WINPEC Working Paper Series No.E1503
June 2015

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Abstract

It has been seen that local equilibrium determinacy can be achieved by applying active monetary policies and passive fiscal policies in discrete time New Keynesian (NK) models that include a fiscal policy rule with a time lag in the policy response. It is also indicated, however, that in models with money-in-the-production-function (MIPF), equilibrium indeterminacy can occur even under these policy actions. In this paper, we first show that the above-mentioned policy implications can be derived from a continuous time NK model that does not introduce a policy lag. Next, we introduce monetary and fiscal policy lags into the model and demonstrate that both or either of these policy lags can resolve the problem of indeterminacy.

JEL Classification: E32; E52

Keywords: continuous time New Keynesian model, policy lag, two-delay differential equations, determinacy analysis

1 Introduction

Optimizing models that consider the stickiness associated with price and nominal wages have often been referred to as New Keynesian (NK) models.[1] In the simplest NK models,

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[1]Introductory textbooks for NK models are provided by Woodford (2003), Galí (2008), and Walsh (2010).

monetary policy is thought to be conducted by financial institutions in accordance with rules that induce responses in the nominal interest rate based on the inflation rate (interest rate rules). If the nominal interest rate is changed by more than one unit in response to each unit of change in the inflation rate, then the policy is referred to as active. Conversely, if the nominal interest rate's response is less than one unit it is referred to as passive. To achieve local equilibrium determinacy, a monetary policy must be active. This proposition is recognized as the "Taylor principle."

The interaction between monetary and fiscal policies has been studied by Leeper (1991), among others. Leeper (1991) evaluated the efficacy of a fiscal policy rule wherein the lump-sum tax responds to total government liabilities (tax collection rule). An active fiscal policy manipulates the amount of tax without abiding by budget discipline. Conversely, a passive fiscal policy seeks to keep total government liabilities in line with income, which implies that the fiscal regime is of a Ricardian type. Leeper (1991) demonstrates that there are two possible combinations of monetary and fiscal policies for local equilibrium determinacy; i.e., for local equilibrium determinacy, an active monetary policy must be combined with a passive fiscal policy, whereas a passive monetary policy must be combined with an active fiscal policy. In the latter case, the treasury budget is balanced by inflation caused by the passive monetary policy.

Although Leeper (1991) assumes flexible prices, Schmitt-Grohé and Uribe (2007) and Kumhof, Nunes, and Yakadina (2010) develop Leeper's model to include price stickiness, i.e., they expand Leeper's model to NK models. They perform similar analyses as in Leeper (1991) by assuming rules that include responses not only of lump-sum taxes but also of the income tax rate to total government liabilities, where the former represents a non-distortionary tax and the latter represents a distortionary tax. Their studies basically confirmed Leeper's results.

These studies use discrete time models. They also assume that tax rates respond to a past (one period earlier) value of government liabilities. In other words, they consider the presence of a policy lag. More generally, a certain policy variable evaluated at period t responds to an endogenous variable evaluated not at time t, x_t, but at time $t-1$, x_{t-1}. Discrete time models have no analytical problem dealing with a time lag. However, how should a time lag be treated in continuous time models? Would the results concerning the effects of monetary and fiscal policies hold in continuous time models with policy lag? The main objective of this study is to answer these questions.

One way of treating the "past" of a variable in continuous time models is to employ a weighted average of such a variable's stream that extends from the infinite past to the present, as in the study of Benhabib, Schmitt-Grohé, and Uribe (2003). However, a time lag in the strict sense could not be represented by the variable's weighted average. Instead, it should be represented by a past value taken at a certain point in time. If we introduce a time lag into a continuous time model, then the system will be described by a delay differential equation (in other words, a differential-difference equation) that includes terms $\dot{x}_t$, x_t, and $x_{t-\theta}$, where θ represents a time lag.

Only a limited amount of research has been conducted on the effects of lag in policy responses using a continuous time model. Although Tsuzuki (2014, 2015) develops continuous time NK models that include an interest rate rule with policy lag, these models do not include a fiscal policy rule. This study develops a continuous time NK model that includes both monetary and fiscal policy rules. Furthermore, we introduce policy lags into these rules and perform an analysis of local equilibrium determinacy.

First, we examine whether the above mentioned results shown by Leeper (1991), Schmitt-Grohé and Uribe (2007), and Kumhof, Nunes, and Yakadina (2010) hold in a continuous time NK model without a policy lag. Next, we investigate the case with the existence of two policy lags.

This report is structured as follows. In Section 2, we discuss the behavior of economic agents in a model economy. Section 3 analyzes the local dynamics for a case where time lag is absent. Section 4 includes an evaluation of a case where monetary and fiscal policies that accompany time lags are introduced. Section 5 provides a conclusion.

2 The model

In this section, we propose a simple continuous time NK model that follows Benhabib, Schmitt-Grohé, and Uribe (2003). The model economy is constructed using monetary and financial institutions and household–firm units that are indexed by i (we normalize their total at unity: i.e., $i \in [0, 1]$). Household–firm units i produce and sell goods i under monopolistic competition. Furthermore, each household–firm unit first aggregates heterogeneous types of goods and then consumes them as composite goods.

We shall describe monetary and fiscal policy implementations following the practice of Leeper (1991) and Schmitt-Grohé and Uribe (2007). Monetary institutions manipulate the nominal interest rate according to fluctuations in the inflation rate; whereas, financial institutions manipulate the income tax rate according to fluctuations in total government liabilities (i.e., money and bonds). Leeper (1991) examines the case of a lump-sum tax, which is a non-distortionary tax, and Schmitt-Grohé and Uribe (2007) examine the case of income tax, which is a distortionary tax. This study deals with the latter, emphasizing the generality of the specification. We also assume that financial institutions spend their revenue, but their expenditures do not affect either production or utility of household–firm units.

First, we will describe the demand for heterogeneous products and the aggregation of such products by household–firm units.

2.1 Simultaneous optimization

Household–firm units aggregate various types of products via the Dixit–Stiglitz function[2]:

$$y = \left[\int_0^1 y_i^{\frac{\phi-1}{\phi}} di\right]^{\frac{\phi}{\phi-1}}, \tag{1}$$

where y represents the quantity of composite products, y_i denotes the quantity of products i, and ϕ (> 1) denotes the elasticity of substitution for the heterogeneous products.[3]

Given the quantity of composite products, the price of composite products (which equals the general price level) p, and prices of products i, p_i, the demand for product i is determined through minimizing cost $\int_0^1 p_i y_i di$ subject to Equation (1). This presents a simultaneous optimization problem (static optimization). The first-order condition for optimization can be obtained as follows[4]:

$$y_i = \left(\frac{p_i}{p}\right)^{-\phi} y, \tag{2}$$

where $p = \left[\int_0^1 p_i^{1-\phi} di\right]^{\frac{1}{1-\phi}}$. Equation (2) is a demand function for products i.

