse it. Moreover, every singleton player has no incentive to merge with any other singleton player, because he gets zero even after the merge. Every player in K has no incentive to deviate alone from K, because he gets $\frac{n-2k+3}{(n-k+1)(n-k+2)}$ $(< \frac{1}{k})$ after his deviation.[*13] Every player in K has no incentive to merge with any singleton player, because he gets $\frac{1}{(n-k+1)(n-k)}$ $(\leq \frac{1}{k}$ for $n \geq 3$, odd) or $\frac{1}{2(n-k+1)(n-k)}$ $(< \frac{1}{k}$ for $n \geq 4$, even) after the merge.[*14] Next, consider a partition which contains a winning coalition whose size is strictly greater than k. The partition $\{N\}$ is individually stable because it is Nash stable. For a partition which contains a winning coalition K^+ with $k^+ > k$, every player i in K^+ has no incentive to deviate from K^+ because $K^+ \setminus \{i\}$ is still winning. Although every singleton player in $N \setminus K^+$ has an incentive to join coalition K^+, all players in K^+ refuse it. □

Proof of Lemma 4.2

Proof. We consider a coalition W deviating from $\mathcal{P}_1 = \{\{1, i^*\} \cup \bar{\mathcal{Q}}_1\}$ or $\mathcal{P}_2 = \{\{1, i^*\} \cup \bar{\mathcal{Q}}_2\}$ to a partition $\mathcal{P}^*$ which contains the coalition W. First, assume that W is a winning coalition in $\mathcal{P}^*$. Note that in both of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, the power index is $\phi_1 = \frac{1}{2}$, $\phi_{i^*} = \frac{1}{2}$ and $\phi_i = 0$ for any $i \in N \setminus \{1, i^*\}$.

- If the apex player 1 is in W, $W = \{1\} \cup S$ for $S \subseteq N \setminus \{1\}$ and $S \neq \emptyset$. Hence, $|W| \geq 2$: $\phi_1(\mathcal{P}^*) \leq \frac{1}{2}$. Although the deviating coalition W contains the apex player 1, he does not agree with this deviation, because $\phi_1(\mathcal{P}_1) = \phi_1(\mathcal{P}_2) = \frac{1}{2} \geq \phi_1(\mathcal{P}^*)$.
- If the apex player 1 is not in W, $W = N \setminus \{1\}$. Hence, in $\mathcal{P}^*$, each minor player obtains $\frac{1}{n-1}$. We have $\frac{1}{n-1} \leq \frac{1}{2}$ when $n \geq 4$. Thus, the minor player i^*, who is paired with the apex player in the original partition $\mathcal{P}_1$ or $\mathcal{P}_2$, does not agree with the deviation.

We next consider W which is not a winning coalition in $\mathcal{P}^*$, namely, $W \in \mathcal{P}^*$, $W \neq \{1\} \cup S$ for $S \subseteq N \setminus \{1\}$, and $W \neq N \setminus \{1\}$. In $\mathcal{P}^*$, the apex player forms his singleton coalition and the minor player i^* is no longer paired with the apex player 1. Let r denote the number of coalitions consisting of minor players in $\mathcal{P}^*$.

[*13] We have the following inequalities: for $n \geq 3$,

$$\frac{1}{k} - \frac{n-2k+3}{(n-k+1)(n-k+2)} = \begin{cases} \frac{2}{n+3} > 0 & \text{if } n \text{ is odd } (k = \frac{n+1}{2}), \\ \frac{2(n-2)}{n(n+2)} > 0 & \text{if } n \text{ is even } (k = \frac{n}{2} + 1). \end{cases}$$

[*14] We have the following inequalities: for $n \geq 3$,

$$\frac{1}{k} - \frac{1}{(n-k+1)(n-k)} = \frac{1}{4k(n-k+1)(n-k)}(n+1)(n-3) \geq 0, \ (n \geq 3);$$
$$\frac{1}{k} - \frac{1}{2(n-k+1)(n-k)} = \frac{1}{4k(n-k+1)(n-k)}(n^2-3n-2) > 0, \ (n \geq 4).$$

- The apex player 1's power in $\mathcal{P}$ is $\phi_1(\mathcal{P}^*) = \frac{r-1}{r+1}$ by Remark 4.1. If $t = 1$ (or, $\bar{\mathcal{Q}}_1$), then $r = 2$. If $t = 2$ (or, $\bar{\mathcal{Q}}_2$), then $r = 2$ or 3. In any case, $\frac{r-1}{r+1} \leq \frac{1}{2}$ as $r = 2, 3$. Thus, the apex player has no incentive to deviate to $\mathcal{P}^*$ alone.
- The power of a coalition consisting of minor players in $\mathcal{P}$, *i.e.*, the sum of each player's power in the coalition, is computed as $\frac{2}{r(r+1)}$ by Remark 4.1. If $t = 1$, then $r = 2$. If $t = 2$, then $r = 2$ or 3. In any case, $\frac{2}{r(r+1)} \leq \frac{1}{2}$ as $r = 2, 3$. Thus, the minor player i^* does not agree with any deviation including him.

