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Consistency and the Core in Games with Externalities

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Abstract

In the presence of externalities across coalitions, Dutta et al. (2010) characterize their value by extending Hart and Mas-Colell reduced game consistency. In the present paper, we provide a characterization result for the core for games with externalities by extending one form of consistency studied by Moulin (1985), which is often refereed to as the complement-reduced game property. Moreover, we analyze another consistency formulated by Davis and Maschler (1965), called the max-reduced game property and a final consistency called the projection-reduced game property. In environments with externalities, we discuss some asymmetric results among these different forms of reduced games.

Keywords: Consistency; Core; Externalities; Reduced game
JEL Classification: C71

1 Introduction

Cooperative game theory is one of the most basic frameworks to analyze coalition formation and to study how we allocate the surplus obtained from the coalition. Many of the "traditional" models of cooperative game theory consider the worth of a coalition as the surplus obtained by the members of the coalition with no help from the other players. This simplification provides a wide variety of sophisticated ideas and insights on allocations such as the Shapley value and the core. Recent works, however, attempt to understand environments in which there is mutual influence among coalitions. In these works, such mutual influence is commonly called *externalities* among coalitions. By using the concept of externalities, we can divide the general field of cooperative games into two classes: games with externalities and games without externalities. Games without externalities, or traditional models, are often refereed to as *coalition function form games*, whereas games with externalities are called *partition function form games*.

In the presence of externalities, the allocation of surplus becomes more complicated. Myerson (1977), Bolger (1989), Macho-Stadler et al. (2004) and Albizuri et al. (2005) propose the allocation rules by generalizing the Shapley value to games with externalities. Moreover, Dutta et al. (2010) characterize

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their value by extending Hart and Mas-Colell consistency to games with externalities.

In contrast to the remarkable progress made in studies on values, there are relatively few works on the core for environments with externalities. One possible reason for this is that a number of types of cores can be defined in the presence of externalities: the definition of the core depends on the "anticipation" of deviating players because of externalities. For example, if some agents who are about to deviate from their original affiliation anticipate the worst reaction from the remaining agents (minimizing the surplus of the deviating agents), the deviating agents may have less incentive to carry out the deviation. The core describing this type of stability is called the *pessimistic core* in Bloch and van den Nouweland (2014) and Abe and Funaki (2015), which is closely associated with the concept known as the α-core introduced by Hart and Kurz (1983). Analogous to the pessimistic core, in the presence of externalities, the definition of each core depends on the anticipation for the reaction of the remaining players. Bloch and van den Nouweland (2014) formulate such anticipations for reactions as *expectation functions* and give axiomatic characterizations to them. However, the axiomatic characterizations for the cores have been left open.

In this paper, we provide characterization results for the cores of games with externalities by using some forms of reduced game consistencies. We show that if an expectation function satisfies a certain condition, then we can axiomatize the core based on the expectation function with some axioms. Instead of Hart and Mas-Colell consistency employed by Dutta et al. (2010), we use the other forms of reduced game consistencies: Complement, Max and Projection consistencies. The objective of this paper is to describe what relationships exist between the cores and the consistencies in the presence of externalities. Our result is summarized in Proposition 3.3 and Table 1.

The remainder of the paper is organized as follows. The next section is devoted to the basic definitions and notations. In Section 3, we describe the axioms and offer the axiomatization result. We discuss the differences among some forms of reduced games in Section 4. Section 5 concludes this paper with some further remarks.

2 Preliminaries

2.1 Games with Externalities

Let $\mathcal{N}$ be a set of all players. We consider a finite player set $N \subsetneq \mathcal{N}$. A *coalition* S is a subset of N. We denote by $|S|$ the number of players in S. For any $S \subseteq N$, a *partition* of S is defined by $\{T_1, ..., T_h\}$ where $1 \leq h \leq |S|$, $T_i \cap T_j = \emptyset$ for $i, j = 1, ..., h$ $(i \neq j)$, $T_i \neq \emptyset$ for $i = 1, ..., h$ and $\bigcup_{i=1}^{h} T_i = S$. We will typically use $\mathcal{P}$ or $\mathcal{Q}$ to denote a partition. Assume that the partition of the empty set $\emptyset$ is $\{\emptyset\}$. For any $S \subseteq N$, let $\Pi(S)$ be the set of all partitions of S. We define an *embedded coalition* of N by $(S, \mathcal{P})$ satisfying $\mathcal{P} \in \Pi(N \setminus S)$. The set of all embedded coalitions of N is given by

$$EC(N) = \{(S, \mathcal{P}) \mid \emptyset \neq S \subseteq N \text{ and } \mathcal{P} \in \Pi(N \setminus S)\}.$$

A *partition function form game* is a pair (N, v), where a *partition function* v is a function that assigns a real number to each embedded coalition, namely, $v : EC(N) \to R$. Let Γ_A be the set of all partition function form games: $\Gamma_A = \{(N, v) \mid \emptyset \neq N \subseteq \mathcal{N},\ |N| < \infty,\ v : EC(N) \to R\}$. For any game, we restrict payoff vectors to the following set: $F(N, v) = \left\{x \in R^N \;\middle|\; \sum_{j \in N} x_j \leq v(N, \{\emptyset\})\right\}$. For a set of games $\Gamma \subseteq \Gamma_A$, a *solution* on Γ is a function σ that associates a subset $\sigma(N,v)$ of $F(N,v)$ with every game $(N,v) \in \Gamma$.

We denote by x_S a *restriction* of $x \in R^N$ on coalition S, *i.e.*, $x_S = (x_j)_{j\in S} \in R^S$. To keep our notation simple, for any coalition S and player i, we typically use $S \cup i$ or $S \setminus i$ to denote $S \cup \{i\}$ or $S \setminus \{i\}$.

2.2 The Reduced Game

We first define the reduced game in the presence of externalities. In Sections 2 and 3, we focus our attention on the reduced game known as the complement-reduced game. The other types of reduced games are discussed in Section 4.

Now, consider $\Gamma \subseteq \Gamma_A$ and $(N,v) \in \Gamma$. Let $S \subseteq N$ $(S \neq \emptyset)$ and $x \in R^N$.

Definition 2.1. The *complement-reduced game with respect to S and x* is the game $(S, v^{S,x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S \setminus T)$,

$$v^{S,x}(T,\mathcal{Q}) = v(T \cup (N \setminus S), \mathcal{Q}) - \sum_{j\in N\setminus S} x_j.$$

The complement-reduced game describes that a coalition T always obtains the help of all leaving players $N \setminus S$ by paying $(x_j)_{j\in N\setminus S}$ for them. The complement-reduced game was initially introduced by Moulin (1985) for games without externalities. Definition 2.1 is the simple extension of the original definition to games with externalities.