2.2 Intertemporal optimization

Next, we provide a description for the consumption behavior, price setting, and production performed by household–firm units. Household–firm units i produce products i using labor

[2]Refer to Dixit and Stiglitz (1977).

[3]When products are completely homogenous (complete substitution), $\phi \to 1$. In this study, we assume that products are heterogeneous, wherein $\phi > 1$.

[4]Refer to Blanchard and Kiyotaki (1987).

forces i. Assuming a linear technology, we specify the production function as follows:

$$y_i = z\ell_i, \tag{3}$$

where z denotes productivity, which is constant and common among all household–firm units, and ℓ_i denotes the labor forces used for producing products i.

Household–firm units i obtain utility from consumption and money holdings and disutility from labor supply and price revisions.[5] We specify the utility function as follows:

$$U(c_i, m_i, \ell_i, \pi_i) = u(c_i, m_i) - \frac{\ell_i^{1+\psi}}{1+\psi} - \frac{\eta}{2}(\pi_i - \pi^*)^2, \tag{4}$$

where c_i denotes the consumption of composite products, m_i denotes real money balances, $\pi_i = \dot{p}_i/p_i$ denotes the price change rate of products i (π^* denotes the steady-state value of the price change rate), $\psi > 0$, and $\eta > 0$.[6] We assume that $u_c = \partial u/\partial c_i > 0$, $u_{cc} = \partial^2 u/\partial c_i^2 < 0$, $u_m = \partial u/\partial m_i > 0$, and $u_{mm} = \partial^2 u/\partial m_i^2 < 0$. Due to the existence of the price revision costs, prices become sticky. Hence, η can be interpreted as representing price stickiness, which increases with increases in η. Furthermore, we formulated price revision costs as proportional to the divergence from the steady-state value. This is an assumption for simplicity and does not affect the arguments made in this study.

Assets of household–firm units i comprise money and bonds: $A_i = M_i + B_i$, where A_i represents nominal assets, M_i represents the nominal money balance, and B_i represents nominal bonds. Assets can be increased based on income and bond interest and can be decreased based on consumption and income taxes. Thus, the following equation holds: $\dot{A}_i = (1-\tau)p_i y_i + RB_i - pc_i$, where R represents the nominal interest rate for bonds, and τ represents the income tax rate. We can rewrite this equation in real terms as follows:

$$\dot{a}_i = (1-\tau)\frac{p_i}{p}y_i + ra_i - c_i - Rm_i, \tag{5}$$

where $a_i = A_i/p$ denotes the real asset balance, and $r = R - \pi$ ($\pi = \dot{p}/p$) denotes the real interest rate.

Household–firm units determine c_i, m_i, and π_i by maximizing the discounted present value of a stream of utility represented by $\int_0^\infty U(c_i, m_i, \ell_i, \pi_i)e^{-\rho t}dt$ (where $\rho > 0$ denotes

[5]The price revision costs can be interpreted as psychological stresses caused by price negotiations.

[6]Here, the price revision cost is specified in a quadratic equation consistent with that outlined by Rotemberg (1982).

the discount rate), subject to the restrictions of Equations (2), (3), (5), and $\pi = \dot{p}/p$. This creates a problem of intertemporal optimization (dynamic optimization).

For this problem, we set a current value Hamiltonian as follows:

$$\mathcal{H} = u(c_i, m_i) - \frac{1}{1+\psi}\left(\frac{1}{z}\left(\frac{p_i}{p}\right)^{-\phi} y\right)^{1+\psi} - \frac{\eta}{2}(\pi_i - \pi^*)^2$$
$$+ \mu_1 \left[(1-\tau)\frac{p_i}{p}\left(\frac{p_i}{p}\right)^{-\phi} y + r a_i - c_i - R m_i\right] + \mu_2 \pi_i p_i,$$

where μ_1 and μ_2 are the co-state variables of the state variables a_i and p_i, respectively. The first-order conditions for optimization can be obtained as follows:

$$\frac{\partial \mathcal{H}}{\partial c_i} = u_c(c_i, m_i) - \mu_1 = 0, \tag{6}$$
$$\frac{\partial \mathcal{H}}{\partial m_i} = u_m(c_i, m_i) - \mu_1 R = 0, \tag{7}$$
$$\frac{\partial \mathcal{H}}{\partial \pi_i} = -\eta(\pi_i - \pi^*) + \mu_2 p_i = 0, \tag{8}$$
$$\dot{\mu}_1 = \rho\mu_1 - \frac{\partial \mathcal{H}}{\partial a_i} = \rho\mu_1 - r\mu_1, \tag{9}$$
$$\dot{\mu}_2 = \rho\mu_2 - \frac{\partial \mathcal{H}}{\partial p_i} = \rho\mu_2 - \left(\frac{y_i}{z}\right)^{\psi}\frac{\phi}{z}\frac{y_i}{p_i} - \mu_1(1-\phi)(1-\tau)\frac{y_i}{p} - \mu_2\pi_i. \tag{10}$$

The second-order conditions are given as follows:

$$u_{cc} < 0;\ D \equiv u_{cc}u_{mm} - u_{cm}^2 > 0. \tag{11}$$

Furthermore, economically significant solutions would require satisfying the transversality conditions expressed as

$$\lim_{t\to\infty} a_i(t)e^{-\rho t} = 0,$$
$$\lim_{t\to\infty} p_i(t)e^{-\rho t} = 0.$$

As all household–firm units' behavior is based on the same equations (i.e., they are symmetric), we can drop subscript i from all variables. Further, we can derive the following equation from Equations (8) and (10):

$$\dot{\pi} = \rho(\pi - \pi^*) - \frac{\phi}{\eta} z^{-(1+\psi)} y^{1+\psi} + \frac{(\phi-1)(1-\tau)}{\eta}\mu_1 y, \tag{12}$$

which is referred to as the NK Phillips Curve (NKPC).

