WINPEC Working Paper Series No.E1504
June 2015

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June 1, 2015

Abstract

We deal with the problem of striking a balance between marginalism and egalitarianism in the class of TU cooperative games. We introduce a new axiom, Weak Surplus Monotonicity. It states that if the marginal contribution of a player increases, the worth of the grand coalition increases and the cooperative surplus increases, then the payoff of the player should also increase. We show that a solution satisfies Efficiency, Symmetry and Weak Surplus Monotonicity if and only if it is a convex combination of the Shapley value, the Equal division and the CIS value. By replacing the new axiom with a stronger axiom and taking the dual, we obtain 11 characterizations of solutions, including the results of Young (1985) or Casajus and Huettner (2014).

JEL classification: C71
Keywords: TU game; Shapley value; Monotonicity; Axiomatization

1 Introduction

In many fair division problems, there is the trade-off between the following two criteria: *marginalism* and *egalitarianism*. Marginalism states that the share of an agent should be determined by marginal contribution of the player. Egalitarianism states that the total surplus should be divided as

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equally as possible. The question is how to strike a balance between the two. We tackle this problem under the assumptions that agents have transferable utility and side payment is allowed. Namely, we consider the class of *TU cooperative games.*

We introduce monotonicity axiom that seems desirable from the two criteria. The new axiom, which we call *Weak Surplus Monotnicity*, states the following: if the marginal contribution of a player increases, the worth of the grand coalition increases and the cooperative surplus increases, then the final payoff of the player also increases. The axiom is marginalistic in the sense that marginal contribution is taken into account, and also has some flavor of egalitarianism since the cooperative surplus and the total payoff matter. We show that a distribution rule satisfies Efficiency, Symmetry and Weak Surplus Monotonicity if and only if it is a convex combination of the Shapley value, the Equal division and the CIS value.[1] Starting from a very weak requirement of monotonicity, we reach a convex combination of well-known solutions. Our result indicates that taking a convex combination is a desirable method to reflect both marginalism and egalitarianism.

The rest of this paper is organized as follows. We first refer to related literature. Section 2 gives preliminary. In Section 3, we define the new axiom, Weak Surplus Monotonicity, and characterize the class of distribution rules that satisfy the axiom. In Section 4, we discuss some complementary issues related to our theorem. Section 5 gives concluding remarks. All proofs are in the Appendix.

Related Literature

Previous works on TU cooperative game theory have developed a variety of distribution rules, or *solutions*. Many solutions are defined either by marginalism or egalitarianism. For example, the (weighted) Shapley value (Shapley (1953)) or the solidarity value (Nowak and Radzik (1994)) are defined based on marginalism; marginal contribution determines the final payoff of a player. The Equal division or the CIS value (Driessen and Funaki (1991)) are defined based on egalitarianism; the total payoff or the cooperative surplus determines the final payoff.

Axiomatizations of a class of solutions that take a convex combination have also been extensively discussed. Ju et al. (2007) defined and characterized the Consensus value, which is a convex combination of the Shapley value and the CIS value. van den Brink and Funaki (2009) characterized the class of solutions that take a convex combination of the Equal division,

[1]The CIS value is often called the Equal Surplus Division rule in the literature.

the CIS value and the ENSC value by using reduced game consistency. van den Brink et al. (2013) proved that the egalitarian Shapley value (Joosten (1996)), which is a convex combination of the Shapley value and the Equal division, is characterized by Efficiency, Symmetry, Weak Monotonicity and Linearity. Casajus and Huettner (2014) proved that the egalitarian Shapley value is characterized by Efficiency, Symmetry and Weak Monotonicity (without Linearity) if there are more than two players.

The result of this paper is closely related to Casajus and Huettner (2014), hereafter C&H. Our marginal contribution is twofold. First, we generalize the result of C&H. Our new axiom, Weak Surplus Monotonicity, is weaker than Weak Monotonicity. Second, we give another proof of C&H. Following Shapley (1953) or Young (1953), we prove the theorem by using a basis of the set of all TU games.[2] Our linear algebraic approach helps readers develop intuition on why the three axioms (Efficiency, Symmetry and a variant of Monotonicity) exclude nonlinear solutions.

2 Preliminary

Let N denote the set of n players, $N = \{1, \cdots, n\}$. A function $v : 2^N \to \mathbb{R}$, satisfying $v(\emptyset) = 0$, is called a game. For each non-empty $S \subseteq N$, $v(S)$ represents the attainable payoff for players in S. Let Γ denote the set of all games.

Let $v \in \Gamma$, $i, j \in N$ be fixed. For each $S \subseteq N \backslash i$, we define the marginal contribution of player i to coalition S by $\Delta_i v(S) = v(S \cup i) - v(S)$.[3] Let $\Delta_i v$ denote the vector of $\Delta_i v(S)$, $S \subseteq N \backslash i$.

A solution is a function from Γ to $\mathbb{R}^n$. Let $v \in \Gamma$. The Shapley value Sh (Shapley (1953)) is defined by

$$Sh_i(v) = \sum_{S \subseteq N : i \notin S} \frac{(n - |S| - 1)!|S|!}{n!} \Delta_i v(S) \text{ for all } i \in N.$$

The Equal division ED is defined by

$$ED_i(v) = \frac{v(N)}{n} \text{ for all } i \in N.$$

[2] Casajus and Huettner (2014) proved their result in a different way; their proof relies on the induction on the number of symmetric players.

[3] $N \backslash i$ and $S \cup i$ are abbreviations of $N \backslash \{i\}$ and $S \cup \{i\}$, respectively. For simplicity, we follow this kind of abbreviations in the remaining part.

The CIS value and the ENSC value (Driessen and Funaki (1991)) are defined by

$$CIS_i(v) = v(i) + \frac{v(N) - \sum_{j\in N} v(j)}{n} \text{ for all } i \in N,$$
$$ENSC_i(v) = v(N) - v(N\backslash i) + \frac{v(N) - \sum_{j\in N}\big(v(N) - v(N\backslash i)\big)}{n} \text{ for all } i \in N.$$

We define two solutions that take a convex combination of the above solutions. For each $\alpha \in [0,1]$, we define the egalitarian Shapley value (Joosten (1996)) ES^α by

$$ES^\alpha(v) = \alpha Sh(v) + (1-\alpha)ED(v) \text{ for all } v \in \Gamma.$$

For each $\alpha \in [0,1]$, we define the Consensus value (Ju et al. (2007)) CV^α by

$$CV^\alpha(v) = \alpha Sh(v) + (1-\alpha)CIS(v) \text{ for all } v \in \Gamma.$$

We say that i and j are substitutes, denoted as $i \sim_v j$, if $\Delta_i v(S) = \Delta_j v(S)$ for all $S \subseteq N\backslash\{i,j\}$. We give two basic axioms satisfied by ψ:

Efficiency (E). $\sum_{i\in N} \psi_i(v) = v(N)$.

Symmetry (S). For any $v \in \Gamma$ and $i, j \in N$, $i \sim_v j$ implies $\psi_i(v) = \psi_j(v)$.

3 Weak Surplus Monotonicity and characterization

Under the assumption that the grand coalition forms, how should the total payoff be divided among players? There are two important criteria: marginalism and egalitarianism. We introduce an axiom that seems desirable from both criteria. Then, we characterize the class of solutions that satisfy the axiom.

We consider the axiom of *monotonicity*, which is one of the most widely accepted axioms in distribution problems. Monotonicity in TU cooperative game states that if the attainable payoffs for coalitions increase in a certain way, then the payoff of a player also increases.

We revisit monotonicity axioms in previous works. Young (1985) characterized the Shapley value by using the following axiom:

Strong Monotonicity (Young (1985)). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i(v) \geq \Delta_i(w)$, then $\psi_i(v) \geq \psi_i(w)$.

This axiom is marginalistic, because only the marginal contribution matters. In view of the situation where players divide the total payoff, it seems reasonable to take the worth of grand coalition into account. Based on this idea, van den Brink et al. (2013) introduced the following axiom:

Weak Monotonicity (WM) (van den Brink et al. (2013)). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i(v) \geq \Delta_i(w)$ and $v(N) \geq w(N)$, then $\psi_i(v) \geq \psi_i(w)$.

Namely, if i's marginal contribution and the worth of the grand coalition increases, then i's payoff increases.

In addition to the worth of grand coalition, there is another important factor in division problems: the stand-alone payoff. It is a convincing reference point for distributing the total payoff. TU cooperative game describes the stand-alone payoff by the worth of singleton coalition.