The complement-reduced game depends neither on the order of leaving players nor on the partition of leaving players. To see this, we offer Lemma 2.2. For notational simplicity, let $v^{-i} := v^{N\setminus i,x}$, *i.e.*, v^{-i} means the complement-reduced game after removing i from the original game. Similarly, we use the following notation:

$$(v^{-i_1})^{-i_2} := (v^{N\setminus i_1,x})^{(N\setminus i_1)\setminus i_2, x_{N\setminus i_1}}.$$

Lemma 2.2. For any $x \in R^N$ and any $i_1, i_2 \in N$ $(i_1 \neq i_2)$,

$$\begin{aligned}(v^{-i_1})^{-i_2} &= (v^{-i_2})^{-i_1} \\ &= v^{N\setminus\{i_1,i_2\},x}.\end{aligned}$$

Proof. For any $T \subseteq N \setminus i_1$ and any $\mathcal{Q} \in \Pi(N \setminus (T \cup i_1))$, we have

$$v^{-i_1}(T,\mathcal{Q}) = v(T \cup i_1, \mathcal{Q}) - x_{i_1}.$$

For any $T' \subseteq N \setminus \{i_1, i_2\}$ and $\mathcal{Q}' \in \Pi(N \setminus (T \cup \{i_1, i_2\}))$,

$$\begin{aligned}(v^{-i_1})^{-i_2}(T',\mathcal{Q}') &= v^{-i_1}(T' \cup i_2, \mathcal{Q}') - x_{i_2} \\ &= v(T' \cup \{i_1,i_2\}, \mathcal{Q}') - x_{i_1} - x_{i_2}. \end{aligned} \tag{2.1}$$

Similarly, we remove them in the order of i_2, i_1 and obtain the same game as (2.1).

Next, assume that players i_1 and i_2 simultaneously leave the game. For any $T' \subseteq N \setminus \{i_1, i_2\}$ and any $\mathcal{Q}' \in \Pi(N \setminus (T \cup \{i_1, i_2\}))$, we have

$$v^{N\setminus\{i_1,i_2\},x}(T',\mathcal{Q}') = v(T' \cup \{i_1,i_2\}, \mathcal{Q}') - x_{i_1} - x_{i_2},$$

which is the same as (2.1). □

Lemma 2.2 shows that the following two properties hold even in the presence of externalities: (i) the complement-reduced game is independent of the order of leaving players; (ii) the game obtained by removing players one by one is equivalent to the game obtained by removing players simultaneously.

In the presence of externalities, the partition of leaving players might influence the worth of a coalition consisting of the remaining players. Definition 2.1 shows that we can ignore this influence in the complement-reduced game, as all leaving players $N \setminus S$ help the remaining players T and form a coalition $T \cup (N \setminus S)$. This property is unique to the complement-reduced game and not true for the other types of reduced games. This difference will be expanded upon in Section 4.

It is straightforward to extend Lemma 2.2. Consider $T = \{i_1, ..., i_t\} \subsetneq N$. For any permutations π, π' of $T = \{i_1, ..., i_t\}$, by repeating Lemma 2.2, we have

$$v^{\pi} = v^{\pi'} = v^{N \setminus T, x},$$

where, for any permutation π'', $v^{\pi''} = (...((v^{-\pi''_1})^{-\pi''_2})...)^{-\pi''_t}$. Player π''_k means the k-th player leaving the game. Hence, we obtain useful notation as follows:

$$v^{-T} := v^{\pi} = v^{\pi'} = v^{N \setminus T, x}.$$

2.3 Expectation Functions

To define the core of games with externalities, we introduce the notion of expectation function formulated by Bloch and van den Nouweland (2014). As noted in Section 1, there are various definitions of a core in the presence of externalities. This diversity can be represented by different expectation functions.

Definition 2.3. An *expectation function* is a mapping ψ associating a partition $\mathcal{P}$ such that $\mathcal{P} \in \Pi(N \setminus S)$, with player set N, partition function v and nonempty coalition $S \subseteq N$, formally,

$$\psi(N, v, S) \in \{\mathcal{P}' | \mathcal{P}' \in \Pi(N \setminus S)\}.$$

We introduce four important expectation functions. An expectation function is:

- *optimistic* if
$$\psi(N, v, S) = \arg\max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}').$$
- *pessimistic* if
$$\psi(N, v, S) = \arg\min_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}').$$
- *disjunctive* if
$$\psi(N, v, S) = \{\{i_{s+1}\}, ..., \{i_n\}\}.$$
- *conjunctive* if
$$\psi(N, v, S) = \{N \setminus S\}.$$

For the optimistic (pessimistic) expectation function, if there are two different partitions attaining the highest (lowest) value, then choose the partition with the smaller index.

Now, we introduce a new property of the expectation functions.

Definition 2.4. Let Γ be a set of games and $(N, v) \in \Gamma$. An expectation function ψ is *complement-coherent (CC)* if, for any $S \subseteq N$ ($|S| \geq 2$), $h \in S$, and $x \in R^N$,

$$\psi(N, v, S) = \psi(N \setminus h, v^{N \setminus h, x}, S \setminus h).$$

Complement-coherence (CC) requires that coalition S's expectation be equal to coalition $S \setminus h$' s expectation. Note that not only $\psi(N, v, S)$ but also $\psi(N \setminus h, v^{N \setminus h, x}, S \setminus h)$ is a partition of $N \setminus S$.

In Definition 2.4, we define CC by removing one player. Below, we consider a slight variant of CC. We call it $\widehat{CC}$ and define it as follows: an expectation function ψ is $\widehat{CC}$ if, for any $S \subseteq N$ and $T \subsetneq S$ ($T \neq \emptyset$),

$$v(S, \psi(N, v, S)) = v(S, \psi(N \setminus T, v^{-T}, S \setminus T)).$$

The following proposition shows that CC is equivalent to $\widehat{CC}$.

Lemma 2.5.

$$CC \iff \widehat{CC}$$

Proof. It is clear that $\widehat{CC} \Rightarrow CC$ holds. We show that $CC \Rightarrow \widehat{CC}$. Let $S \subseteq N$ and $T \subsetneq S$ with $T \neq \emptyset$. Define $T = \{h_1, ..., h_t\}$. By CC, we have

$$\begin{aligned}\psi(N, v, S) &= \psi(N \setminus h_1, v^{-h_1}, S \setminus h_1)\\ &= \psi(N \setminus \{h_1, h_2\}, (v^{-h_1})^{-h_2}, S \setminus \{h_1, h_2\})\\ &\dots\\ &= \psi(N \setminus T, (...(v^{-h_1})...)^{-h_t}, S \setminus T)\\ &= \psi(N \setminus T, v^{-T}, S \setminus T),\end{aligned}$$

where the last equality holds by Lemma 2.2. □

The four expectation functions listed above are all CC. For the proof, see Proposition A.1 and Corollary A.5 in the Appendix.