If $t \geq 3$ ($r \geq 4$), then the apex player always has an incentive to deviate alone, namely, $\frac{r-1}{r+1} > \frac{1}{2}$. □

Proof of Proposition 4.3

Proof. For any partition $\mathcal{P}$, consider the following three cases:

A. $\mathcal{P}$ contains a coalition which is a proper superset of $\{1, i^*\}$ ($i^* \in N \setminus \{1\}$);
B. $\mathcal{P}$ contains $\{1, i^*\}$ ($i^* \in N \setminus \{1\}$);
C. $\mathcal{P}$ is neither A nor B (the apex player forms his singleton coalition).

Note that the condition (i) is Case B, and the condition (ii) is a subcase of Case C. We show that the partitions in B and the specific case of C are in the pessimistic core and the other partitions are not.

Case A. Let $\mathcal{P}$ be a partition in Case A. Partition $\mathcal{P}$ contains a coalition which is a proper superset of $\{1, i^*\}$ ($i^* \in N \setminus \{1\}$). We denote the partition by K^+, therefore, $|K^+| \geq 3$. The coalition K^+ is winning in $\mathcal{P}$ and each player in K^+ obtains at most $\frac{1}{3}$. Hence, the coalition $\{1, i^*\} \subsetneq K^+$ has an incentive to deviate and obtain $(\frac{1}{2}, \frac{1}{2})$ in the pessimistic view.

Case B. In $\mathcal{P}$, players 1 and i^* obtain $\frac{1}{2}$ respectively, and the other minor players obtain 0. Now, we offer Table 5, which shows a list of power indices in the pessimistic view with respect to each deviation. Player i means a minor player. Assuming $n \geq 4$, in view of Table 5, each of 1 and i^* cannot obtain more

Table 5 A list of power indices in the pessimistic view

S	ϕ_S	$\phi_S/\lvert S\rvert$	S	ϕ_S	$\phi_S/\lvert S\rvert$
$\{1\}$	0	0	$\{i\}$	0	0
$\{1, i\}$	1	$\frac{1}{2}$	$\{i_1, i_2\}$	0	0
$\{1, i_1, ..., i_{n-2}\}$	1	$\frac{1}{n-1}$	$\{i_1, ..., i_{n-2}\}$	0	0
$\{1, i_1, ..., i_{n-1}\} = N$	1	$\frac{1}{n}$	$\{i_1, ..., i_{n-1}\}$	1	$\frac{1}{n-1}$

than $\frac{1}{2}$ per capita by any deviation. The other $n - 2$ minor players cannot obtain a positive value by any deviation consisting of at most $n - 2$ minor players. Thus, the partitions in B are in the pessimistic core.

Case C. Let $\mathcal{P} = \{1\} \cup \mathcal{Q}_r$, where r is the number of the coalitions consisting of minor players. We divide Case C into three small cases: C1, $r = 1$; C2, $r = 2$; and C3, $r \geq 3$.

- For C1, the apex player obtains 0 and each minor player obtains $\frac{1}{n-1} < \frac{1}{2}$ in $\mathcal{P}$. Hence, the apex player and one of the minor players can form their coalition and deviate to get $(\frac{1}{2}, \frac{1}{2})$.

- For C2, by Remark 4.1, the apex player obtains $\frac{1}{3}$ and each coalition of minor players gets $\frac{1}{3}$ in $\mathcal{P}$. Hence, the apex player and one of the minor players can form their coalition and deviate to get $(\frac{1}{2}, \frac{1}{2})$.
- For C3, by Remark 4.1, the apex player obtains at least $\frac{1}{2}$. Hence, we rule out deviations including the apex player. If there exists $S \in \mathcal{Q}_r$ such that $|S| \leq \frac{2(n-1)}{r(r+1)}$, then for any $i \in S$, we have $\phi_i(\mathcal{P}) \geq \frac{1}{n-1} (= \frac{2}{r(r+1)} / \frac{2(n-1)}{r(r+1)})$. Thus, any possible deviation should contain at most $n-2$ minor players. However, in view of Table 5, such deviations give zero to the deviating minor players. Thus, in the pessimistic view, no coalition has an incentive to deviate from this partition. If there is no $S \in \mathcal{Q}_r$ such that $|S| \leq \frac{2(n-1)}{r(r+1)}$, then for any $S \in \mathcal{Q}_r$ and any $i \in S$, we have $\phi_i(\mathcal{P}) < \frac{1}{n-1}$. Thus, all minor players form the winning coalition and jointly deviate to C1.

