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Abstract

In this paper, we give a necessary and sufficient condition under which the Shapley value belongs to, or coincides with another solution concept which satisfies Weak Strategic Invariance. Our approach is based on the linear basis which has the following property: when we express a game by a linear combination of the linear basis, the Shapley value appears in the coefficients. We mainly discuss the relationship between the Shapley value and the Core, the prenucleolus.

JEL classification: C71
Keywords: Cooperative games; Shapley value; Relationship among solutions

1 Introduction

The purpose of this paper is to clarify the relationship between the Shapley value and other solution concepts. Regarding this approach, the relationship between the Shapley value and the Core has mainly been discussed. For instance, Hoffmann and Sudhölter (2007) proved that the the Shapley value belongs to the Core in exact assignment games. This result can be interpreted as one of the sufficient conditions under which the Shapley value belongs to the Core. The research about the condition can be traced back to Shapley (1971), who proved that the Shapley value belongs to the Core in convex games. Since the convexity of a game is not a necessary condition, many variations of convex games have also been proposed by weakening the requirement. Iñarra and Usategui (1993) defined average convex games (which include convex games) and partially average convex games (which include average convex games) and proved that the Shapley value belongs to the Core in both class of games. Izawa and Takahashi (1998) defined totally convex games (which include average convex games), and proved that their requirement was necessary and sufficient condition. We also provide a necessary and sufficient condition, but our result is an extension of the previous works. In Theorem 3, we give a necessary and sufficient condition under which the Shapley value belongs to, or coincides with another solution concept which satisfies Weak Strategic Invariance.

Weak Strategic Invariance is a very weak condition, and we can deal with many solution concepts. In this paper, we mainly focus on the Core and the prenucleolus. In Theorem 5, we show that, in the game where the Shapley value does not belong to the Core, when the Shapley value first belongs to the Core if the worth of the grand coalition alone continues to increase.

Regarding the coincidence of the Shpaley value and the prenucleolus, the result of this paper will be clear in Corollary 2. We give the necessary and sufficient condition of the coincidence for 3-person game as the condition of the worths of coalitions. In particular, we show that the set of all 0-normalized games where the Shapley value and the prenucleolus coincide is equivalent to the union of all symmetric games and the set of all games satisfying PS property, which was introduced by Kar, Mitra and Mutuswami (2009).

This paper is organized as follows. Section 2 contains notations and definitions. In Section 3, we give the definition of a linear basis of the set of games introduced by Yokote (2013), which plays a crucial role in the following sections. In Section 4, we focus on the relationship between the Shapley value and other solution concepts by using the linear basis. Section 5 gives concluding remarks.

2 Notations and Definitions

2.1 Game and Coalition

For any two sets A and B, $A \subset B$ means that A is a proper subset of B. $A \subseteq B$ means that $A \subset B$ or $A = B$. $|A|$ denote the cardinality of A. Let $N \subset \mathbb{N}$ be a finite set of players, and let $S \subseteq N$ be a coalition of N. We define $|N| = n$. The characteristic function $v : 2^N \to \mathbb{R}$ assigns a real number to each coalition of N, and satisfies $v(\emptyset) = 0$. $v(S)$ can be considered to be the worth of a coalition. The pair (N, v) is called a game, and the set of all games is denoted as Γ. For any $(N, v) \in \Gamma$, let $(S, v), S \subseteq N, S \neq \emptyset$ denote the restriction of (N, v) on S. For any $(N, v), (N, w) \in \Gamma$, we define the sum of games $(N, v + w) \in \Gamma$ as follows: $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N, S \neq \emptyset$. A game $(N, v) \in \Gamma$ is convex if the following property is satisfied: $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. A game $(N, v) \in \Gamma$ is simple if the following condition is satisfied: $v(S) = 0$ or 1 for all $S \subseteq N$. A game (N, v) is 0-normalized if $v(\{i\}) = 0$ for all $i \in N$.

Let $(N, v) \in \Gamma$ and $x \in \mathbb{R}^n$. Suppose that each coordinate $i, i = 1, \ldots, n$, of x corresponds to the amount player i receives in x. Then, we define the excess of coalition S with respect to x as follows: $e(S, x) = v(S) - \sum_{i \in S} x_i$. We say that the family of coalitions $\mathscr{S} \subseteq 2^N, \emptyset \notin \mathscr{S}$ is balanced if the following condition is satisfied: there exists a collection of positive numbers $(\delta_S)_{S \subseteq N, S \neq \emptyset}$ such that

$$\sum_{S: i \in S, S \in \mathscr{S}} \delta_S = 1,$$

for all $i \in N$.

2.2 Value, Solution and Axiom

Let (N, v) be a game. We define the preimputation set as follows:

$$X(N, v) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)\}.$$

A value σ on Γ prescribes an element of $X(N, v)$ to each game $(N, v) \in \Gamma$. A solution ψ on Γ prescribes a subset of $X(N, v)$ to each game $(N, v) \in \Gamma$.

