WINPEC Working Paper Series No.E1601
April 2016

Efficiency and the Core in Games with Positive and Negative Externalities

Takaaki Abe

Waseda INstitute of Political EConomy
Waseda University
Tokyo,Japan

Efficiency and the Core in Games with Positive and Negative Externalities

Takaaki Abe *

March 26, 2016

Abstract

In this paper, we study games with externalities. In the presence of externalities, the worth of a coalition depends on the coalition itself and a partition of the player set. In the presence of externalities, *superadditivity* is no longer sufficient for the *efficiency of the grand coalition*: some partitions may yield more surplus than the grand coalition. This result was indicated by Hafalir (GEB 61:242-258, 2007), who proposed *convexity* as the sufficient condition. We attempt to extend the results of Hafalir and explore certain weaker sufficient conditions to achieve the efficiency of the grand coalition. Our approach features two types of externalities: positive and negative. Furthermore, we examine a condition for the non-emptiness of the *core*. In the presence of externalities, the definition of the core is not unique. In this light, we specifically analyze the *optimistic core* which is the smallest of all types of cores. We prove that a combination of negative externalities and particular conditions can induce the nonempty optimistic core.

Keywords: Coalition formation; Core; Efficiency; Externalities; Partition function games
JEL Classification: C71

1 Introduction

The purpose of this paper is to offer certain conditions in which the coalition among all players—the *grand coalition*—becomes efficient in environments with externalities. Furthermore, we study a condition for stability in terms of deviations from the grand coalition.

In the cooperative game theory literature, many methods have been proposed to allocate the surplus generated by a group of players. Such allocations are often referred to as "solutions." In attempts to explore solutions, many studies have focused on the "fairness" of solutions and

* Graduate School of Economics, Waseda University
1-6-1, Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, Japan
JSPS Research Fellow
E-mail: takatomo3639@asagi.waseda.jp

have (at least somewhat) taken their "efficiency" for granted: almost all solutions allocate the surplus of the grand coalition.*1 Therefore, previous studies that have studied solutions have assumed the formation of the grand coalition.

As an assumption that justifies the formation of the grand coalition, *superadditivity* is well known. Justifying the grand coalition is based on the plausible fact that superadditivity implies the grand coalition's efficiency—the surplus of the grand coalition is greater than the entire aggregated surplus of each partitioned coalition (Definition 2.1). If superadditivity holds and the surplus is transferable among players, the players certainly derive more benefit by forming larger coalitions and derive the most benefit, as a consequence, by forming the largest coalition: the grand coalition. However, this reasoning is no longer valid when there are externalities among coalitions.

In games without externalities (often referred to as *coalition function form games*), it is clear that superadditivity makes the grand coalition efficient. Nevertheless, the same is not always true in games with externalities (referred to as *partition function form games*) in which the worth of each coalition depends not only on the coalition but also on the partition, *i.e.*, the entire structure of coexisting coalitions. Surprisingly, in the presence of externalities, there may be certain coalition structures that are more efficient than the grand coalition, even in superadditive settings. This problem has been highlighted by Hafalir (2007), who defined the *convexity* of partition functions and proved that such convexity is sufficient for the efficiency of the grand coalition in environments with externalities.

In this paper, we attempt to extend the results of Hafalir (2007). In our approach, we focus on two types of externalities: positive externalities and negative externalities. Many games describing economic situations are basically classified as characterized by the presence of either positive or negative externalities.*2 On one hand, under positive externalities, each coalition enjoys the benefits of a merger among other coalitions, and coalitions are therefore more likely to cooperate with one another. On the other hand, under negative externalities, each coalition is damaged by a connection among other coalitions. Such an environment is believed to be more competitive. This difference in externalities yields an asymmetry of tendencies toward the formation of the grand coalition. We will argue this subject more extensively in Section 3.

Moreover, it is important to note that efficiency does not necessarily mean the "stability" of the grand coalition. In other words, as we discuss below, efficiency is not sufficient for the non-emptiness of the *core*. The situation in which we have the efficient grand coalition and the empty core can be considered as an analog of the "prisoners' dilemma": even if a Pareto

*1 Maskin (2003) proposed a solution that may dispose of some amount of the surplus obtained from the grand coalition and is thus an important exception in this regard.

*2 Nevertheless, there is a class of environments in which externalities are neither positive nor negative.

optimal outcome was temporarily achieved, some players might have an incentive to deviate from it. In the presence of externalities, players attempting to deviate from the grand coalition may consider various reactions of the other players and coalitions. This diversity of conjecture yields many definitions of the core in partition function form. In this paper, we focus mainly on two types of cores: the optimistic core and the pessimistic core. We expound upon this topic in Section 4.

The remainder of this paper is organized as follows. We first recall the settings of partition function form games in Section 2. Then, we introduce three basic conditions for games: the efficiency of the grand coalition, superadditivity and convexity. In Section 3, we introduce our two conditions and prove that they are sufficient for the efficiency of the grand coalition with respect to positive and negative externalities. The results regarding the non-emptiness of the optimistic core is presented in Section 4.

2 Preliminaries

2.1 Partition Function Form Games

Let $N = \{1, ..., n\}$ be a set of players. A *coalition* of players is a subset of the player set, *i.e.*, $S \subseteq N$. For any coalition S, $|S|$ denotes the number of players in S. A partition of the player set N is defined by $\mathcal{P} = \{S_1, ..., S_h\}$, where $1 \leq h \leq n$, $S_i \cap S_j = \emptyset$ for $i, j = 1, ..., h$ $(i \neq j)$, and $\bigcup_{i=1}^{h} S_i = N$. Let Π be the set of all partitions of N. For any partition $\mathcal{P}$, we denote by $|\mathcal{P}|$ the number of coalitions in $\mathcal{P}$. We define an *embedded coalition* by $(S, \mathcal{P})$, satisfying $S \in \mathcal{P}$. The set of all embedded coalitions $\mathscr{L}$ is given by $\mathscr{L} = \{(S, \mathcal{P}) \in 2^N \times \Pi \mid S \in \mathcal{P}\}$. A *partition function* v is a function that assigns a real number to each embedded coalition, $v : \mathscr{L} \to R$. By convention, $\emptyset \in \mathcal{P}$ and $v(\emptyset, \mathcal{P}) = 0$ for any partition $\mathcal{P}$. Let (N, v) denote a partition function form game.

2.2 Efficiency

As discussed briefly in Section 1, this paper aims to clarify the sufficient conditions for the efficiency of the grand coalition. We define the efficiency of the grand coalition as follows.

Definition 2.1. A partition function v satisfies the *efficiency of the grand coalition* (*EFF*) if for any partition $\mathcal{P}$ of N,

$$v(N, \{N\}) \geq \sum_{S \in \mathcal{P}} v(S, \mathcal{P}).$$

In words, we can consider the summation of each coalition's surplus for any partition. The efficiency of the grand coalition describes that situation in which the surplus of the grand coalition is greater than this summation of each partition.

