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Abstract

We introduce a new axiom, the balanced contributions property for equal contributors. This axiom is defined by restricting the requirement of the balanced contributions property (Myerson (1980)) to two players whose contributions to the grand coalition are the same. We prove that this axiom, together with efficiency and weak covariance, characterizes a new class of solutions, the **r**-egalitarian Shapley values. This class subsumes many variants of the Shapley value, e.g., the egalitarian Shapley values or the discounted Shapley values. Our characterization uncovers a common property for many variants of the Shapley value, as well as provides a new method for characterizing them. Based on our new axiom, we also provide a non-cooperative implementation of the **r**-egalitarian Shapley values. Our new mechanism clarifies the difference among variants of the Shapley value from the bargaining power of players affected by the cost of continuing a bargaining.

Keywords: TU games, Balanced contributions property, Shapley value, Axiomatization, Implementation

JEL classification: C71, C72

1. Introduction

Since the pioneering work by Shapley (1953), the analysis of solutions in the cooperative game theory has made a considerable progress. Shapley proved that his new solution, the Shapley value, is the unique solution satisfying the following

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four axioms: efficiency, symmetry, the null player property and additivity. Later on, researchers pointed out the limitation of some of his axioms, and replaced them with alternative axioms. This approach has developed new variants of the Shapley value.

In the literature, different variants of the Shapley value are typically discussed separately. In this study, on the other hand, we introduce a new class of solutions that subsumes many variants of the Shapley value. By providing a new axiomatization and a non-cooperative implementation of the class of solutions, we discuss the difference among variants in a unified manner.

The key step of our analysis is to introduce a weakening of the balanced contributions property by Myerson (1980). This axiom states that for any game and any two players i and j, the effect of player i leaving the game on the payoff of j is the same as the effect of player j leaving the game on the payoff of i.[1] Our main finding is that, by relaxing the condition "for any two players", we can discuss more equitable solutions, e.g., the equal division value, in a similar manner. Our new weaker axiom, the balanced contributions property for equal contributors, restricts the requirement of the balanced contributions property to two players whose contributions to the grand coalition are the same.

We prove that our new axiom and two standard axioms characterize the class of $\mathbf{r}$-*egalitarian Shapley values*. An $\mathbf{r}$-egalitarian Shapley value is parameterized by an infinite sequence of real numbers $\mathbf{r}$. The class of the solutions has the advantage of subsuming many variants of the Shapley value, e.g., the egalitarian Shapley values (Joosten (1996), van den Brink et al. (2013)), the discounted Shapley values (Joosten (1996), van den Brink and Funaki (2015)) or the generalized solidarity values (Casajus and Huettner (2014a)).

We point out two contributions of our characterization. First, we uncovered a common property for many variants of the Shapley value. This is a notable result in light of the fact that each variant is derived from a different perspective. Second, our new axiom offers a new method for characterizing variants of the Shapley value. This is comparable to previous studies where variants of the Shapley value have been characterized by replacing the null player property in Shapley's (1953) axiomatization (see van den Brink and Funaki (2015)), or by weakening strong monotonicity in Young's (1985) axiomatization (see van den Brink et al. (2013), Casajus and Huettner (2014b) or Yokote and Funaki (2015)).

There is a line of literature on the balanced contributions property. A weakening of the property (Kamijo and Kongo (2010)), the weighted version (Hart and Mas-Colell (1989), Calvo and Santos (2000)), a parallel property with respect to

[1]We follow the explanation of the property by van den Brink and Chun (2012).

players nullification (Gómez-Rúa and Vidal-Puga (2010), Béal et al. (2016)), generalizations to games with coalition/levels structures (Calvo et al. (1996), Vidal-Puga and Bergantiños (1996), Gómez-Rúa and Vidal-Puga (2011), Kamijo (2013)) and an extension to NTU games (Hart and Mas-Colell (1996)) have been introduced. The basic idea of the property has been applied to several allocation problems, e.g., the exchange economies (Gudiño (2015)), the discrete cost allocation problems (Calvo and Santos (2006)), the sequencing problems (van den Brink and Chun (2012)), and the bankruptcy problems (Hwang (2015)). To the best of our knowledge, this study is the first to argue that the balanced contributions property can be used to discuss equitable solutions.

Based on our new axiom, we also provide a non-cooperative implementation of the **r**-egalitarian Shapley values. More specifically, we modify Pérez-Castrillo and Wettstein's (2001) bidding mechanism by introducing the cost of proceeding to the next round. The cost affects the bargaining power of players, which causes them to behave in a different way. We prove that, when the cost is parametrized by an exogenously given sequence **r**, the **r**-egalitarian Shapley value is implemented.

The reminder of the paper is organized as follows. Section 2 introduces basic concepts and our new axiom, the balanced contributions property for equal contributors. In Section 3, we define and characterize the class of r-egalitarian Shapley values. Section 4 introduces a new bidding mechanism and implements the **r**-egalitarian Shapley values. Section 5 concludes this paper. All proofs are provided in Section 6.

2. The balanced contributions property for equal contributors

We first introduce basic concepts.[2] Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{N}$ the set of natural numbers, with the convention that $0 \notin \mathbb{N}$. Fix a countable infinite set $\mathcal{U}$,[3] the universe of players, and let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathcal{U}$. For $S \in \mathcal{N}$, let $|S|$ denote the cardinality of S. For $S, N \in \mathcal{N}$, let s and n denote $|S|$ and $|N|$, respectively. A (TU)-game is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a coalition function $v \in \mathbb{V}(N) := \{f : 2^N \to \mathbb{R} : f(\emptyset) = 0\}$. Let Γ denote the set of all games. For $(N, v) \in \Gamma$ and $S \subseteq N$, $S \neq \emptyset$, with a slight abuse of notation, let (S, v) denote the game in which the domain of v is restricted from 2^N to 2^S.

Given that the grand coalition forms, we investigate the problem of how to fairly divide the total payoff among players. A solution is a function ψ that assigns a payoff

[2]We follow notations of Casajus and Huettner (2014a).

[3]All the theorems in this paper remain valid even if we assume that $\mathcal{U}$ is finite.

vector $\psi(N,v) \in \mathbb{R}^N$ to each game $(N,v) \in \Gamma$. We introduce two basic solutions. The Shapley value (Shapley (1953)) is given by[4]

$$Sh_i(N,v) = \sum_{S \subseteq N \setminus i} p_{n,s} \big[v(S \cup i) - v(S) \big] \text{ for all } (N,v) \in \Gamma, i \in N,$$

where

$$p_{n,s} = \frac{s!(n-s-1)!}{n!}.$$

The equal division value is given by

$$ED_i(N,v) = \frac{v(N)}{n} \text{ for all } (N,v) \in \Gamma, i \in N.$$

An axiom is a requirement for solutions, and is intended to capture our idea of desirability of solutions like equity or fairness. We first introduce a fundamental axiom in the context of dividing the total payoff $v(N)$.

Efficiency, E $\sum_{i \in N} \psi_i(N,v) = v(N)$ for all $(N,v) \in \Gamma$.

The following axiom introduced by Myerson (1980) is a widely-used fairness criterion in cooperative games:

Balanced contributions property, BC For any $(N,v) \in \Gamma$ with $n \geq 2$ and $i,j \in N$, $i \neq j$,

$$\psi_i(N,v) - \psi_i(N \setminus j, v) = \psi_j(N,v) - \psi_j(N \setminus i, v). \qquad (1)$$

This axiom states that the effect of player i leaving the game on the payoff of j is the same as the effect of player j leaving the game on the payoff of i.

The balanced contributions property has been intensively discussed in the literature. This axiom, however, is flawed by its too strong requirement. Under efficiency, the only solution satisfying the balanced contributions property is the Shapley value. The Shapley value determines final payoffs only by the contributions of players, and does not allow for egalitarian principles or solidarity among players.