We define three values on Γ. The Shapley value ϕ, introduced by Shapley (1953), is defined as follows:

$$\phi_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{(n - |S|)!(|S| - 1)!}{n!} \Big(v(S) - v(S \backslash \{i\}) \Big),$$

for all $i \in N$. Note that the Shapley value is a linear function. Namely, $\phi(N, \alpha v + \beta w) = \alpha \phi(N, v) + \beta \phi(N, w)$ for all $(N, v), (N, w) \in \Gamma$, $\alpha, \beta \in \mathbb{R}$.

The dividend of $(N, v) \in \Gamma$, introduced by Harsanyi (1959), is defined as follows:

$$D(S, v) = \sum_{k=0}^{|S|-1} (-1)^k \sum_{T \subseteq S: |S| - |T| = k} v(T),$$

for all $S \subseteq N, S \neq \emptyset$. The Shapley value can be calculated by using the dividend.

$$\phi_i(N,v) = \sum_{T \subseteq N : i \in T} \frac{1}{|T|} D(T,v),$$

for all $i \in N$.

For any two vectors $x, y \in \mathbb{R}^n$, $y \geq_{lex} x$ means that y is greater than x in the lexicographic ordering of $\mathbb{R}^n$. Let $\theta(x) = (\theta_1(x), \theta_2(x), \ldots, \theta_{2^n-2}(x)) \in \mathbb{R}^{2^n-2}$ denote the sequence of excess of $S \subset N, S \neq \emptyset$ with respect to x, where $\theta_t(x) \geq \theta_{t+1}(X)$ for all $t, 1 \leq t \leq 2^n - 3$. By dropping the individual rationality of the Nucleolus introduced by Schmeidler (1969), the prenucleolus can be defined as follows:

$$PN(N,v) = \{x \in X(N,v) : \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in X(N,v)\}.$$

It is known that the set $\{x \in X(N,v) : \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in X(N,v)\}$ is non-empty and singleton, so we can identify the prenucleolus as a value on Γ.

For any game (N,v), we define

$$m_i(N,v) := v(N) - v(N \backslash \{i\}),$$

and

$$b(N,v) := \frac{v(N) - \sum_{i \in N} m_i}{n}.$$

The ENSC value was defined by Driessen and Funaki (1991).

$$ENSC_i(N,v) = m_i(N,v) + b(N,v),$$

for all $i \in N$.

We give one solution defined on Γ. The Core is defined as follows:

$$C(N,v) = \Big\{x \in X(N,v) : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N, S \neq \emptyset\Big\}.$$

A game (N,v) is called balanced if $C(N,v) \neq \emptyset$.

For any game $(N,v) \in \Gamma$ and $\beta \in \mathbb{R}^n$, we define the game $(N, v+\beta) \in \Gamma$ as follows:

$$(v+\beta)(S) = v(S) + \sum_{i \in S} \beta_i,$$

for all $S \subseteq N, S \neq \emptyset$. A value σ on Γ satisfies Weak Strategic Invariance if the following condition is satisfied:

$$\sigma(N, v+\beta) = \sigma(N,v) + \beta,$$

for all $(N, v) \in \Gamma$ and $\beta \in \mathbb{R}^n$. A solution ψ on Γ satisfies Weak Strategic Invariance if the following condition is satisfied:

$$\psi(N, v + \beta) = \{x + \beta : x \in \psi(v)\} \text{ if } \psi(N, v) \neq \emptyset,$$
$$\psi(N, v + \beta) = \emptyset \text{ if } \psi(N, v) = \emptyset,$$

for all $(N, v) \in \Gamma$ and $\beta \in \mathbb{R}^n$.

3 Linear basis, coefficients and the dividend

In this section, we give the definition of a new linear basis of the set of games with player set N. The most famous one is unanimity games $(u_S)_{S \subseteq N, S \neq \emptyset}$, which were first introduced by Shapley (1953).

$$u_S(T) = \begin{cases} 1 \text{ if } S \subseteq T, \\ 0 \text{ otherwise.} \end{cases}$$

Here, we define the one-join game for some coalition $S \subseteq N, S \neq \emptyset$ as follows:

$$\bar{u}_S(T) = \begin{cases} 1 \text{ if } |T \cap S| = 1, \\ 0 \text{ otherwise.} \end{cases}$$

This game was introduced by Yokote (2013).[1] Let us interpret and compare the above two games. First, take any game $(N, v) \in \Gamma$ and coalition $S \subseteq N, S \neq \emptyset$, and suppose that players in $N \backslash S$ live in a region. There is a pie which yields a payoff of 1 in the region, but players in $N \backslash S$ cannot get the pie (namely, they are null players). The players S are trying to go to the region and get the pie. Then, unanimity game u_S captures the situation in which players in S can get the pie if and only if all players in S enter the region. On the other hand, one-join game $\bar{u}_S$ captures the situation in which a player in S can get the pie if and only if he is the first player who enters the region. The cooperation of players in $N \backslash S$ has nothing to do with getting the pie.

The following theorem was proved by Yokote (2013).[2]

Theorem 1 (Lemma 3 of Yokote (2013)) *Let N be a set of players. Then, the set of games $(\bar{u}_S)_{S \subseteq N, S \neq \emptyset}$ is a linear basis of the set of all games with player set N.*

[1] From the definition, one-join game is a simple game.