□

Proof of Lemma 4.4

Proof. Let $\mathcal{P} = \{1\} \cup \mathcal{Q}_r$ be a partition satisfying (ii) of Proposition 4.3. First assume that the subpartition $\mathcal{Q}_r$ contains a coalition S such that $|S| \geq 2$. We show that each player in S has an incentive to deviate alone form $\mathcal{P}$. For any player $i \in S$, $\phi_i(\mathcal{P}) = \frac{2}{r(r+1)|S|}$. After his solo deviation, he gets $\frac{2}{(r+1)(r+2)}$. Now, we have

$$\begin{aligned}\frac{2}{(r+1)(r+2)} - \frac{2}{r(r+1)|S|} &= \frac{2r|S| - (r+2)}{r(r+1)(r+2)|S|} \\ &= \frac{r(2|S|-1)-2}{r(r+1)(r+2)} \\ &> 0,\end{aligned}$$

where the final inequality follows form $r \geq 3$ and $|S| \geq 2$. This implies that each player in S has an incentive to deviate from $\mathcal{P}$.

If $\mathcal{Q}_r$ is a partition of $N \setminus \{1\}$ into singletons, then we have $\phi_i(\mathcal{P}) = \frac{2}{n(n-1)} < \frac{1}{n-1}$ for any $i \in N \setminus \{1\}$, as $n \geq 4$. Thus, all minor players jointly deviate to $\{\{1\}, N \setminus \{1\}\}$ and obtain $\frac{1}{n-1}$ per capita. □

Proof of Proposition 4.6

Proof. The following Table 6 shows a list of power indices in the optimistic view with respect to each deviation. Player i means a minor player. We focus on the partitions in the projective core because of $C^{\text{opt}} \subseteq C^{\text{pro}}$. Let $\mathcal{P}$ be a partition in the projective core. There exist $n-2$ minor players who obtain zero in $\mathcal{P}$. In view of Table 6, one of them has an incentive to deviate alone and get $\frac{1}{3}$ in the optimistic view. If all minor players who get zero are singletons in $\mathcal{P}$, then they jointly deviate and get $\frac{1}{3(n-2)} > 0$ per capita. □

Table 6 A list of power indices in the optimistic view

S	ϕ_S	$\phi_S/\lvert S\rvert$	S	ϕ_S	$\phi_S/\lvert S\rvert$
$\{1\}$	$\frac{n-2}{n}$	$\frac{n-2}{n}$	$\{i\}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{1, i\}$	1	$\frac{1}{2}$	$\{i_1, i_2\}$	$\frac{1}{3}$	$\frac{1}{3\cdot 2}$
$\{1, i_1, ..., i_{n-2}\}$	1	$\frac{1}{n-1}$	$\{i_1, ..., i_{n-2}\}$	$\frac{1}{3}$	$\frac{1}{3(n-2)}$
$\{1, i_1, ..., i_{n-1}\} = N$	1	$\frac{1}{n}$	$\{i_1, ..., i_{n-1}\}$	1	$\frac{1}{n-1}$

Proof of Proposition 4.7

Proof. For any partition $\mathcal{P}$, consider the following three cases:

A. $\mathcal{P}$ contains a coalition which is a proper superset of $\{1, i^*\}$ ($i^* \in N \setminus \{1\}$);
B. $\mathcal{P}$ contains $\{1, i^*\}$ ($i^* \in N \setminus \{1\}$);
C. $\mathcal{P}$ is neither A nor B (the apex player forms his singleton coalition).

Let $n = 4$ and $V = \mathrm{B} \cup [N]$. The following table nay be useful. Note that the minor players 2, 3 and 4 are symmetric.