The main objective of this study is to modify the axiom in such a way that more equitable solutions can be discussed. To this end, we focus on the equal division

[4]We denote a singleton set $\{i\}$ simply by i.

value, which is a reference point of equitable solutions. To see why the equal division value violates BC, consider a game $(N, v) \in \Gamma$ such that $v(N\backslash i) \neq v(N\backslash j)$ for some $i, j \in N$. Then, $\psi_i(N, v) = \psi_j(N, v)$, while $\psi_i(N\backslash j, v) \neq \psi_j(N\backslash i, v)$, violating (1).

The above discussion points out a limitation of the balanced contributions property, but at the same time, suggests a direction for modifying the axiom. Let us consider a weakening of BC in which we require (1) only for two players $i, j \in N$ who satisfy $v(N\backslash i) = v(N\backslash j)$. This seems to be a right direction for accommodating more equitable solutions, because at least the equal division value satisfies it. Moreover, the new axiom can be naturally interpreted as follows: if the total payoff changes by the same amount, i.e., $v(N) - v(N\backslash i) = v(N) - v(N\backslash j)$, then their payoffs also change by the same amount, i.e., (1) holds.

Balanced contributions property for equal contributors, BCEC Let $(N, v) \in \Gamma$ and $i, j \in N$, $i \neq j$. If $v(N\backslash i) = v(N\backslash j)$, then

$$\psi_i(N, v) - \psi_i(N\backslash j, v) = \psi_j(N, v) - \psi_j(N\backslash i, v).$$

Here, "equal contributors" refers to the fact that the two players' contributions to the grand coalition are the same. BCEC is a substantial weakening of BC, and many efficient solutions satisfy it. For the sake of characterizing solutions, we introduce additional axioms.

For $N \in \mathcal{N}$ and $i \in N$, we define $u_i \in \mathbb{V}(N)$ by $u_i(T) = 1$ if $i \in T$ and 0 otherwise. For $(N, v) \in \Gamma$, $i \in N$ and $\lambda \in \mathbb{R}$, we define $v + \lambda u_i \in \mathbb{V}(N)$ by $(v + \lambda u_i)(S) = v(S) + \lambda u_i(S)$ for all $S \subseteq N$, $S \neq \emptyset$. We intend to capture the situation in which the worth of coalitions including i increases by λ.

The following is a standard axiom in the literature:

Covariance, COV For any $(N, v) \in \Gamma$, $i \in N$ and $\lambda \in \mathbb{R}$, $\psi_i(N, v + \lambda u_i) = \psi_i(N, v) + \lambda$ and $\psi_j(N, v + \lambda u_i) = \psi_j(N, v)$ for all $j \neq i$.

One can check that the Shapley value satisfies this axiom, while the equal division value does not. Hence, we consider the following weakening:

Weak covariance, COV$^-$ For any $(N, v) \in \Gamma$, $i \in N$ and $\lambda \in \mathbb{R}$, $\psi(N, v + \lambda u_i) = \psi(N, v) + \lambda\psi(N, u_i)$.

This axiom states that ψ is linear with respect to the addition of u_i.

Up to this point, we focused on axioms satisfied by the Shapley value and modified them so that the equal division value satisfies them. The two solutions satisfy E, BCEC and COV$^-$, but they are not the only solutions satisfying the axioms. In the next section, we identify the boundary of solutions satisfying the three axioms by providing an axiomatization.

3. The r-egalitarian Shapley value and its axiomatization

We first introduce the idea of rescaling the worth of coalitions. Let $\mathcal{S}$ denote the set of infinite sequences of real numbers, i.e.,

$$\mathcal{S} = \big\{\{r_k\}_{k=1}^{\infty} : r_k \in \mathbb{R} \text{ for all } k = 1, 2, \cdots\big\}.$$

For $\mathbf{r} \in \mathcal{S}$ and $(N, v) \in \Gamma$, we define $v^{\mathbf{r}} \in \mathbb{V}(N)$ by

$$v^{\mathbf{r}}(S) = r_s v(S) \text{ for all } S \subseteq N, S \neq \emptyset.$$

In the game $v^{\mathbf{r}}$, we "rescale" the worth of each coalition by multiplying the r-th entry of the sequence $\mathbf{r}$, where s is the size of coalition S. This kind of rescaling is often discussed in the context of the per-capita measure or discounting. We interpret $v^{\mathbf{r}}$ as generalizing these ideas by allowing for any sequence of real numbers.

We define the **r-egalitarian Shapley value** $ESh^{\mathbf{r}}$ by

$$ESh_i^{\mathbf{r}}(N, v) = \frac{v(N) - v^{\mathbf{r}}(N)}{n} + Sh_i(N, v^{\mathbf{r}}) \text{ for all } (N, v) \in \Gamma, i \in N. \qquad (2)$$

The interpretation of this solution is as follows. Given a sequence $\mathbf{r}$, we first rescale the worth of coalitions and construct an imaginary game $v^{\mathbf{r}}$. Then, we apply the Shapley value to the game $v^{\mathbf{r}}$. By efficiency of the Shapley value, the payoff vector obtain by this procedure sums up to $v^{\mathbf{r}}(N) = r_n v(N)$, which is different from the total payoff $v(N)$ whenever $r_n \neq 1$. To fill in the gap between the two, we equally divide the difference $v(N) - v^{\mathbf{r}}(N)$ among players.

We remark that when $r_n \neq 0$, $ESh^{\mathbf{r}}$ can be written in the form of a convex combination:

$$ESh_i^{\mathbf{r}}(N, v) = (1 - r_n) \cdot ED_i(N, v) + r_n \cdot Sh_i\Big(N, \frac{v^{\mathbf{r}}}{r_n}\Big) \text{ for all } (N, v) \in \Gamma, i \in N,$$

$$\text{where } \frac{v^{\mathbf{r}}}{r_n} \in \mathbb{V}(N) \text{ is defined by } \frac{v^{\mathbf{r}}}{r_n}(S) = \frac{1}{r_n} \cdot v^{\mathbf{r}}(S) \text{ for all } S \subseteq N, S \neq \emptyset.$$

In particular, we can see the inclusion of two "polar" solutions, the Shapley value and the equal division value.

The class of **r**-egalitarian Shapley values has the advantage of subsuming many variants of the Shapley value. First, the α-egalitarian Shapley value (Joosten (1996), van den Brink et al. (2013)) is an **r**-egalitarian Shapley value with the sequence $r_k = \alpha$ for all $k = 1, 2, \cdots$. Second, the δ-discounted Shapley value (Joosten (1996), van den Brink and Funaki (2015)) is also an **r**-egalitarian Shapley value in a sense.

To see this point, let $\delta \in [0,1]$ and $m \in \mathbb{N}$, and consider a sequence $\mathbf{r} \in \mathcal{S}$ such that $r_k = \delta^{m-k}$ for all $k = 1, 2, \cdots, m$. Then, for any $(N, v) \in \Gamma$ with $n = m$, $ESh^{\mathbf{r}}(N, v)$ coincides with the δ-discounted Shapley value of (N, v). Moreover, for any $\xi \in [0, 1]$, the ξ-generalized solidarity value (Casajus and Huettner (2014a)) is also an $\mathbf{r}$-egalitarian Shapley value with the sequence

$$r_k = 1 - \frac{k \cdot \xi}{(k-1) \cdot \xi + 1} \text{ for all } k = 1, 2, \cdots .$$

As proven by Casajus and Huettner (2014a), $\xi = \frac{1}{2}$ corresponds to the solidarity value (Nowak and Radzik (1994)).

We now show that the class of $\mathbf{r}$-egalitarian Shapley values is characterized by the axioms discussed in Section 2.

Theorem 1. *A solution ψ satisfies E, COV⁻ and BCEC if and only if there exists $\mathbf{r} \in \mathcal{S}$ such that $\psi = ESh^{\mathbf{r}}$.*

Proof. See Subsection 6.2. □

In Theorem 1, we allow for any sequence of real numbers. On the other hand, all the variants of the Shapley value mentioned above satisfy $r_k \in [0, 1]$ for all $k = 1, 2, \cdots$. This restriction is also consistent with the idea of generalizing the per-capita measure or discounting. Hence, it is worthwhile to characterize the class of $\mathbf{r}$-egalitarian Shapley values satisfying $r_k \in [0, 1]$ for all $k = 1, 2, \cdots$.