[2] Although Yokote (2013) proved the theorem by using the weighted version of the linear basis, the result here can be easily obtained by letting $\omega = (1, \cdots, 1)$.

From Theorem 1, any game $(N, v) \in \Gamma$ can be expressed by a linear combination of $(\bar{u}_S)_{S \subseteq N, S \neq \emptyset}$. Let $d_N(S, v)$ be the coefficient of $\bar{u}_S, S \subseteq N, S \neq \emptyset$ in the linear combination. Namely,

$$v = \sum_{S \subseteq N : S \neq \emptyset} d_N(S, v) \bar{u}_S. \tag{1}$$

Let $d_{N,v} \in \mathbb{R}^{2^n - 1}$ denote the column vector whose coordinate is $d_N(S, v), S \subseteq N, S \neq \emptyset$. We can rewrite equation (1) in the matrix form. Let $\bar{Q}_N$ denote the $(2^n - 1, 2^n - 1)$ matrix whose column vector is $\bar{u}_S, S \subseteq N, S \neq \emptyset$. Then,

$$v = \bar{Q}_N d_{N,v}. \tag{2}$$

We can easily prove that $\bar{Q}_N$ is a symmetric matrix.

Example 1 We give an example of $\bar{Q}_N$ for 3-person game, $N = \{1, 2, 3\}$.

	$\bar{u}_{\{1\}}$	$\bar{u}_{\{2\}}$	$\bar{u}_{\{3\}}$	$\bar{u}_{\{12\}}$	$\bar{u}_{\{13\}}$	$\bar{u}_{\{23\}}$	$\bar{u}_N$
$\{1\}$	1	0	0	1	1	0	1
$\{2\}$	0	1	0	1	0	1	1
$\{3\}$	0	0	1	0	1	1	1
$\{1, 2\}$	1	1	0	0	1	1	0
$\{1, 3\}$	1	0	1	1	0	1	0
$\{2, 3\}$	0	1	1	1	1	0	0
N	1	1	1	0	0	0	0

And the example of $d_{N,v}$ is given as follows.

$$d_{N,v} = \begin{pmatrix} d_N(\{1\}, v) \\ d_N(\{2\}, v) \\ d_N(\{3\}, v) \\ d_N(\{1, 2\}, v) \\ d_N(\{1, 3\}, v) \\ d_N(\{2, 3\}, v) \\ d_N(N, v) \end{pmatrix}.$$

■

The reason why we focus on this linear basis is that the coefficients of $\bar{u}_S, |S| = 1$, is precisely the Shapley value.

Theorem 2 (Theorem 2 of Yokote, Funaki and Kamijo (2013)) *Let (N, v) be a game. Then, we have*

$$d_N(S, v) = (-1)^{|S|-1} \sum_{T \subseteq N : S \subseteq T} \frac{1}{|T|} D(T, v),$$

for all $S \subseteq N, S \neq \emptyset$. In particular,

$$d_N(\{i\}, v) = \sum_{T \subseteq N : i \in T} \frac{1}{|T|} D(T, v) = \phi_i(N, v),$$

for all $i \in N$.

Note that we can calculate the coefficients $d_N(S, v), S \subseteq N, S \neq \emptyset$ by following the same calculation of the Shapley value.

4 Relationship between the Shapley value and other solution concepts

In this section, we focus on the relationship between the Shapley value and other solution concepts which satisfy Weak Strategic Invariance. We first introduce some additional notations.

For any $(N, v) \in \Gamma$, let $d^0_{N,v} \in \mathbb{R}^{2^n-1}$ denote the column vector whose i-th coordinate, $1 \leq i \leq n$, is 0 and whose j-th coordinate, $n+1 \leq j \leq 2^n - 1$, is equal to $d_N(S, v)$ for some $S \subseteq N, |S| \geq 2$. We define[3]

$$v^{Sh}_N := \bar{Q}_N d^0_{N,v}.$$

Similarly, let $f_{N,v} \in \mathbb{R}^{2^n-1}$ denote the column vector whose i-th coordinate, $1 \leq i \leq n$, is $\phi_i(N, v)$ and whose j-th coordinate, $n+1 \leq j \leq 2^n - 1$, is 0. Then, together with equation (2), we have

$$\begin{aligned} v &= \bar{Q}_N d_{N,v} \\ &= \bar{Q}_N (d^0_{N,v} + f_{N,v}) \\ &= \bar{Q}_N d^0_{N,v} + \bar{Q}_N f_{N,v}. \end{aligned}$$

Example 2 If we use the same 3-person game of Example 1, then

$$d^0_{N,v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_N(\{1,2\}, v) \\ d_N(\{1,3\}, v) \\ d_N(\{2,3\}, v) \\ d_N(N, v) \end{pmatrix}, f_{N,v} = \begin{pmatrix} \phi_1(N, v) \\ \phi_2(N, v) \\ \phi_3(N, v) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

[3]Take any $(N, v) \in \Gamma$, and we remark the definition of v^{Sh}_S for $S \subset N, S \neq \emptyset$. The game is given as follows: first, express the game (S, v) by a linear combination of $(\bar{u}_T)_{T \subseteq S, T \neq \emptyset}$. Second,let the coefficients of $\bar{u}_T, |T| = 1$ be 0. So, v^{Sh}_S is not a restriction of v^{Sh}_N on S.