2.4 The Core Based on an Expectation Function

In this subsection, we introduce the core based on an expectation function. Define $X(N, v) = \left\{x \in R^N \;\middle|\; \sum_{j \in N} x_j = v(N, \{\emptyset\})\right\}$. Then, the core based on an expectation function is given as the following definition.

Definition 2.6. Let Γ be a set of games and $(N, v) \in \Gamma$. Given an expectation function ψ, the *ψ-core* of game (N, v) is defined as follows:

$$C^{\psi}(N, v) = \left\{ x \in X(N, v) \;\middle|\; \text{for any nonempty } S \subseteq N, \; \sum_{j \in S} x_j \geq v(S, \psi(N, v, S)) \right\}.$$

If expectation function ψ is optimistic, pessimistic, disjunctive or conjunctive, then the ψ-core means the optimistic core C^{opt}, the pessimistic core C^{pes}, the disjunctive core C^{dis}, or the conjunctive core C^{con}, respectively. For any expectation function ψ, let Γ_{C^ψ} denote the class of games in which the nonempty ψ-core exists.[*1]

3 An Axiomatic Approach

We introduce the axioms for our characterization results.

Axiom 1 (comp-RGP)**.** Let Γ be a set of games, $(N,v) \in \Gamma$ and $S \subseteq N$. A solution σ on Γ satisfies the *complement-reduced game property (comp-RGP)* if for every $x \in \sigma(N,v)$, we have $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.

Axiom 2 (NE on Γ)**.** Let Γ be a set of games. A solution σ on Γ satisfies *non-emptiness on* Γ *(NE on* Γ*)* if for every $(N,v) \in \Gamma$, we have $\sigma(N,v) \neq \emptyset$.

Axiom 3 (ψ-IR)**.** Let ψ be an expectation function. A solution σ on Γ satisfies *ψ-individual rationality (ψ-IR)* if for every $(N,v) \in \Gamma$, any $x \in \sigma(N,v)$, and every player $i \in N$, we have $x_i \geq v(\{i\}, \psi(N,v,\{i\}))$.

Note that Axiom 1 is independent of expectation function ψ. Axiom 3 directly depends on ψ. Axiom 2 may depend on ψ if Γ is specified by ψ. For any ψ, the ψ-core satisfies ψ-IR because the expectation function ψ is common to both ψ-core and ψ-IR. It is also clear that ψ-core is nonempty on Γ if $\Gamma = \Gamma_{C^\psi}$. For comp-RGP, we have the following result.

Proposition 3.1. If an expectation function ψ is CC, the ψ-core satisfies comp-RGP on Γ_{C^ψ}.

Proof. Let $C^\psi(N,v)$ be the ψ-core of (N,v) and $x \in C^\psi(N,v)$. For every nonempty $S \subseteq N$, it suffices to show that $x_S \in C^\psi(S, v^{S,x})$. By Definition 2.1, for any $T \subseteq S$ $(T \neq \emptyset)$, we have

$$
\begin{aligned}
&\sum_{j\in T} x_j - v^{S,x}(T, \psi(S, v^{S,x}, T)) \\
&= \sum_{j\in T} x_j - \left[v(T \cup (N\setminus S), \psi(S, v^{S,x}, T)) - \sum_{j \in N\setminus S} x_j \right] \\
&= \sum_{j \in T\cup(N\setminus S)} x_j - v(T \cup (N\setminus S), \psi(S, v^{S,x}, T)) \\
&\geq v(T \cup (N\setminus S), \psi(N, v, T\cup(N\setminus S))) - v(T \cup (N\setminus S), \psi(S, v^{S,x}, T)) \qquad (3.1) \\
&= v(T \cup (N\setminus S), \psi(S, v^{S,x}, T)) - v(T \cup (N\setminus S), \psi(S, v^{S,x}, T)) \qquad (3.2) \\
&= 0,
\end{aligned}
$$

where (3.1) holds because of $x \in C^\psi(N,v)$, and we have (3.2) because ψ is CC. □

Now, we offer the axiomatization below. We first show that a well-known result holds even in games with externalities (Lemma 3.2). This result will be used in the proof of the axiomatization (Proposition 3.3).

[*1] Abe and Funaki (2015) generalize the Bondareva-Shapley condition and define the class Γ_{C^ψ}. The balancedness of each type of core is also studied.

Lemma 3.2. Let ψ be an expectation function and σ be a solution on a set of games Γ. If σ satisfies comp-RGP and ψ-IR, then σ satisfies *efficiency*: for any $x \in \sigma(N, v)$

$$\sum_{j \in N} x_j = v(N, \{\emptyset\}).$$

Proof. This is a simple extension of Peleg (1986) and Tadenuma (1992). Let $(N, v) \in \Gamma$ and $x \in \sigma(N, v)$. Assume that σ is not efficient. Then, there exists $x \in \sigma(N, v)$ such that $\sum_{j \in N} x_j < v(N, \{\emptyset\})$. Let i be a player in N. By comp-RGP, we have $x_i \in \sigma\left(\{i\}, v^{\{i\},x}\right)$. For any ψ, by ψ-IR, we have $x_i \geq v^{\{i\},x}(\{i\}, \{\emptyset\}) = v(N, \{\emptyset\}) - \sum_{j \in N \setminus i} x_j$. Hence, $\sum_{j \in N} x_j \geq v(N, \{\emptyset\})$, and the desired contradiction has been obtained. □

Proposition 3.3. Let ψ be an expectation function. If ψ is CC, then ψ-core C^ψ is the unique function on Γ_{C^ψ} that satisfies comp-RGP, NE on Γ_{C^ψ}, and ψ-IR.

Proof. We prove uniqueness next. Let σ be a solution satisfying the three conditions. The proof consists of two parts: $\sigma \subseteq C^\psi$ and $C^\psi \subseteq \sigma$.

Part 1:

First, we show that $\sigma(N, v) \subseteq C^\psi(N, v)$ for any $(N, v) \in \Gamma_{C^\psi}$. From Lemma 3.2, it follows that σ satisfies efficiency.

Induction base:

For $|N| = 1$, $\sigma(N, v) \subseteq C^\psi(N, v)$ because of efficiency. For $|N| = 2$, let $N = \{i, j\}$. By efficiency, $x_i + x_j = v(N, \{\emptyset\})$ for any $x \in \sigma(N, v)$. By ψ-IR, $x_i \geq v(\{i\}, \{\{j\}\})$ and $x_j \geq v(\{j\}, \{\{i\}\})$. Hence, $\sigma(N, v) \subseteq C^\psi(N, v)$.