Further, solving Equation (7) for m, we obtain

$$\begin{aligned} & m = m(c, \mu_1, R); \qquad (13) \\ & m_c = \frac{\partial m}{\partial c} = -\frac{u_{cm}}{u_{mm}};\ u_\mu = \frac{\partial m}{\partial \mu_1} = \frac{R}{u_{mm}} < 0;\ m_R = \frac{\partial m}{\partial R} = \frac{\mu_1}{u_{mm}} < 0. \end{aligned}$$

Substituting this equation into Equation (6) and solving for c, we obtain

$$\begin{aligned} & c = c(\mu_1, R); \qquad (14) \\ & c_\mu = \frac{\partial c}{\partial \mu_1} = \frac{u_{mm} - u_{cm}R}{D};\ c_R = \frac{\partial c}{\partial R} = -\frac{u_{cm}\mu_1}{D}. \end{aligned}$$

2.3 Monetary policy

Interest rate rule

Monetary institutions manipulate the nominal interest rate according to fluctuations in the inflation rate as follows:

$$R = R(\pi);\ R'(\pi) > 0;\ R(\pi^*) = \bar{R}, \qquad (15)$$

where $\bar{R}$ represents a nominal interest rate that corresponds with the target inflation rate. Here, the target inflation rate shall be its steady-state value. Furthermore, $R'(\pi^*) > 1$ shall be an active monetary policy, and $R'(\pi^*) < 1$ shall be a passive monetary policy as per the terminology set forth in Leeper (1991).

Generalized interest rate rule

The generalized interest rate rule, which posits that the nominal interest rate responds to a weighted stream of inflation rates, can be represented as follows:

$$R(t) = R(\pi^g(t));\ \pi^g(t) \equiv \int_{-\infty}^{t} \delta(s)\pi(s)ds, \qquad (16)$$

where $\delta(s)$ is a weighting factor for inflation rate stream, $\int_{-\infty}^{t} \pi(s)ds$, and is defined as follows:

$$\delta(s) = \left(\frac{n}{\theta_1}\right)^n \frac{(t-s)^{n-1}}{(n-1)!} e^{-\frac{n}{\theta_1}(t-s)},$$

where n takes positive integer numbers, $\theta_1 > 0$, and $\int_{-\infty}^{t} \delta(s)ds = 1$. The mean of this function is given by θ_1 and the variance is given by θ_1^2/n.

When $n \to 1$, then $\delta(s)$ becomes an exponential function, $(1/\theta_1)e^{-(1/\theta_1)(t-s)}$, which implies that monetary authorities place the greatest emphasis on the present (see Fig. 1).[7] The backward-looking interest rate rule proposed by Benhabib, Schmitt-Grohé, and Uribe (2003) corresponds to this case, wherein θ_1 measures the degree to which the monetary authority is backward looking. When $n \geq 2$, $\delta(s)$ becomes a unimodal function that achieves maximum at $s = t - (n-1)\theta_1/n$, and when $n \to \infty$, it becomes a vertical line at $t - \theta_1$.[8]

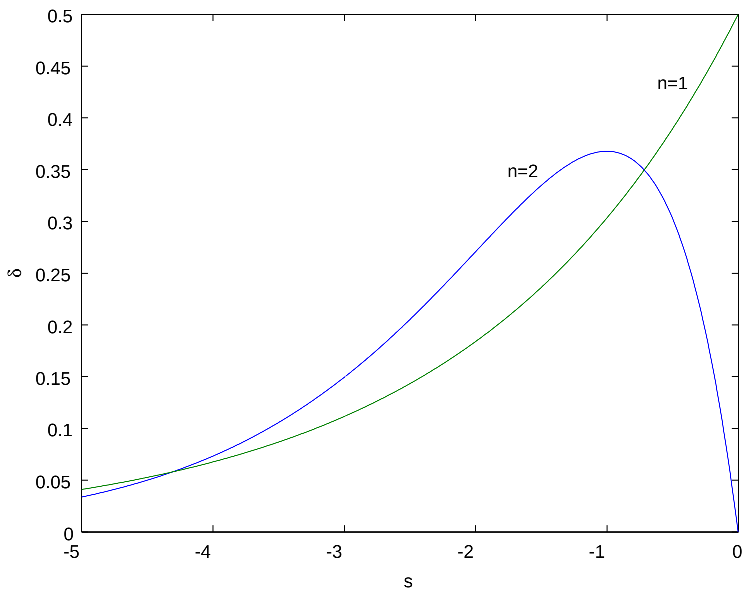

Figure 1: Function $\delta(s)$

Interest rate rule with a delay

When delays exist in the interest rate's response to fluctuations in the inflation rate, the interest rate rule can be represented as follows:

$$R(t) = R(\pi(t - \theta_1)), \tag{17}$$

[7]We assumed $\theta_1 = 2.0$ and $t = 0$.

[8]The standard interest rate rule in Equation (15) corresponds to the case where $n \to \infty$ and $\theta_1 \to 0$.

which corresponds to the case where $n \to \infty$ in the generalized interest rate rule. θ_1 represents delays in monetary policy.

2.4 Fiscal policy

The budget constraint equation for the public sector is provided as follows: $\dot{B} = RB - \dot{M} - \tau py + pg$, where g represents real government spending and is assumed to be constant. Rewriting this equation in real terms, we obtain

$$\dot{a} = ra - Rm - \tau y + g. \tag{18}$$

Tax rate rule

Financial institutions manipulating the income tax rate according to fluctuations in the total of real government liabilities, a, can be expressed as

$$\tau = \tau(a);\ \tau'(a) > 0;\ \tau(a^*) = \bar{\tau}, \tag{19}$$

where $\bar{\tau}$ is an income tax rate that corresponds with the target level of total government liabilities. Here, the target level of total government liabilities shall be its steady-state value. Substituting Equation (19) into Equation (18) and ignoring the responses of other endogenous variables to variations in a, the dynamic path of total government liabilities becomes locally stable if

$$\frac{r^*}{y^*} - \tau'(a^*) < 0.$$

Therefore, following the terminology of Leeper (1991), $\tau'(a^*) > r^*/y^*$ represents a passive fiscal policy and $\tau'(a^*) < r^*/y^*$ represents an active fiscal policy.

Tax rate rule with delay

When delays are present in financial institutions' responses to fluctuations in total government liabilities, the tax rate rule can be expressed as follows:

$$\tau(t) = \tau(a(t - \theta_2)), \tag{20}$$

where θ_2 represents delays in fiscal policy.