2.3 Superadditivity

In the presence of externalities, the definition of superadditivity is no longer unique. We first introduce the definition that is employed by Bloch and van den Nouweland (2014) and Hafalir (2007).

Definition 2.2. A partition function v satisfies *superadditivity* if for any $S, T \subseteq N$ with $S \cap T = \emptyset$ and any partition $\mathcal{Q}$ of $N \setminus (S \cup T)$,

$$v(S \cup T, \{(S \cup T)\} \cup \mathcal{Q}) \geq v(S, \{S, T\} \cup \mathcal{Q}) + v(T, \{S, T\} \cup \mathcal{Q}).$$

It is commonly known that superadditivity yields the efficient grand coalition in the absence of externalities. However, in games with externalities, superadditivity fails to accomplish the efficiency of the grand coalition, which is illustrated in the following example.

Example 2.3. This example was introduced by Hafalir (2007). Let $N = \{1, 2, 3\}$. Consider the following symmetric partition function:

$$\begin{aligned}
&v(N, \{N\}) = 11; \\
&v(\{i\}, \{\{i\}, \{j, k\}\}) = 1 \quad and \quad v(\{j, k\}, \{\{i\}, \{j, k\}\}) = 9; \\
&v(\{i\}, \{\{1\}, \{2\}, \{3\}\}) = 4 \quad for\ any\ i \in N.
\end{aligned}$$

This game satisfies superadditivity. However, the grand coalition is not efficient because we have $v(N, \{N\}) = 11 < 12 = \sum_{i=1}^{3} v(\{i\}, \{\{1\}, \{2\}, \{3\}\})$.

2.4 Convexity

Because of the difficulty described in Example 2.3, Hafalir (2007) defined convexity in partition function form and showed that convexity implies the efficiency of the grand coalition.

Definition 2.4. A partition function v satisfies *convexity* if for any $S, T \subseteq N$, and any partition $\mathcal{Q}$ of $N \setminus (S \cup T)$,

$$\begin{aligned}
v(S \cup T, \{(S \cup T)\} \cup \mathcal{Q}) + v(S \cap T, \{(S \cap T), S \setminus T, T \setminus S\} \cup \mathcal{Q}) \\
\geq v(S, \{S, T \setminus S\} \cup \mathcal{Q}) + v(T, \{T, S \setminus T\} \cup \mathcal{Q}).
\end{aligned}$$

Moreover, Hafalir (2007) examined another extension of convexity to partition functions as follows.[*3]

[*3] See Appendix A in Hafalir (2007) for detailed discussions.

Definition 2.5. A partition function v satisfies *weak convexity* if for any $S, T \subseteq N$ with $|S \setminus T| = |T \setminus S| \leq 1$, and any partition $\mathcal{Q}$ of $N \setminus (S \cup T)$,

$$v(S \cup T, \{(S \cup T)\} \cup \mathcal{Q}) + v(S \cap T, \{(S \cap T), S \setminus T, T \setminus S\} \cup \mathcal{Q})$$
$$\geq v(S, \{S, T \setminus S\} \cup \mathcal{Q}) + v(T, \{T, S \setminus T\} \cup \mathcal{Q}).$$

Notably, in games with no externalities, it is commonly known that weak convexity is equivalent to convexity. In contrast, under externalities, weak convexity becomes literally weaker than convexity. As the following example illustrates, weak convexity does not achieve the efficiency of the grand coalition.

Example 2.6. This example was introduced by Hafalir (2007). Consider the following 5-player game: $N = \{1, 2, 3, 4, 5\} = \{i, j, k, l, m\}$;

$$v(N, \{N\}) = 25;$$
$$v(\{i,j,k,l\}, \{\{i,j,k,l\}, \{m\}\}) = 18 \ \ \textit{and} \ \ v(\{m\}, \{\{i,j,k,l\}, \{m\}\}) = 3;$$
$$v(\{i,j,k\}, \{\{i,j,k\}, \{l,m\}\}) = 17 \ \ \textit{and} \ \ v(\{l,m\}, \{\{i,j,k\}, \{l,m\}\}) = 6;$$
$$v(\{i,j,k\}, \{\{i,j,k\}, \{l\}, \{m\}\}) = 12 \ \ \textit{and} \ \ v(\{l\}, \{\{i,j,k\}, \{l\}, \{m\}\}) = 3;$$
$$v(\{i,j\}, \{\{i,j\}, \{k,l\}, \{m\}\}) = 9 \ \ \textit{and} \ \ v(\{m\}, \{\{i,j\}, \{k,l\}, \{m\}\}) = 8;$$
$$v(\{i,j\}, \{\{i,j\}, \{k\}, \{l\}, \{m\}\}) = 7 \ \ \textit{and} \ \ v(\{k\}, \{\{i,j\}, \{k\}, \{l\}, \{m\}\}) = 3;$$
$$v(\{i\}, \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}) = 3 \ \ \textit{for any } i \in N.$$

This game satisfies weak convexity but violates the efficiency of the grand coalition because we have

$$v(N, \{N\}) = 25 < 26 = 9 + 9 + 8$$
$$= v(\{i,j\}, \{\{i,j\}, \{k,l\}, \{m\}\}) + v(\{k,l\}, \{\{i,j\}, \{k,l\}, \{m\}\}) + v(\{m\}, \{\{i,j\}, \{k,l\}, \{m\}\}).$$

Furthermore, this game is not convex because for $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$,

$$34 = v(S \cup T, \{S \cup T\}) + v(S \cap T, \{(S \cap T), S \setminus T, T \setminus S\})$$
$$< v(S, \{S, T \setminus S\}) + v(T, \{T, S \setminus T\}) = 35.$$

3 Conditions for the Efficiency of the Grand Coalition

3.1 Environments with Externalities

In this section, we provide two new conditions (Definitions 3.2 and 3.4) to accomplish the efficiency of the grand coalition. We aim to obtain efficiency by conditions weaker than convexity. To this end, we focus on the properties of externalities. Externalities can be classified as positive or negative. A partition function is said to have positive (negative) externalities when each coalition becomes better (worse) off whenever other coalitions combine. Positive externalities and negative externalities are formally defined as follows.

A partition function v has *positive externalities* if for any mutually disjoint $C, S, T \subseteq N$ ($C \neq \emptyset$, $S \neq \emptyset$, $T \neq \emptyset$), and any partition $\mathcal{Q}$ of $N \setminus (S \cup T \cup C)$,

$$v(C, \{C, (S \cup T)\} \cup \mathcal{Q}) > v(C, \{C, S, T\} \cup \mathcal{Q}).$$

Similarly, a partition function v has *negative externalities* if for any mutually disjoint $C, S, T \subseteq N$ ($C \neq \emptyset$, $S \neq \emptyset$, $T \neq \emptyset$) and any partition $\mathcal{Q}$ of $N \setminus (S \cup T \cup C)$,

$$v(C, \{C, (S \cup T)\} \cup \mathcal{Q}) < v(C, \{C, S, T\} \cup \mathcal{Q}).$$

In this paper, we follow the definitions of Hafalir (2007) and use the strict inequalities ($>$) and ($<$), which can be replaced with ($\geq$) and ($\leq$), respectively, and the essentials of our results do not change in these settings.