From the viewpoint of stand-alone payoff, we can find room for modification in Weak Monotonicity. Let us give an illustrative example. Consider the 3-person game w given by

$$w(1) = w(2) = w(3) = 0, w(12) = w(13) = w(23) = 10, w(N) = 60.$$

Suppose that the game changes to the following game v:

$$v(1) = 0, v(2) = v(3) = 50, v(12) = v(13) = v(23) = 60, v(N) = 110.$$

Note that $\Delta_i w = \Delta_i v$ and $w(N) \leq v(N)$. Thus, Weak Monotonicity concludes that 1's payoff increases, i.e., $\psi_1(w) \leq \psi_1(v)$. However, to determine the change in 1's payoff does not seem straightforward in this case. In both games, the worths of 2-person coalitions are symmetric. On the other hand, as for the 1-person coalitions, only the worths of 2 and 3 increase. Thus, we can interpret that 1 is in a weaker position in v than in w. Although the worth of grand coalition increases, the increase in the stand-alone payoffs is much larger. In other words, the total payoff minus the sum of stand-alone payoffs, namely *cooperative surplus*, decreases. So, it seems also possible that 1's payoff decreases from w to v.

Motivated by the above argument, we give a weaker axiom than Weak Monotonicity. In order to conclude that i's payoff increases, we additionally require that the cooperative surplus also increases.

Weak Surplus Monotonicity (M). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i(v) \geq \Delta_i(w)$, $v(N) \geq w(N)$ and $v(N) - \sum_{j \in N} v(j) \geq w(N) - \sum_{j \in N} w(j)$, then $\psi_i(v) \geq \psi_i(w)$.

We are now ready to state the main result. The class of solutions that satisfy E, S and M is equivalent to the class of solutions that take a convex combination of ES and CV.

Theorem 1 *Let $n \geq 6$. Then, ψ satisfies E, S and M if and only if there exist $\alpha, \beta, \gamma \in [0,1]$ such that $\psi = \alpha ES^{\beta} + (1-\alpha)CV^{\gamma}$.*

We give a remark on the assumption $n \geq 6$ in Section 5. The proof of Theorem 1 is given in Appendix A. We briefly explain the sketch of the proof. We first decompose the set of games Γ into several linear subspaces by using a basis introduced by Yokote and Funaki (2014). We first show that, if ψ satisfies the three axioms, then it is a convex combination of the solutions on each linear subspace. We also show that ψ is linear with respect to the addition of games in some linear subspaces. By unifying the result for each linear subspace, we obtain the result on the whole space Γ.

Weak Monotonicity seems to be a very weak condition; it states the change in i's payoff only when the three factors (marginal contribution, total payoff and cooperative surplus) change as stated in the axiom. For other cases, the axiom says nothing. Despite its weak requirement, Weak Surplus Monotonicity suffices to characterize the class of convex combinations of well-known solutions. Theorem 1 indicates that taking a convex combination is a reasonable way to reflect marginalism and egalitarianism.

4 Discussions on Theorem 1

In this section, we discuss some complementary issues related to Theorem 1.

4.1 Another proof of Casajus and Huettner (2014)

Let us revisit the following theorem:

Theorem 2 (Casajus and Huettner (2014)) *Let $n \geq 3$. Then, a solution ψ satisfies E, S and WM if and only if there exists $\alpha \in [0,1]$ such that $\psi = ES^{\alpha}$.*

In Appendix B, we give another proof of Theorem 2 by following the same line of proof of Theorem 1; see also 'outline of proof' in Appendix A.

The merit of giving another proof is to clarify the mathematical structure behind the theorem. We prove Theorem 2 by using a basis of the set of games. Our linear algebraic method helps readers understand why the three axioms exclude non-linear functions. The proof also helps us compare Theorem 2 with the theorem of Young (1985), who axiomatized the Shapley value by using a basis.

4.2 Axiomatization of other classes of solutions

By replacing Weak Mononiticty with a stronger axiom, we obtain characterizations of other solutions.

Surplus Monotonicity (SM). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i(v) \geq \Delta_i(w)$ and $v(N) - \sum_{j\in N} v(j) \geq w(N) - \sum_{j\in N} w(j)$, then $\psi_i(v) \geq \psi_i(w)$.

Strong Surplus Monotonicity (SSM). Let $v, w \in \Gamma$ and $i \in N$. If $v(i) \geq w(i)$ and $v(N) - \sum_{j\in N} v(j) \geq w(N) - \sum_{j\in N} w(j)$, then $\psi_i(v) \geq \psi_i(w)$.

Weak Grand Coalition Monotonicity (WGM). Let $v, w \in \Gamma$ and $i \in N$. If $v(N) \geq w(N)$ and $v(N) - \sum_{j\in N} v(j) \geq w(N) - \sum_{j\in N} w(j)$, then $\psi_i(v) \geq \psi_i(w)$.

Theorem 3 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and SM if and only if there exists $\alpha \in [0, 1]$ such that $\psi = CV^\alpha$.*

Theorem 4 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and SSM if and only if $\psi = CIS$.*

Theorem 5 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and WGM if and only if there exists $\alpha \in [0, 1]$ such that $\psi = \alpha ED + (1 - \alpha)CIS$.*

For the proofs of Theorems 3, 4 and 5, see Appendix C.

4.3 Dual of axioms

Another direction of extending Theorem 1 is to take the dual of Weak Surplus Monotonicity. For a detailed survey on the duals of axioms, see Oishi et al. (2013).

Consider the following axiom:

Dual Weak Surplus Monotonicity (DM). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i(v) \geq \Delta_i(w)$, $v(N) \geq w(N)$ and $v(N) - \sum_{j\in N}(v(N) - v(N\backslash j)) \geq w(N) - \sum_{j\in N}(w(N) - w(N\backslash j))$, then $\psi_i(v) \geq \psi_i(w)$.

From the duality between the CIS value and the ENSC value, we obtain the following theorem:

Theorem 6 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and DM if and only if there exist $\alpha, \beta \in [0, 1]$ such that $\psi = \alpha ES^\beta + (1 - \alpha)ENSC$.*

For the proof, see Appendix D. We can also consider the duals of SM, SSM and WGM in the same way.

Dual Surplus Monotonicity (DSM). Let $v, w \in \Gamma$ and $i \in N$. If $\Delta_i(v) \geq \Delta_i(w)$ and $v(N) - \sum_{j \in N}(v(N) - v(N \backslash j)) \geq w(N) - \sum_{j \in N}(w(N) - w(N \backslash j))$, then $\psi_i(v) \geq \psi_i(w)$.

Dual Strong Surplus Monotonicity (DSSM). Let $v, w \in \Gamma$ and $i \in N$. If $v(N) - v(N \backslash i) \geq w(N) - w(N \backslash i)$ and $v(N) - \sum_{j \in N}(v(N) - v(N \backslash j)) \geq w(N) - \sum_{j \in N}(w(N) - w(N \backslash j))$, then $\psi_i(v) \geq \psi_i(w)$.

Dual Weak Grand Coalition Monotonicity (DWGM). Let $v, w \in \Gamma$ and $i \in N$. If $v(N) \geq w(N)$ and $v(N) - \sum_{j \in N}(v(N) - v(N \backslash j)) \geq w(N) - \sum_{j \in N}(w(N) - w(N \backslash j))$, then $\psi_i(v) \geq \psi_i(w)$.

Theorem 7 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and DSM if and only if there exist $\alpha, \beta \in [0, 1]$ such that $\psi = \alpha Sh + (1 - \alpha)ENSC$.*

Theorem 8 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and DSSM if and only if $\psi = ENSC$.*

Theorem 9 *Let $n \geq 6$. Then, a solution ψ satisfies E, S and DWGM if and only if there exist $\alpha, \beta \in [0, 1]$ such that $\psi = \alpha ED + (1 - \alpha)ENSC$.*

We can prove Theorems 7, 8, 9 in the same way as Theorem 6.

Overall, based on Theorem 1, we obtain 11 characterizations of solutions that take a convex combination. The following figure summarizes the results:

Axioms \ Solutions	Sh	ED	CIS	ENSC
Weak Surplus Monotonicity (Th. 1)	√	√	√	
Dual Weak Surplus Monotonicity (Th. 6)	√	√		√
Weak Monotonicity (Th. 2)	√	√		
Surplus Monotonicity (Th. 3)	√		√	
Dual Surplus Monotonicity (Th. 7)	√			√
Weak Grand Coalition Monotonicity (Th. 5)		√	√	
Dual Weak Grand Coalition Monotonicity (Th. 9)		√		√
Strong Monotonicity (Young (1985))	√			
Strong Surplus Monotonicity (Th. 4)			√	
Dual Strong Surplus Monotonicity (Th. 8)				√
Grand Coalition Monotonicity (Casajus and Huettner (2014))		√		

Each axiom characterizes the class of solutions that take a convex combination of solutions marked by √.