3 Cases with no policy lags

In this section, we consider local dynamics of a model economy with no policy lags. Using the equilibrium condition $y = c + g$ for the goods market, the model economy expressed in Equations (9), (12)–(15), and (19) can be summarized in the following three variable differential equation system:

$$\begin{aligned}
\dot{\mu}_1 &= [\rho - R(\pi) + \pi]\mu_1, \\
\dot{\pi} &= \rho(\pi - \pi^*) - \frac{\phi}{\eta} z^{-(1+\psi)}[c(\mu_1, R(\pi)) + g]^{1+\psi} + (1 - \tau(a))\frac{\phi - 1}{\eta}\mu_1[c(\mu_1, R(\pi)) + g], \\
\dot{a} &= [R(\pi) - \pi]a - R(\pi)m(c(\mu_1, R(\pi)), \mu_1, R(\pi)) - \tau(a)[c(\mu_1, R(\pi)) + g] + g.
\end{aligned} \tag{21}$$

The stationary point of System (21) can be expressed as (μ_1^*, π^*, a^*) that satisfies the simultaneous equations as follows:

$$\begin{aligned}
\pi^* &= \bar{R} - \rho, \\
\mu_1^*[c(\mu_1^*, \bar{R}) + g]^{-\psi} &= \frac{\phi}{(1 - \bar{\tau})(\phi - 1)} z^{-(1+\psi)}, \\
a^* &= \frac{\bar{R}m(c(\mu_1^*, \bar{R}), \mu_1^*, \bar{R}) + \bar{\tau}(c(\mu_1^*, \bar{R}) + g) - g}{\rho}.
\end{aligned} \tag{22}$$

The Jacobian matrix for System (21) evaluated at the stationary point is expressed as follows:

$$J_1 = \begin{bmatrix} 0 & -(R' - 1)\mu_1^* & 0 \\ P_1 & \rho - P_2 R' & -\tau' \frac{\phi - 1}{\eta}\mu_1^* y^* \\ P_3 & (R' - 1)a^* + P_4 R' & r^* - \tau' y^* \end{bmatrix},$$

where[9]

$$\begin{aligned}
P_1 &= -\psi \frac{\phi}{\eta} z^{-(1+\psi)}[c(\mu_1^*, \bar{R}) + g]^{\psi} c_\mu + \frac{(1 - \bar{\tau})(\phi - 1)}{\eta}[c(\mu_1^*, \bar{R}) + g], \\
P_2 &= \psi \frac{\phi}{\eta} z^{-(1+\psi)}[c(\mu_1^*, \bar{R}) + g]^{\psi} c_R, \\
P_3 &= -\bar{R}(m_c c_\mu + m_\mu) - \bar{\tau} c_\mu, \\
P_4 &= -[m(c(\mu_1^*, \bar{R}), \mu_1^*, \bar{R}) + \bar{R}(m_c c_R + m_R) + \bar{\tau} c_R].
\end{aligned}$$

[9] We used Equation (22) for the derivations of P_1 and P_2.

From Equation (14), we can see that the following relationship holds: $P_2 \gtreqless 0 \iff c_R \gtreqless 0 \iff u_{cm} \lesseqgtr 0$. As shown by Feenstra (1986) and Carlstrom and Fuerst (2003), money-in-the-utility-function (MIUF) model with a negative correlation between consumption and real money balances (i.e., $u_{cm} < 0$) is equivalent to a money-in-the-production-function (MIPF) model. Therefore, we can consider the case of $P_2 > 0$ as an equivalent case to an MIPF model.

The characteristic equation for System (21) can be expressed as

$$\Delta_1(\lambda) \equiv |\lambda I - J_1| = \lambda^3 + v_1\lambda^2 + v_2\lambda + v_3 = 0, \tag{23}$$

where

$$\begin{aligned}
v_1 &= -(\rho - P_2R') - (r^* - \tau' y^*),\\
v_2 &= P_1(R'-1)\mu_1^* + (\rho - P_2R')(r^* - \tau' y^*) + [(R'-1)a^* + P_4R']\tau'\frac{\phi-1}{\eta}\mu_1^* y^*,\\
v_3 &= -P_1(R'-1)\mu_1^*(r^* - \tau' y^*) - P_3(R'-1)\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^*.
\end{aligned}$$

Because c and π are both jump variables and a is a non-jump variable, the equilibrium can only be locally determinate when Equation (23) possesses exactly two roots with positive real parts.

We investigate the conditions for local determinacy. Recalling that $-v_3 = detJ_1$ equals the product of the roots, a necessary condition for determinacy can be provided by $v_3 > 0$. If $v_3 > 0$, then the signs of the roots therein will be expressed as $++-$ or $---$. In addition, if at least one of the conditions for Routh–Hurwitz stability[10] (which provides necessary and sufficient conditions for the real parts of all the roots to be negative: $v_1 > 0$; $v_2 > 0$; $v_3 > 0$; and $v_4 \equiv v_1v_2 - v_3 > 0$) is *not* satisfied, then we can identify the pattern of the signs as $++-$. In this case, the equilibrium is locally determinate.

We now focus on R' and τ', which indicate the responsiveness of monetary and fiscal policies, respectively. Define the sets V_1–V_4 as

$$\begin{aligned}
V_1 &= \{(R',\tau') : v_1 < 0\},\\
V_2 &= \{(R',\tau') : v_2 < 0\},\\
V_3 &= \{(R',\tau') : v_3 > 0\},\\
V_4 &= \{(R',\tau') : v_4 < 0\}.
\end{aligned}$$

[10]For example, refer to Chapter 18 in Gandolfo (2010).

Then, we can characterize the set (R', τ') that achieves local determinacy as follows:

$$Determinacy = \{V_1 \cup V_2 \cup V_4\} \cap V_3.$$

Numerical simulation

We now provide a numerical simulation. We specify the utility function as follows:

$$u(c, m) = \frac{(cm)^{1-\sigma} - 1}{1 - \sigma};\ \sigma > 0.$$

Furthermore, each parameter is set as follows in conjunction with those outlined in Benhabib, Schmitt-Grohé, and Uribe (2003) (quarterly data): $\phi = 21$; $\psi = 1$; $\sigma = 2$; $\eta = 350$; $\rho = 0.005$; and $\bar{R} = 0.015$. When $\sigma > 1$, $C_R < 0$ (hence, $P_2 > 0$) holds. Parameter σ represents the inverse of the elasticity of substitution for intertemporal consumption and is generally assumed to be greater than unity. Therefore, in the above numerical example, our model becomes equivalent to an MIPF model.

The steady-state value of the income tax rate shall be set to equal its average level of 0.2. In addition, g is set at 0.073 for the share of government expenditure in the national income, g/y^*, to match its realistic value of 0.19. Finally, the constant productivity z is set at unity. On these assumptions, we obtain the sets V_1–V_4 described on the R'-τ' plane as shown in Fig. 2.

The results implied by Fig. 2 are partly consistent with those of Kumhof, Nunes, and Yakadina (2010); i.e., a local equilibrium determinacy could be realized using a passive monetary policy ($R' < 1$) accompanied by an active fiscal policy ($\tau' < r^*/y^*$)[11]. However, unlike in Kumhof, Nunes, and Yakadina's (2010) model, local determinacy cannot be achieved when monetary policy is active ($R' > 1$) and fiscal policy is passive ($\tau' > r^*/y^*$). In this case, the signs of the real parts of all roots are negative. Thus, indeterminacy occurs.