In the literature of cooperative games, several early works addressed certain economic situations in which externalities exist among coalitions. We can categorize the externalities as either positive or negative. A leading example of such economic models with positive externalities is Cournot quantity competition in partition function form, which was initially introduced by Ray and Vohra (1999). Other examples of positive externalities include the "public goods games" studied by Yi (1997) and Ray and Vohra (1997) and the "tragedy of the commons" examined by Funaki and Yamato (1999). It might be argued that a competitive situation such as a Cournot oligopoly should be compatible with negative—rather than positive—externalities. This intuition is captured by Cournot competition with joint ventures studied by Bloch (1995) and Yi (1998). Another example involving negative externalities includes "customs unions" studied by Yi (1996).

3.2 Positive Externalities

In this subsection, we focus on games with positive externalities. First, by slightly changing superadditivity given in Definition 2.2, we define a stronger condition.

Definition 3.1. A partition function v satisfies *strong-superadditivity* (*strong-SA*) if for any mutually disjoint $S, T, S', T' \subseteq N$ such that $S \neq \emptyset, T \neq \emptyset$ and any $\mathcal{P} \in \Pi$ with $S, T, S', T' \in \mathcal{P}$, we have

$$\begin{aligned} &v(S \cup T, \{(S \cup T), (S' \cup T')\} \cup \mathcal{Q}) \\ &\geq v(S, \{S, T, S', T'\} \cup \mathcal{Q}) + v(T, \{S, T, S', T'\} \cup \mathcal{Q}), \end{aligned}$$

where $\mathcal{Q}$ is the partition of $N \setminus (S \cup T \cup S' \cup T')$ defined by $\mathcal{Q} = \mathcal{P} \setminus \{S, T, S', T'\}$.

Strong-SA describes the situation in which two coalitions S' and T' form their coalition $(S' \cup T')$ in response to the first formation of coalition $(S \cup T)$. In fact, strong-SA requires that the merging coalitions S and T can benefit from any following merger of the other two coalitions S' and T'. Note that S' and T' may be empty and that strong-SA implies superadditivity in both positive and negative externalities.

Next, we consider a variant of strong-SA in which responsive coalitions S' and T' are restricted to be nonempty. With this restriction, each partition requires at least four nonempty coalitions: S, T, S', T'. The later part of the following definition covers those exceptional cases in which the cardinality of a partition is two or three.

Definition 3.2. A partition function v satisfies *quasi strong-superadditivity* (*QS-SA*) if for any mutually disjoint $S, T, S', T' \subseteq N$ such that $S \neq \emptyset, T \neq \emptyset, S' \neq \emptyset, T' \neq \emptyset$ and any $\mathcal{P} \in \Pi$ such that $S, T, S', T' \in \mathcal{P}$ and $|\mathcal{P}| \geq 4$, we have

$$\begin{aligned} &v(S \cup T, \{(S \cup T), (S' \cup T')\} \cup \mathcal{Q}) \\ &\geq v(S, \{S, T, S', T'\} \cup \mathcal{Q}) + v(T, \{S, T, S', T'\} \cup \mathcal{Q}), \end{aligned}$$

where $\mathcal{Q}$ is the partition of $N \setminus (S \cup T \cup S' \cup T')$ defined by $\mathcal{Q} = \mathcal{P} \setminus \{S, T, S', T'\}$; otherwise, *i.e.*, $|\mathcal{P}| = 2$ *or* 3, for any nonempty $S, T \in \mathcal{P}$ with $S \cap T = \emptyset$, we have

$$v(S \cup T, \{(S \cup T)\} \cup \mathcal{Q}') \geq v(S, \{S, T\} \cup \mathcal{Q}') + v(T, \{S, T\} \cup \mathcal{Q}'),$$

where $\mathcal{Q}'$ is the partition of $N \setminus (S \cup T)$ defined by $\mathcal{Q}' = \mathcal{P} \setminus \{S, T\}$.

Notably, in games under positive externalities, QS-SA is weaker than superadditivity because a merger not only between S and T but also between S' and T' is beneficial for S and T. However, we show in the next subsection that QS-SA is not necessarily weaker than superadditivity under negative externalities.

The following proposition posits that QS-SA can accomplish the efficiency of the grand coalition.

Proposition 3.3. For any partition function v with positive externalities, if v satisfies QS-SA, then v satisfies the efficiency of the grand coalition.

Proof. The proof is implemented by induction on the cardinality of partition $\mathcal{P}$.

Induction hypothesis: For any partition $\mathcal{P}$,

$$v(N, \{N\}) \geq \sum_{S \in \mathcal{P}} v(S, \mathcal{P}).$$

Induction base: For $|\mathcal{P}| = 2$, let $\mathcal{P} = \{S, T\}$. From the definition (of the part of $|\mathcal{P}| \leq 3$) of QS-SA, we have

$$v(N, \{N\}) \geq v(S, \mathcal{P}) + v(T, \mathcal{P}).$$

For $|\mathcal{P}| = 3$, let $\mathcal{P} = \{S, T, C\}$. By QS-SA, we have

$$v(S \cup T, \{(S \cup T), C\}) \geq v(S, \{S, T, C\}) + v(T, \{S, T, C\}).$$

As v has positive externalities, we obtain

$$v(C, \{(S \cup T), C\}) > v(C, \{S, T, C\}).$$

Using the result of $|\mathcal{P}| = 2$, we obtain

$$\begin{aligned} v(N, \{N\}) &\geq v((S \cup T), \{(S \cup T), C\}) + v(C, \{(S \cup T), C\}) \\ &> v(S, \{S, T, C\}) + v(T, \{S, T, C\}) + v(C, \{S, T, C\}). \end{aligned}$$

Induction proof: We assume that the induction hypothesis holds for $|\mathcal{P}| \leq k - 1$, and we show that the induction hypothesis holds for $|\mathcal{P}| = k$. Thus, we assume hereinafter that $|\mathcal{P}| = k \geq 4$.