Note that the worth of each coalition is given as follows.

$$\begin{aligned}
v(\{1\}) &= \phi_1(N,v) + d_N(\{1,2\},v) + d_N(\{1,3\},v) + d_N(N,v),\\
v(\{2\}) &= \phi_2(N,v) + d_N(\{1,2\},v) + d_N(\{2,3\},v) + d_N(N,v),\\
v(\{3\}) &= \phi_3(N,v) + d_N(\{1,3\},v) + d_N(\{2,3\},v) + d_N(N,v),\\
v(\{1,2\}) &= \phi_1(N,v) + \phi_2(N,v) + d_N(\{1,3\},v) + d_N(\{2,3\},v),\\
v(\{1,3\}) &= \phi_1(N,v) + \phi_3(N,v) + d_N(\{1,2\},v) + d_N(\{2,3\},v),\\
v(\{2,3\}) &= \phi_2(N,v) + \phi_3(N,v) + d_N(\{1,2\},v) + d_N(\{1,3\},v),\\
v(N) &= \phi_1(N,v) + \phi_2(N,v) + \phi_3(N,v).
\end{aligned}$$

If we omit $\phi_i(N,v), i \in N$ from the equations above, then we have the game $\bar{Q}_N d^0_{N,v}$. On the other hand, if we omit $d_N(S,v), S \subseteq N, S \neq \emptyset$, then we have the game $\bar{Q}_N f_{N,v}$. ∎

Since $\bar{Q}_N f_{N,v} = \sum_{i \in N} \phi_i(N,v) \cdot u_{\{i\}}$, we have $v = v^{Sh}_N + \phi(N,v)$. Now, take any value σ which satisfies Weak Strategic Invariance. Weak Strategic Invariance implies

$$\begin{aligned}
\sigma(N,v) &= \sigma(N, v^{Sh}_N + \phi(N,v))\\
&= \sigma(N, v^{Sh}_N) + \phi(N,v). \qquad (3)
\end{aligned}$$

We can apply the same argument for any solution ψ which satisfies Weak Strategic Invariance. If $\psi(N, v^{Sh}_N) \neq \emptyset$, then we have

$$\psi(N,v) = \{x + \phi(N,v) : x \in \psi(N, v^{Sh}_N)\}. \qquad (4)$$

The above two equations (3) and (4) lead us to the following theorem.

Theorem 3 *Let σ be a value on Γ which satisfies Weak Strategic Invariance. Then,*

$$\phi(N,v) = \sigma(N,v) \text{ if and only if } \sigma(N, v^{Sh}_N) = \mathbf{0}.$$

Let ψ be a solution on Γ which satisfies Weak Strategic Invariance. Then,

$$\phi(N,v) \in \psi(N,v) \text{ if and only if } \mathbf{0} \in \psi(N, v^{Sh}_N).$$

Proof. From equation (3), $\phi(N,v) = \sigma(N,v)$ if and only if $\sigma(N, v^{Sh}_N) = \mathbf{0}$. The same argument holds for a solution ψ if we use equation (4). □

Note that (N, v^{Sh}_N) is the game which is strategically equivalently transformed with respect to the Shapley value. In other words, (N, v^{Sh}_N) is the excess game of (N,v) at $\phi(N,v)$. The point is that, from Theorem 2, we can get the game v^{Sh}_N without calculating the Shapley value.

4.1 The Shapley value and the Core

Let $\Gamma_{SC} := \{(N,v) \in \Gamma : \phi(N,v) \in C(N,v)\}$. First, we give a variation of Theorem 3.

Theorem 4 *$(N,v) \in \Gamma_{SC}$ if and only if*

$$v_N^{Sh}(S) \leq 0,$$

for all $S \subseteq N$.

The proof is obvious from the definition of the Core and Theorem 3.

We apply this theorem and give a corollary for 0-normalized 3-person game.

Corollary 1 *Let $(N,v), N = \{1,2,3\}$, be a 0-normalized balanced game which satisfies $v(N) \geq 0$. Then, $\phi(N,v) \in C(N,v)$ if and only if*

$$\begin{aligned} v(\{1,2\}) + v(\{1,3\}) &\leq 4\big(v(N) - v(\{2,3\})\big), \\ v(\{1,2\}) + v(\{2,3\}) &\leq 4\big(v(N) - v(\{1,3\})\big), \\ v(\{1,3\}) + v(\{2,3\}) &\leq 4\big(v(N) - v(\{1,2\})\big). \end{aligned}$$

Proof. The condition of Theorem 4 is equivalent to

$$\bar{Q}_N d^0_{N,v} = \begin{pmatrix} 1001101 \\ 0101011 \\ 0010111 \\ 1100110 \\ 1011010 \\ 0111100 \\ 1110000 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_N(\{1,2\},v) \\ d_N(\{1,3\},v) \\ d_N(\{2,3\},v) \\ d_N(N,v) \end{pmatrix} \leq \mathbf{0}.$$