Induction proof:

We assume that $\sigma(N, v') \subseteq C^\psi(N, v')$ for any $(N, v') \in \Gamma^\psi$ with $|N| \leq k$ $(k \geq 2)$. We show that for any $(M, v) \in \Gamma^\psi$ with $|M| = k + 1$, we have $\sigma(M, v) \subseteq C^\psi(M, v)$.

Let $x \in \sigma(M, v)$ and $h \in M$. By comp-RGP, we have $x_{M \setminus h} \in \sigma(M \setminus h, v^{M \setminus h, x})$. By the assumption of induction, $\sigma(M \setminus h, v^{M \setminus h, x}) \subseteq C^\psi(M \setminus h, v^{M \setminus h, x})$. Hence, for any nonempty $S \subseteq M \setminus h$,

$$\begin{aligned}\sum_{j \in S} x_j &\geq v^{M \setminus h, x}(S, \psi(M \setminus h, v^{M \setminus h, \mathcal{P}, x}, S)) \\ &= v(S \cup h, \psi(M \setminus h, v^{M \setminus h, \mathcal{P}, x}, S)) - x_h \\ &= v(S \cup h, \psi(M, v, S \cup h)) - x_h, \end{aligned} \tag{3.3}$$

where (3.3) holds because ψ is CC. Thus, we obtain

$$\sum_{j \in S \cup h} x_j \geq v(S \cup h, \psi(M, v, S \cup h))$$

for any nonempty $S \subseteq M \setminus h$. In addition, by ψ-IR, we have $x_i \geq v(\{i\}, \psi(M, v, \{i\}))$. Hence, $\sigma(M, v) \subseteq C^\psi(M, v)$. By induction, it follows that $\sigma(N, v) \subseteq C^\psi(N, v)$ for all (N, v) in Γ^ψ.

Part 2:

Next, we show that $C^{\psi}(N,v) \subseteq \sigma(N,v)$ for all $(N,v) \in \Gamma_{C^{\psi}}$. To prove this, we construct a game (M,u) by using a game $(N,v) \in \Gamma_{C^{\psi}}$ and a payoff vector $x \in C^{\psi}(N,v)$. Fix $(N,v) \in \Gamma_{C^{\psi}}$ and $x \in C^{\psi}(N,v)$. We define $M := N \cup h$, where $h \in \mathcal{N}$ and $h \notin N$. Define u as follows:

$$\begin{array}{ll} u(\{h\}, \mathcal{P}') = 0 & \text{for all } \mathcal{P}' \in \Pi(M \setminus h), \\ u(S \cup h, \mathcal{P}'') = v(S, \mathcal{P}'') & \text{for all } \mathcal{P}'' \in \Pi(M \setminus (S \cup h)), \\ u(S, \mathcal{P}''') = \sum_{j \in S} x_j, & \text{for all } \mathcal{P}''' \in \Pi(M \setminus S). \end{array} \tag{3.4}$$

Now, consider $y = (x, 0) \in R^M$. We will prove the following claims.

Claim 1 $y \in C^{\psi}(M,u)$.

Proof. By the definition of y and u, we have

$$\sum_{j \in M} y_j = \sum_{j \in N} x_j = v(N, \{\emptyset\}) = u(M, \{\emptyset\}).$$

First, we show that $v = u^{M \setminus h, y}$. For any $S \subseteq N = M \setminus h$ and any $\mathcal{P}'' \in \Pi(N \setminus S)$, we have

$$\begin{aligned} u^{M \setminus h, y}(S, \mathcal{P}'') &= u(S \cup h, \mathcal{P}'') - y_h \\ &= u(S \cup h, \mathcal{P}'') \\ &= v(S, \mathcal{P}''), \end{aligned}$$

where the last equality holds because of the second line of (3.4).

Now, for any $S \subseteq N = M \setminus h$, we have

$$\begin{aligned} \sum_{j \in S \cup h} y_j = \sum_{j \in S} x_j &\geq v(S, \psi(N, v, S)) \\ &= u(S \cup h, \psi(M, u, S \cup h)). \end{aligned}$$

The last equality holds because ψ is CC and v is a complement-reduced game of u. In addition, by the third line of (3.4), for any $S \subseteq N = M \setminus h$ and any $\mathcal{P}''' \in \Pi(M \setminus S)$, we have

$$\sum_{j \in S} y_j = \sum_{j \in S} x_j = u(S, \mathcal{P}''').$$

This completes the proof of Claim 1. □

Claim 2 $\{y\} = C^{\psi}(M,u)$.

Proof. If there exists $z \in C^{\psi}(M,u)$ such that $z \neq y$, we must have $\sum_{j \in M} z_j = u(M, \{\emptyset\}) = v(N, \{\emptyset\}) = \sum_{j \in N} x_j = u(N, \{\{h\}\}) \leq \sum_{j \in N} z_j$, and $z_h \geq u(h, \mathcal{P}') = 0$ for any $\mathcal{P}' \in \Pi(M \setminus h)$. Hence, $z_h = 0$.

For any $i \in N$ and any $\mathcal{P}''' \in \Pi(M \setminus i)$, we have $z_i \geq u(i, \mathcal{P}''') = x_i = y_i$ and, also, $\sum_{j \in N} z_j = \sum_{j \in M} z_j = u(M, \{\emptyset\}) = \sum_{j \in M} y_j = \sum_{j \in N} y_j$. Thus, we obtain $z_i = y_i$ for all $i \in N$, *i.e.*, $z = y$. This completes the proof of Claim 2. □

Now, consider $x \in C^{\psi}(N,v)$ and (M,u) again. By the first half of this proof, $\sigma(M,u) \subseteq C^{\psi}(M,u)$. As mentioned above, $C^{\psi}(M,u) = \{y\}$. By connecting them, $\sigma(M,u) \subseteq C^{\psi}(M,u) = \{y\}$. By NE on Γ^{ψ}, we obtain $\sigma(M,u) = C^{\psi}(M,u) = \{y\}$. Furthermore, by comp-RGP and $v = u^{M \setminus h, y}$, we have $x = y_N \in \sigma(N, u^{M \setminus h, y}) = \sigma(N,v)$. Thus, $C^{\psi}(N,v) \subseteq \sigma(N,v)$. □

Proposition 3.3 states that we can generalize the axiomatization of the core of games without externalities by using expectation function ψ. Note that if we "remove" externalities, this axiomatization coincides with Tadenuma's approach. To see this, consider an expectation function ψ as a function transforming a game with externalities v into a game without externalities w, by setting $w(S) := v(S, \psi(N, v, S))$. Proposition 3.3 shows that if ψ is CC, then this transformation ψ keeps the core's axiomatic characterization unchanged. As we have mentioned, the four expectation functions are all CC. Therefore, we have the following corollary.