Take four coalitions: $S, T, S', T' \in \mathcal{P}$. Let $\mathcal{Q}$ denote the partition of $N \setminus (S \cup T \cup S' \cup T')$ such that $\mathcal{Q} = \mathcal{P} \setminus \{S, T, S', T'\}$. Let $\mathcal{P}' = \{(S \cup T), (S' \cup T')\} \cup \mathcal{Q}$. By QS-SA, we have

$$\begin{aligned} v((S \cup T), \mathcal{P}') &\geq v(S, \mathcal{P}) + v(T, \mathcal{P}), \ \textit{and} \\ v((S' \cup T'), \mathcal{P}') &\geq v(S', \mathcal{P}) + v(T', \mathcal{P}). \end{aligned} \tag{3.1}$$

As $\mathcal{P} \setminus \{S, T, S', T'\} = \mathcal{Q} = \mathcal{P}' \setminus \{(S \cup T), (S' \cup T')\}$, for any $C \in \mathcal{Q}$, $v(C, \mathcal{P}') > v(C, \mathcal{P})$ holds under positive externalities. Therefore, we have $\sum_{C \in \mathcal{Q}} v(C, \mathcal{P}') > \sum_{C \in \mathcal{Q}} v(C, \mathcal{P})$. Thus, in view of (3.1), we obtain

$$\sum_{C' \in \mathcal{P}'} v(C', \mathcal{P}') > \sum_{C' \in \mathcal{P}} v(C', \mathcal{P}).$$

Since the cardinality of $\mathcal{P}'$ is $k - 2$, we have

$$v(N, \{N\}) \geq \sum_{C \in \mathcal{P}'} v(C, \mathcal{P}') > \sum_{C \in \mathcal{P}} v(C, \mathcal{P}).$$

This completes the induction proof. $\square$

The last strict inequality, ($>$), stems from the definition of positive externalities. Thus, it can be replaced with a non-strict inequality ($\geq$), by replacing ($>$) with ($\geq$) in the definition of positive externalities as well.

We note that, under positive externalities, convexity is excessively strong for the efficiency of the grand coalition, and that superadditivity is sufficient for such efficiency. Proposition 3.3 shows that even QS-SA, which is weaker than superadditivity, is sufficient for the efficiency of the grand coalition.

Now, let us examine another type of superadditivity. This is the second condition of our main discussion.

Definition 3.4. A partition function v satisfies *optimistic-superadditivity* (*optimistic-SA*) if for any $S, T \subseteq N$ with $S \cap T = \emptyset$,

$$\max_{\substack{\mathcal{P}\in\Pi:\\(S\cup T)\in\mathcal{P}}} v(S\cup T, \mathcal{P}) \geq \max_{\substack{\mathcal{P}'\in\Pi:\\S\in\mathcal{P}'}} v(S, \mathcal{P}') + \max_{\substack{\mathcal{P}''\in\Pi:\\T\in\mathcal{P}''}} v(T, \mathcal{P}'').$$

The idea behind optimistic-SA is a reduction of partition function to coalition function. In general, we can generate a coalition function from a partition function by fixing a partition of outsiders for each coalition. Considering this transformation, a partition function v satisfies optimistic-SA if and only if the "optimistically reduced coalition function, *i.e.*, $v^{opt}(S) := \max_{\mathcal{P}'\in\Pi:S\in\mathcal{P}'} v(S, \mathcal{P}')$ for any $S \subseteq N$," satisfies ordinary superadditivity in coalition function form. As described in the following proposition, optimistic-SA accomplishes the efficiency of the grand coalition.

Proposition 3.5. For any partition function v, if v satisfies optimistic-SA then v satisfies the efficiency of the grand coalition.

Proof. For any partition $\mathcal{P}$, we have

$$v(N, \{N\}) - \sum_{S\in\mathcal{P}} v(S, \mathcal{P}) \geq v(N, \{N\}) - \sum_{S\in\mathcal{P}} \left[\max_{\substack{\mathcal{P}'\in\Pi:\\S\in\mathcal{P}'}} v(S, \mathcal{P}')\right].$$

Since $\mathcal{P}$ is a partition of N, by the definition of optimistic-SA, we obtain

$$v(N, \{N\}) - \sum_{S\in\mathcal{P}} \left[\max_{\substack{\mathcal{P}'\in\Pi:\\S\in\mathcal{P}'}} v(S, \mathcal{P}')\right] \geq 0.$$

□

Proposition 3.5 showed that optimistic-SA implies the efficiency of the grand coalition under both positive and negative externalities;[*4] meanwhile, the relationship with convexity varies depending on the type of externalities. The following example illustrates that optimistic-SA is not weaker than convexity.

Example 3.6. Consider the following 4-player game: $N = \{1, 2, 3, 4\}$;

$v(N, \{N\}) = 8;$
$v(\{i, j, k\}, \{\{i, j, k\}, \{h\}\}) = 5.1 \textit{ and } v(\{h\}, \{\{i, j, k\}, \{h\}\}) = 2 \;\; \textit{for } \{i, j, k, h\} = N;$
$v(\{i, j\}, \{\{i, j\}, \{k, h\}\}) = 3.5 \;\; \textit{for } \{i, j, k, h\} = N;$
$v(\{i, j\}, \{\{i, j\}, \{k\}, \{h\}\}) = 3 \textit{ and } v(\{k\}, \{\{i, j\}, \{k\}, \{h\}\}) = 1.5 \;\; \textit{for } \{i, j, k, h\} = N;$
$v(\{i\}, \{\{1\}, \{2\}, \{3\}, \{4\}\}) = 1 \;\; \textit{for } i = 1, 2, 3, 4.$

[*4] Moreover, this proposition continues to hold when externalities are neither positive nor negative. However, "pessimistic-SA" defined in the same manner as Definition 3.4 fails to imply the efficiency of the grand coalition.

This game satisfies convexity while violating optimistic-SA because for any coalition S, the optimistic value $v^{opt}(S) = \max_{\mathcal{P}' \in \Pi : S \in \mathcal{P}'} v(S, \mathcal{P}')$ is determined as follows:

$$v^{opt}(S) = \begin{cases} 8 & if \quad |S| = 4 \\ 5.1 & if \quad |S| = 3 \\ 3.5 & if \quad |S| = 2 \\ 2 & if \quad |S| = 1 \end{cases}.$$

3.3 Negative Externalities

First, let us focus on QS-SA. The role of QS-SA under negative externalities is completely different from that under positive externalities. The following example shows that QS-SA fails to achieve efficiency in games with negative externalities.

Example 3.7. Consider the following 4-player game: $N = \{1, 2, 3, 4\}$;

$v(N, \{N\}) = 11;$
$v(\{i, j, k\}, \{\{i, j, k\}, \{h\}\}) = 10.2 \ and \ v(\{h\}, \{\{i, j, k\}, \{h\}\}) = 0 \quad for \ \{i, j, k, h\} = N;$
$v(\{i, j\}, \{\{i, j\}, \{k, h\}\}) = 5 \quad for \ \{i, j, k, h\} = N;$
$v(\{i, j\}, \{\{i, j\}, \{k\}, \{h\}\}) = 9.1 \ and \ v(\{k\}, \{\{i, j\}, \{k\}, \{h\}\}) = 1 \quad for \ \{i, j, k, h\} = N;$
$v(\{i\}, \{\{1\}, \{2\}, \{3\}, \{4\}\}) = 2 \quad for \ i = 1, 2, 3, 4.$

This game satisfies QS-SA. However, the grand coalition is not efficient because for partition $\{\{i, j\}, \{k\}, \{h\}\}$,

$$\sum_{S \in \{\{i,j\},\{k\},\{h\}\}} v(S, \{\{i, j\}, \{k\}, \{h\}\}) = 9.1 + 1 + 1 > 11 = v(N, \{N\}).$$

Notably, Example 3.7 is not trivial, as QS-SA is not weaker than superadditivity under negative externalities,[*5] which can be checked in the following Example 3.8. Furthermore, Example 3.8 simultaneously shows that QS-SA is not weaker than convexity.