We introduce some notations and axioms. We say that a game $(N, v) \in \Gamma$ is monotonic if $v(S) \geq v(T)$ for all $S, T \subseteq N$ with $S \supseteq T$. This means that if a coalition weakly increases (in the sense of set inclusion), then the worth of the coalition weakly increases. In a monotonic game, we can say that all players are not non-productive.

Desirability, D For any $(N, v) \in \Gamma$ and $i, j \in N$ such that $v(S \cup i) - v(S) \geq v(S \cup j) - v(S)$ for all $S \subseteq N \backslash \{i, j\}$, $\psi_i(N, v) \geq \psi_j(N, v)$.

Positivity, P For any $(N, v) \in \Gamma$ and $i \in N$ such that (N, v) is monotonic, $\psi_i(N, v) \geq 0$.

The desirability axiom states that if i's contributions are greater than or equal to j's contributions, then i should receive at least j's payoff. The positivity axiom states that if all players are not non-productive, then no one should end up with negative payoff.

Corollary 1. *A solution ψ satisfies E, COV⁻, BCEC, D and P if and only if there exists $\mathbf{r} \in \mathcal{S}$ with $r_k \in [0, 1]$ for all $k \in \mathbb{N}$ such that $\psi = ESh^{\mathbf{r}}$.*

Proof. See Subsection 6.3. □

We point out two contributions of our characterization. First, Theorem 1 indicates that many variants of the Shapley value satisfy in common the balanced contributions property for equal contributors. This is a notable result in light of the fact that each variant of the Shapley value is derived from a different perspective. Second, Theorem 1 offers a new method for characterizing specific variants of the Shapley value. We can characterize specific variants by introducing additional axioms in such a way that only the corresponding sequences survive. For example, if we replace COV$^-$ with COV, then only the sequence corresponding to the Shaspley value survives.

Corollary 2. *A solution ψ satisfies E, COV and BCEC if and only if $\psi = Sh$.*

Proof. See Subsection 6.4. □

Recall that the Shapley value is the unique solution satisfying E and BC. Thus, Corollary 2 means that the effect of weakening BC to BCEC is compensated by introducing COV.

4. Implementation of the r-egalitarian Shapley value

This section is devoted to a non-cooperative implementation of the **r**-egalitarian Shapley values. As discussed in Pérez-Castrillo and Wettstein (2001), the implementation theory is a part of Nash program, and is intended to answer the problem of how to achieve a cooperative solution through non-cooperative behavior.

Our approach is based on the bidding mechanism developed by Pérez-Castrillo and Wettstein (2001). The mechanism has been adapted to the egalitarian Shapley values and the discounted Shapley values by van den Brink et al. (2013) and van den Brink and Funaki (2015), respectively. In this section, we modify the bidding mechanism in such a way that the **r**-egalitarian Shapley values are implemented. Our new mechanism enables us to explain the difference among variants of the Shapley value from the bargaining power of players affected by the cost of continuing a bargaining.

We say that a game $(N, v) \in \Gamma$ is zero-monotonic if $v(S \cup i) - v(S) \geq v(i)$ for all $i \in N$, $S \subseteq N \backslash i$. This means that player i's contribution to a coalition is always no less than i's stand-alone payoff. Fix $m \in \mathbb{N}$ and we restrict our attention to zero-monotonic games with non-negative worth of coalitions and with no more than m players. Namely, we consider the class $\hat{\Gamma}^m$ defined by

$$\hat{\Gamma}^m = \{(N, v) \in \Gamma : n \leq m, (N, v) \text{ is zero-monotonic}, v(S) \geq 0 \text{ for all } S \subseteq N\}.$$

Since the number of players is finite, it suffices to consider finite sequences. We define

$$\hat{\mathcal{S}} = \big\{\{\hat{r}_k\}_{k=1}^m : \hat{r}_k \in [0,1] \text{ for all } k = 1, \cdots, m\big\}.$$

For $\hat{\mathbf{r}} \in \hat{\mathcal{S}}$, we define the $\hat{\mathbf{r}}$-egalitarian Shapley value $ESh^{\hat{\mathbf{r}}}$ on $\hat{\Gamma}^m$ by

$$ESh_i^{\hat{\mathbf{r}}}(N,v) = (1-\hat{r}_n)\cdot\frac{v(N)}{n} + Sh_i(N, v^{\hat{r}}) \text{ for all } (N,v) \in \hat{\Gamma}^m, i \in N,$$

where $v^{\hat{r}} \in \mathbb{V}(N)$ is defined by $v^{\hat{r}}(S) = \hat{r}_s v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Similar to the bidding mechanism by Pérez-Castrillo and Wettstein (2001), each round of our mechanism consists of three stages: (1) all players make bids to each other, and a proposer is determined; (2) the proposer proposes a payoff distribution among the remaining players; (3) the players other than the proposer sequentially accept or reject the offer. Meanwhile, different from the bidding mechanism, after a rejection of a proposal, we assume that the players other than the proposer incur a fixed amount of cost before proceeding to the next round. The cost is proportional to the total payoff over which the remaining players bargain.

We formally describe our mechanism by following the notation of van den Brink and Funaki (2015). Fix $(N,v) \in \hat{\Gamma}^m$, $\hat{\mathbf{r}} \in \hat{\mathcal{S}}$. Let N_t be the player set of the game with which each round $t \in \{1, \cdots, m\}$ will start, so $N_1 = N$.

$\hat{\boldsymbol{r}}$-modified bidding mechanism.
Round $\boldsymbol{t}$, $t \in \{1, \cdots, m-1\}$.

Stage 1 Each player $i \in N_t$ makes bids $b_j^i \in \mathbb{R}$ for every $j \neq i$. For each $i \in N_t$, let $B^i = \sum_{j \in N_t \backslash i}(b_j^i - b_i^j)$ be the net bid of player i. Let α_t be the player with the highest net bid of round t (in case of a non-unique maximizer we choose any of these maximal bidders to be the "winner" with equal probability). Once α_t has been chosen, player α_t pays every other player $j \in N_t \backslash \alpha_t$ its offered bid $b_j^{\alpha_t}$. Player α_t becomes the proposer in the next stage. Go to Stage 2.

Stage 2 Player α_t proposes an offer $y_j^{\alpha_t} \in \mathbb{R}$ to every player $j \in N_t \backslash \alpha_t$ (this offer is additional of the bids paid at Stage 1). Go to Stage 3.

Stage 3 The players other than α_t, sequentially, either accept or reject the offer. If at least one player rejects it, then the offer is rejected. In this case, player α_t leaves the game and obtains $v(\alpha_t)$. Every player in $N_t \backslash \alpha_t$ incurs the same cost $\frac{1-\hat{r}_{n-t}}{n-t} v(N_t \backslash \alpha_t)$ and proceeds to Round $t+1$ to bargain over $v(N_t \backslash \alpha_t)$. If all the players accept the offer, then the offer is implemented and the mechanism stops.