Since the first three columns of $\bar{Q}_N$ are irrelevant, by omitting them, and by multiplying 6 to $d^0_{N,v}$, we have

$$\begin{pmatrix} 1101 \\ 1011 \\ 0111 \\ 0110 \\ 1010 \\ 1100 \\ 0000 \end{pmatrix} \begin{pmatrix} d_N(\{1,2\},v) \\ d_N(\{1,3\},v) \\ d_N(\{2,3\},v) \\ d_N(N,v) \end{pmatrix} \leq \mathbf{0}. \tag{5}$$

This condition is equivalent to

$$\begin{aligned}
&d_N(\{1,2\},v) + d_N(\{1,3\},v) + d_N(N,v) \leq 0,\\
&d_N(\{1,2\},v) + d_N(\{2,3\},v) + d_N(N,v) \leq 0,\\
&d_N(\{1,3\},v) + d_N(\{2,3\},v) + d_N(N,v) \leq 0,\\
&d_N(\{1,3\},v) + d_N(\{2,3\},v) \leq 0,\\
&d_N(\{1,2\},v) + d_N(\{2,3\},v) \leq 0,\\
&d_N(\{1,2\},v) + d_N(\{1,3\},v) \leq 0.
\end{aligned} \tag{6}$$

From Theorem 2, we have

$$d_N(N,v) = -\frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{3} - \frac{v(\{2,3\})}{3} + \frac{v(N)}{3}. \tag{7}$$

Since (N,v) is balanced, from the Bondareva-Shapley theorem, we have

$$-v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + 2v(N) \leq 0.$$

Since $v(N) \geq 0$ from the assumption, we have

$$-v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(N) \leq 0,$$

which implies $d_N(N,v) \leq 0$ from equation (7). As a result, the condition of (10) is equivalent to

$$\begin{aligned}
&d_N(\{1,3\},v) + d_N(\{2,3\},v) \leq 0,\\
&d_N(\{1,2\},v) + d_N(\{2,3\},v) \leq 0,\\
&d_N(\{1,2\},v) + d_N(\{1,3\},v) \leq 0.
\end{aligned} \tag{8}$$

From Theorem 2, we have

$$d_N(\{1,2\},v) = -\frac{v(\{1,2\})}{6} + \frac{v(\{1,3\})}{3} + \frac{v(\{2,3\})}{3} - \frac{v(N)}{3}, \tag{9}$$

$$d_N(\{1,3\},v) = \frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{6} + \frac{v(\{2,3\})}{3} - \frac{v(N)}{3}, \tag{10}$$

$$d_N(\{2,3\},v) = \frac{v(\{1,2\})}{3} + \frac{v(\{1,3\})}{3} - \frac{v(\{2,3\})}{6} - \frac{v(N)}{3}. \tag{11}$$

By substituting these equations into the condition (8), we get the result. $\square$

Corollary 1 states that the necessary and sufficient condition can be checked by comparing the following two values: one is the sum of player i's marginal contribution to $\{i,j\},\{i,k\}$, and the other is 4 times player i's marginal contribution to the grand coalition, where $i,j,k \in N, i \neq j \neq k$.

Remark 1 In the setting of Corollary 1, the coefficients of singleton coalitions can be calculated as follows:

$$
\begin{aligned}
d_N(\{1\},v) &= \frac{v(\{1,2\})}{6} + \frac{v(\{1,3\})}{6} - \frac{v(\{2,3\})}{3} + \frac{v(N)}{3},\\
d_N(\{2\},v) &= \frac{v(\{1,2\})}{6} - \frac{v(\{1,3\})}{3} + \frac{v(\{2,3\})}{6} + \frac{v(N)}{3},\\
d_N(\{3\},v) &= -\frac{v(\{1,2\})}{3} + \frac{v(\{1,3\})}{6} + \frac{v(\{2,3\})}{6} + \frac{v(N)}{3}.
\end{aligned}
$$

Moreover, we have

$$
\bar{Q}_N d_{N,v} = \begin{pmatrix} 1001101 \\ 0101011 \\ 0010111 \\ 1100110 \\ 1011010 \\ 0111100 \\ 1110000 \end{pmatrix}
\begin{pmatrix}
\frac{v(\{1,2\})}{6} + \frac{v(\{1,3\})}{6} - \frac{v(\{2,3\})}{3} + \frac{v(N)}{3} \\
\frac{v(\{1,2\})}{6} - \frac{v(\{1,3\})}{3} + \frac{v(\{2,3\})}{6} + \frac{v(N)}{3} \\
-\frac{v(\{1,2\})}{3} + \frac{v(\{1,3\})}{6} + \frac{v(\{2,3\})}{6} + \frac{v(N)}{3} \\
-\frac{v(\{1,2\})}{6} + \frac{v(\{1,3\})}{3} + \frac{v(\{2,3\})}{3} - \frac{v(N)}{3} \\
\frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{6} + \frac{v(\{2,3\})}{3} - \frac{v(N)}{3} \\
\frac{v(\{1,2\})}{3} + \frac{v(\{1,3\})}{3} - \frac{v(\{2,3\})}{6} - \frac{v(N)}{3} \\
-\frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{3} - \frac{v(\{2,3\})}{3} + \frac{v(N)}{3}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ v(\{1,2\}) \\ v(\{1,3\}) \\ v(\{2,3\}) \\ v(N) \end{pmatrix}.
$$

which is equal to the original game. ∎

Remark 2 As Shapley (1971) proved, any convex game belongs to Γ_{SC}. We can prove this theorem by using Theorem 4. In other words, we can prove the following proposition without appealing to the Shapley's theorem.