5 Concluding remarks

In the proofs of some lemmas, we borrow the ideas of previous works. The proofs of Lemmas 6 and 13 are based on the proof of Claim 5 of Casajus and Huettner (2014). The proof of Lemma 19 is based on the proof of Theorem 2 of Young (1985).

In Theorem 1, we assumed that there are at least 6 players. We give a remark on this assumption. For $n = 1$, ψ is uniquely determined by E. For $n = 2$, there exists a solution that satisfies E, S and M but is not a convex combination of the three solutions; see Appendix B of Casajus and Huettner (2014). For $n = 3, 4, 5$, the validity of Theorem 1 remains an open question.

It is known that, in the characterization of Young (1985), we can replace Strong Monotonicity with *Marginality.* This axiom is weaker than Strong Monotonicity, and states the following: if the marginal contribution of a player is the same for two different games, then the payoff of the player for the games is the same. Namely, we replace inequality in the definition of Strong Monotonicity with equality. If we modify Weak Surplus Monotonicity in this way, then Theorem 1 does not hold. Consider the following axiom: if i's marginal contribution, the worth of the grand coalition and the cooperative surplus are the same for two games, then i's payoff is the same. This axiom and the two basic axioms (Efficiency and Symmetry) cannot exclude a solution that takes a *linear combination* of the three solutions. For example, a solution given by $2Sh(v) - ED(v)$ satisfies the three axioms.

We conjecture that some other classes of solutions in TU games can also be characterized by monotonicity. For example, it seems possible to characterize the class of solutions that take a convex combination of the egalitarian Shapley value, the CIS value and the ENSC value by using a further weaker axiom than Weak Surplus Monotonicity. As we show in Appendix A, the key of proof is to construct a basis of the set of games and properly decompose the set. In previous studies, the unanimity basis by Shapley (1953) has played a central role. Our reslut indicates that, by finding another basis, we can obtain new characterizations of solutions.

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Appendix A

We prove Theorem 1. Let us give some preliminaries. For each $v, w \in \Gamma$ and $\alpha \in \mathbb{R}$, we define addition and scalar multiplication as follows: $(v+w)(S) = v(S) + w(S)$ for all $S \subseteq N$, $(\alpha v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. Then, we can identify Γ as a linear space $\mathbb{R}^{2^n-1}$.

For each $T \subseteq N$, $T \neq \emptyset$, we define the T-unanimity game u_T (Shapley (1953)) by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

For each $T \subseteq N$, $|T| \geq 2$, we define $\bar{u}_T$ by

$$\bar{u}_T(S) = \begin{cases} 1 & \text{if } |S \cap T| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $u_T = \bar{u}_T$ for all $T \subseteq N$, $|T| = 2$.

Theorem 10 (Yokote and Funaki (2014)) *The following set*

$$\{u_T : T \subseteq N, 1 \leq |T| \leq 2\} \cup \{\bar{u}_T : T \subseteq N, |T| \geq 3\}$$

is a basis of Γ.

Define $u^1 = \sum_{i \in N} u_i$, $u^2 = \sum_{T \subseteq N : |T|=2} u_T$. Then, from Theorem 10, the following set

$$\begin{aligned} &\left\{u^1\right\} \cup \{u_1 - u_i : i \in N, i \neq 1\} \cup \{u^2\} \\ &\cup \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1,2\}\} \cup \{\bar{u}_T : |T| \geq 3\} \end{aligned}$$

is also a basis. We give an example for $n = 3$:

	u^1	$u_1 - u_2$	$u_1 - u_3$	u^2	$u_{12} - u_{13}$	$u_{12} - u_{23}$	$\bar{u}_N$
1	1	1	1	0	0	0	0
2	1	−1	0	0	0	0	0
3	1	0	−1	0	0	0	0
12	2	0	1	1	1	1	1
13	2	1	0	1	−1	0	1
23	2	−1	−1	1	0	−1	1
123	3	0	0	3	0	0	0

For a finite set of games $V \subseteq \Gamma$, let $\mathrm{Sp}(V)$ denote the set of games expressed by a linear combination of the games in V.[4] Define

$$\begin{aligned} V^1 &= \{u^1\} \cup \{u^2\} \cup \{\bar{u}_T : |T| \geq 3\}, & \Gamma^1 &= \mathrm{Sp}(V^1), \\ V^2 &= \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1,2\}\}, & \Gamma^2 &= \mathrm{Sp}(V^2), \\ V^3 &= \{u_1 - u_i : i \in N, i \neq 1\}, & \Gamma^3 &= \mathrm{Sp}(V^3). \end{aligned}$$

Note that $\mathrm{Sp}(V^1 \cup V^2 \cup V^3) = \Gamma$.

Outline of proof

Let ψ be a solution that satisfies E, S and M. The proof consists of 5 steps. We first prove that, in Step 1, ψ is additive (in particular, linear) with respect to the addition of games in Γ^3, i.e.,

$$\psi(v + v^3) = \psi(v) + \psi(v^3) \text{ for all } v \in \Gamma, v^3 \in \Gamma^3. \tag{A}$$

Next, we show that, in Step 2, ψ is additive (in particular, linear) with respect to the addition of games in Γ^2, i.e.,

$$\psi(v + v^2) = \psi(v) + \psi(v^2) \text{ for all } v \in \Gamma, v^2 \in \Gamma^2. \tag{B}$$

In Step 3, we endogenously derive the coefficients and show that ψ is a convex combination of the solutions on Γ^2 and Γ^3. In steps 4, we show that ψ coincides with ED on Γ^1, i.e.,

$$\psi_i(v^1) = \frac{v^1(N)}{n} \text{ for all } i \in N, v^1 \in \Gamma^1. \tag{C}$$

In step 5, we unify all the results and complete the proof. We can express any game v by $v = v^1 + v^2 + v^3$, where $v^j \in \Gamma^j$, $j = 1, 2, 3$. Then,

$$\begin{aligned} \psi_i(v) &= \psi_i(v^1 + v^2 + v^3) \overset{\text{(A)}}{=} \psi_i(v^1 + v^2) + \psi(v^3) \\ &\overset{\text{(B)}}{=} \psi_i(v^1) + \psi_i(v^2) + \psi_i(v^3) \overset{\text{(C)}}{=} \psi_i(v^2) + \psi_i(v^3) + \frac{v(N)}{n}. \end{aligned}$$

Since ψ is a convex combination on Γ^2 and Γ^3 from step 3, we conclude that ψ is a convex convex combination on the whole space.

[4] *Sp* refers to *Span*. Mathematically speaking, $\mathrm{Sp}(V)$ is the linear subspace in $\mathbb{R}^{2^n-1}$ spanned by the vectors in V.

Proof of Theorem 1

We use the following abbreviations:

Lemma $\to$ L, Case 1 $\to$ C1, Induction Hypothesis $\to$ IH.

The proof consists of 5 steps.
Step 1: We show the linearity of ψ with respect to the addition of $v \in \Gamma^3$.