Table 7 The three cases in the four-player apex game

	$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4
A	1234	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
A	123|4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
B (V)	12|34	$\frac{1}{2}$	$\frac{1}{2}$	0	0
B (V)	12|3|4	$\frac{1}{2}$	$\frac{1}{2}$	0	0
C	1|234	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
C	1|23|4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
C (V)	1|2|3|4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Internal stability. Let $\mathcal{P}, \mathcal{P}'$ be any two different partitions in B. Let $\{1, i^*\} \in \mathcal{P}$ and $\{1, i^{**}\} \in \mathcal{P}'$ ($i^* \neq i^{**}$). We have $\{1, i^*\} \cap \{1, i^{**}\} = \{1\}$. Now, assume that there exists a path from $\mathcal{P}$ to $\mathcal{P}'$ satisfying the both conditions of Definition 2.7. Player i^{**} must prefer $\mathcal{P}'$ to $\mathcal{P}$. However, both 1 and i^* have no incentive to deviate together with i^{**} because they obtain $\frac{1}{2}$ in $\mathcal{P}$. Moreover, any deviation without 1 and i^* does not make the path to $\mathcal{P}'$. This is a contradiction.

Let $\mathcal{P}$ be a partition in B and $\mathcal{P}' = [N]$. Two minor players prefer $\mathcal{P}' = [N]$. However, any path starting from these two minor players does not reach $\mathcal{P}' = [N]$.[*15] Moreover, conversely, there is no path from $\mathcal{P}' = [N]$ to any partition in B, because the apex player disagrees with the direct deviation and at least one minor player disagrees with any indirect deviating path.

[*15] For example, we call the deviating minor players 3 and 4 as in Table 7. The path stops at $\{\{1, 2\}, \{3, 4\}\}$ if it starts from $\{\{1, 2\}, \{3\}, \{4\}\}$; the path stops at $\{\{1, 2\}, \{3\}, \{4\}\}$ if it starts from $\{\{1, 2\}, \{3, 4\}\}$.

External stability. Let $\mathcal{P}$ be a partition in A or C$\backslash[N]$. In view of Table 7, a coalition $\{1, i^*\}$ can directly deviates from $\mathcal{P}$ to a partition in B$\subsetneq V$. □

Proof of Proposition 4.9

Proof. First, the partition $\{N\}$ is Nash stable, because if a player deviates alone from $\{N\}$ (namely, changes his affiliation from $\{N\}$ to $\emptyset$), then he gets zero which is strictly less than $\frac{1}{n}$.

Next, we show that any partition $\mathcal{P} \in \Pi(N) \setminus \{N\}$ is not Nash stable. If $\mathcal{P}$ has a winning coalition consisting of the apex player and at least one minor player, then there is at least one minor player i who gets zero outside the winning coalition. This minor player i has an incentive to merge with the winning coalition and obtains a positive value.

We show that if the apex player forms his singleton coalition in $\mathcal{P}$, every minor player has an incentive to merge with the apex player. Let $\mathcal{Q}_r$ denote a partition of $N \setminus \{1\}$ consisting of r coalitions. For any coalition $S \in \mathcal{Q}_r$ and any $i \in S$, we have, by Remark 4.1,

$$\phi_i(\mathcal{P}) = \frac{2}{r(r+1)|S|} < \frac{1}{2}, \ (r \geq 2),$$

where $\frac{1}{2}$ is the power which the minor player i can obtain by merging with 1 and forming coalition $\{1, i\}$. When $r = 1$, $|S| = n - 1$. Hence, we have $\frac{2}{r(r+1)(n-1)} = \frac{1}{n-1} < \frac{1}{2}$ for any $n \geq 4$. □

Proof of Proposition 4.10

Proof. First, $\{N\}$ is individually stable for any n, because it is Nash stable.

Next, let $\mathcal{P}$ be a partition in the projective core: $\mathcal{P} = \{1, i^*\} \cup \bar{\mathcal{Q}}_1$ or $\{1, i^*\} \cup \bar{\mathcal{Q}}_2$, where $\bar{\mathcal{Q}}_t$ is a partition of $N \setminus \{1, i^*\}$ consisting of t coalitions (these are in the projective core; see Proposition 4.5.). There are six possible moves: (i) the apex player 1 deviates alone to be a singleton; (ii) the apex player 1 moves to one of minor players' coalitions; (iii) the minor player i^* deviates alone to be a singleton; (iv) the minor player i^* moves to one of minor players' coalitions; (v) a minor player i ($\neq i^*$) such that $\phi_i(\mathcal{P}) = 0$ moves to the winning coalition $\{1, i^*\}$; (vi) a minor player i ($\neq i^*$) such that $\phi_i(\mathcal{P}) = 0$ moves to another minor players' coalition.