Proposition 1 *Let (N, v) be a convex game. Then,*

$$v_N^{Sh}(S) \leq 0,$$

for all $S \subseteq N$.

The proof needs two lemmas.

Lemma 1 (Lemma 7 of Yokote, Funaki and Kamijo (2013)) *Let (N, v), $n \geq 2$ be a game and let $j \in N$. Then, we have*

$$d_{N\backslash\{j\}}(S, v) = d_N(S, v) + d_N(S \cup \{j\}, v),$$

for all $S \subseteq N\backslash\{j\}, S \neq \emptyset$.

And the following lemma was proved by Mas-Colell, Whinston and Green (1995).[4]

[4]See Proposition 18.AA.1 on page 683.

Lemma 2 *Let (N, v) be a convex game. Then, we have*

$$\phi_i(S, v) \leq \phi_i(T, v),$$

for all $S \subset T \subseteq N, S \neq \emptyset$, and for all $i \in S$.

We skip the precise proof since it is a bit cumbersome and the result itself is already known. ∎

Up to now, we focused on games where the Shapley value belongs to the Core. Here, we change our perspective. Consider the game (N, v) such that $\phi(N, v) \notin C(N, v)$ (it also includes the case $C(N, v) = \emptyset$). If we increase the worth of the grand coalition alone, when will the Shapley value first belong to the Core? We give a theorem to answer this question.

For any game $(N, v) \in \Gamma$, we define three additional notations. First, let $M(N, v) :=\max\{e\big(S, \phi(N, v)\big) : S \in 2^N \backslash (\{N\} \cup \emptyset)\}$. $M(N, v)$ is the maximal excess with respect to the Shapley value. Second, let $\eta(N, v) := |S^e|$, where $e(S^e, \phi(N, v)) = M(N, v)$ and $|S^e| \leq |T|$ for all T such that $e(T, \phi(N, v)) = M(N, v)$. $\eta(N, v)$ is the smallest cardinality of a coalition whose excess with respect to the Shapley value is maximum.

Theorem 5 *Let $(N, v) \in \Gamma$ be a game such that $\phi(N, v) \notin C(N, v)$. Then,*

$$\min\Big\{\lambda \in \mathbb{R} : \phi(N, v + \lambda u_N) \in C\Big(N, v + \lambda u_N\Big)\Big\} = \frac{nM(N, v)}{\eta(N, v)}.$$

Proof. The dividend of $(N, \lambda u_N)$ is given as follows:

$$D(S, \lambda u_N) = \begin{cases} \lambda & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 2,

$$d_N(S, \lambda u_N) = (-1)^{|S|-1}\frac{\lambda}{n}, \tag{12}$$

for all $S \subseteq N, S \neq \emptyset$. Let us fix $S \subseteq N, S \neq \emptyset$ and calculate $(\lambda u_N)^{Sh}_N(S), S \subseteq N, S \neq \emptyset$. We need to add all $d_N(T, v), T \subseteq N$ such that $|T \cap S| = 1, |T| \geq 2$. Choosing T is equivalent to choosing $i \in S$ and $R \subseteq N \backslash S, R \neq \emptyset$, since we can express $T = \{i\} \cup R$. Taking this view into consideration, together with equation (11), we have

$$\begin{aligned} (\lambda u_N)^{Sh}_N(S) &= \frac{\lambda|S|}{n} \sum_{k=1}^{n-|S|} (-1)^k \binom{n-|S|}{k} \\ &= -\frac{\lambda|S|}{n}, \end{aligned}$$

for all $S \subseteq N, S \neq \emptyset$, where the second equality holds from the binomial theorem.

We prove a lemma.

Lemma 3 *For any $(N, v), (N, w) \in \Gamma$, $\alpha, \beta \in \mathbb{R}$, we have $(\alpha v + \beta w)_N^{Sh} = \alpha v_N^{Sh} + \beta w_N^{Sh}$.*