Example 3.8. Consider the following 4-player game: $N = \{1, 2, 3, 4\}$;

$v(N, \{N\}) = 60;$
$v(\{i, j, k\}, \{\{i, j, k\}, \{h\}\}) = 45 \ and \ v(\{h\}, \{\{i, j, k\}, \{h\}\}) = 13 \quad for \ \{i, j, k, h\} = N;$
$v(\{i, j\}, \{\{i, j\}, \{k, h\}\}) = 29 \quad for \ \{i, j, k, h\} = N;$
$v(\{i, j\}, \{\{i, j\}, \{k\}, \{h\}\}) = 30 \ and \ v(\{k\}, \{\{i, j\}, \{k\}, \{h\}\}) = 14 \quad for \ \{i, j, k, h\} = N;$
$v(\{i\}, \{\{1\}, \{2\}, \{3\}, \{4\}\}) = 15 \quad for \ i = 1, 2, 3, 4.$

[*5] Superadditivity and even strong-SA are not sufficient for the efficiency of the grand coalition.

This game satisfies convexities and, hence, superadditivity. However, it violates QS-SA because for i, j, k, h with $\{i, j, k, h\} = N$, we have

$$\begin{aligned} v(\{i,j\}, \{\{i,j\},\{k,h\}\}) = 29 < 30 = 15 + 15 \\ = v(\{i\}, \{\{1\},\{2\},\{3\},\{4\}\}) + v(\{j\}, \{\{1\},\{2\},\{3\},\{4\}\}). \end{aligned}$$

Next, we analyze optimistic-SA. As described by Proposition 3.5, optimistic-SA achieves the efficiency of the grand coalition, regardless of the type of externalities, which raises another question regarding the relationship between convexity and optimistic-SA. The answer, under negative externalities, is as follows: convexity implies optimistic-SA. This result vividly contrasts with that of positive externalities. The following proposition states that even weak convexity, which is always weaker than convexity, implies optimistic-SA under negative externalities.

Proposition 3.9. For any partition function v with negative externalities, if v satisfies weak convexity, then v satisfies optimistic-SA.

Proof. Let $\mathcal{I}(S) = \{\{i_1\}, ..., \{i_{|S|}\}\}$ denote the partition of S to singletons. As the game v has negative externalities, it suffices to show that for any nonempty $S \subseteq N$ and any $T_1, T_2 \subseteq S$ with $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = S$, $T_1 \neq \emptyset$, $T_2 \neq \emptyset$,

$$v(S, \{S\} \cup \mathcal{I}(N \setminus S)) \geq v(T_1, \{T_1\} \cup \mathcal{I}(N \setminus T_1)) + v(T_2, \{T_2\} \cup \mathcal{I}(N \setminus T_2)).$$

To maintain the clarity of our notation, define $v^{\mathcal{I}}(S) = v(S, \{S\} \cup \mathcal{I}(N \setminus S))$ for any $S \subseteq N$. With this notation, the inequality above can be written as follows:

$$v^{\mathcal{I}}(S) \geq v^{\mathcal{I}}(T_1) + v^{\mathcal{I}}(T_2). \tag{3.2}$$

The proof is implemented by induction on the cardinality of coalition S. The induction hypothesis is developed by (3.2).

Induction base: For $|S| = 2$, let $S = \{i, j\}$. By weak convexity, we obtain

$$v^{\mathcal{I}}(\{i,j\}) \geq v^{\mathcal{I}}(\{i\}) + v^{\mathcal{I}}(\{j\}).$$

For $|S| = 3$, let $S = \{i, j, h\}$. By weak convexity, we have

$$\begin{aligned} v^{\mathcal{I}}(\{i,j,h\}) &\geq v^{\mathcal{I}}(\{i,j\}) - v^{\mathcal{I}}(\{j\}) + v^{\mathcal{I}}(\{j,h\}) \\ &\geq v^{\mathcal{I}}(\{i\}) + v^{\mathcal{I}}(\{j,h\}), \end{aligned}$$

where the last inequality holds by the result of $|S| = 2$.

Induction proof: Assume that the induction hypothesis is true for any $|S| \leq k - 1$. We show that it is true for $|S| = k$ as well. For any $T_1, T_2 \subseteq S$ such that $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = S$, $T_1 \neq \emptyset$, $T_2 \neq \emptyset$, let $i_1 \in T_1$ and $j_1 \in T_2$. By weak convexity, we have

$$v^{\mathcal{I}}(S) \geq v^{\mathcal{I}}(S \setminus \{i_1\}) + v^{\mathcal{I}}(S \setminus \{j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, j_1\}). \tag{3.3}$$

First, we focus on $S \setminus \{i_1\}$ in (3.3). If $T_1 = \{i_1\}$, inequality (3.3) can be written as follows:

$$v^{\mathcal{I}}(S) \geq v^{\mathcal{I}}(T_2) + v^{\mathcal{I}}(S \setminus \{j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, j_1\}).$$

We note that the cardinality of $S \setminus \{j_1\}$ is $k-1$. Because the inequality $v^{\mathcal{I}}(S \setminus \{j_1\}) \geq v^{\mathcal{I}}(\{i_1\}) + v^{\mathcal{I}}(S \setminus \{i_1, j_1\})$ holds by induction hypothesis, we obtain

$$\begin{aligned} v^{\mathcal{I}}(S) &\geq v^{\mathcal{I}}(T_2) + v^{\mathcal{I}}(\{i_1\}) \\ &= v^{\mathcal{I}}(T_2) + v^{\mathcal{I}}(T_1). \end{aligned}$$

We now assume $|T_1| \geq 2$. If there is a player $i_2 \in T_1$ such that $i_2 \neq i_1$, then, by weak convexity, we have

$$v^{\mathcal{I}}(S \setminus \{i_1\}) \geq v^{\mathcal{I}}(S \setminus \{i_1, i_2\}) + v^{\mathcal{I}}(S \setminus \{i_1, j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, i_2, j_1\}).$$

Similarly, if there is a player $i_3 \in T_1$, such that $i_3 \neq i_2$ and $i_3 \neq i_1$, then we have

$$v^{\mathcal{I}}(S \setminus \{i_1, i_2\}) \geq v^{\mathcal{I}}(S \setminus \{i_1, i_2, i_3\}) + v^{\mathcal{I}}(S \setminus \{i_1, i_2, j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, i_2, i_3, j_1\}).$$

We can iterate this process exactly $|T_1| - 1$ $(= |T_1 \setminus \{i_1\}|)$ times.