Lemma 1 *Let $i \in N$, $v \in \Gamma$. Then, for any $p, q \in \mathbb{N}$ and $j \in N$, $j \neq i$,*

$$\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) = q\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].$$

Proof. Let p be fixed.
Induction base: If $q = 1$, the result holds.
Induction step: Suppose that the result holds for $q = r - 1$, and we prove the result for $q = r$, where $r \geq 2$. Choose $k \in N \backslash \{i, j\}$. Define $w = v + \frac{r}{p}u_i - \frac{r-1}{p}u_j - \frac{1}{p}u_k$. Then,

$$\begin{aligned}
\psi_j(w) - \psi_j(v) &\overset{\text{M}}{=} \psi_j\Big(v + \frac{r-1}{p}(u_i - u_j)\Big) - \psi_j(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_i\Big(v + \frac{r-1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\
&\overset{\text{IH}}{=} -(r-1)\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big], \\
\psi_k(w) - \psi_k(v) &\overset{\text{M}}{=} \psi_k\Big(v + \frac{1}{p}(u_i - u_k)\Big) - \psi_k(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_k)\Big) - \psi_i(v)\Big] \\
&\overset{\text{M}}{=} -\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi_i(w) - \psi_i(v) &\overset{\text{E,M}}{=} -\{\psi_j(w) - \psi_j(v)\} - \{\psi_k(w) - \psi_k(v)\} \\
&= (r-1)\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\
&+ \psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v) \\
&= r\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big]. \qquad (1)
\end{aligned}$$

On the other hand,

$$\psi_i(w) - \psi_i(v) \overset{\text{M}}{=} \psi_i\Big(v + \frac{r}{p}(u_i - u_j)\Big) - \psi_i(v). \tag{2}$$

Equations (1) and (2) complete the proof. □

Lemma 2 *Let $i \in N$, $v \in \Gamma$. Then, for any $p, q \in \mathbb{N}$ and $j \in N$, $j \neq i$,*

$$\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) = \frac{q}{p}\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big].$$

Proof. Let p be fixed. Then, by letting $q = p$ in L1,

$$\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) \overset{\text{L1}}{=} p\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big],$$

$$\frac{1}{p}\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big] = \psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v).$$

It follows that, for any $q \in \mathbb{N}$,

$$\begin{aligned}\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) &\overset{\text{L1}}{=} q\Big[\psi_i\Big(v + \frac{1}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\ &= \frac{q}{p}\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big],\end{aligned}$$

which completes the proof. □

Lemma 3 *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, and $p, q \in \mathbb{N}$. Then,*

$$\psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) = -\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].$$

Proof. Choose $k \in N \backslash \{i, j\}$. Define $w = v + \frac{q}{p}(u_i + u_j) - \frac{2q}{p}u_k$. Then,

$$\begin{aligned}\psi_j(w) - \psi_j(v) &\overset{\text{M}}{=} \psi_j\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_j(v) \\ &\overset{\text{E,M}}{=} -\Big[\psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big], \\ \psi_k(w) - \psi_k(v) &\overset{\text{M}}{=} \psi_k\Big(v + \frac{2q}{p}(u_i - u_k)\Big) - \psi_k(v) \\ &\overset{\text{L2}}{=} 2 \cdot \frac{q}{p}\Big[\psi_k\big(v + (u_i - u_k)\big) - \psi_k(v)\Big] \\ &\overset{\text{L2}}{=} 2\Big[\psi_k\Big(v + \frac{q}{p}(u_i - u_k)\Big) - \psi_k(v)\Big] \\ &\overset{\text{E,M}}{=} -2\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_k)\Big) - \psi_i(v)\Big] \\ &\overset{\text{M}}{=} -2\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big].\end{aligned}$$

Thus,

$$\begin{aligned}\psi_i(w) - \psi_i(v) &\overset{\text{E,M}}{=} -\{\psi_j(w) - \psi_j(v)\} - \{\psi_k(w) - \psi_k(v)\} \\ &= \Big[\psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\ &\quad + 2\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big]. \end{aligned} \tag{3}$$

On the other hand,

$$\psi_i(w) - \psi_i(v) \overset{\text{M}}{=} \psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v). \tag{4}$$

Equations (3) and (4) complete the proof. □

Lemma 4 *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$. Then, for any $s \in \mathbb{Q}$,*

$$\psi_i\big(v + s(u_i - u_j)\big) - \psi_i(v) = s\big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\big].$$

Proof. If $s \geq 0$, L2 completes the proof. Suppose that $s < 0$. Then, we can write $s = -\frac{q}{p}$ for some $p, q \in \mathbb{N}$.

$$\begin{aligned}\psi_i\big(v + s(u_i - u_j)\big) - \psi_i(v) &= \psi_i\Big(v - \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v) \\ &\overset{\text{L3}}{=} -\Big[\psi_i\Big(v + \frac{q}{p}(u_i - u_j)\Big) - \psi_i(v)\Big] \\ &\overset{\text{L2}}{=} -\frac{q}{p}\Big[\psi_i\Big(v + (u_i - u_j)\Big) - \psi_i(v)\Big] \\ &= s\big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\big],\end{aligned}$$

which completes the proof. □

Lemma 5 *Let $i \in N$, $v \in \Gamma$. Then, for any $\lambda \in \mathbb{R}$ and $j \in N$, $j \neq i$,*

$$\psi_i\big(v + \lambda(u_i - u_j)\big) - \psi_i(v) = \lambda\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big].$$

Proof. Let $\lambda \in \mathbb{R}$. Choose sequences of rational numbers $\{r_t\}$ and $\{s_t\}$ that converge to λ from below and above, respectively. Then,

$$\begin{aligned}r_t\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big] &\overset{\text{L4}}{=} \psi_i\Big(v + r_t(u_i - u_j)\Big) - \psi_i(v) \\ &\overset{\text{M}}{\leq} \psi_i\Big(v + \lambda(u_i - u_j)\Big) - \psi_i(v) \\ &\overset{\text{M}}{\leq} \psi_i\Big(v + s_t(u_i - u_j)\Big) - \psi_i(v) \\ &\overset{\text{L4}}{=} s_t\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big].\end{aligned}$$

By letting $t \to \infty$, we obtain the result. □

Lemma 6 *Let $v, w \in \Gamma$. Then,*

$$\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) = \psi_i\big(w + (u_i - u_j)\big) - \psi_i(w).$$

Proof. Choose a game $z \in \Gamma$ such that

$$z(N) \geq v(N), z(N) - \sum_{m \in N} z(m) \geq v(N) - \sum_{m \in N} v(m), \Delta_i z \geq \Delta_i v,$$
$$z(N) \geq w(N), z(N) - \sum_{m \in N} z(m) \geq w(N) - \sum_{m \in N} w(m), \Delta_i z \geq \Delta_i w.$$

Suppose that

$$\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) \neq \psi_i\big(z + (u_i - u_j)\big) - \psi_i(z).$$

For any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\psi_i\big(v + \lambda(u_i - u_j)\big) - \psi_i(v) - \Big\{\psi_i\big(z + \lambda(u_i - u_j)\big) - \psi_i(z)\Big\} \\
\overset{\text{L5}}{=}&\lambda\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) - \Big\{\psi_i\big(z + (u_i - u_j)\big) - \psi_i(z)\Big\}\Big], \\
&\psi_i\big(v + \lambda(u_i - u_j)\big) - \psi_i\big(z + \lambda(u_i - u_j)\big) \\
=&\lambda\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) - \Big\{\psi_i\big(z + (u_i - u_j)\big) - \psi_i(z)\Big\}\Big] \\
&+ \psi_i(v) - \psi_i(z).
\end{aligned}$$

So, by appropriately choosing λ, we obtain

$$\psi_i\big(v + \lambda(u_i - u_j)\big) - \psi_i\big(z + \lambda(u_i - u_j)\big) > 0.$$

This contradicts M. So, we must have

$$\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v) = \psi_i\big(z + (u_i - u_j)\big) - \psi_i(z).$$

By applying the same argument to the games w and z, we obtain the desired equality. □

Lemma 7 *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, and $\lambda \in \mathbb{R}$. Then,*

$$\psi\big(v + \lambda(u_i - u_j)\big) = \psi(v) + \lambda\psi(u_i - u_j).$$

Proof. By letting $w = \mathbf{0}$ in L6,

$$\begin{aligned}
\lambda\psi_i(u_i - u_j) &\overset{\text{L6}}{=} \lambda\Big[\psi_i\big(v + (u_i - u_j)\big) - \psi_i(v)\Big] \\
&\overset{\text{L5}}{=} \psi_i\big(v + \lambda(u_i - u_j)\big) - \psi_i(v).
\end{aligned}$$

For player $k \in N\backslash\{i,j\}$,

$$\psi_k\big(v+\lambda(u_i-u_j)\big)-\psi_k(v) \overset{\text{M}}{=} 0 \overset{\text{E,S}}{=} \lambda\psi_k(\mathbf{0}) \overset{\text{M}}{=} \lambda\psi_k(u_i-u_j).$$

E completes the proof. □

Step 2: We show the linearity of ψ with respect to the addition of $v \in \Gamma^2$. For any $v \in \Gamma$ and $i,j \in N$, $i \neq j$, define

$$\psi_{ij}(v) = \psi_i(v) + \psi_j(v).$$

Lemma 8 *Let $v \in \Gamma$ and $i,j \in N$, $i \neq j$, $k,l \in N\backslash\{i,j\}$, $k \neq l$. Then, for any $p,q \in \mathbb{N}$,*

$$\psi_{ij}\Big(v+\frac{q}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v) = q\Big[\psi_{ij}\Big(v+\frac{1}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)\Big].$$