Proof. For any $S \subseteq N, S \neq \emptyset$, from linearity of the Shapley value, we have

$$\begin{aligned}
\alpha v_N^{Sh}(S) + \beta w_N^{Sh}(S) &= \alpha\Big\{v(S) - \sum_{i\in S}\phi_i(N, v)\Big\} + \beta\Big\{w(S) - \sum_{i\in S}\phi_i(N, w)\Big\} \\
&= (\alpha v + \beta w)(S) - \sum_{i\in S}\phi_i(N, \alpha v + \beta w) \\
&= (\alpha v + \beta w)_N^{Sh}(S),
\end{aligned}$$

which proves the desired property. □

From this lemma, we have

$$(v + \lambda u_N)_N^{Sh}(S) = v_N^{Sh}(S) + (\lambda u_N)_N^{Sh}(S) = v_N^{Sh}(S) - \frac{\lambda|S|}{n},$$

for all $S \subseteq N, S \neq \emptyset$. Since $\phi(N, v) \notin C(N, v)$, from Theorem 4, there exists $S \subseteq N, S \neq \emptyset$ such that $v_N^{Sh}(S) > 0$. We need to reduce the worth of all such coalitions until $v_N^{Sh}(S) \leq 0$. The smaller the size of a coalition is, the harder it is to reduce the worth of the coalition. Hence, among coalitions with the largest worth, the worth of coalition with the smallest cardinality must be equal to 0. Since $v_N^{Sh}(S)$ is the excess of $S \subseteq N, S \neq \emptyset$ with respect to the Shapley value, the largest worth is equal to $M(N, v_N^{Sh})$. It follows that, we must have

$$\begin{aligned}
\frac{\lambda\eta(N, v_N^{Sh})}{n} &= M(N, v_N^{Sh}), \\
\lambda &= \frac{nM(N, v_N^{Sh})}{\eta(N, v_N^{Sh})} = \frac{nM(N, v)}{\eta(N, v)},
\end{aligned}$$

where the last equality holds since $M(N, v)$ and $\eta(N, v)$ are independent from strategically equivalent transformation. In this case, the Shapley value belongs to the Core, and the value $\frac{nM(N,v)}{\eta(N,v)}$ is obviously the minimum value as the statement requires.

□

Let us emphasize the importance of this theorem. Consider a social planner who is trying to implement the Shapley value as the distribution rule of resources in a society. Suppose also that the social planner could notice that the Shapley value does not belong to the Core. Then, the natural consequence of implementing the value is that some coalitions will deviate from the society and enjoy their own worths of coalitions. In order for the social planner to avoid this consequence, he needs to increase the total resources and widen the Core, but how many? Theorem 5 can be used to answer this question.

Corollary 2 *Let $(N, v), N = \{1, 2, 3\}$, be a 0-normalized game which satisfies $v(N) \geq 0$ and $v(\{1, 2\}) \geq v(\{1, 3\}) \geq v(\{2, 3\})$. If*

$$v(\{1, 3\}) + v(\{2, 3\}) > 4(v(N) - v(\{1, 2\})),$$

then $\phi(N, v) \notin C(N, v)$. Moreover,

Case 1 *If $v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) > v(N)$, then*

$$\begin{aligned}&\min\Big\{\lambda \in \mathbb{R} : \phi(N, v + \lambda u_N) \in C\Big(N, v + \lambda u_N\Big)\Big\}\\&= v(\{1, 2\}) + \frac{v(\{1, 3\})}{4} + \frac{v(\{2, 3\})}{4} - v(N).\end{aligned}$$

Case 2 *If $v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) \leq v(N)$, then*

$$\begin{aligned}&\min\Big\{\lambda \in \mathbb{R} : \phi(N, v + \lambda u_N) \in C\Big(N, v + \lambda u_N\Big)\Big\}\\&= v(\{1, 2\}) - \frac{v(\{1, 3\})}{2} - \frac{v(\{2, 3\})}{2} - v(N).\end{aligned}$$

Proof. We first show that if $\phi(N, v) \in C(N, v)$, then $v(\{1, 3\}) + v(\{2, 3\}) \leq 4(v(N) - v(\{1, 2\}))$. Since (N, v) is balanced and $v(N) \geq 0$, from Corollary 1, if $\phi(N, v) \in C(N, v)$, then we have

$$\begin{aligned}v(\{1, 2\}) + v(\{1, 3\}) + 4v(\{2, 3\}) &\leq 4v(N),\\ v(\{1, 2\}) + v(\{2, 3\}) + 4v(\{1, 3\}) &\leq 4v(N),\\ v(\{1, 3\}) + v(\{2, 3\}) + 4v(\{1, 2\}) &\leq 4v(N).\end{aligned}$$

which implies

$$v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) + \max\{3v(\{1, 2\}, 3v(\{1, 3\}), 3v(\{2, 3\})\} \leq 4v(N).$$

Since $v(\{1, 2\}) \geq v(\{1, 3\}) \geq v(\{2, 3\})$, we must have

$$v(\{1, 3\}) + v(\{2, 3\}) + 4v(\{1, 2\}) \leq 4v(N).$$

Note that $v(\{1,2\}) \geq v(\{1,3\}) \geq v(\{2,3\})$ implies the following inequality.

$$\begin{aligned}
&v(\{1,2\}) + v(\{1,3\}) \geq v(\{1,2\}) + v(\{2,3\}) \geq v(\{1,3\}) + v(\{2,3\}),\\
&2v(\{1,2\}) + 2v(\{1,3\}) + v(\{2,3\})\\
\geq &2v(\{1,2\}) + v(\{1,3\}) + 2v(\{2,3\})\\
\geq &v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}).
\end{aligned}$$