Corollary 3.4. The four types of cores, *i.e.*, C^{opt}, C^{pes}, C^{dis} and C^{con}, can be axiomatized with axioms 1-3 on each class: $\Gamma_{C^{opt}}, \Gamma_{C^{pes}}, \Gamma_{C^{dis}}$ and $\Gamma_{C^{con}}$, respectively.

An example of expectation function that is not CC is

$$\psi(N, v, S) = \begin{cases} \{N \setminus S\} & \text{if } |S| \geq 2, \\ \{\{i_{s+1}\}, ..., \{i_n\}\} & \text{if } |S| = 1. \end{cases}$$

This expectation function can be seen as the combination of the conjunctive expectation and the disjunctive expectation. Expectation functions consisting of different expectation rules are, typically, not CC.

In view of Proposition 3.3, one might consider that the analogous proof can be adapted for the other types of reduced games. However, this conjecture is not necessarily true. We will see this fact in the following section.

4 The Other Reduced Games

In this section, we will extend and analyze the max-reduced game and the projection-reduced game, which were formulated by Davis and Maschler (1965) and Funaki and Yamato (2001), respectively. This extension includes two technical difficulties. First, we need a partition as the additional specifier to define the reduced game. We use $v^{S,\mathcal{P},x}$ to denote the reduced game instead of the previous notation $v^{S,x}$. Second, the generalization of the max-reduced game yields two possible extensions: max-I and max-II. The difference between the two is the domain of maximization. For any coalition $S \subseteq N$, the former ignores the partition structure of $N \setminus S$ and chooses $C \subseteq N \setminus S$, whereas the latter chooses C in the partition of $N \setminus S$.

Formally, we consider a set of games $\Gamma \subseteq \Gamma_A$ and a game $(N, v) \in \Gamma$. Let $S \subseteq N$ ($S \neq \emptyset$), $\mathcal{P} \in \Pi(N \setminus S)$, and $x \in R^N$.

Definition 4.1. The *max-reduced game (I) with respect to* $S, \mathcal{P}$ *and* x is the game $(S, v_{m1}^{S,\mathcal{P},x})$ defined as follows: for any $T \subseteq S$ ($T \neq \emptyset$) and any $\mathcal{Q} \in \Pi(S \setminus T)$,

$$v_{m1}^{S,\mathcal{P},x}(T, \mathcal{Q}) = \begin{cases} \max\limits_{C \subseteq N \setminus S} \left[v(T \cup C, \mathcal{Q} \cup (\mathcal{P}|_{(N \setminus S) \setminus C})) - \sum\limits_{j \in C} x_j \right], & if \;\; T \subsetneq S \\ v(N, \{\emptyset\}) - \sum_{j \in N \setminus S} x_j, & if \;\; T = S \end{cases}.$$

The *max-reduced game (II)*, $(S, v_{m2}^{S,\mathcal{P},x})$, is also defined by replacing the domain of the maximization $C \subseteq N \setminus S$ with $C \subseteq \mathcal{P}$.

Definition 4.2. The *projection-reduced game with respect to* $S, \mathcal{P}$ *and* x is the game $(S, v_p^{S,\mathcal{P},x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S \setminus T)$,

$$v_p^{S,\mathcal{P},x}(T,\mathcal{Q}) = \begin{cases} v(T, \mathcal{Q} \cup \mathcal{P}), & if \;\; T \subsetneq S \\ v(N, \{\emptyset\}) - \sum_{j \in N \setminus S} x_j, & if \;\; T = S \end{cases}.$$

If players leave the game one by one, the max-reduced games (both I and II) and the projection-reduced game are all independent of the order of the leaving players and the complement-reduced game. However, if two or more players simultaneously leave the game as a single group, the max-reduced games (I, II) and the projection-reduced game may depend on the partition of the leaving players.[*2] This contrasts with the fact that the complement-reduced game is independent of the partition of the leaving players.

The gap between "one-by-one leaving" and "at-once leaving" yields two RGPs. We call them "one-by-one RGP" and "at-once RGP." It is clear that the one-by-one RGP is weaker than the at-once RGP. We restrict our attention to the weaker RGP, *i.e.*, one-by-one RGP, and denote it, simply, RGP hereafter.

Now, we return to the main question of this section: can we adapt the technique of Proposition 3.3 to the proof of the axiomatizations for the max and the projection-reduced game? Table 1 describes the relationship between the cores and RGPs. It shows that the pessimistic core, the disjunctive core and the conjunctive core do not satisfy all RGPs except comp-RGP. All propositions and examples of Table 1 are found in the Appendix.

Table 1 The relationship between the cores and RGPs

	RGP			
	Max(I)	Max(II)	Projection	Complement
Optimistic core	Yes	Yes	Yes	Yes
Pessimistic core	-	-	-	Yes
Disjunctive core	-	-	-	Yes
Conjunctive core	-	-	-	Yes

Moreover, RGP based on one type of reduced game is not necessarily compatible with coherence based on the same type of reduced game. To see this, we define max(I, II)-coherence and projection-coherence in the same manner as CC. There is no ambiguity in defining these coherences. We define them by replacing $v^{N\setminus h,x}$ in Definition 2.4 with $v_{m1}^{N\setminus h,\{h\},x}$, $v_{m2}^{N\setminus h,\{h\},x}$ or $v_p^{N\setminus h,\{h\},x}$.[*3] Table 2 shows the relationship among the four expectation functions and coherences. All propositions and examples of Table 2 are found in the Appendix.

It is important that Table 1 and 2 indicate the difficulty of the generalization for the axiomatization using reduced game consistency. For simplicity, we focus on, for example, the max(I)-reduced game. The following discussion also holds for the max(II) and the projection-reduced games. Now, the following statement is true: even if an expectation function ψ satisfies max(I)-coherence, the ψ-core does

[*2] Formally, as in Lemma 2.2, we have $(v_{m1}^{-i_1})_{m1}^{-i_2} = (v_{m1}^{-i_2})_{m1}^{-i_1}$. However, there possibly exist partitions $\mathcal{P}$ and $\mathcal{P}'$ such that $v_{m1}^{N\setminus\{i_1,i_2\},\mathcal{P},x} \neq v_{m1}^{N\setminus\{i_1,i_2\},\mathcal{P}',x}$, where (m1) can be replaced with (m2) or (p).

[*3] In other words, we use the weaker definition based on "one-by-one leaving."

Table 2 The relationship between expectation functions and coherence

	Coherence			
	Max(I)	Max(II)	Projection	Complement
Optimistic expectation	-	-	-	Yes
Pessimistic expectation	-	-	-	Yes
Disjunctive expectation	Yes	Yes	Yes	Yes
Conjunctive expectation	Yes	Yes	Yes	Yes

not necessarily satisfy max(I)-RGP. Moreover, the disjunctive expectation satisfies max(I)-coherence, whereas the disjunctive core does not satisfy max(I)-RGP. This statement contrasts with that of Proposition 3.1. The max(I)-version of Proposition 3.3, therefore, does not hold. Moreover, any additional axioms no longer support the axiomatization based on reduced game consistency because the ψ-core violates RGP. The completion of our axiomatization in Proposition 3.3 is ascribed to the property of the complement-reduced game, which enables us to combine its coherence with its RGP.