Proof. Let p be fixed.
Induction base: If $q=1$, the result holds.
Induction step: Suppose that the result holds for $q=r-1$, and we prove the result for $q=r$, where $r \geq 2$. Without loss of generality, suppose that $1=i$, $2=j$, $3=k$ and $4=l$. Choose players $5,6$ and define $w = v+\frac{r}{p}u_{12}-\frac{r-1}{p}u_{34}-\frac{1}{p}u_{56}$. Then,

$$\begin{aligned}
\psi_{34}(w)-\psi_{34}(v) &\overset{\text{M}}{=} \psi_{34}\Big(v+\frac{r-1}{p}(u_{12}-u_{34})\Big)-\psi_{34}(v)\\
&\overset{\text{E,M}}{=} -\Big[\psi_{12}\Big(v+\frac{r-1}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v)\Big]\\
&\overset{\text{IH}}{=} -(r-1)\Big[\psi_{12}\Big(v+\frac{1}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v)\Big],\\
\psi_{56}(w)-\psi_{56}(v) &\overset{\text{M}}{=} \psi_{56}\Big(v+\frac{1}{p}(u_{12}-u_{56})\Big)-\psi_{56}(v)\\
&\overset{\text{E,M}}{=} -\Big[\psi_{12}\Big(v+\frac{1}{p}(u_{12}-u_{56})\Big)-\psi_{12}(v)\Big]\\
&\overset{\text{M}}{=} -\Big[\psi_{12}\Big(v+\frac{1}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}\psi_{12}(w)-\psi_{12}(v) &\overset{\text{E,M}}{=} -\{\psi_{34}(w)-\psi_{34}(v)\}-\{\psi_{56}(w)-\psi_{56}(v)\}\\ &=(r-1)\Big[\psi_{12}\Big(v+\frac{1}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v)\Big]\\ &+\Big[\psi_{12}\Big(v+\frac{1}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v)\Big]\\ &=r\Big[\psi_{12}\Big(v+\frac{1}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v)\Big]. \end{aligned}\tag{5}$$

On the other hand,

$$\psi_{12}(w)-\psi_{12}(v)\overset{\text{M}}{=}\psi_{12}\Big(v+\frac{r}{p}(u_{12}-u_{34})\Big)-\psi_{12}(v). \tag{6}$$

Equations (5) and (6) complete the proof. □

Lemma 9 *Let $v\in\Gamma$ and $i,j\in N$, $i\neq j$, $k,l\in N\backslash\{i,j\}$, $k\neq l$. Then, for any $p,q\in\mathbb{N}$,*

$$\psi_{ij}\Big(v+\frac{q}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)=\frac{q}{p}\Big[\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\Big].$$

Proof. Let p be fixed. Then, by letting $q=p$ in L8,

$$\begin{aligned}&\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\overset{\text{L8}}{=}p\Big[\psi_{ij}\Big(v+\frac{1}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)\Big],\\ &\frac{1}{p}\Big[\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\Big]=\psi_{ij}\Big(v+\frac{1}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v).\end{aligned}$$

It follows that, for any $q\in\mathbb{N}$,

$$\begin{aligned}\psi_{ij}\Big(v+\frac{q}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)&\overset{\text{L8}}{=}q\Big[\psi_{ij}\Big(v+\frac{1}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)\Big]\\ &=\frac{q}{p}\Big[\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\Big],\end{aligned}$$

which completes the proof. □

Lemma 10 *Let $v\in\Gamma$, $i,j\in N$, $i\neq j$, $k,l\in N\backslash\{i,j\}$, $k\neq l$, and $p,q\in\mathbb{N}$. Then,*

$$\psi_{ij}\Big(v-\frac{q}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)=-\Big[\psi_{ij}\Big(v+\frac{q}{p}(u_{ij}-u_{kl})\Big)-\psi_{ij}(v)\Big].$$

Proof. Without loss of generality, suppose that $1 = i$, $2 = j$, $3 = k$ and $4 = l$. Choose players $5, 6$ and define $w = v + \frac{q}{p}(u_{12} + u_{34}) - \frac{2q}{p}u_{56}$. Then,

$$\begin{aligned}
\psi_{34}(w) - \psi_{34}(v) &\overset{\text{M}}{=} \psi_{34}\Big(v - \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{34}(v) \\
&\overset{\text{E,M}}{=} -\Big[\psi_{12}\Big(v - \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big], \\
\psi_{56}(w) - \psi_{56}(v) &\overset{\text{M}}{=} \psi_{56}\Big(v + \frac{2q}{p}(u_{12} - u_{56})\Big) - \psi_{56}(v) \\
&\overset{\text{L9}}{=} 2 \cdot \frac{q}{p}\Big[\psi_{56}\Big(v + (u_{12} - u_{56})\Big) - \psi_{56}(v)\Big] \\
&\overset{\text{L9}}{=} 2\Big[\psi_{56}\Big(v + \frac{q}{p}(u_{12} - u_{56})\Big) - \psi_{56}(v)\Big] \\
&\overset{\text{E,M}}{=} -2\Big[\psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{56})\Big) - \psi_{12}(v)\Big] \\
&\overset{\text{M}}{=} -2\Big[\psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi_{12}(w) - \psi_{12}(v) &\overset{\text{E,M}}{=} -\{\psi_{34}(w) - \psi_{34}(v)\} - \{\psi_{56}(w) - \psi_{56}(v)\} \\
&= \Big[\psi_{12}\Big(v - \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big] \\
&\quad + 2\Big[\psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v)\Big]. \qquad (7)
\end{aligned}$$

On the other hand,

$$\psi_{12}(w) - \psi_{12}(v) \overset{\text{M}}{=} \psi_{12}\Big(v + \frac{q}{p}(u_{12} - u_{34})\Big) - \psi_{12}(v). \qquad (8)$$

Equations (7) and (8) complete the proof. □

Lemma 11 *Let $v \in \Gamma$, $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then, for any $s \in \mathbb{Q}$,*

$$\psi_{ij}\big(v + s(u_{ij} - u_{kl})\big) - \psi_{ij}(v) = s\big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\big].$$

Proof. If $s \geq 0$, L9 completes the proof. Suppose that $s < 0$. Then, we can write $s = -\frac{q}{p}$ for some $p, q \in \mathbb{N}$.

$$\begin{aligned}\psi_{ij}\big(v + s(u_{ij} - u_{kl})\big) - \psi_{ij}(v) &= \psi_{ij}\Big(v - \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{L10}}{=} -\Big[\psi_{ij}\Big(v + \frac{q}{p}(u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big] \\ &\overset{\text{L9}}{=} -\frac{q}{p}\Big[\psi_{ij}\Big(v + (u_{ij} - u_{kl})\Big) - \psi_{ij}(v)\Big] \\ &= s\big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\big],\end{aligned}$$

which completes the proof. □

Lemma 12 *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$ and $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then,*

$$\psi_{ij}\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_{ij}(v) = \lambda\big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\big].$$

Proof. Let $\lambda \in \mathbb{R}$. Choose sequences of rational numbers $\{r_t\}$ and $\{s_t\}$ that converge to λ from below and above, respectively. Then,

$$\begin{aligned}r_t\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big] &\overset{\text{L11}}{=} \psi_{ij}\Big(v + r_t(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{M}}{\leq} \psi_{ij}\Big(v + \lambda(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{M}}{\leq} \psi_{ij}\Big(v + s_t(u_{ij} - u_{kl})\Big) - \psi_{ij}(v) \\ &\overset{\text{L11}}{=} s_t\Big[\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v)\Big].\end{aligned}$$

By letting $t \to \infty$, we obtain the result. □

Lemma 13 *Let $v, w \in \Gamma$ and $i, j \in N$, $i \neq j$, $k, l \in N \backslash \{i, j\}$, $k \neq l$. Then,*

$$\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v) = \psi_{ij}\big(w + (u_{ij} - u_{kl})\big) - \psi_{ij}(w).$$

Proof. Choose a game $z \in \Gamma$ such that

$$z(N) \geq v(N), z(N) - \sum_{m \in N} z(m) \geq v(N) - \sum_{m \in N} v(m), \Delta_i z \geq \Delta_i v, \Delta_j z \geq \Delta_j v,$$
$$z(N) \geq w(N), z(N) - \sum_{m \in N} z(m) \geq w(N) - \sum_{m \in N} w(m), \Delta_i z \geq \Delta_i w, \Delta_j z \geq \Delta_j w.$$