Kumhof, Nunes, and Yakadina (2010) introduce money via a cash-in-advance constraint. This assumption creates a similar situation to an MIUF model that corresponds to the case of $P_2 < 0$ in our model. If we assume $\sigma < 1$ in the utility function, then $P_2 < 0$ holds. For example, assuming $\sigma = 0.9$ and describing the sets V_1–V_4 on the R'-τ' plane,

[11] In the case of distortionary taxes as the income tax, the dynamic system becomes indecomposable; i.e., the law of motion of a is affected by the other endogenous variables μ_1 and π. Hence, the bifurcation value of τ' (0.019) found in Fig. 2 slightly exceeds the value of $r^*/y^* = 0.013$.

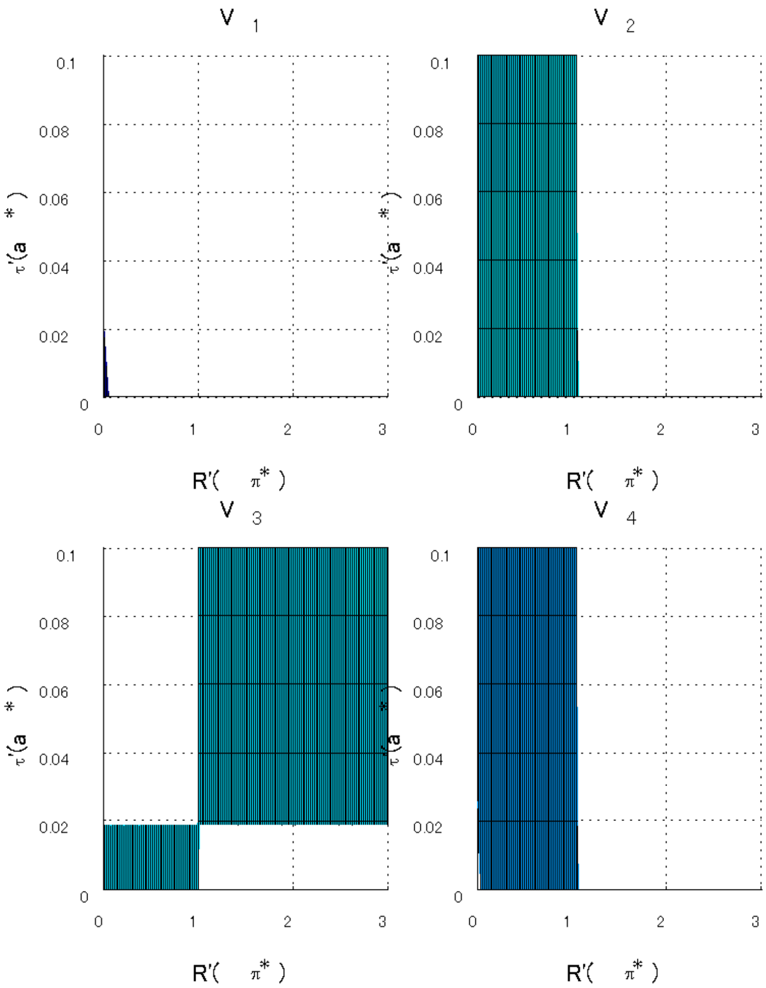

Figure 2: Case in which $\sigma = 2$

we obtain Fig. 3. This figure demonstrates that our result becomes completely consistent with that by Kumhof, Nunes, and Yakadina (2010).

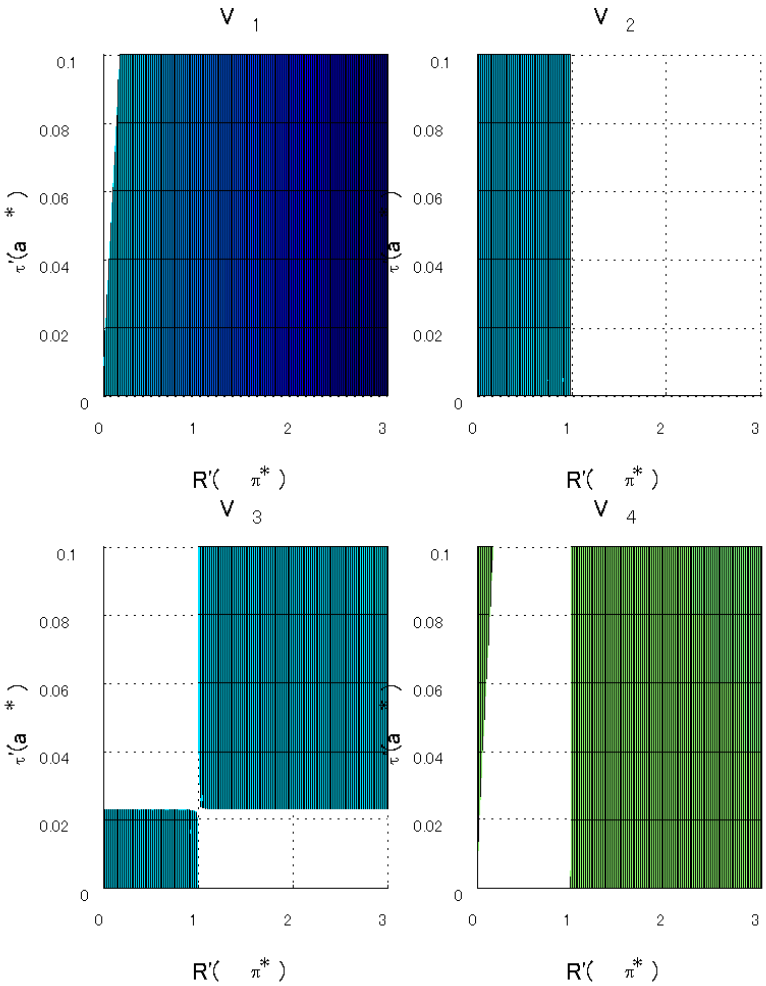

Figure 3: Case in which $\sigma = 0.9$

Furthermore, as suggested by the studies of Benhabib, Schmitt-Grohé, and Uribe (2003) and Carlstrom and Fuerst (2003), equilibrium indeterminacy can occur even under an active monetary policy in an MIPF model, regardless of whether it is a continuous time model or a discrete time model. Although the main contribution of the study of Benhabib, Schmitt-Grohé, and Uribe (2003) is the finding that a Hopf bifurcation occurs (which indicates the presence of *global* indeterminacy) in a continuous time NK model, we emphasize here the aspect that an active monetary policy and local indeterminacy are not mutually exclusive. This suggests that the case of Fig. 2 is considered as the correspondence of the models examined by Benhabib, Schmitt-Grohé, and Uribe (2003) and Carlstrom and Fuerst (2003).