We can apply this iteration to $S \setminus \{j_1\}$ in (3.3) as well.*6 As a result of both iterations, we obtain the following (in)equalities from (3.3):

$$\begin{aligned} v^{\mathcal{I}}(S) &\geq v^{\mathcal{I}}(S \setminus \{i_1\}) + v^{\mathcal{I}}(S \setminus \{j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, j_1\}) \\ &= v^{\mathcal{I}}(T_1) + v^{\mathcal{I}}(T_2) - v^{\mathcal{I}}(S \setminus \{i_1, j_1\}) \\ &\quad + \sum_{l=2}^{|T_1|} \left[v^{\mathcal{I}}(S \setminus \{i_1, ..., i_{l-1}, j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, ..., i_l, j_1\}) \right] \\ &\quad + \sum_{l'=2}^{|T_2|} \left[v^{\mathcal{I}}(S \setminus \{j_1, ..., j_{l'-1}, i_1\}) - v^{\mathcal{I}}(S \setminus \{j_1, ..., j_{l'}, i_1\}) \right] \\ &= v^{\mathcal{I}}(T_1) + v^{\mathcal{I}}(T_2) - v^{\mathcal{I}}(S \setminus \{i_1, j_1\}) \\ &\quad + v^{\mathcal{I}}(S \setminus \{i_1, j_1\}) - v^{\mathcal{I}}(S \setminus \{i_1, ..., i_{|T_1|}, j_1\}) \\ &\quad + v^{\mathcal{I}}(S \setminus \{j_1, i_1\}) - v^{\mathcal{I}}(S \setminus \{j_1, ..., j_{|T_2|}, i_1\}). \end{aligned} \tag{3.4}$$

Since $\{i_1, ..., i_{|T_1|}\} = T_1$ and $\{j_1, ..., j_{|T_2|}\} = T_2$, (3.4) can be written as follows:

$$\begin{aligned} &v^{\mathcal{I}}(T_1) + v^{\mathcal{I}}(T_2) + v^{\mathcal{I}}(S \setminus \{i_1, j_1\}) - v^{\mathcal{I}}(S \setminus (T_1 \cup \{j_1\})) - v^{\mathcal{I}}(S \setminus (T_2 \cup \{i_1\})) \\ &\geq v^{\mathcal{I}}(T_1) + v^{\mathcal{I}}(T_2). \end{aligned}$$

where the inequality holds by the induction hypothesis because the cardinality of $S \setminus \{i_1, j_1\}$ equals $k-2$. This completes the induction proof. □

As the result of our exploration, we can summarize the relationship of our conditions with convexity and the efficiency of the grand coalition in Table 1.

*6 Thus, analogous to T_1, we consider the case of $|T_2| \geq 2$ hereafter.

Table 1 Relationship among QS-SA, optimistic-SA, efficiency and convexity

	positive externalities	negative externalities
convexity $\Rightarrow$ QS-SA	Yes	-
convexity $\Rightarrow$ optimistic-SA	-	Yes
QS-SA $\Rightarrow$ EFF	Yes	-
optimistic-SA$\Rightarrow$ EFF	Yes	Yes

Furthermore, the implications of relations involving superadditivity are illustrated in Figure 1. It might be questioned whether Definition 2.2 is an appropriate extension of superadditivity to partition functions. In fact, another definition of superadditivity is proposed by de Clippel and Serrano (2008), who consider a stronger definition than Definition 2.2 and offer the following definition: for any mutually disjoint $S_1, ..., S_k \subseteq N$ $(1 \leq k \leq n)$ and any partition $\mathcal{Q}$ of $N \setminus (\cup_{j=1}^k S_j)$, we have

$$v\left(\cup_{j=1}^k S_j, \{(\cup_{j=1}^k S_j)\} \cup \mathcal{Q}\right) \geq \sum_{j=1}^k v(S_j, \{S_1, ..., S_k\} \cup \mathcal{Q}). \tag{3.5}$$

Clearly, condition (3.5) accomplishes the efficiency of the grand coalition because the grand coalition can be taken as N as $\cup_{j=1}^k S_j$ with $k = 1$. In the absence of externalities, both definitions coincide. Notably, in Figure 1, condition (3.5) is located between superadditivity (of Definition 2.2) and convexity under both types of externalities. In addition, optimistic-SA is not weaker than condition (3.5): it is possible to find a game satisfying (3.5) and violating optimistic-SA under both types of externalities.

4 Non-emptiness of the Core

In the presence of externalities, players attempting to deviate from the grand coalition can anticipate a variety of reactions from the other players and coalitions. This diversity of anticipation yields many definitions of the core in partition function form. In this paper, we focus mainly on a contrastive pair of core(s): the optimistic core and the pessimistic core.

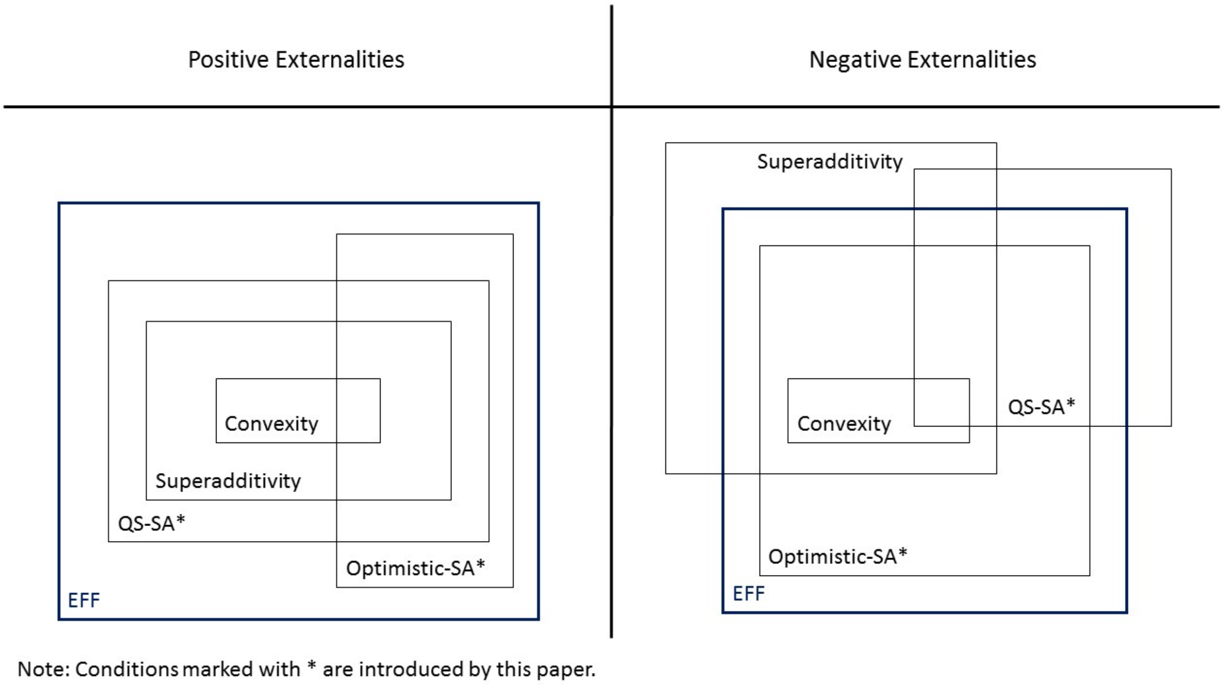
Definition 4.1. Let x be a payoff vector in R^N. The *optimistic-core* is a set of payoff vectors such that for any coalition $S \subseteq N$,