Suppose that

$$\psi_{ij}\big(v + (u_{ij} - u_{kl})\big) - \psi_{ij}(v) \neq \psi_{ij}\big(z + (u_{ij} - u_{kl})\big) - \psi_{ij}(z).$$

For any $\lambda \in \mathbb{R}$,

$$\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}(v)-\Big\{\psi_{ij}\big(z+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}(z)\Big\}$$
$$\overset{\text{L12}}{=}\lambda\Big[\psi_{ij}(v+(u_{ij}-u_{kl}))-\psi_{ij}(v)-\Big\{\psi_{ij}\big(z+(u_{ij}-u_{kl})\big)-\psi_{ij}(z)\Big\}\Big],$$
$$\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}\big(z+\lambda(u_{ij}-u_{kl})\big)$$
$$=\lambda\Big[\psi_{ij}(v+(u_{ij}-u_{kl}))-\psi_{ij}(v)-\Big\{\psi_{ij}\big(z+(u_{ij}-u_{kl})\big)-\psi_{ij}(z)\Big\}\Big]$$
$$+\psi_{ij}(v)-\psi_{ij}(z).$$

So, by appropriately choosing λ, we obtain

$$\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}\big(z+\lambda(u_{ij}-u_{kl})\big)>0.$$

This contradicts M. So, we must have

$$\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)=\psi_{ij}\big(z+(u_{ij}-u_{kl})\big)-\psi_{ij}(z).$$

By applying the same argument to the games w and z, we obtain the desired equality. □

Lemma 14 *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$, and $i,j \in N$, $i \neq j$, $k,l \in N\backslash\{i,j\}$, $k \neq l$. If $v(i)=v(j)$, then,*

$$\psi_i\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_i(v)=\lambda\psi_i(u_{ij}-u_{kl}).$$

Proof. **Case 1:** Suppose that $i \sim j$ in v. Then,

$$\begin{aligned}2\Big[\psi_i\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_i(v)\Big] &\overset{\text{S}}{=}\psi_{ij}\big(v+\lambda(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\\ &\overset{\text{L12}}{=}\lambda\Big[\psi_{ij}\big(v+(u_{ij}-u_{kl})\big)-\psi_{ij}(v)\Big]\\ &\overset{\text{L13}}{=}\lambda\psi_{ij}(u_{ij}-u_{kl})\\ &\overset{\text{S}}{=}2\lambda\psi_i(u_{ij}-u_{kl}).\end{aligned}$$

Case 2: Suppose that $i \nsim j$ in v. By Pinter (2012), there exists a game $w \in \Gamma$ such that

$$\Delta_i v=\Delta_i w, i \sim j \text{ in } w.$$

Note that $w(j)=w(i)=v(i)=v(j)$. Choose $k,l \in N\backslash\{i,j\}$, $k \neq l$, and define $z \in \Gamma$ by

$$\begin{aligned}z=w&+\sum_{m\in N\backslash\{i,j\}}\big(v(m)-w(m)\big)u_m\\ &+\Big(v(N)-w(N)-\sum_{m\in N\backslash\{i,j\}}\big(v(m)-w(m)\big)\Big)u_{kl}.\end{aligned}$$

Then, $\Delta_i v = \Delta_i z$, $v(m) = z(m)$ for all $m \in N$, $v(N) = z(N)$ and $i \sim j$ in z.

$$\begin{aligned}\psi_i\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_i(v) &\overset{\text{M}}{=} \psi_i\big(z + \lambda(u_{ij} - u_{kl})\big) - \psi_i(z) \\ &\overset{\text{C1}}{=} \lambda\psi_i(u_{ij} - u_{kl}),\end{aligned}$$

which completes the proof. □

Lemma 15 *Let $v \in \Gamma$, $\lambda \in \mathbb{R}$, and $S, T \subseteq N$, $|S| = |T| = 2$, $S \neq T$. If $v(i) = v(j)$ for all $i, j \in N$, then*

$$\psi\big(v + \lambda(u_S - u_T)\big) - \psi(v) = \lambda\psi(u_S - u_T).$$

Proof. **Case 1:** Suppose that $S \cap T = \emptyset$. Let $S = \{i, j\}$, $T = \{k, l\}$. Then, for $m = i, j$,

$$\psi_m\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_m(v) \overset{\text{L14}}{=} \lambda\psi_m(u_{ij} - u_{kl}).$$

For $m = k, l$,

$$\begin{aligned}\psi_m\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_m(v) &= \psi_m\big(v - \lambda(u_{kl} - u_{ij})\big) - \psi_m(v) \\ &\overset{\text{L14}}{=} -\lambda\psi_m(u_{kl} - u_{ij}) \\ &\overset{\text{L14}}{=} \lambda\psi_m(u_{ij} - u_{kl}).\end{aligned}$$

For player $m \in N \backslash \{i, j, k, l\}$,

$$\psi_m\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_m(v) \overset{\text{M}}{=} 0 \overset{\text{E,S}}{=} \psi_m(\mathbf{0}) \overset{\text{M}}{=} \lambda\psi_m(u_{ij} - u_{kl}).$$

Case 2: Suppose that $S \cap T \neq \emptyset$. Let $S = \{i, j\}$, $T = \{j, k\}$. Choose $l \in N \backslash \{i, j, k\}$. For player i,

$$\begin{aligned}\psi_i\big(v + \lambda(u_{ij} - u_{jk})\big) - \psi_i(v) &\overset{\text{M}}{=} \psi_i\big(v + \lambda(u_{ij} - u_{kl})\big) - \psi_i(v) \\ &\overset{\text{L14}}{=} \lambda\psi_i(u_{ij} - u_{kl}) \\ &\overset{\text{M}}{=} \lambda\psi_i(u_{ij} - u_{jk}).\end{aligned}$$

For player k,

$$\begin{aligned}\psi_k(v + \lambda(u_{ij} - u_{jk})) - \psi_k(v) &= \psi_k\big(v - \lambda(u_{jk} - u_{ij})\big) - \psi_k(v) \\ &\overset{\text{M}}{=} \psi_k\big(v - \lambda(u_{jk} - u_{il})\big) - \psi_k(v) \\ &\overset{\text{L14}}{=} -\lambda\psi_k(u_{jk} - u_{il}) \\ &\overset{\text{L14}}{=} \lambda\psi_k(u_{il} - u_{jk}) \\ &\overset{\text{M}}{=} \lambda\psi_k(u_{ij} - u_{jk}).\end{aligned}$$

For player $m \in N\backslash\{i,j,k\}$,

$$\psi_m\big(v+\lambda(u_{ij}-u_{jk})\big)-\psi_m(v) \overset{\text{M}}{=} 0 \overset{\text{E,S}}{=} \psi_m(\mathbf{0}) \overset{\text{M}}{=} \lambda\psi_m(u_{ij}-u_{jk}).$$

E completes the proof. □

Step 3 We endogenously derive the coefficients of Sh and ED.

Lemma 16 *For distinct players $i,j,k \in N$,*

$$\psi_j(u_{ij}-u_{jk})=0.$$

Proof. Without loss of generality, suppose that $1=i$, $2=j$, $3=k$. Choose players $4,5$. Then,

$$\begin{aligned}
\psi_1(u_{12}-u_{23}) &\overset{\text{M}}{=} \psi_1(u_{12}+u_{23}-2u_{45}),\\
\psi_3(u_{23}-u_{12}) &\overset{\text{M}}{=} \psi_3(u_{12}+u_{23}-2u_{45}),\\
\psi_1(u_{12}+u_{23}-2u_{45}) &\overset{\text{S}}{=} \psi_3(u_{12}+u_{23}-2u_{45}).
\end{aligned}$$

The above equations imply

$$\psi_1(u_{12}-u_{23})=\psi_3(u_{23}-u_{12}). \tag{9}$$

For player $m \in N\backslash\{1,2,3\}$,

$$\psi_m(u_{12}-u_{23}) \overset{\text{M}}{=} \psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0.$$