In each case, the final payoffs for the players are given as follows:

Acceptance: each player $j \in N_t \backslash \alpha_t$ receives $y_j^{\alpha_t}$ and player α_t obtains the remainder $v(N_t) - \sum_{j \in N_t \backslash \alpha_t} y_j^{\alpha_t}$. The final payoff to player $j \in N_t \backslash \alpha_t$ is

$$y_j^{\alpha_t} + b_j^{\alpha_t} + \sum_{k=1}^{t-1} \Big(b_j^{\alpha_k} - \frac{1 - \hat{r}_{n-k}}{n-k} v(N_k \backslash \alpha_k) \Big),$$

while player α_t receives

$$v(N_t) - \sum_{j \in N_t \backslash \alpha_t} (y_j^{\alpha_t} + b_j^{\alpha_t}) + \sum_{k=1}^{t-1} \Big(b_{\alpha_t}^{\alpha_k} - \frac{1 - \hat{r}_{n-k}}{n-k} v(N_k \backslash \alpha_k) \Big).$$

Rejection: the final payoff for player α_t is

$$v(\alpha_t) + \sum_{k=1}^{t-1} \Big(b_{\alpha_t}^{\alpha_k} - \frac{1 - \hat{r}_{n-k}}{n-k} v(N_k \backslash \alpha_k) \Big).$$

Round $\boldsymbol{m}$. $N_m = N_{m-1} \backslash \alpha_{m-1}$. Since N_m is a singleton coalition it is a one-player game in this round. The game immediately stops such that player $i \in N_m$ gets $v(N_m)$. Its final payoff thus is

$$v(N_m) + \sum_{k=1}^{t-1} \Big(b_i^{\alpha_k} - \frac{1 - \hat{r}_{n-k}}{n-k} v(N_k \backslash \alpha_k) \Big).$$

Theorem 2. *The $\hat{\mathbf{r}}$-modified bidding mechanism implements $ESh^{\hat{\mathbf{r}}}(N, v)$ in any subgame perfect equilibrium.*

Proof. See Subsection 6.5. □

The essence of our new mechanism is to control the bargaining power of players by appropriately determining the cost of proceeding to the next round. Let us informally define "bargaining power" as the payoff a player can guarantee in the next round. Then, higher (resp. lower) cost implies lower (resp. higher) bargaining power.

As an extreme case, consider the equal division value implemented by the sequence $\hat{r}_k = 0$ for all $k = 1, \cdots, m$. In this case, after a rejection of a proposal, the total attainable payoff for the remaining players in the next round is equal to the total cost, which forces their bargaining power to be essentially 0. Indeed, in an equilibrium, the proposer at Stage 2 declares that he receives the total payoff $v(N)$, and all the other players accept the offer. Given this fact, at Stage 1, all players behave anonymously. As the opposite extreme case, consider the sequence $\hat{r}_k = 1$ for all $k = 1, \cdots, m$, which corresponds to the original bidding mechanism by Pérez-Castrillo and Wettstein (2001). In this case, players can fully exert their bargaining power. Thus, it is intuitive that all players receive the Shapley value, which is the measure of a player's *own* contribution in the game. All the variants of the Shapley value discussed in Section 3 lie between the two cases.

5. Conclusion

In this study, we first introduced a weakening of the balanced contributions property by Myerson (1980), in an attempt to discuss more equitable solutions. By using this axiom, we characterized the class of **r**-egalitarian Shapley values that subsumes many variants of the Shapley value. We also provided a non-cooperative implementation of the values. A notable finding is that, with a slight modification of the existing axiom and the rule of the mechanism, many existing solutions can be discussed in a unified manner.

Both the balanced contributions property and the bidding mechanism have been tailored to incorporate asymmetric solutions; see Hart and Mas-Colell (1989) and Pérez-Castrillo and Wettstein (2001). When we consider a convex combination of solutions, its asymmetric version is complex because each solution has its own asymmetric (or "weighted") version. It remains as a topic for future work to incorporate asymmetric solutions in our framework.

Our new axiom requires the same effect of leaving a game between two players whose contributions to the grand coalition are the same. One can consider a further weaker axiom in which the same effect is required only for two symmetric players. We will discuss this axiom in a separate paper.

6. Proofs

6.1. Preliminary results and axioms

We first provide two propositions.

Proposition 1. *Let* $\mathbf{r} \in \mathcal{S}$, $(N, v) \in \Gamma$ *with* $n \geq 2$, *and* $i, j \in N$. *Then,*

$$\begin{aligned} &ESh_i^{\mathbf{r}}(N, v) - ESh_i^{\mathbf{r}}(N \backslash j, v) + \frac{1 - r_{n-1}}{n-1} v(N \backslash j) \\ =&ESh_j^{\mathbf{r}}(N, v) - ESh_j^{\mathbf{r}}(N \backslash i, v) + \frac{1 - r_{n-1}}{n-1} v(N \backslash i). \end{aligned}$$

Proof. The statement follows from the following equations:

$$\begin{aligned}
&\big[ESh_i^{\mathbf{r}}(N,v) - ESh_i^{\mathbf{r}}(N\backslash j,v)\big] - \big[ESh_j^{\mathbf{r}}(N,v) - ESh_j^{\mathbf{r}}(N\backslash i,v)\big]\\
=&\Big[(1-r_n)\frac{v(N)}{n} + Sh_i(N,v^{\mathbf{r}}) - (1-r_{n-1})\frac{v(N\backslash j)}{n-1} - Sh_i(N\backslash j,v^{\mathbf{r}})\Big]\\
-&\Big[(1-r_n)\frac{v(N)}{n} + Sh_j(N,v^{\mathbf{r}}) - (1-r_{n-1})\frac{v(N\backslash i)}{n-1} - Sh_j(N\backslash i,v^{\mathbf{r}})\Big]\\
=&\big[Sh_i(N,v^{\mathbf{r}}) - Sh_i(N\backslash j,v^{\mathbf{r}})\big] - \big[Sh_j(N,v^{\mathbf{r}}) - Sh_j(N\backslash i,v^{\mathbf{r}})\big]\\
&+ (1-r_{n-1})\cdot\frac{v(N\backslash i) - v(N\backslash j)}{n-1}\\
=&(1-r_{n-1})\cdot\frac{v(N\backslash i) - v(N\backslash j)}{n-1},
\end{aligned}$$

where the last equality follows from BC of the Shapley value. □

Proposition 2. *Let* $\mathbf{r}\in\mathcal{S}$, $(N,v)\in\Gamma$ *with* $n\geq 2$, *and* $i\in N$. *Then,*

$$ESh_i^{\mathbf{r}}(N,v) = \frac{v(N) - r_{n-1}v(N\backslash i)}{n} + \frac{1}{n}\sum_{j\neq i}\Big[ESh^{\mathbf{r}}(N\backslash j,v) - \frac{(1-r_{n-1})v(N\backslash j)}{n-1}\Big].$$

Proof. The statement follows from the following equations:

$$\begin{aligned}
ESh^{\mathbf{r}}(N,v) &= \frac{(1-r_n)v(N)}{n} + Sh_i(N,v^{\mathbf{r}})\\
&= \frac{(1-r_n)v(N)}{n} + \frac{v^{\mathbf{r}}(N) - v^{\mathbf{r}}(N\backslash i)}{n} + \frac{1}{n}\sum_{j\neq i} Sh(N\backslash j,v^{\mathbf{r}})\\
&= \frac{(1-r_n)v(N)}{n} + \frac{r_n v(N) - r_{n-1}v(N\backslash i)}{n}\\
&+ \frac{1}{n}\sum_{j\neq i}\Big[ESh^{\mathbf{r}}(N\backslash j,v) - \frac{(1-r_{n-1})v(N\backslash j)}{n-1}\Big]\\
&= \frac{v(N) - r_{n-1}v(N\backslash i)}{n} + \frac{1}{n}\sum_{j\neq i}\Big[ESh^{\mathbf{r}}(N\backslash j,v) - \frac{(1-r_{n-1})v(N\backslash j)}{n-1}\Big],
\end{aligned}$$

where the second equality follows from the recursive formula of the Shapley value.[5]
□

[5]See (5.1) of Maschler and Owen (1989).

We introduce additional notations.[6] Let $(N,v) \in \Gamma$ and $\pi : N \to \mathcal{U}$ be an injection. We define $(\pi N, \pi v) \in \Gamma$ by $\pi v(\pi S) = v(S)$ for all $S \subseteq N$, $S \neq \emptyset$. For any $x \in \mathbb{R}^N$, we define $y = \pi(x) \in \mathbb{R}^{\pi(N)}$ by $y_{\pi(i)} = x_i$ for all $i \in N$.