Thus, we can state that the results derived from a discrete time model are consistent with those derived from a continuous time model without a policy lag. In the next section, we introduce monetary and fiscal policy lags into the model and reconsider the local equilibrium determinacy. Specifically, we show that monetary and financial authorities can avoid equilibrium indeterminacy by utilizing policy "delays." Therefore, in the following discussion, we restrict the analysis to the case where $-v_3 = detJ_1 < 0$.

4 Cases with policy lags

The model economy system when Equations (17) and (20) are used as the interest rate rule and the tax rate rule, respectively, is expressed as follows:

$$\begin{aligned}
\dot{\mu}_1(t) &= [\rho - R(\pi(t-\theta_1)) + \pi(t)]\mu_1(t), \\
\dot{\pi}(t) &= \rho(\pi(t) - \pi^*) - \frac{\phi}{\eta} z^{-(1+\psi)} [c(\mu_1(t), R(\pi(t-\theta_1))) + g]^{1+\psi} \\
&\quad + (1 - \tau(a(t-\theta_2)))\frac{\phi - 1}{\eta}\mu_1(t)[c(\mu_1(t), R(\pi(t-\theta_1))) + g], \\
\dot{a}(t) &= [R(\pi(t-\theta_1)) - \pi(t)]a(t) - R(\pi(t-\theta_1))m(c(\mu_1(t), R(\pi(t-\theta_1))), \mu_1(t), R(\pi(t-\theta_1))) \\
&\quad - \tau(a(t-\theta_2))[c(\mu_1(t), R(\pi(t-\theta_1))) + g] + g.
\end{aligned} \tag{24}$$

This is a two-delay differential equation system.

The steady-state values of System (24) are expressed as shown in Equation (22). By linearizing System (24) around the stationary point, the equations become as follows:

$$\begin{aligned}
\dot{\hat{\mu}}_1(t) &= -[R'\hat{\pi}(t-\theta_1) - \hat{\pi}(t)]\mu_1^*, \\
\dot{\hat{\pi}}(t) &= \rho\hat{\pi}(t) - P_2 R'\hat{\pi}(t-\theta_1) + P_1\hat{\mu}_1(t) - \tau'\frac{\phi-1}{\eta}\mu_1^*[c(\mu_1^*, \bar{R}) + g]\hat{a}(t-\theta_2), \\
\dot{\hat{a}}(t) &= P_3\hat{\mu}_1(t) - a^*\hat{\pi}(t) + (P_4 + a^*)\hat{\pi}(t-\theta_1) + r^*\hat{a}(t) - \tau'[c(\mu_1^*, \bar{R}) + g]\hat{a}(t-\theta_2),
\end{aligned} \tag{25}$$

where $\hat{\mu}_1(t) \equiv \mu_1(t) - \mu_1^*$, $\hat{\pi}(t) \equiv \pi(t) - \pi^*$, and $\hat{a}(t) \equiv a(t) - a^*$. Assuming the exponential functions $\hat{\mu}_1(t) = C_1 e^{\lambda t}$, $\hat{\pi}(t) = C_2 e^{\lambda t}$, and $\hat{a}(t) = C_3 e^{\lambda t}$ (where C_1, C_2, and C_3 are arbitrary constants, and λ is an eigenvalue) as the solutions to this system and plugging these functions into System (25), we can obtain the following:

$$\begin{bmatrix} \lambda & -\mu_1^* + R'\mu_1^* e^{-\theta_1\lambda} & 0 \\ -P_1 & \lambda - \rho + P_2 R' e^{-\theta_1\lambda} & \tau' \frac{\phi-1}{\eta}\mu_1^* y^* e^{-\theta_2\lambda} \\ -P_3 & a^* - (P_4 + a^*)R' e^{-\theta_1\lambda} & \lambda - r^* + \tau' y^* e^{-\theta_2\lambda} \end{bmatrix} \begin{bmatrix} \hat{\mu}_1(t) \\ \hat{\pi}(t) \\ \hat{a}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the matrix on the left side, which we denote as $\Delta_2(\lambda)$, must be zero for non-trivial solutions to exist in the above system: i.e.,

$$\Delta_2(\lambda) \equiv s_0(\lambda) + s_1(\lambda)e^{-\theta_1\lambda} + s_2(\lambda)e^{-\theta_2\lambda} + s_3(\lambda)e^{-(\theta_1+\theta_2)\lambda} = 0, \tag{26}$$

where

$$\begin{aligned} s_0(\lambda) &= \lambda^3 - (\rho + r^*)\lambda^2 + (\rho r^* - P_1\mu_1^*)\lambda + P_1\mu_1^* r^*, \\ s_1(\lambda) &= P_2R'\lambda^2 - P_2R'r^*\lambda + P_1R'\mu_1^*\lambda - P_1R'\mu_1^* r^*, \\ s_2(\lambda) &= \tau' y^*\lambda^2 - \rho\tau' y^*\lambda - a^*\tau'\frac{\phi-1}{\eta}\mu_1^* y^*\lambda + P_3\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^* - P_1\mu_1^*\tau' y^*, \\ s_3(\lambda) &= P_2R'\tau' y^*\lambda + (P_4 + a^*)R'\tau'\frac{\phi-1}{\eta}\mu_1^* y^*\lambda - P_3R'\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^* + P_1R'\mu_1^*\tau' y^*. \end{aligned}$$

Equation (26) is a characteristic equation for System (25).

It is well-known that Equation (26) possesses an infinite number of roots due to the existence of terms that include the exponential functions $e^{-\theta_1\lambda}$, $e^{-\theta_2\lambda}$, and $e^{-(\theta_1+\theta_2)\lambda}$.[12] In addition, unlike ordinary differential equations, delay differential equations require the initial conditions for the endogenous variables that are evaluated not only in time $t = t_0$ (time zero) but also in time $t_0 - \theta_1 \leq t \leq t_0$ or $t_0 - \theta_2 \leq t \leq t_0$. As $\mu_1(t)$ and $\pi(t)$ are the jump variables, their initial values should be determined by economic agents. However, the values they can determine in time t_0 are only $\mu_1(t_0)$ and $\pi(t_0)$. This is because it is impossible to go back and set the past values. Therefore, if there are exactly two roots that possess positive real parts among the infinite number of roots, then the initial conditions are uniquely determined; i.e., the equilibrium is locally determinate. However, the equilibrium will be indeterminate if there are less than two roots with positive real parts, and the equilibrium will become unstable if there are more than two roots with positive real parts (an equilibrium will not be present).