$$\sum_{j \in S} x_j \geq \max_{\mathcal{P}' \in \Pi : S \in \mathcal{P}'} v(S, \mathcal{P}'), \quad and \quad \sum_{j \in N} x_j = v(N, \{N\}).$$

Similarly, the *pessimistic-core* is a set of payoff vectors such that for any coalition $S \subseteq N$,

$$\sum_{j \in S} x_j \geq \min_{\mathcal{P}' \in \Pi : S \in \mathcal{P}'} v(S, \mathcal{P}'), \quad and \quad \sum_{j \in N} x_j = v(N, \{N\}).$$

Figure 1   Conditions for the efficiency of the grand coalition



It is commonly known that the optimistic core is the smallest of all types of cores, whereas the pessimistic core is the largest.[*7] On one hand, under optimistic views, players (or, a coalition) find it easy to deviate from their original affiliation because they expect the best reaction from the other players, *i.e.*, $\arg\max_{\mathcal{P}':S\in\mathcal{P}'} v(S,\mathcal{P}')$. On the other hand, under pessimistic views, players are less likely to deviate because they anticipate the worst case from other players, *i.e.*, $\arg\min_{\mathcal{P}':S\in\mathcal{P}'} v(S,\mathcal{P}')$. Unfortunately, neither optimistic-SA nor QS-SA discussed in Section 3 guarantee the non-emptiness of each type of core. Therefore, we require another condition that is stronger than both of these conditions and simultaneously weaker than convexity.

Now, we claim that weak convexity is sufficient for both the efficiency of the grand coalition and the non-emptiness of the core. As discussed in Example 2.6, weak convexity does not always achieve the efficiency of the grand coalition. However, weak convexity does achieve efficiency by focusing on negative externalities.

Proposition 4.2. For any partition function v with negative externalities, if v satisfies weak convexity, then v satisfies the efficiency of the grand coalition.

[*7] Abe and Funaki (2015) generalized the Bondareva-Shapley condition for partition function form games.

Proof. The proof follows from Proposition 3.9 and Proposition 3.5. □

Notably, weak convexity has not been considered as a candidate for the sufficient condition of the nonempty core because of the difficulty illustrated in Example 2.6. However, Proposition 4.2 states that weak convexity can become a candidate if we focus on negative externalities. The following proposition shows the truth of this conjecture.

Proposition 4.3. For any partition function v with negative externalities, if v satisfies weak convexity, then the optimistic core is nonempty.

Proof. The basic approach behind this proof is similar to Hafalir (2007). However, in this proof, we restrict our attention to weak convexity and games with negative externalities because of Proposition 4.2.

For any $S \subseteq N$, let $\mathcal{I}(S)$ denote the partition of S to singletons. Since v satisfies weak convexity, for any $S, T \subseteq N$ with $|S \setminus T| = |T \setminus S| = 1$, we have

$$\begin{aligned} & v(S \cup T, \{S \cup T\} \cup \mathcal{I}(N \setminus (S \cup T))) + v(S \cap T, \{S \cap T, S \setminus T, T \setminus S\} \cup \mathcal{I}(N \setminus (S \cup T))) \\ & \geq v(S, \{S, T \setminus S\} \cup \mathcal{I}(N \setminus (S \cup T))) + v(T, \{T, S \setminus T\} \cup \mathcal{I}(N \setminus (S \cup T))). \end{aligned} \tag{4.1}$$

For notational simplicity, define $v^{\mathcal{I}}(S) = v(S, \{S\} \cup \mathcal{I}(N \setminus S))$. In view of $|S \setminus T| = |T \setminus S| = 1$, we can rewrite (4.1) as follows:

$$v^{\mathcal{I}}(S \cup T) + v^{\mathcal{I}}(S \cap T) \geq v^{\mathcal{I}}(S) + v^{\mathcal{I}}(T).$$

Now, we can consider $v^{\mathcal{I}}$ as a coalition function. Because weak convexity is equivalent to convexity for coalition functions, $v^{\mathcal{I}}$ is convex in the sense of a coalition function game. Let us take a payoff vector $x \in R^N$ such that for any $i \in N$ and any $S \subseteq N \setminus \{i\}$, $x_i = v^{\mathcal{I}}(S \cup \{i\}) - v^{\mathcal{I}}(S)$. By the convexity of coalition function $v^{\mathcal{I}}$, the vector x is in the core of the coalition function $v^{\mathcal{I}}$, *i.e.*, $\sum_{j \in S} x_j \geq v^{\mathcal{I}}(S)$ for any $S \subseteq N$ and $\sum_{j \in N} x_j = v^{\mathcal{I}}(N)$. Under the definition of $v^{\mathcal{I}}(S)$ and negative externalities, we have

$$v^{\mathcal{I}}(S) = v(S, \{S\} \cup \mathcal{I}(N \setminus S)) = \max_{\mathcal{P}' \in \Pi : S \in \mathcal{P}'} v(S, \mathcal{P}'), \tag{4.2}$$

which implies that x is in the optimistic core of partition function v. □

Corollary 4.4. For any game v with positive externalities, if v is weak convexity then the pessimistic core is nonempty.

Proof. The first half of this proof is the same as Proposition 4.3. In games with positive externalities, we have $v^{\mathcal{I}}(S) = v(S, \{S\} \cup \mathcal{I}(N \setminus S)) = \min_{\mathcal{P}' \in \Pi : S \in \mathcal{P}'} v(S, \mathcal{P}')$, instead of (4.2). Hence, x is in the pessimistic core. □

If externalities are positive, we cannot obtain the parallel result of Proposition 4.2, and the grand coalition can be inefficient even when weak convexity is satisfied. Bloch and van den

Nouweland (2014) discussed the case in which some partitions are more efficient than the grand coalition and then introduced the notion of "block deviation", i.e., a deviating coalition does not split after the deviation. It is possible to define each type of core as the set of payoff vectors that are not block-deviated via any coalition. For the formal definition of the core based on the deviation, see Abe and Funaki (2015).