We say that (N', w) is equivalent to (N,v) if there exists an injection $\pi : N \to \mathcal{U}$ such that $\pi(N) = N'$ and $\pi v = w$. For each game $(N,v) \in \Gamma$, we define the binary relation $\sim_{(N,v)}$ on N as follows:

$$i \sim_{(N,v)} j \;\Leftrightarrow\; v(S \cup i) = v(S \cup j) \text{ for all } S \subseteq N \backslash \{i,j\}.$$

For each game $(N,v) \in \Gamma$, we define the binary relation $\sim^*_{(N,v)}$ on N as follows:

$$i \sim^*_{(N,v)} j \;\Leftrightarrow\; v(N \backslash i) = v(N \backslash j). \tag{3}$$

We introduce two axioms.

Symmetry, S Let $(N,v) \in \Gamma$. If $i \sim_{(N,v)} j$, then $\psi_i(N,v) = \psi_j(N,v)$.

Anonymity, A For any $(N,v) \in \Gamma$ and an injection $\pi : N \to \mathcal{U}$, $\psi(\pi(N), \pi(v)) = \pi(\psi(N,v))$.

Let $N \in \mathcal{N}$. For each $T \subseteq N$, $T \neq \emptyset$, we define the T-unanimity game $u_T \in \mathbb{V}(N)$ by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise.

6.2. Proof of Theorem 1

If part: Since $ESh^{\mathbf{r}}$ is linear, $ESh^{\mathbf{r}}$ satisfies COV$^-$. We prove that $ESh^{\mathbf{r}}$ satisfies E. For any $(N,v) \in \Gamma$,

$$\begin{aligned}\sum_{i \in N} ESh^{\mathbf{r}}_i(N,v) &= (1 - r_n)v(N) + \sum_{i \in N} Sh_i(N, v^{\mathbf{r}}) \\ &= (1 - r_n)v(N) + r_n v(N) \\ &= v(N),\end{aligned}$$

where the second equality follows from E of the Shapley value. By Proposition 1, $ESh^{\mathbf{r}}$ satisfies BCEC.

Only if part: For each $m \in \mathbb{N}$, we define $\Gamma^m = \{(N,v) \in \Gamma : n = m\}$. We first prove three lemmas.

[6] We follow notations of Peleg and Sudhölter (2007).

Lemma 1. *Let $m \geq 2$. Suppose that ψ satisfies BCEC and A on Γ^{m-1}. Then ψ satisfies S on Γ^m.*

Proof. Let $(N, v) \in \Gamma^m$ and $i, j \in N$, $i \neq j$, with $i \sim_{(N,v)} j$. Since $v(N\backslash i) = v(N\backslash j)$, by BCEC,

$$\psi_i(N, v) - \psi_j(N, v) = \psi_i(N\backslash j, v) - \psi_j(N\backslash i, v). \tag{4}$$

Since $i \sim_{(N,v)} j$, $(N\backslash j, v)$ is equivalent to $(N\backslash i, v)$. By A on Γ^{m-1}, $\psi_i(N\backslash j, v) = \psi_j(N\backslash i, v)$. Together with (4), we obtain $\psi_i(N, v) = \psi_j(N, v)$. □

Lemma 2. *Let $m \geq 2$. Suppose that ψ satisfies E, COV$^-$ and S on Γ^m. Then for any $N \in \mathcal{N}$ with $n = m$ and $i, j \in N$, $\psi_i(N, u_i) = \psi_j(N, u_j)$.*

Proof. Let $x = \psi_i(N, u_i)$, $y = \psi_j(N, u_j)$. Then,

$$\psi_i(N, u_i) + \psi_i(N, u_j) \stackrel{\text{E,S}}{=} x + \frac{1-y}{n-1},$$
$$\psi_j(N, u_i) + \psi_j(N, u_j) \stackrel{\text{E,S}}{=} \frac{1-x}{n-1} + y.$$

Together with

$$\psi_i(N, u_i) + \psi_i(N, u_j) \stackrel{\text{COV}^-}{=} \psi_i(N, u_i + u_j) \stackrel{\text{S}}{=} \psi_j(N, u_i + u_j) \stackrel{\text{COV}^-}{=} \psi_j(N, u_i) + \psi_j(N, u_j),$$

we obtain

$$x + \frac{1-y}{n-1} = \frac{1-x}{n-1} + y.$$

This equation implies $x = y$. □

Lemma 3. *Let $m \geq 2$. Suppose that ψ satisfies BCEC and S on Γ^m. Then for any $N \in \mathcal{N}$ with $n = m$, $i, j \in N$ with $i \neq j$, and $k \in \mathbb{N}\backslash N$,*

$$\psi_i(N, u_i) = \psi_i\big((N\backslash j) \cup k, u_i\big).$$

Proof. Let $M = N \cup k$. Define $v = u_i + u_{M\backslash i}$ and consider the $n+1$-person game (M, v). Note that $v(M\backslash l) = v(M\backslash l')$ for all $l, l' \in M$.

Let $x = \psi_i(M\backslash k, u_i)$, $y = \psi_i(M\backslash j, u_i)$. Then,

$$\psi_i(M,v) - \psi_j(M,v) \overset{\text{BCEC}}{=} \psi_i(M\backslash j, v) - \psi_j(M\backslash i, v) = y - \psi_j(M\backslash i, u_{M\backslash i}), \tag{5}$$

$$\psi_i(M,v) - \psi_k(M,v) \overset{\text{BCEC}}{=} \psi_i(M\backslash k, v) - \psi_k(M\backslash i, v) = x - \psi_k(M\backslash i, u_{M\backslash i}). \tag{6}$$

By taking (5) – (6),

$$\psi_k(M,v) - \psi_j(M,v) \overset{\text{BCEC}}{=} y - \psi_j(M\backslash i, u_{M\backslash i}) - x + \psi_k(M\backslash i, u_{M\backslash i}) \overset{\text{S}}{=} y - x. \tag{7}$$

On the other hand,

$$\begin{aligned}\psi_j(M,v) - \psi_k(M,v) &\overset{\text{BCEC}}{=} \psi_j(M\backslash k, u_i) - \psi_k(M\backslash j, u_i) \\ &\overset{\text{S}}{=} \frac{1-x}{n-1} - \frac{1-y}{n-1} \\ &= \frac{y-x}{n-1}.\end{aligned} \tag{8}$$

By (7) and (8), we obtain $x = y$. □

To prove the only-if part, it suffices to prove that there exists $\mathbf{r} = \{r_k\}_{k=1}^{\infty} \in \mathcal{S}$ such that, for any $(N,v) \in \Gamma$ and $i \in N$,

$$\psi_i(N,v) = r_n \cdot \frac{v(N)}{n} + \sum_{S \subseteq N\backslash i} p_{n,s}\Big[(1 - r_{s+1})v(S \cup i) - (1 - r_s)v(S)\Big]. \tag{9}$$

For 1-person games, ψ is uniquely determined by E. We focus on 2-person games. Let $\{i,j\} \in \mathcal{N}$. Since ψ satisfies A on Γ^1, by Lemma 1, ψ satisfies S on Γ^2. By E and S on Γ^2, for any $\{i,j\} \in \mathcal{N}$ and $\lambda \in \mathbb{R}$,

$$\psi(\{i,j\}, \lambda u_{ij}) = \Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big). \tag{10}$$

For each $\{i,j\} \in \mathcal{N}$, by E and Lemma 2,

$$\psi_i(\{i,j\}, u_j) = \psi_j(\{i,j\}, u_i).$$

For each $\{i,j\} \in \mathcal{N}$, let $r(\{i,j\}) \in \mathbb{R}$ denote the above equal value. For any $\{i,j\} \in$

$\mathcal{N}$ and $k \in \mathbb{N} \backslash \{i,j\}$,

$$r(\{i,j\}) \overset{\text{E}}{=} 1 - \psi_i(\{i,j\}, u_i) \overset{\text{L3}}{=} 1 - \psi_i(\{i,k\}, u_i) \overset{\text{E}}{=} r(\{i,k\}). \tag{11}$$