[12]Refer to Chapter 3 in Bellman and Cooke (1963).

A numerical method has been developed by Lin and Wang (2012) that can be used to investigate an equation that includes exponential functions, as in Equation (26).[13] The analysis performed hereafter adheres to Lin and Wang (2012).

4.1 Preconditions

First, to apply the Lin and Wang's (2012) method, some preconditions should be checked. According to their study, Equation (26) should satisfy the following conditions:

(i) $\deg(s_0(\lambda)) \geq \max\{\deg(s_1(\lambda)), \deg(s_2(\lambda)), \deg(s_3(\lambda))\}$;

(ii) $\Delta_2(0) \neq 0$;

(iii) A solution common to all four polynomials $s_0(\lambda) = 0$, $s_1(\lambda) = 0$, $s_2(\lambda) = 0$, and $s_3(\lambda) = 0$ does not exist (i.e., these are coprime);

(iv) $\lim_{\lambda\to\infty}(|s_1(\lambda)/s_0(\lambda)| + |s_2(\lambda)/s_0(\lambda)| + |s_3(\lambda)/s_0(\lambda)|) < 1$.

Condition (i) is satisfied by $3 > \max\{2, 2, 1\}$. Condition (ii) is also satisfied by $\Delta_2(0) = P_1\mu_1^*(r^* - \tau' y^*)(1 - R') + P_3\mu_1^{*2}\tau'\frac{\phi-1}{\eta}y^*(1 - R') = v_3 = -detJ_1 \neq 0$. Condition (iii) is numerically confirmed as follows: Solving equation $s_0(\lambda) = 0$ for λ, we obtain $\lambda = -0.103,\ 0.005,\ 0.108$. These values cannot be the roots of equation $s_3(\lambda) = 0$ for $\{(R', \tau') : R' \in (0, 3), \tau' \in (0, 0.1)\}$. Therefore, Condition (iii) is satisfied. Finally, Condition (iv) is satisfied by $\lim_{\lambda\to\infty}(|s_1(\lambda)/s_0(\lambda)| + |s_2(\lambda)/s_0(\lambda)| + |s_3(\lambda)/s_0(\lambda)|) = 0$.

Now, we examine the effects of lags (θ_1, θ_2) on local equilibrium determinacy. The procedure for the analysis is as follows:

(1) The points where pure imaginary roots appear, i.e., the points where dynamics can change, are characterized (if they are present).[14] These points are referred to as the crossing points.

(2) We describe the sets of the crossing points (called the crossing curves) on the $(\theta_1, \theta_2) \in \mathbb{R}_+^2$ plane using numerical simulation.

(3) We indicate the existence of regions where local determinacy is achieved.

[13] The merit of Lin and Wang's (2012) method is that it is applicable to the case where an equation includes not only the exponential functions as $e^{-\theta_1\lambda}$ and $e^{-\theta_2\lambda}$ but also the function as $e^{-(\theta_1+\theta_2)\lambda}$. If $s_3(\lambda) = 0$, then we can use a different method developed by Gu, Niculescu, and Chen (2005).

[14] It is ensured from Condition (ii) that a zero real root cannot appear.

4.2 Crossing curves

We denote a pure imaginary root as $\lambda = i\omega$ (where ω = imaginary part[15] > 0, and $i = \sqrt{-1}$). Plugging this expression into Equation (26), we obtain the following:

$$\Delta_2(i\omega) = s_0(i\omega) + s_1(i\omega)e^{-i\omega\theta_1} + s_2(i\omega)e^{-i\omega\theta_2} + s_3(i\omega)e^{-i\omega(\theta_1+\theta_2)} = 0. \tag{27}$$

We first characterize the values of ω that satisfy Equation (27). According to Lemma 3.2 in Lin and Wang (2012), $\Delta_2(i\omega) = 0$ holds for $\omega \in \mathbb{R}_+$ that satisfy the following:

$$F(\omega) \equiv (|s_0|^2 + |s_1|^2 - |s_2|^2 - |s_3|^2)^2 - 4(M_1^2 + N_1^2) < 0,$$

where

$$\begin{aligned} M_1(\omega) &= \mathrm{Re}(s_2\bar{s}_3) - \mathrm{Re}(s_0\bar{s}_1), \\ N_1(\omega) &= \mathrm{Im}(s_2\bar{s}_3) - \mathrm{Im}(s_0\bar{s}_1). \end{aligned}$$

We denote the set of $\omega > 0$ that satisfies condition $F(\omega) < 0$ as Ω (crossing set). For $\omega \in \Omega$, the sets (θ_1, θ_2) satisfying Equation (27) (crossing points) can be expressed as follows (Equation 17 in Lin and Wang 2012):

$$\begin{aligned} \Theta^{\pm} &\equiv \{(\theta_1^{\pm}(\omega), \theta_2^{\mp}(\omega)) \in \mathbb{R}_+^2\} \\ &= \left\{ \left(\frac{\pm\delta_1(\omega) - \varphi_1(\omega) + 2n_1\pi}{\omega}, \frac{\mp\delta_2(\omega) - \varphi_2(\omega) + 2n_2\pi}{\omega} \right);\ n_1, n_2 \in \mathbb{Z} \right\}, \end{aligned} \tag{28}$$

[15] We can assume that $\omega > 0$ without losing generality because the pure imaginary roots will always be conjugated.

where

$$
\begin{aligned}
\delta_1(\omega) &= \cos^{-1}\left(\frac{|s_0|^2+|s_1|^2-|s_2|^2-|s_3|^2}{2\sqrt{M_1^2+N_1^2}}\right);\ \delta_1 \in [0,\pi],\\
\varphi_1(\omega) &= \arg\{s_2\bar{s}_3 - s_0\bar{s}_1\}\\
&= \tan^{-1}\left(\frac{\operatorname{Im}(s_2\bar{s}_3 - s_0\bar{s}_1)}{\operatorname{Re}(s_2\bar{s}_3 - s_0\bar{s}_1)}\right),\\
\delta_2(\omega) &= \cos^{-1}\left(\frac{|s_0|^2-|s_1|^2+|s_2|^2-|s_3|^2}{2\sqrt{M_2^2+N_2^2}}\right);\ \delta_2 \in [0,\pi],\\
M_2(\omega) &= \operatorname{Re}(s_1\bar{s}_3) - \operatorname{Re}(s_0\bar{s}_2),\\
N_2(\omega) &= \operatorname{Im}(s_1\bar{s}_3) - \operatorname{Im}(s_0\bar{s}_2),\\
\varphi_2(\omega) &= \arg\{s_1\bar{s}_3 - s_0\bar{s}_2\}\\
&= \tan^{-1}\left(\frac{\operatorname{Im}(s_1\bar{s}_3 - s_0\bar{s}_2)}{\operatorname{Re}(s_1\bar{s}_3 - s_0\bar{s}_2)}\right).
\end{aligned}
$$

Lin and Wang (2012) also demonstrate that Θ^+ and Θ^- form a class of continuous curves on $\mathbb{R}^2_+$. We call these curves crossing curves. In the next section, we illustrate an example of crossing curves using a numerical simulation.