Now, Case (v) does not occur, as the players 1 and i^* refuse his move. Case (vi) does not occur, because as long as the winning coalition $\{1, i^*\}$ holds, any minor player i obtains 0. Case (ii) does not occur, as this move gives the apex player at most $\frac{1}{2}$. Below, we show that (i), (iii) or (iv) do not occur.

- We show that Case (i) does not occur. Let $\mathcal{P}^*$ be a partition after the apex player's deviation from $\{1, i^*\} \in \mathcal{P}$ to $\{1\} \in \mathcal{P}^*$. Let r denote the number of coalitions consisting of minor players in $\mathcal{P}^*$. We have $\phi_1(\mathcal{P}^*) = \frac{r-1}{r+1}$ by Remark 4.1. If $t = 1$ (or, $\hat{\mathcal{Q}}_1$), then $r = 2$. If $t = 2$ (or, $\hat{\mathcal{Q}}_2$), then $r = 2$ or 3. In any case, $\frac{r-1}{r+1} \leq \frac{1}{2}$ when $r = 2, 3$. Thus, the apex player does not take the move of (i).
- We show that Cases (iii) and (iv) do not occur. In both cases, the apex player forms his singleton coalition after the i^*'s move. In the same manner with Case (i), let $\mathcal{P}^*$ be a partition after the

i^*'s move and r denote the number of all minor players' coalitions in $\mathcal{P}^*$. For any $S \in \mathcal{P}^* \setminus \{1\}$, we have $\phi_S(\mathcal{P}^*) = \frac{2}{r(r+1)}$ by Remark 4.1. If $t = 1$, then $r = 2$. If $t = 2$, then $r = 2$ or 3. In any case, $\frac{2}{r(r+1)} \leq \frac{1}{2}$ when $r = 2, 3$. Thus, neither Cases (iii) nor (iv) occur.

Moreover, assuming $n \geq 5$, we show that $[N]$ is individually stable. For the apex player, we have

$$\phi_1([N]) = \frac{(n-1)-1}{(n-1)+1} = \frac{n-2}{n} > \frac{1}{2},$$

which implies that the apex player has no incentive to form any winning coalition and he refuses any offer from minor players to form a winning coalition. Moreover, for any minor player i, we have

$$\phi_i([N]) = \frac{2}{(n-1)n} > \frac{2}{2(n-2)(n-1)},$$

where the right hand side of the inequality is the power of the player i after merging with another minor player. Hence, every minor player does not form any two-player coalition with another minor player.

Finally, we show that the other partitions are not individually stable. If $\mathcal{P} = \{1, i^*\} \cup \hat{\mathcal{Q}}_t$ and $t \geq 3$, then the apex player always has an incentive to deviate alone, namely, $r \geq 4$ and obtains $\frac{r-1}{r+1} > \frac{1}{2}$. If $\mathcal{P}$ contains no winning coalition and $\mathcal{P} \neq [N]$, then a minor player who belongs to a coalition S consisting of two or more minor players has an incentive to deviate alone as follows: for $|S| \geq 2$ and any $r < n-1$,

$$\frac{2}{(r+1)(r+2)} > \frac{2}{r(r+1)|S|} = \phi_i(\mathcal{P}).$$

This completes the proof.

Note that if $n = 4$, the apex player does not refuse the offer from minor players to form a winning coalition and each minor player has incentive to form a winning coalition with the apex player (see Table 3). □

Table 8 The five-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
12345	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$					+	+
1234|5	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0						+
123|45	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	+	+		+		+
123|4|5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	+	+		+		+
12|34|5	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$						
12|3|4|5	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
1|2|3|4|5	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$						

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Table 9 The six-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
123456	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$					+	+
12345|6	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0						+
1234|56	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	+	+		+		+
1234|5|6	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	+	+		+		+
123|456	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
123|45|6	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$						
123|4|5|6	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$						
12|34|56	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
12|34|5|6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
12|3|4|5|6	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$						
1|2|3|4|5|6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						

Table 10 The seven-player symmetric majority game

$\mathcal{P}$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	C^{pro}	C^{pes}	C^{opt}	V	Nash	IS
1234567	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$					+	+
123456|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0						+
12345|67	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0						+
12345|6|7	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0						+
1234|567	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0		+		+		
1234|56|7	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	+	+		+		+
1234|5|6|7	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	+	+		+		+
123|456|7	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$						
123|45|67	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
123|45|6|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$						
123|4|5|6|7	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$						
12|34|56|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0						+
12|34|5|6|7	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$						
12|3|4|5|6|7	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$						
1|2|3|4|5|6|7	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$						

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