For any $\{i,j\} \in \mathcal{N}$, by replacing i and j with an outside player, we can obtain an arbitrary 2-person player set $\{i', j'\} \in \mathcal{N}$. This observation and (11) imply that $r(\{i,j\}) \in \mathbb{R}$ does not depend on the choice of $\{i,j\} \in \mathcal{N}$. Choose an arbitrary $\{i,j\} \in \mathcal{N}$ and we define r_1 by

$$r_1 = 2r(\{i,j\}). \tag{12}$$

By COV$^-$ and E, for any $\lambda \in \mathbb{R}$,

$$\psi(\{i,j\}, \lambda u_j) = \Big(\frac{\lambda r_1}{2}, \lambda - \frac{\lambda r_1}{2}\Big), \tag{13}$$

$$\psi(\{i,j\}, \lambda u_i) = \Big(\lambda - \frac{\lambda r_1}{2}, \frac{\lambda r_1}{2}\Big). \tag{14}$$

Let $(\{i,j\}, v) \in \Gamma^2$. For each $S \subseteq \{i,j\}$, $S \neq \emptyset$, let d_S denote the dividend of S in $(\{i,j\}, v)$.[7] Then,

$$\begin{aligned}
\psi_i(\{i,j\}, v) &\overset{\text{COV}^-}{=} \psi_i(\{i,j\}, d_{\{ij\}} u_{\{ij\}}) + \psi_i(\{i,j\}, d_i u_i) + \psi_i(\{i,j\}, d_j u_j) \\
&\overset{(10),(13),(14)}{=} \frac{d_{\{ij\}}}{2} + d_i - d_i \cdot \frac{r_1}{2} + d_j \cdot \frac{r_1}{2} \\
&= \frac{1}{2}\big\{v(\{ij\}) - v(i) - v(j)\big\} + v(i) - v(i) \cdot \frac{r_1}{2} + v(j) \cdot \frac{r_1}{2}.
\end{aligned}$$

Thus for any $r_2 \in \mathbb{R}$, we get

$$\psi_i(\{i,j\}, v) = r_2 \cdot \frac{v(\{ij\})}{2} + \frac{1}{2}(1 - r_1)v(i) + \frac{1}{2}\big[(1 - r_2)v(\{ij\}) - (1 - r_1)v(j)\big].$$

It follows that, when r_1 is given by (12) and r_2 is arbitrary, ψ coincides with (9) for 2-person games.

We proceed by an induction on the number of players. It suffices to prove that for each $t \in \mathbb{N}$ with $t \geq 3$, the following claim holds:

Claim t **.** *Suppose that there exist real numbers $\{r_1, \cdots, r_{t-2}\}$ such that, for any*

[7] The dividend d_S is defined by $d_i = v(i)$, $d_j = v(j)$, and $d_{\{ij\}} = v(\{ij\}) - v(i) - v(j)$.

$(N,v) \in \Gamma^m$ with $1 \leq m \leq t-1$, and any $r_{t-1} \in \mathbb{R}$, $\psi(N,v)$ coincides with (9).

Then, there exists $r_{t-1} \in \mathbb{R}$ such that, for any $(N,v) \in \Gamma^t$ and any $r_t \in \mathbb{R}$, $\psi(N,v)$ coincides with (9).

The proof of Claim t consists of two steps. In Step 1, we endogenously derive r_{t-1}. In Step 2, we prove that the real number r_{t-1} derived in Step 1 satisfies the desired condition.

Step 1: By the induction hypothesis, ψ satisfies A on Γ^{t-1}. Thus, by Lemma 1, ψ satisfies S on Γ^t. For any $N \in \mathcal{N}$ with $n = t$, $i, j \in N$ with $i \neq j$, and $i', j' \in N$ with $i' \neq j'$,

$$\psi_j(N, u_i) \overset{\text{E,S}}{=} \frac{1 - \psi_i(N, u_i)}{n-1} \overset{\text{L2}}{=} \frac{1 - \psi_{i'}(N, u_{i'})}{n-1} \overset{\text{E,S}}{=} \psi_{j'}(N, u_{i'})$$

For each $N \in \mathcal{N}$, let $r(N) \in \mathbb{R}$ denote the above equal value. For any $N \in \mathcal{N}$ with $n = t$, $i, j \in N$ and $k \in \mathbb{N} \backslash N$,

$$r(N) \overset{\text{E,S}}{=} \frac{1 - \psi_i(N, u_i)}{n-1} \overset{\text{L3}}{=} \frac{1 - \psi_i\big((N\backslash j) \cup k, u_i\big)}{n-1} \overset{\text{E,S}}{=} r\big((N\backslash j) \cup k\big). \tag{15}$$

For any $N \in \mathcal{N}$, by repeatedly replacing a player in N with an outside player, we can obtain an arbitrary t-person player set $N' \in \mathcal{N}$. This observation and (15) imply that $r(N) \in \mathbb{R}$ does not depend on the choice of $N \in \mathcal{N}$. Choose an arbitrary player set $N \in \mathcal{N}$ and we define r_{t-1} by

$$r_{t-1} = n(n-1)r(N) - \sum_{m=1}^{t-2} r_m. \tag{16}$$

Step 2: We prove that, when r_{t-1} is given by (16), $\psi(N,v)$ coincides with (9) for all $(N,v) \in \Gamma^t$.

For each $(N,v) \in \Gamma$, the binary relation $\sim^*_{(N,v)}$ defined by (3) is an equivalent relation and induces a partition on N. Let $\mathcal{P}(N,v)$ denote the partition and set

$$\#(N,v) = \max_{S \in \mathcal{P}(N,v)} |S|.$$

We proceed by an induction on $\#(N,v)$.[8]

[8] We remark that, in the remaining part, $n = t$. Namely, we restrict our attention to player sets N with t players.

Induction base: Consider a game $(N, v) \in \Gamma^t$ with $\#(N, v) = n$, i.e.,

$$v(N\backslash i) = v(N\backslash j) \text{ for all } i, j \in N.$$

Fix $i \in N$. Then,

$$\begin{aligned}
&\psi_i(N, v) - \psi_j(N, v) \overset{\text{BCEC}}{=} \psi_i(N\backslash j, v) - \psi_j(N\backslash i, v) \text{ for all } j \in N\backslash i,\\
&\psi_i(N, v) - \psi_i(N, v) = 0.
\end{aligned}$$

By taking the sum of both sides of the above equations, together with E, we get

$$\begin{aligned}
&n\psi_i(N, v) - v(N)\\
&= \sum_{j\in N\backslash i} \Big[\psi_i(N\backslash j, v) - \psi_j(N\backslash i, v)\Big]\\
&\overset{\text{IH}}{=} \sum_{j\in N\backslash i} \Big[\sum_{S\subseteq N\backslash\{i,j\}} p_{n-1,s}\Big\{(1 - r_{s+1})v(S \cup i) - (1 - r_s)v(S)\Big\}\\
&- \sum_{S\subseteq N\backslash\{i,j\}} p_{n-1,s}\Big\{(1 - r_{s+1})v(S \cup j) - (1 - r_s)v(S)\Big\}\Big]\\
&= \sum_{j\in N\backslash i} \Big[\sum_{S\subseteq N\backslash\{i,j\}} p_{n-1,s}(1 - r_{s+1})\{v(S \cup i) - v(S \cup j)\}\Big]\\
&= \sum_{S\subsetneq N\backslash i} (n - s - 1)p_{n-1,s}(1 - r_{s+1})v(S \cup i) - \sum_{S\subseteq N\backslash i : S\neq\emptyset} s \cdot p_{n-1,s-1}(1 - r_s)v(S)\\
&= \sum_{S\subsetneq N\backslash i} \frac{s!(n - s - 1)!}{(n - 1)!}(1 - r_{s+1})v(S \cup i) - \sum_{S\subseteq N\backslash i} \frac{s!(n - s - 1)!}{(n - 1)!}(1 - r_s)v(S)\\
&= n \cdot \sum_{S\subseteq N\backslash i} p_{n,s}\Big[(1 - r_{s+1})v(S \cup i) - (1 - r_s)v(S)\Big] - (1 - r_n)v(N),
\end{aligned}$$

where r_t is an arbitrary real number. It follows that, for any $r_t \in \mathbb{R}$,

$$n\psi_i(N, v) = r_n v(N) + n \sum_{S\subseteq N\backslash i} p_{n,s}\Big[(1 - r_{s+1})v(S \cup i) - (1 - r_s)v(S)\Big]. \qquad (17)$$

Induction step: Suppose that the result holds for any $(N, v) \in \Gamma^t$ with $\#(N, v) = l + 1$, and we prove the result for $(N, v) \in \Gamma^t$ with $\#(N, v) = l$, where $1 \leq l \leq n - 1$.