5 Concluding Remarks

In this paper, we analyzed the relationship between reduced games and cores in the presence of externalities. We showed that if an expectation function ψ is CC, then the ψ-core can be axiomatized. In this section, we add three remarks.

The first remark concerns the relationship between CC and *subset consistency* studied by Bloch and van den Nouweland (2014).[*4] We change the original definition slightly to suit our framework as follows: for any $S \subseteq N$ and $T \subseteq S$,

$$\psi(N, v, S) = \psi(N, v, T)|_{(N \setminus S)},$$

where $\psi(N, v, T)|_{(N \setminus S)}$ is a partition of $N \setminus S$, the elements of which are the same as $\psi(N, v, T)$.[*5] Subset consistency describes that all players within S share the expectation on the behavior of outside players $N \setminus S$. Subset consistency is not the sufficient condition for the axiomatization and is logically independent of CC: there is an expectation function that satisfies subset consistency but violates CC, and vice versa.

Second, one might believe that the condition for the axiomatization, such as CC, should not depend on the specific reduced game, such as the complement-reduced game. We note that it is actually possible to define a condition that is free from any form of reduced game and is a sufficient condition for the axiomatization as follows. Let (N, v) be a game. For any $h \in N$, consider $w : EC(N \setminus h) \to R$ such that $v(S, \mathcal{P}) \geq v(S, \mathcal{P}') \iff w(S \setminus h, \mathcal{P}) \geq w(S \setminus h, \mathcal{P}')$ for any $S \ni h$ and any $\mathcal{P}, \mathcal{P}' \in \Pi(N \setminus S)$. Then $\psi(N, S, v) = \psi(N \setminus h, S \setminus h, w)$. It is clear that this condition does not depend on any specific type of reduced game and is a sufficient condition for the axiomatization. Therefore, although CC is weaker

[*4] Bloch and van den Nouweland (2014) define their *coherence*, which describes coherence within a partition. We, however, focus on subset consistency because our CC is closer to subset consistency rather than the coherence.

[*5] Formally, for any partition $\mathcal{P}$ and coalition $S \subseteq N$, let $\mathcal{P}|_S$ be given by $\mathcal{P}|_S = \{S \cap C \mid C \in \mathcal{P},\ S \cap C \neq \emptyset\} \in \Pi(S)$.

than this condition, one can replace CC with this condition. The four expectation functions listed earlier all satisfy this condition.

Third, throughout this paper we used expectation functions associated with partitions, namely, $\psi(N, v, S) \in \Pi(N \setminus S)$. However, the probabilistic approach, as is sometimes seen in the papers studying the generalization of the Shapley value, is also possible, in which $\psi(N, v, S)$ is a probability distribution over $\Pi(N \setminus S)$. Each definition and result in this paper can be straightforwardly adjusted to the probabilistic framework. It is notable that any convex combination of the four expectation functions listed earlier, for instance, 50% for the best partition $\arg\max_{\mathcal{P}' \in \Pi(N\setminus S)} v(S, \mathcal{P}')$ and 50% for the worst partition $\arg\min_{\mathcal{P}' \in \Pi(N\setminus S)} v(S, \mathcal{P}')$, is still CC.

Appendix

To distinguish each form of reduced game, in this appendix, we use symbols $v_{m1}^{S,\mathcal{P},x}$, $v_{m2}^{S,\mathcal{P},x}$, $v_{p}^{S,\mathcal{P},x}$ and $v_{c}^{S,x}$ to denote max(I), max(II), projection and complement-type of reduced game, respectively. Table A.1 and Table A.2 correspond to Table 1 and Table 2, respectively. The number assigned to each cell represents the proposition or example describing the cell, *e.g.*, for the proposition showing that the optimistic core satisfies Max-I RGP, see Proposition A.8.

Table A.1 The relationship between expectation cores and RGP (corresponding to Table 1)

	RGP			
	Max(I)	Max(II)	Projection	Complement
Optimistic core	Yes $_{A.8}$	Yes $_{A.8}$	Yes $_{A.8}$	Yes $_{\text{Prop.3.1}}$
Pessimistic core	- $_{A.9}$	- $_{A.9}$	- $_{A.11}$	Yes $_{\text{Prop.3.1}}$
Disjunctive core	- $_{A.10}$	- $_{A.10}$	- $_{A.11}$	Yes $_{\text{Prop.3.1}}$
Conjunctive core	- $_{A.9}$	- $_{A.9}$	- $_{A.12}$	Yes $_{\text{Prop.3.1}}$

Table A.2 The relationship between expectation functions and coherence (corresponding to Table 2)

	Coherence			
	Max(I)	Max(II)	Projection	Complement
Optimistic expectation	- $_{A.6}$	- $_{A.6}$	- $_{A.6}$	Yes $_{A.1}$
Pessimistic expectation	- $_{A.6}$	- $_{A.6}$	- $_{A.6}$	Yes $_{A.1}$
Disjunctive expectation	Yes $_{A.5}$	Yes $_{A.5}$	Yes $_{A.5}$	Yes $_{A.5}$
Conjunctive expectation	Yes $_{A.5}$	Yes $_{A.5}$	Yes $_{A.5}$	Yes $_{A.5}$

Proposition A.1. If ψ is optimistic or pessimistic, then ψ is CC.

Proof. We denote by ψ^{opt} the optimistic expectation function. Let (N, v) be a game, and $S \subseteq N$

($|S| \geq 2$). We define $\mathcal{P}^*$ as follows:

$$\mathcal{P}^* := \psi^{opt}(N, v, S) = \underset{\mathcal{P}' \in \Pi(N \setminus S)}{\arg\max}\ v(S, \mathcal{P}'). \tag{A.1}$$

For any $h \in S$ and $x \in R^N$, we have

$$\begin{aligned} v_c^{N \setminus h, x}(S \setminus h, \mathcal{P}^*) &= v(S, \mathcal{P}^*) - x_h \\ &= \max_{\mathcal{P}' \in \Pi(N \setminus S)} [v(S, \mathcal{P}') - x_h] \\ &= \max_{\mathcal{P}' \in \Pi(N \setminus S)} [v_c^{N \setminus h, x}(S \setminus h, \mathcal{P}')], \end{aligned}$$

where the first equality holds by the definition of complement reduced games, the second by (A.1) and the last by the definition of complement reduced games. Hence, we obtain

$$\mathcal{P}^* = \underset{\mathcal{P}' \in \Pi(N \setminus S)}{\arg\max}\ v_c^{N \setminus h, x}(S \setminus h, \mathcal{P}') = \psi^{opt}(N \setminus h, v_c^{N \setminus h, x}, S \setminus h),$$

and, then, $\psi^{opt}(N, v, S) = \psi^{opt}(N \setminus h, v_c^{N \setminus h, x}, S \setminus h)$, which implies ψ^{opt} is CC.