4.3 Numerical simulation

We assume the same parameter values and functional form of the utility function as in the previous section. Furthermore, we suppose that monetary authorities implement an active policy ($R' = 1.5$) and fiscal authorities implement a passive policy ($\tau' = 0.09$). Then, if monetary and fiscal policy lags are not present (i.e., $\theta_1 = \theta_2 = 0$), indeterminacy occurs, wherein the signs of the roots are $---$.

The crossing set Ω is given by $\omega \in (0.126, 0.460)$ (Fig. 4). For $\omega \in \Omega$, we can describe Θ^+ and Θ^- as shown in Fig. 5. The solid curves represent Θ^+, and the dashed curves represent Θ^-.

We call the direction of the curve corresponding to increasing ω as the positive direction. When we move in the positive direction along curves Θ^+ (Θ^-), the region on the left of Θ^+ (Θ^-) has two more (less) roots with positive real parts (Theorem 4.1 in Lin and Wang 2012). Considering this, we use arrows to indicate the crossing directions to which roots with positive real parts increase when lags (θ_1, θ_2) intersect with these curves (Fig. 5). The region on the end of an arrow has two more roots with positive real parts.

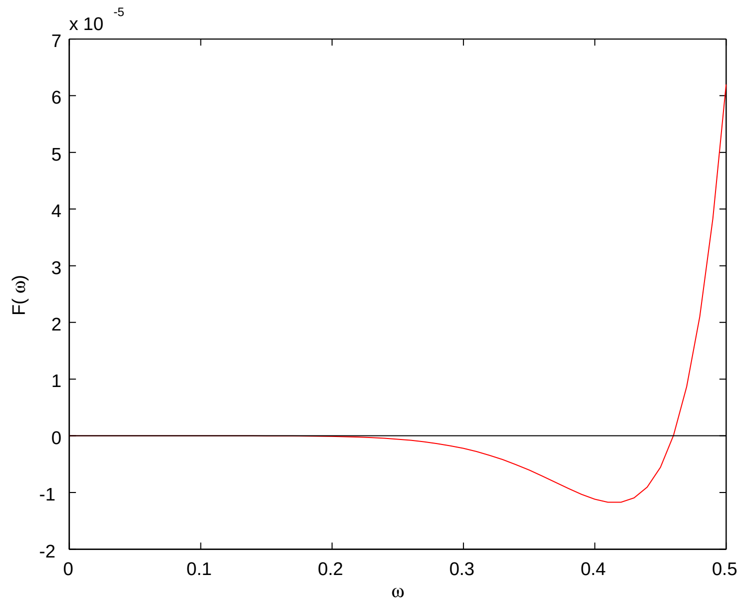

Figure 4: Graph of $F(\omega)$

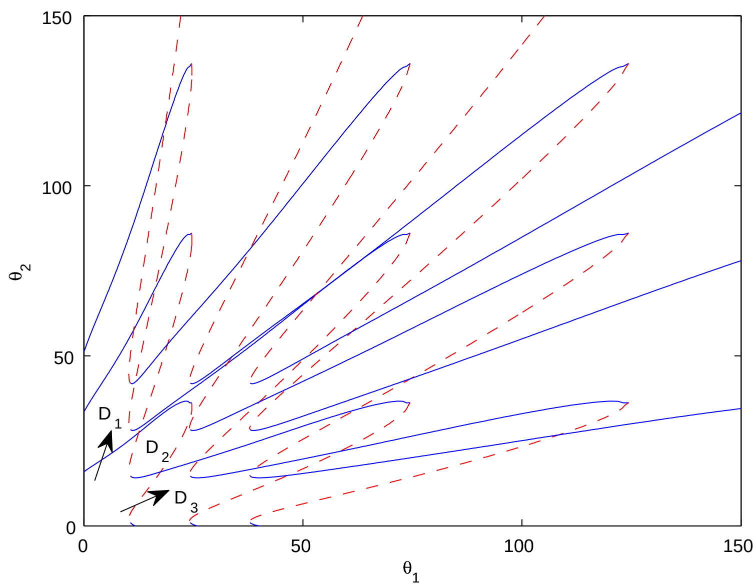

Figure 5: Crossing curves

In the three regions indicated by D_1, D_2, and D_3 in Fig. 5, exactly two roots with positive real parts exist: therefore, the equilibrium is locally determinate. Accordingly, we can make the following proposition:

Proposition 4.1 Monetary and fiscal authorities can establish local equilibrium determinacy by introducing lags into policy responses when the equilibrium is indeterminate and where the signs of the roots are $---$ under policies without lags.

This proposition indicates that "desirable lags" of monetary and fiscal policies exist. If lags are too short, then equilibrium indeterminacy will occur, and if they are too long, the equilibrium will become unstable.

Furthermore, under any configuration of crossing curves, as the number of roots that possess positive real parts changes by two at a time when θ_1 or/and θ_2 intersect a crossing curve, it is impossible for monetary and fiscal authorities to achieve local determinacy by policy lags if the signs of the roots are $+--$ under policies without a lag.

5 Conclusion

In this report, we considered how policy lags can affect local equilibrium determinacy by developing a continuous time NK model that introduces time lags in monetary and fiscal policy responses. We demonstrated that not only the combination of monetary and fiscal policies' activeness but also the timings of their implementation play an important role in achieving local determinacy.

On the assumption of a plausible parameter set, indeterminacy can arise under the combination of active monetary policy and passive fiscal policy if no delays occur in policy implementation by monetary and fiscal authorities. However, such indeterminacy can be resolved by setting lags in the desirable regions, as indicated by D_1–D_3 in Fig. 5.

According to Friedman (1948), there are three types of general policy lags, i.e., recognition, implementation, and diffusion lag. Unlike recognition lag and diffusion lag, it may be possible to purposefully adjust the amount of implementation lag to a certain degree. Thus, the research in this paper suggests that adjusting the timing of policy implementation may be used as a method for a policy aimed at stabilization.

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