Let $(N, v) \in \Gamma^t$ with $\#(N, v) = l$. Choose a coalition $T \in \mathcal{P}(N, v)$ with $|T| = l$. Since $l \leq n - 1$, $N\backslash T \neq \emptyset$. Choose players $i \in T$ and $j \in N\backslash T$. Define $\delta =$

$v(N\backslash j) - v(N\backslash i)$ and consider the game $(N, v + \delta u_j)$. In this game,

$$
\begin{aligned}
(v + \delta u_j)(N\backslash j) &= v(N\backslash j),\\
(v + \delta u_j)(N\backslash k) &= v(N\backslash j) \text{ for all } k \in T.
\end{aligned}
$$

It follows that $\#(N, v + \delta u_j) = l + 1$. Thus, we can apply the induction hypothesis. For any player $k \in N\backslash j$,

$$
\begin{aligned}
&\psi_k(N, v + \delta u_j)\\
&= r_n \cdot \frac{v(N) + \delta}{n} + \sum_{S \subseteq N\backslash\{j,k\}} p_{n,s}\Big[(1 - r_{s+1})v(S \cup k) - (1 - r_s)v(S)\Big]\\
&+ \sum_{S \subseteq N\backslash k : S \ni j} p_{n,s}\Big[(1 - r_{s+1})\big(v(S \cup k) + \delta\big) - (1 - r_s)\big(v(S) + \delta\big)\Big]\\
&= r_n \cdot \frac{v(N)}{n} + \sum_{S \subseteq N\backslash k} p_{n,s}\Big[(1 - r_{s+1})v(S \cup k) - (1 - r_s)v(S)\Big]\\
&+ \delta\Big[\frac{r_n}{n} + \sum_{S \subseteq N\backslash k, S \ni j} p_{n,s}(-r_{s+1} + r_s)\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{r_n}{n} + \sum_{S \subseteq N\backslash k, S \ni j} p_{n,s}(-r_{s+1} + r_s)\\
&= \frac{r_n}{n} + \sum_{q=2}^{n-1}\Big\{\frac{(n-2)!}{\big((n-2) - (q-1)\big)!} \cdot p_{n,q} - \frac{(n-2)!}{\big((n-2) - (q-2)\big)!} \cdot p_{n,q-1}\Big\}r_q + \frac{r_1}{n(n-1)} - \frac{r_n}{n}\\
&= \sum_{q=2}^{n-1}\Big\{\frac{q}{n(n-1)} - \frac{q-1}{n(n-1)}\Big\}r_q + \frac{r_1}{n(n-1)}\\
&= \frac{1}{n(n-1)}\sum_{q=1}^{n-1} r_q,
\end{aligned}
$$

we obtain

$$\psi_k(N, v+\delta u_j) = r_n \cdot \frac{v(N)}{n} + \sum_{S\subseteq N\backslash k} p_{n,s}\Big[(1-r_{s+1})v(S\cup k) - (1-r_s)\cdot v(S)\Big] \\ + \frac{\delta}{n(n-1)}\sum_{q=1}^{n-1} r_q. \tag{18}$$

By COV$^-$,

$$\begin{aligned}\psi_k(N,v) &= \psi_k(N, v+\delta u_j) - \psi_k(N,\delta u_j)\\ &\overset{(16),(18)}{=} r_n \cdot \frac{v(N)}{n} + \sum_{S\subseteq N\backslash k} p_{n,s}\Big[(1-r_{s+1})v(S\cup k) - (1-r_s)\cdot v(S)\Big].\end{aligned}$$

Thus, the desired equation holds for all $k \in N\backslash j$. E completes the proof. □

6.3. Proof of Corollary 1

If part: Since $ESh^{\mathbf{r}}$ is a linear solution satisfying E and S, by Theorem 2 of Radzik and Driessen (2013), $ESh^{\mathbf{r}}$ satisfies D and P.
Only if part: Since ψ satisfies E, COV$^-$ and BCEC, by Theorem 1, there exists $\mathbf{r} \in \mathcal{S}$ such that $\psi = ESh^{\mathbf{r}}$. In particular, ψ is a linear solution satisfying E and S. The desired condition follows from Theorem 2 of Radzik and Driessen (2013).

6.4. Proof of Corollary 2

If part: By Theorem 1, the Shapley value satisfies E and BCEC. One can easily check that the Shapley value satisfies COV.
Only if part: For each $N \in \mathcal{N}$, let $(N, \mathbf{0})$ denote the null game, i.e., $\mathbf{0}(S) = 0$ for all $S \subseteq N$. Since COV is stronger than COV$^-$, by Theorem 1, ψ is an $\mathbf{r}$-egalitarian Shapley value. Thus,

$$\psi_i(N, \mathbf{0}) = 0 \text{ for all } N \in \mathcal{N}, i \in N. \tag{19}$$

We go back to the proof of only-if part of Theorem 1. By an induction argument, we prove that ψ is represented in the form of (9) with $r_t = 0$ for all $t \in \mathbb{N}$. Let $\{i,j\} \in \mathcal{N}$. Then, r_1 defined by (12) satisfies

$$r_1 = \psi_j(\{i,j\}, u_i) = \psi_j(\{i,j\}, \mathbf{0}) = 0,$$

where the second equality follows from COV and the last equality follows from (19).

Suppose that $r_m = 0$ for all $m \in \mathbb{N}$ with $m \leq t-2$, and we prove that r_{t-1} defined by (16) satisfies $r_{t-1} = 0$, where $t \geq 3$. Choose $N \in \mathcal{N}$ with $n = t$ and $i, j \in N$, $i \neq j$. Then,

$$r_{t-1} = n(n-1)\psi_j(N, u_i) - \sum_{m=1}^{t-2} r_m = n(n-1)\psi_j(N, \mathbf{0}) - \sum_{m=1}^{t-2} r_m = 0,$$

where the second equality follows from COV, and the last equality follows from the induction hypothesis and (19).

It follows that $r_t = 0$ for all $t \in \mathbb{N}$. By substituting this equation into (9), we conclude that ψ is the Shapley value. □

6.5. Proof of Theorem 2

We mimic the proof of Pérez-Castrillo and Wettstein (2001). We fix $(N, v) \in \hat{\Gamma}^m$ and $\hat{\mathbf{r}} \in \hat{\mathcal{S}}$.

The proof proceeds by an induction. For round $m-1$, it is clear that every equilibrium outcome coincides with $ESh^{\hat{\mathbf{r}}}$. We now assume that every equilibrium outcome coincides with $ESh^{\hat{\mathbf{r}}}$ for round $t+1$ and show it also holds for round t, where $t \in \{1, \cdots, m-2\}$.