By replacing max with min, we complete the proof of the pessimistic expectation function ψ^{pes} as well. □

Proposition A.2. If ψ satisfies the following condition: for any games (N, v), (M, w), and nonempty coalitions $S \subseteq N$, $T \subseteq M$,

$$N \setminus S = M \setminus T \Longrightarrow \psi(N, v, S) = \psi(M, w, T), \tag{A.2}$$

then ψ satisfies all four types of coherence: Max-I, Max-II, Projection and Complement.

Proof. We prove CC (or, complement coherence). The other types of coherence are obtained in the same way. Fix a game (N, v). For any $x \in R^N$ and $h \in N$, we can specify the complement reduced game $(N \setminus h, v_c^{N \setminus h, x})$. For any S such that $h \in S \subseteq N$, we have

$$N \setminus S = (N \setminus h) \setminus (S \setminus h).$$

Using (A.2), we obtain $\psi(N, v, S) = \psi(N \setminus h, v_c^{N \setminus h, x}, S \setminus h,)$ □

Lemma A.3. If ψ is disjunctive, then ψ satisfies (A.2).

Proof. We denote by ψ^{dis} the disjunctive expectation function. For any nonempty T and S with $T \in S \subseteq N$, and any $w : EC(N \setminus T) \to R$, we have $\psi^{dis}(N, v, S) = \{\{i\} | i \in N \setminus S\} = \psi^{dis}(N \setminus T, w, S \setminus T)$. □

Lemma A.4. If ψ is conjunctive, then ψ satisfies (A.2).

Proof. This is similar to Lemma A.3. Let ψ^{con} denote the conjunctive expectation function. We have $\psi^{con}(N, v, S) = \{N \setminus S\} = \psi^{con}(N \setminus T, w, S \setminus T)$. □

Corollary A.5. If ψ is disjunctive or conjunctive, then ψ satisfies all four types of coherence.

Proof. See Lemmas A.3, A.4 and Proposition A.2. □

Example A.6. Consider the following 4-player game: $N = \{i_1, i_2, i_3, i_4\}$;

$$\begin{aligned}
&v(N, \{\emptyset\}) = 12;\\
&v(\{i,j,k\}, \{\{h\}\}) = 5 \textit{ and } v(\{h\}, \{\{i,j,k\}\}) = 0, \quad \textit{for } \{i,j,k,h\} = N;\\
&v(\{i,j\}, \{\{k,h\}\}) = 4, \quad \textit{for } \{i,j,k,h\} = N;\\
&v(\{i,j\}, \{\{k\},\{h\}\}) = 3 \textit{ and } v(\{k\}, \{\{i,j\},\{h\}\}) = 1, \quad \textit{for } \{i,j,k,h\} = N;\\
&v(\{i\}, \{\{j\},\{k\},\{h\}\}) = 2, \quad \textit{for } \{i,j,k,h\} = N.
\end{aligned}$$

Let $x = (3,3,3,3)$, $S = \{i_1, i_2\}$ and player $h = 1$. For the optimistic expectation function, we have

$$\arg\max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}') = \{\{i_3, i_4\}\},$$

because $\max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}') = \max\{v(S, \{\{i_3, i_4\}\}), v(S, \{\{i_3\}, \{i_4\}\})\} = \max\{4, 3\}$. However, in the Max-I reduced game, we have

$$\arg\max_{\mathcal{P}' \in \Pi(N \setminus S)} v_{m1}^{-h}(S \setminus h, \mathcal{P}') = \{\{i_3\}, \{i_4\}\},$$

because

$$\begin{aligned}
\max_{\mathcal{P}' \in \Pi(N \setminus S)} v_{m1}^{-h}(S \setminus h, \mathcal{P}') &= \max \left\{ \begin{array}{ll} v(S, \{\{i_3, i_4\}\}) - x_h, & v(S \setminus h, \{\{i_1\}, \{i_3, i_4\}\}), \\ v(S, \{\{i_3\}, \{i_4\}\}) - x_h, & v(S \setminus h, \{\{i_1\}, \{i_3\}, \{i_4\}\}) \end{array} \right\} \qquad \text{(A.3)}\\
&= \max\{4-3, 1, 3-3, 2\}\\
&= 2,
\end{aligned}$$

which is the worth of the bottom-right element in (A.3). Hence, $\psi^{opt}(N, v, S) = \{\{i_3, i_4\}\} \neq \{\{i_3\}, \{i_4\}\} = \psi^{opt}(N \setminus h, v_{m1}^{-h}, S \setminus h)$. For the optimistic expectation function, this example is still valid for Max-II and Projection coherence as well. For the pessimistic expectation function, we can generate the example by swapping $v(\{i,j\}, \{\{i,j\}, \{k,h\}\})$ for $v(\{i,j\}, \{\{i,j\}, \{k\}, \{h\}\})$.

Lemma A.7. Let $(N, v) \in \Gamma$. Let $S \subseteq N$, $\mathcal{P} \in \Pi(N \setminus S)$ and $x \in R^N$. We denote each type of reduced game by $v_{m1}^{S,\mathcal{P},x}$, $v_{m2}^{S,\mathcal{P},x}$, $v_{p}^{S,\mathcal{P},x}$ and $v_{c}^{S,x}$, respectively. Then, for any $T \subseteq S$ ($T \neq \emptyset$) and $\mathcal{Q} \in \Pi(S \setminus T)$, we have

$$\begin{aligned}
v_{m1}^{S,\mathcal{P},x}(T, \mathcal{Q}) &\geq v_{m2}^{S,\mathcal{P},x}(T, \mathcal{Q}),\\
v_{m2}^{S,\mathcal{P},x}(T, \mathcal{Q}) &\geq v_{p}^{S,\mathcal{P},x}(T, \mathcal{Q}),\\
v_{m2}^{S,\mathcal{P},x}(T, \mathcal{Q}) &\geq v_{c}^{S,x}(T, \mathcal{Q}).
\end{aligned}$$

Proof. The first inequality follows from the domain of maximization: in view of the definitions, for any $\mathcal{P} \in \Pi(N \setminus S)$,

$$\{C | C \in \mathcal{P}\} \text{ (or, Max-II) } \subseteq \{C | C \subseteq N \setminus S\} \text{ (or, Max-I) }.$$

The second (third) inequality holds because we can take $\emptyset$ ($N \setminus S$) as maximizer C. □

Proposition A.8. The optimistic-core satisfies all types of RGP on $\Gamma_{C^{opt}}$: maxI-RGP, maxII-RGP, projection-RGP and comp-RGP.