Thus,

$$\begin{aligned}
\psi_2(u_{12}-u_{23}) &\overset{\text{E}}{=} -\psi_1(u_{12}-u_{23})-\psi_3(u_{12}-u_{23})\\
&\overset{\text{L15}}{=} -\psi_1(u_{12}-u_{23})+\psi_3(u_{23}-u_{12})\\
&=0,
\end{aligned}$$

where the last equality holds from equation (9). □

Define

$$y=\psi_3(nu_1), \tag{10}$$
$$z=\psi_3(nu_{12}). \tag{11}$$

Then,

$$\begin{aligned}
0 &\overset{\text{E,S}}{=} \psi_3(\mathbf{0}) \overset{\text{M}}{\leq} y \overset{\text{M}}{\leq} \psi_3(nu_N) \overset{\text{E,S}}{=} 1,\\
0 &\overset{\text{E,S}}{=} \psi_3(\mathbf{0}) \overset{\text{M}}{\leq} z \overset{\text{M}}{\leq} \psi_3(nu_N) \overset{\text{E,S}}{=} 1.
\end{aligned}$$

In addition,

$$y = \psi_3(nu_1) \overset{\text{M}}{\leq} \psi_3(nu_{12}) = z.$$

Define $x = 1 - z$. Note that $0 \leq x \leq 1$, $0 \leq x + y \leq 1$. Define a solution $\Phi^{x,y}$ by

$$\Phi^{x,y}(v) = xSh(v) + yED(v) + (1 - x - y)CIS(v) \text{ for all } v \in \Gamma. \tag{12}$$

Lemma 17 *Let $i \in N$, $i \neq 1$. Then,*

$$\psi(u_1 - u_i) = \Phi^{x,y}(u_1 - u_i).$$

Proof. For player 1,

$$\begin{aligned}
1 &\overset{\text{E,S}}{=} \psi_1(u^1) \\
&= \psi_1\Big(nu_1 - \sum_{j \neq 1}(u_1 - u_j)\Big) \\
&\overset{\text{L7}}{=} \psi_1(nu_1) - \sum_{j \neq 1} \psi_1(u_1 - u_j) \\
&\overset{\text{E,S}}{=} \big(n - (n-1)y\big) - \sum_{j \neq 1} \psi_1(u_1 - u_j) \\
&\overset{\text{M}}{=} \big(n - (n-1)y\big) - (n-1)\psi_1(u_1 - u_i).
\end{aligned}$$

By rearranging,

$$\psi_1(u_1 - u_i) = 1 - y = \Phi_1^{x,y}(u_1 - u_i).$$

For player $j \in N \backslash \{1, i\}$,

$$\psi_j(u_1 - u_i) \overset{\text{M}}{=} \psi_j(\mathbf{0}) \overset{\text{E,S}}{=} 0 = \Phi_j^{x,y}(u_1 - u_i).$$

E completes the proof. □

Lemma 18 *Let $S \subseteq N$, $|S| = 2$, $S \neq \{1, 2\}$. Then,*

$$\psi(u_{12} - u_S) = \Phi^{x,y}(u_{12} - u_S).$$

Proof. **Case 1:** Suppose that $2 \notin S$. From the definition of u^2, we have

$$u^2(N) = \frac{n(n-1)}{2} \text{ and } u^2 = \frac{n(n-1)}{2}u_{12} - \sum_{T \subseteq N: |T|=2, T \neq \{1,2\}} (u_{12} - u_T).$$

With this in mind, for player 2,

$$\begin{aligned}
1 &\overset{\text{E,S}}{=} \psi_2\Big(\frac{2}{n-1}u^2\Big) \\
&= \psi_2\Big(nu_{12} - \frac{2}{n-1} \sum_{T \subseteq N: |T|=2, T \neq \{1,2\}} (u_{12} - u_T)\Big) \\
&\overset{\text{L15}}{=} \psi_2(nu_{12}) - \frac{2}{n-1} \sum_{T \subseteq N: |T|=2, T \neq \{1,2\}} \psi_2(u_{12} - u_T) \\
&\overset{\text{E,S}}{=} \frac{1}{2}\big\{n - (n-2)z\big\} - \frac{2}{n-1} \sum_{T \subseteq N: |T|=2, T \neq \{1,2\}} \psi_2(u_{12} - u_T) \\
&\overset{\text{L16}}{=} \frac{1}{2}\big\{n - (n-2)z\big\} - \frac{2}{n-1} \sum_{T \subseteq N: |T|=2, 2 \notin T} \psi_2(u_{12} - u_T) \\
&\overset{\text{M}}{=} \frac{1}{2}\big\{n - (n-2)z\big\} - \frac{2}{n-1} \cdot \frac{(n-1)(n-2)}{2} \psi_2(u_{12} - u_S) \\
&= \frac{1}{2}\big\{n - (n-2)z\big\} - (n-2)\psi_2(u_{12} - u_S).
\end{aligned}$$

By rearranging

$$\psi_2(u_{12} - u_S) = \frac{1-z}{2} = \frac{x}{2} = \Phi_2^{x,y}(u_{12} - u_S).$$

For player 1, if $1 \notin S$,

$$\psi_1(u_{12} - u_S) \overset{\text{S}}{=} \psi_2(u_{12} - u_S) = \Phi_2^{x,y}(u_{12} - u_S) \overset{\text{S}}{=} \Phi_1^{x,y}(u_{12} - u_S).$$

If $1 \in S$,

$$\psi_1(u_{12} - u_S) \overset{\text{L16}}{=} 0 = \Phi_1^{x,y}(u_{12} - u_S).$$

For $m \in N \backslash (S \cup \{1,2\})$,

$$\psi_m(u_{12} - u_S) \overset{\text{M}}{=} \psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0 = \Phi_m^{x,y}(u_{12} - u_S).$$

E and S yield the desired equation.
Case 2: Suppose that $2 \in S$. We can write $S = \{2, i\}$ for some $i \in N \backslash \{1,2\}$.

For player 1,

$$\psi_1(u_{12} - u_{2i}) \overset{\text{M}}{=} \psi_1(u_{12} - u_{34}) \overset{\text{C1}}{=} \Phi_1^{x,y}(u_{12} - u_{34}) = \Phi_1^{x,y}(u_{12} - u_{2i}).$$

For player 2,

$$\psi_2(u_{12} - u_{2i}) \overset{\text{L16}}{=} 0 = \Phi_2^{x,y}(u_{12} - u_{2i}).$$

For $m \in N \backslash \{1, 2, i\}$,

$$\psi_m(u_{12} - u_{2i}) \overset{\text{M}}{=} \psi_m(\mathbf{0}) \overset{\text{E,S}}{=} 0 = \Phi_m^{x,y}(u_{12} - u_{2i}).$$

E completes the proof. □

Step 4 We show that ψ coincides with ED on Γ^1.

Lemma 19 *Let $v \in \Gamma^1$. Then,*

$$\psi_i(v) = \Phi_i^{x,y}(v) = \frac{v(N)}{n} \text{ for all } i \in N.$$

Proof. Since $v \in \Gamma^1$, there exist unique real numbers α, β, γ_T, $T \subseteq N$, $|T| \geq 3$, such that

$$v = \alpha u^1 + \beta u^2 + \sum_{T \subseteq N : |T| \geq 3} \gamma_T \bar{u}_T.$$

Let $\mathcal{C} = \{T \subseteq N : |T| \geq 3, \gamma_T \neq 0\}$. We proceed by induction.
Induction base: If $|\mathcal{C}| = 0$, then $v = \gamma w$, so the result holds from E and S.
Induction step: Suppose that the result holds for $|\mathcal{C}| = t - 1$, and we prove the result for $|\mathcal{C}| = t$, where $t \geq 1$.
Consider a player $j \in N \backslash (\cap_{R \in \mathcal{C}} R)$. By choosing a coalition R such that $R \in \mathcal{C}$ and $j \notin R$, we have

$$\psi_j(v') \overset{\text{M}}{=} \psi_j(v' - \gamma_R \bar{u}_R) \overset{\text{IH}}{=} \frac{v(N)}{n}.$$

Consider a player $i \in \cap_{R \in \mathcal{C}} R$. Note that the payoff of player $j \in N \backslash (\cap_{R \in \mathcal{C}} R)$ is already determined. Since the players in $\cap_{R \in \mathcal{C}} R$ are substitutes, E and S uniquely determine the payoffs of all players. □

Step 5 We show that $\psi(v) = \Phi^{x,y}(v)$ for all $v \in \Gamma$.