Surprisingly, according to Corollary 4.4, even when the grand coalition is not efficient, a payoff distribution of the grand coalition may belong to the pessimistic core. The next example describes the case.

Example 4.5. Let $N = \{1, 2, 3\}$. Consider a partition function v defined as follows:

$$\begin{aligned}
&v(N, \{N\}) = 5;\\
&v(\{i\}, \{\{i\}, \{j, k\}\}) = 4 \ and\ v(\{j, k\}, \{\{i\}, \{j, k\}\}) = 2 \quad for\ \{i, j, k\} = N;\\
&v(\{i\}, \{\{1\}, \{2\}, \{3\}\}) = 0 \quad for\ i = 1, 2, 3.
\end{aligned}$$

In this game, the grand coalition is not efficient because

$$v(\{i\}, \{\{i\}, \{j, k\}\}) + v(\{j, k\}, \{\{i\}, \{j, k\}\}) = 4 + 2 > 5 = v(N, \{N\}).$$

However, assuming block deviation, payoff vector $(\frac{5}{3}, \frac{5}{3}, \frac{5}{3})$ is in the pessimistic core of this game. Thus, any individual player does not deviate from $(\frac{5}{3}, \frac{5}{3}, \frac{5}{3})$ because each player considers the worst case: $\min\{v(\{i\}, \{\{i\}, \{j, k\}\}), v(\{i\}, \{\{1\}, \{2\}, \{3\}\})\} = \min\{4, 0\}$. Without assuming block deviation, the payoff vector is out of the pessimistic core (because the grand coalition itself can deviate from $\{N\}$ by splitting into $\{\{i\}, \{j, k\}\}$) and then obtains, for example, payoff $(2, 2, 2)$.

In contrast to the pessimistic core, the optimistic core must become empty whenever the grand coalition is not efficient. The following proposition proves this fact.

Proposition 4.6. If the optimistic core is nonempty, then v is the efficiency of the grand coalition.

Proof. For any partition $\mathcal{P}$, we have

$$v(N, \{N\}) - \sum_{S \in \mathcal{P}} v(S, \mathcal{P}) \geq v(N, \{N\}) - \sum_{S \in \mathcal{P}} \left[\max_{\substack{\mathcal{P}' \in \Pi: \\ S \in \mathcal{P}'}} v(S, \mathcal{P}') \right].$$

Because a payoff vector x exists in the optimistic core defined as Definition 4.1, we obtain

$$\begin{aligned} v(N,\{N\}) - \sum_{S\in\mathcal{P}} \left[\max_{\substack{\mathcal{P}'\in\Pi:\\ S\in\mathcal{P}'}} v(S,\mathcal{P}') \right] &= \sum_{j\in N} x_j - \sum_{S\in\mathcal{P}} \left[\max_{\substack{\mathcal{P}'\in\Pi:\\ S\in\mathcal{P}'}} v(S,\mathcal{P}') \right] \\ &= \sum_{S\in\mathcal{P}} \left[\sum_{j\in S} x_j - \max_{\substack{\mathcal{P}'\in\Pi:\\ S\in\mathcal{P}'}} v(S,\mathcal{P}') \right] \\ &\geq 0. \end{aligned}$$

□

The results of this section are summarized in Table 2.

Table 2 The relationship between weak-convexity and two types of cores

	positive externalities	negative externalities
weak-convexity $\Rightarrow$ optimistic-core$\neq \emptyset$	-	Yes
weak-convexity $\Rightarrow$ pessimistic-core$\neq \emptyset$	Yes	Yes
weak-convexity $\Rightarrow$ EFF	-	Yes

5 Concluding Remarks

In this paper, we used inequalities to define both the optimistic core and the pessimistic core (see Definition 4.1). However, in the general field of cooperative game theory, the core is often defined as the set of payoff vectors that are not dominated by any coalition (see, for example, Abe and Funaki (2015) for the formal definition in partition function form games). We call the former the *inequality core* and the latter the *domination core.* Then, as a general rule, the inequality core always becomes a subset of the domination core. It might be possible that the "gap" between them matters. We note that the gap does not arise in the class we have analyzed. To see this, consider the following condition: for any nonempty coalition $S \subseteq N$,

$$v(N,\{N\}) \geq v(S,\mathcal{P}^*) + v(\{i_{s+1}\},\mathcal{P}^*) + ... + v(\{i_n\},\mathcal{P}^*), \tag{5.1}$$

where $\mathcal{P}^* = \{S, \{i_{s+1}\}, ..., \{i_n\}\}$. The condition (5.1) can be seen as a generalization of zero monotonicity. It is clear that the domination core thus become smaller than the inequality core if and only if (5.1) holds and that the efficiency of the grand coalition implies (5.1). Thus, the gap is eliminated in the class of games considered in this paper.

Furthermore, we considered the grand coalition as the coalition structure that is the most efficient. However, in some environments, another partition may possibly be the most efficient coalition structure. To analyze the conditions of such cases, we must introduce the notion of block deviation and splitting deviation more formally.

References

[1] Abe, T. and Y. Funaki (2015), "A Necessary and Sufficient Condition for Non-emptiness of the Core of a Partition Function Form Game," *WINPEC Working Paper Series* No.E1502, Institute of Political Economy, Waseda University.

[2] Bloch, F. (1995), "Endogenous Structures of Association in Oligopolies," *Rand Journal of Economics* **26**, 537-556.

[3] Bloch, F. and A. van den Nouweland (2014), "Expectation Formation Rules and the Core of Partition Function Games," *Games and Economic Behavior* **88**, 339-353.

[4] de Clippel, G. and R. Serrano (2006), "Marginal Contributions and Externalities in the Value," *Econometrica* **76**, 1413-1436.

[5] Funaki, Y. and T. Yamato (1999), "The Core of an Economy with a Common Pool Resource: A Partition Function Form Approach," *International Journal of Game Theory* **28**, 157-171.

[6] Hafalir, I. (2007), "Efficiency in Coalition Games with Externalities," *Games and Economic Behavior* **61**, 242-258.

[7] Maskin, E. (2003), "Bargaining, Coalitions and Externalities," Presidential Address to the Economic Society, Institute for Advanced Study, Princeton University.

[8] Ray, D. and R. Vohra (1997), "Equilibrium Binding Agreements," *Journal of Economic Theory* **73**, 30-78.

[9] Ray, D. and R. Vohra (1999), "A Theory of Endogenous Coalition Structures," *Games and Economic Behavior* **26**, 286-336.

[10] Yi, S. S. (1996), "Endogenous Formation of Customs Unions under Imperfect Competition: Open Regionalism Is Good," *Journal of International Economics* **41**, 153-177.

[11] Yi, S. S. (1997), "Stable Coalition Structures with Externalities," *Games and Economic Behavior* **20**, 201-237.

[12] Yi, S. S. (1998), "Endogenous Formation of Joint Ventures with Efficiency Gains," *Rand Journal of Economics* **29**, 610-631.