Proof. Let $C^{opt}(N,v)$ be the optimistic core of (N,v) and $x \in C^{opt}(N,v)$. We show that the optimistic-core satisfies maxI-RGP. For any $S \subseteq N$, $T \subsetneq S$ ($T \neq \emptyset$) and $\mathcal{P} \in \Pi(N \setminus S)$, we have

$$\begin{aligned}
&\sum_{j \in T} x_j - \max_{\mathcal{Q} \in \Pi(S \setminus T)} v_{m1}^{S,\mathcal{P},x}(T, \mathcal{Q}) \\
&= \sum_{j \in T} x_j - \max_{\mathcal{Q} \in \Pi(S \setminus T)} \max_{C \subseteq N \setminus S} \left[v(T \cup C, \mathcal{Q} \cup (\mathcal{P}|_{(N \setminus S) \setminus C})) - \sum_{j \in C} x_j \right] \\
&= \sum_{j \in T} x_j - \left[v(T \cup C^*, \mathcal{Q}^* \cup (\mathcal{P}|_{(N \setminus S) \setminus C^*})) - \sum_{j \in C^*} x_j \right] \qquad \text{(A.4)} \\
&= \sum_{j \in T \cup C^*} x_j - v(T \cup C^*, \mathcal{Q}^* \cup (\mathcal{P}|_{(N \setminus S) \setminus C^*})) \\
&\geq \max_{\mathcal{P}' \in \Pi(N \setminus (T \cup C^*))} v(T, \mathcal{P}') - v(T \cup C^*, \mathcal{Q}^* \cup (\mathcal{P}|_{(N \setminus S) \setminus C^*})) \qquad \text{(A.5)} \\
&\geq 0,
\end{aligned}$$

where $C^*, \mathcal{Q}^*$ in (A.4) are maximizers of the target formula, and (A.5) holds because $x \in C^{opt}(N,v)$. Similarly, for $T = S$, we have

$$\begin{aligned}
\sum_{j \in S} x_j - v^{S,\mathcal{P},x}(S, \{\emptyset\}) &= \sum_{j \in S} x_j - \left(v(N, \{\emptyset\}) - \sum_{j \in N \setminus S} x_j \right) \\
&= \sum_{j \in N} x_j - v(N, \{\emptyset\}) \\
&= 0.
\end{aligned}$$

By Lemma A.7, we can replace $v_{m1}^{S,\mathcal{P},x}$ with $v_{m2}^{S,\mathcal{P},x}$, $v_{p}^{S,\mathcal{P},x}$ and $v_{c}^{S,\mathcal{P},x}$, respectively. Then, we obtain the desired proposition. □

Example A.9. Consider the following 4-player game: $N = \{1,2,3,4\}$,

$$v(S,\mathcal{P}) = \begin{cases} 12 & \text{if } (S,\mathcal{P}) = (N, \{\emptyset\}), \\ 6 & \text{if } (S,\mathcal{P}) = (\{i,j\}, \{\{k\},\{h\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (1,3,4,4)$. Then, $x \in C^{pes}(N,v) = C^{con}(N,v)$. Now, for $S = \{1,2\}$ and $\mathcal{P} = \{\{3\},\{4\}\}$, we have the following Max-I reduced game:

$$\begin{aligned}
v_{m1}^{S,P,x}(\{1,2\},\{\emptyset\}) &= 12 - (4+4) = 4, \\
v_{m1}^{S,P,x}(\{1\},\{\{2\}\}) &= 6 - 4 = 2, \\
v_{m1}^{S,P,x}(\{2\},\{\{1\}\}) &= 6 - 4 = 2.
\end{aligned}$$

The restriction of x, $x_S = (1,3)$, is out of the pessimistic core (and the conjunctive core) of the reduced game: $x_S = (1,3) \notin \{(2,2)\} = C^{pes}(S, v_{m1}^{S,P,x}) = C^{con}(S, v_{m1}^{S,P,x})$. We have the Max-II reduced game as well as Max-I.

Example A.10. Consider the following 5-player game: $N = \{1,2,3,4,5\}$,

$$v(S,\mathcal{P}) = \begin{cases} 15 & \text{if } (S,\mathcal{P}) = (N, \{\emptyset\}), \\ 7 & \text{if } (S,\mathcal{P}) = (\{i,j\}, \{\{k\},\{h,l\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4, x_5) = (2, 2, 4, 4, 3)$. Then, $x \in C^{dis}(N, v)$. For $S = \{3, 4\}$ (who obtain 4 in x) and $\mathcal{P} = \{\{1\}, \{2, 5\}\}$, we have the following Max-I reduced game:

$$v_{m1}^{S,P,x}(\{3,4\}, \{\emptyset\}) = 15 - (2 + 2 + 3) = 8,$$
$$v_{m1}^{S,P,x}(\{3\}, \{\{4\}\}) = 7 - 2 = 5,$$
$$v_{m1}^{S,P,x}(\{4\}, \{\{3\}\}) = 7 - 2 = 5.$$

Hence, the disjunctive core is empty. We have the same result in Max-II as well as Max-I.

Example A.11. Consider the following 4-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k, h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k\}, \{h\}\}), \\ 3 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j\}, \{k\}, \{h\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (3, 3, 3, 3)$. Then, $x \in C^{pes}(N, v) = C^{dis}(N, v)$. For $S = \{1, 2\}$ and $\mathcal{P} = \{\{3, 4\}\}$, we have the following projection reduced game:

$$v_p^{S,P,x}(\{1,2\}, \{\emptyset\}) = 12 - (3 + 3) = 6,$$
$$v_p^{S,P,x}(\{1\}, \{\{2\}\}) = 4,$$
$$v_p^{S,P,x}(\{2\}, \{\{1\}\}) = 4.$$

Hence, the pessimistic core and the disjunctive core are empty in the reduced game.

Example A.12. Consider the following 4-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 & \text{if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 3 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k, h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k\}, \{h\}\}), \\ 4 & \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j\}, \{k\}, \{h\}\}), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (3, 3, 3, 3)$. Then, $x \in C^{con}(N, v)$. For $S = \{1, 2\}$ and $\mathcal{P} = \{\{3, 4\}\}$, we have the following projection reduced game:

$$v_p^{S,P,x}(\{1,2\}, \{\emptyset\}) = 12 - (3 + 3) = 6,$$
$$v_p^{S,P,x}(\{1\}, \{\{2\}\}) = 4,$$
$$v_p^{S,P,x}(\{2\}, \{\{1\}\}) = 4.$$

Hence, the conjunctive core of the reduced game becomes empty.

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