Lemma 20 *Let $v^1 \in \Gamma^1$, $v^2 \in \Gamma^2$. Then, $\psi(v^1 + v^2) = \Phi^{x,y}(v^1 + v^2)$.*

Proof. Since $v^2 \in \Gamma^2$, there exist unique real numbers γ_T, $T \subseteq N$, $|T| = 2$, $T \neq \{1, 2\}$, such that

$$v^2 = \sum_{T \subseteq N : |T|=2, T \neq \{1,2\}} \gamma_T (u_{12} - u_T).$$

Since $v^1(i) = v^1(j)$ for all $i, j \in N$,[5]

$$\begin{aligned}
\psi(v^1 + v^2) &\overset{\text{L15}}{=} \psi(v^1) + \sum_{T \subseteq N : |T|=2, T \neq \{1,2\}} \gamma_T \psi(u_{12} - u_T) \\
&\overset{\text{L19,L18}}{=} \Phi^{x,y}(v^1) + \sum_{T \subseteq N : |T|=2, T \neq \{1,2\}} \gamma_T \Phi^{x,y}(u_{12} - u_T) \\
&= \Phi^{x,y}(v).
\end{aligned}$$

□

We resume the proof of Theorem 1. Let $v \in \Gamma$. Then, there exist $v^1 \in \Gamma^1$, $v^2 \in \Gamma^2$ and $v^3 \in \Gamma^3$ such that

$$v = v^1 + v^2 + v^3.$$

Since $v^3 \in \Gamma^3$, there exist unique real numbers γ_i, $i \in N$, such that

$$v^3 = \sum_{i \in N} \gamma_i (u_1 - u_i).$$

Then,

$$\begin{aligned}
\psi(v^1 + v^2 + v^3) &= \psi\Big(v^1 + v^2 + \sum_{i \in N} \gamma_i (u_1 - u_i)\Big) \\
&\overset{\text{L7}}{=} \psi(v^1 + v^2) + \sum_{i \in N} \gamma_i \psi(u_1 - u_i) \\
&\overset{\text{L20, L17}}{=} \Phi^{x,y}(v^1 + v^2) + \sum_{i \in N} \gamma_i \Phi^{x,y}(u_1 - u_i) \\
&= \Phi^{x,y}(v^1 + v^2 + v^3).
\end{aligned}$$

Thus, $\psi = \Phi^{x,y}$. Equivalently, it is a convex combination of the egalitarian Shapley value and the Consensus value, which completes the proof. □

[5] Note that $(u^1 + u^2)(i) = (u^1 + u^2)(j)$ for all $i, j \in N$ and $\bar{u}_T(i) = 0$ for all $T \subseteq N$, $|T| \geq 3$, $i \in N$.

Appendix B

Based on the proof in Appendix A, we give another proof of Theorem 2. For each $T \subseteq N$, $T \neq \emptyset$, we define $\bar{w}_T$ by

$$\bar{w}_T(S) = \begin{cases} 1 & \text{if } |S \cap T| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 11 (Yokote et al. (2014)) *The following set*

$$\{u_i : i \in N\} \cup \{\bar{w}_T : T \subseteq N, |T| \geq 2\}$$

is a basis of Γ.

From Theorem 11, we can construct the following basis:

$$\{u^1\} \cup \{u_1 - u_i : i \in N, i \neq 1\} \cup \{\bar{w}_T : |T| \geq 2\}. \tag{13}$$

Proof of Theorem 2. Choose $i \in N \backslash \{1\}$ and define $x = \psi_1(u_1 - u_i)$. Here, x does not depend on the choice of $i \in N\backslash\{1\}$. To see this, let $i, j \in N\backslash\{1\}$. Then,

$$\psi_1(u_1 - u_i) \overset{\text{M}}{=} \psi_1\Big(u_1 - \frac{1}{2}u_i - \frac{1}{2}u_j\Big) \overset{\text{M}}{=} \psi_1(u_1 - u_j).$$

x satisfies

$$0 \overset{\text{E,S}}{=} \psi_1(\mathbf{0}) \overset{\text{WM}}{\leq} x \overset{\text{WM}}{\leq} \psi_1(u^1) \overset{\text{E,S}}{=} 1.$$

The following equality holds: for any $i \in N\backslash\{1\}$,

$$\psi(u_1 - u_i) = ES^x(u_1 - u_i). \tag{14}$$

Indeed, for player $j \in N\backslash\{1, i\}$,

$$\psi_j(u_1 - u_i) \overset{\text{WM}}{=} \psi_j(\mathbf{0}) \overset{\text{E,S}}{=} 0.$$

Since $\psi_1(u_1 - u_i) = ES^x_1(u_1 - u_i)$, E implies $\psi(u_1 - u_i) = ES^x(u_1 - u_i)$.

Let us go back to Step 1 of proof of Theorem 1. In order to prove Lemma 7, it suffices to assume that $n \geq 3$. Since WM is stronger than M, we can use the lemma:

$$\psi\big(v + \lambda(u_1 - u_i)\big) = \psi(v) + \lambda\psi(u_1 - u_i) \text{ for all } v \in \Gamma, i \neq 1, \text{ and } \lambda \in \mathbb{R}. \tag{15}$$

Define

$$\Gamma^4 = \text{Sp}(u^1 \cup \{\bar{w}_T : T \subseteq N, |T| \geq 2\}).$$

By following the same line of proof of Lemma 19, we obtain

$$\psi_i(v) = ES_i^x(v) = \frac{v(N)}{n} \text{ for all } i \in N, v \in \Gamma^4. \tag{16}$$

Let $v \in \Gamma$. Then, from the fact that the set of games in (13) is a basis, there exist $\gamma_i \in \mathbb{R}$, $i \neq 1$, and $v^4 \in \Gamma^4$ such that

$$v = \sum_{i \in N : i \neq 1} \gamma_i (u_1 - u_i) + v^4.$$

We conclude that

$$\begin{aligned}\psi(v) &\overset{(15)}{=} \sum_{i \in N : i \neq 1} \gamma_i \psi(u_1 - u_i) + \psi(v^4) \\ &\overset{(14),\,(16)}{=} \sum_{i \in N : i \neq 1} \gamma_i ES^x(u_1 - u_i) + ES^x(v^4) \\ &= ES^x(v).\end{aligned}$$

□

Appendix C

We prove Theorems 3, 4 and 5. Note that all the reasonings in the proof of Theorem 1 remain valid since SM, SSM and WGM are stronger than M.

Proof of Theorem 3. Define y and z as we did in equations (10), (11) and let $x = 1 - z$. Then,

$$0 \overset{\text{E,S}}{=} \psi_3(\mathbf{0}) \overset{\text{SM}}{=} \psi_3(nu_1) = y.$$

Thus, equation (12) reduces to

$$\Phi^{x,y}(v) = xSh(v) + (1-x)CIS(v) \text{ for all } v \in \Gamma.$$

□

Proof of Theorem 4. Define y and z as we did in equations (10), (11) and let $x = 1 - z$. Then,

$$0 \overset{\text{E,S}}{=} \psi_3(\mathbf{0}) \overset{\text{SSM}}{=} \psi_3(nu_1) = y,$$
$$z = \psi_3(nu_{12}) \overset{\text{SSM}}{=} \psi_3(nu_N) = 1.$$

It follows that $x = y = 0$. Thus, equation (12) reduces to

$$\Phi^{x,y}(v) = CIS(v) \text{ for all } v \in \Gamma.$$

□

Proof of Theorem 5. Define y and z as we did in equations (10), (11) and let $x = 1 - z$. Then,

$$z = \psi_3(nu_{12}) \overset{\text{WGM}}{=} \psi_3(nu_N) = 1.$$

It follows that $x = 0$. Thus, equation (12) reduces to

$$\Phi^{x,y}(v) = yCIS(v) + (1 - y)ED(v) \text{ for all } v \in \Gamma.$$

□

Appendix D

We prove Theorem 6. For a game $v \in \Gamma$, we define the dual game v^d by

$$v^d(S) = v(N) - v(N \backslash S) \text{ for all } S \subseteq N, S \neq \emptyset.$$

Proof of Theorem 6. Define $\psi^d : \Gamma \to \mathbb{R}^n$ by

$$\psi^d(v) = \psi(v^d).$$

Then, ψ^d satisfies E, S and M. From Theorem 1, there exist $\alpha, \beta \in [0, 1]$ such that

$$\psi^d = \alpha ES^\beta + (1 - \alpha)CIS.$$

For any $v \in \Gamma$,

$$\begin{aligned} \psi(v) &= \psi^d(v^d) \\ &= \alpha ES^\beta(v^d) + (1 - \alpha)CIS(v^d) \\ &= \alpha ES^\beta(v) + (1 - \alpha)ENSC(v), \end{aligned}$$

which complete the proof. □