For $i, j \in N_t$, set

$$x_i := \frac{1 - \hat{r}_{n-t}}{n-t} v(N_t \backslash i),\ \ \phi_i(N_t) := ESh_i^{\hat{\mathbf{r}}}(N_t, v),\ \ \phi_i(N_t \backslash j) := ESh_i^{\hat{\mathbf{r}}}(N_t \backslash j, v).$$

We first prove that $ESh^{\hat{\mathbf{r}}}$ is indeed an equilibrium outcome. We explicitly construct an SPE that yields $ESh^{\mathbf{r}}$ as an SPE outcome. Consider the following strategies:

Stage 1: each player i, $i \in N_t$, announces $b_j^i = \phi_j(N_t) - \phi_j(N_t \backslash i) + x_i$.

Stage 2: player $i \in N_t$, if he is the proposer, offers $y_j^i = \phi_j(N_t \backslash i) - x_i$ to every $j \neq i$.

Stage 3: player $i \in N_t$, if player $j \neq i$ is the proposer, accepts any offer greater than or equal to $\phi_i(N_t \backslash j) - x_i$ and rejects any offer strictly smaller than $\phi_i(N_t \backslash j) - x_j$.

It is clear that these strategies yield $ESh^{\hat{\mathbf{r}}}$ for any player who is not the proposer. Given that the coalition N_t is formed, the proposer also receives the payoff assigned by $ESh^{\hat{\mathbf{r}}}$.

We now show that all net bids B^i are equal to zero. Following the above mentioned strategies,

$$\begin{aligned}
\sum_{j\neq i} b_j^i &= \sum_{j\neq i}\Big[\phi_j(N_t) - \phi_j(N_t\backslash i) + x_i\Big] \\
&= \sum_{j\neq i}\Big[\phi_i(N_t) - \phi_i(N_t\backslash j) - (1-\hat{r}_{n-t})\cdot\frac{v(N_t\backslash i) - v(N_t\backslash j)}{n-t}\Big] + (1-\hat{r}_{n-t})v(N_t\backslash i) \\
&= \sum_{j\neq i}\Big[\phi_i(N_t) - \phi_i(N_t\backslash j) + x_j\Big] \\
&= \sum_{j\neq i} b_i^j,
\end{aligned}$$

where the second equality follows from Proposition 1.[9]

To check that the previous strategies constitute an SPE, note first that the strategies at Stages 2 and 3 are best responses if, for any $i \in N$,

$$\begin{aligned}
&v(N_t) - \sum_{j\neq i}\Big[\phi_j(N_t\backslash i) - \frac{1-\hat{r}_{n-t}}{n-t}v(N_t\backslash i)\Big] \geq v(i) \\
\Rightarrow &v(N_t) - v(N_t\backslash i) + (1-\hat{r}_{n-t})v(N_t\backslash i) \geq v(i) \\
\Rightarrow &v(N_t) - \hat{r}_{n-t}v(N_t\backslash i) \geq v(i).
\end{aligned}$$

The last inequality follows from zero-monotonicity, $\hat{r}_{n-t} \in [0,1]$ and $v(N_t\backslash i) \geq 0$. Moreover, the strategies at Stage 1 are also best responses; this part can be proved by following the same argument of Pérez-Castrillo and Wettstein (2001).

We now show that any SPE yields $ESh^{\hat{\mathbf{r}}}$.

Claim 1. *In any SPE, at Stage 3, all players other than the proposer α_t accept the offer if $y_i^{\alpha_t} > \phi_i(N_t\backslash\alpha_t) - x_{\alpha_t}$ for every player $i \neq \alpha_t$. Moreover, if $y_i^{\alpha_t} < \phi_i(N\backslash\alpha_t) - x_{\alpha_t}$ for at least some $i \neq \alpha_t$, then the offer is rejected.*

Proof. In the case of a rejection, by the induction argument the payoff to a player $i \neq \alpha_t$ is $\phi_i(N_t\backslash\alpha_t) - x_{\alpha_t}$. This establishes the desired condition. □

Claim 2. *Choose any SPE and a subgame starting from $t = 2$. Let α_t be the proposer. Then, the final payoffs to player α_t and $i \neq \alpha_t$ are $v(N_t) - \hat{r}_{n-t}v(N_t\backslash\alpha_t) - \sum_{j\neq\alpha_t} b_j^{\alpha_t}$ and $\phi_i(N_t\backslash\alpha_t) - x_{\alpha_t} + b_i^{\alpha_t}$, respectively.*

[9]Note that $n - t = |N_t| - 1$. So, Proposition 1 is applied to the game (N_t, v).

Proof. We consider two cases.
Case 1: Suppose that the offer by α_t is accepted. In this case, an offer such that $y_j^{\alpha_t} > \phi_j(N_t \backslash \alpha_t) - x_{\alpha_t}$ for some $j \neq \alpha_t$ cannot be part of an SPE, since α_t could still offer $\phi_i(N_t \backslash \alpha_t) - x_{\alpha_t} + \epsilon$ to every $i \neq \alpha_t$, with $\epsilon < y_j^{\alpha_t} - \phi_j(N_t \backslash \alpha_t) + x_{\alpha_t}$ and $\epsilon > 0$. Thus, the offer satisfies $y_j^{\alpha_t} = \phi_j(N_t \backslash \alpha_t) - x_{\alpha_t}$ for all $j \neq \alpha_t$, which establish the desired condition.
Case 2: Suppose that the offer by α_t is rejected. In this case, player α_t receives $v(\alpha_t)$. Suppose to the contrary that $v(N_t) - \hat{r}_{n-t} v(N_t \backslash \alpha_t) > v(\alpha_t)$. Then, player α_t can improve his payoff by offering $\phi_i(N_t \backslash \alpha_t) - x_{\alpha_t} + \epsilon/(n-t)$ to every $i \neq \alpha_t$, with $\epsilon < v(N_t) - \hat{r}_{n-t} v(N_t \backslash \alpha_t) - v(\alpha_t)$, since

$$\begin{aligned} v(N_t) - \sum_{i \neq \alpha_t} \Big[\phi_i(N_t \backslash \alpha_t) - x_{\alpha_t} + \frac{\epsilon}{n-t}\Big] &= v(N_t) - v(N_t \backslash \alpha_t) + (1 - \hat{r}_{n-t}) v(N_t \backslash \alpha_t) - \epsilon \\ &= v(N_t) - \hat{r}_{n-t} v(N_t \backslash \alpha_t) - \epsilon \\ &> v(\alpha_t). \end{aligned}$$

By the above contradiction, we must have $v(N_t) - \hat{r}_{n-t} v(N_t \backslash \alpha_t) = v(\alpha_t)$. This equation and the induction hypothesis establishes the desired condition. □

The following claims can be proved by mimicking the proof of the corresponding claims in Pérez-Castrillo and Wettstein (2001).

Claim 3. *In any SPE, $B^i = B^j$ for all i and j and hence $B^i = 0$ for all $i \in N$.*

Claim 4. *In any SPE, each player's payoff is the same regardless of who is chosen as the proposer.*

The following claim completes the proof:

Claim 5. *In any SPE, every player receives the payoff assigned by $ESh^{\hat{\mathbf{r}}}$.*

Proof. Note first that if player i is the proposer, his final payoff is given by $z_i^i = v(N_t) - v(N_t \backslash i) - \sum_{j \neq i} b_j^i$. On the other hand, if player $j \neq i$ is the proposer, the final payoff of player j is given by $z_i^j = \phi_i(N_t \backslash j) + b_i^j$.

$$\begin{aligned} \sum_j z_i^j &= \Big(v(N_t) - v(N_t \backslash i) - \sum_{j \neq i} b_j^i\Big) + \sum_{j \neq i} \Big(\phi_i(N_t \backslash j) - \frac{1 - \hat{r}_{n-t}}{n-t} v(N_t \backslash j) + b_i^j\Big) \\ &= v(N_t) - v(N_t \backslash i) + \sum_{j \neq i} \phi_i(N_t \backslash j) - \frac{1 - \hat{r}_{n-t}}{n-t} \sum_{j \neq i} v(N_t \backslash j) \\ &= n\phi_i(N_t), \end{aligned}$$

where the last equality follows from Proposition 2. Since player i is indifferent to all possible choices of the proposer, we obtain the desired condition. □

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