Together with $-2v(\{2,3\}) \geq -2v(\{1,3\}) \geq -2v(\{1,2\})$, we have

$$\begin{aligned}
&2v(\{1,2\}) + 2v(\{1,3\}) - v(\{2,3\})\\
\geq &2v(\{1,2\}) - v(\{1,3\}) + 2v(\{2,3\})\\
\geq &-v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}).
\end{aligned}$$

From equations (9) to (11), we have

$$d_N(\{2,3\}, v) \geq d_N(\{1,3\}, v) \geq d_N(\{1,2\}, v). \tag{13}$$

In order to calculate excesses, we need to calculate the worths of coalitions of (N, v_N^{Sh}).

$$\begin{aligned}
v_N^{Sh}(\{1\}) &= d_N(\{1,2\}, v) + d_N(\{1,3\}, v) + d_N(N, v), && (14)\\
v_N^{Sh}(\{2\}) &= d_N(\{1,2\}, v) + d_N(\{2,3\}, v) + d_N(N, v), && (15)\\
v_N^{Sh}(\{3\}) &= d_N(\{1,3\}, v) + d_N(\{2,3\}, v) + d_N(N, v), && (16)\\
v_N^{Sh}(\{1,2\}) &= d_N(\{1,3\}, v) + d_N(\{2,3\}, v), && (17)\\
v_N^{Sh}(\{1,3\}) &= d_N(\{1,2\}, v) + d_N(\{2,3\}, v), && (18)\\
v_N^{Sh}(\{2,3\}) &= d_N(\{1,2\}, v) + d_N(\{1,3\}, v), && (19)\\
v_N^{Sh}(N) &= 0.
\end{aligned}$$

From equation (13), we have

$$v_N^{Sh}(\{1,2\}) \geq v_N^{Sh}(\{1,3\}) \geq v_N^{Sh}(\{2,3\}), \tag{20}$$
$$v_N^{Sh}(\{3\}) \geq v_N^{Sh}(\{2\}) \geq v_N^{Sh}(\{1\}). \tag{21}$$

Now, suppose that Case 1 is satisfied. Then, $-v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(N) < 0$. Together with equation (7), we have $d_N(N, v) < 0$. Equations (14) to (19) imply that the

$$M(N, v) = v_N^{Sh}(\{1,2\}), \eta(N, v) = 2.$$

From equations (10) and (11), the following equality holds.[5]

$$v_N^{Sh}(\{1,2\}) = \frac{2v(\{1,2\})}{3} + \frac{v(\{1,3\})}{6} + \frac{v(\{2,3\})}{6} - \frac{2v(N)}{3}.$$

Together with Theorem 5, the condition of the statement can be obtained.

Suppose that Case 2 is satisfied. Then, by following the same discussion, we have $d_N(N,v) \geq 0$. It follows that

$$M(N,v) = v_N^{Sh}(\{3\}), \eta(N,v) = 1.$$

From equations (7), (10), (11), the following equality holds.[6]

$$v_N^{Sh}(\{3\}) = \frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{6} - \frac{v(\{2,3\})}{6} - \frac{v(N)}{3}.$$

Together with Theorem 5, the condition of the statement can be obtained.

4.2 The Shapley value and the prenucleolus

We first give the necessary and sufficient condition under which the Shapley value coincides with the prenucleolus. The proof is obvious from Theorem 3 and Kohlberg's (1971) theorem.

Theorem 6 *Let (N,v) be a game. Then, $\phi(N,v) = PN(N,v)$ if and only if for any $\alpha \in \mathbb{R}, \{S \subseteq N, S \neq \emptyset : v_N^{Sh}(S) \geq \alpha\} \neq \emptyset$ implies that the family of coalitions is balanced.*

By using this theorem, we precisely characterize the coincidence of the two values in the case of 0-normalized 3-person game.

Corollary 3 *Let $(N,v), N = \{1,2,3\}$, be a 0-normalized game. Then, $\phi(N,v) = PN(N,v)$ if and only if one of the following two conditions holds.*

Condition 1: $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\})$
Condition 2: $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) = v(N)$.

[5]Note that under the condition $v(\{1,3\})+v(\{2,3\}) > 4(v(N)-v(\{1,2\})$, $v_N^{Sh}(\{1,2\}) > 0$. This condition must hold since otherwise the maximum excess is no greater than 0, which implies that the Shapley value belongs to the Core.

[6]Since $v_N^{Sh}(\{1,2\}) > 0$ and $d_N(N,v) \geq 0$, we also have $v_N^{Sh}(\{3\}) = v_N^{Sh}(\{1,2\}) + d_N(N,v) > 0$.

Proof. **If Part:** Under Condition 1, the game is symmetric, so the coincidence is obvious.[7]

Suppose that Condition 2 holds. Then, the following property holds.

$$v(\{1\}) - v(\emptyset) + v(N) - v(\{2,3\}) = v(\{1,2\}) + v(\{1,3\}),$$
$$v(\{1,2\}) - v(\{2\}) + v(\{1,3\}) - v(\{3\}) = v(\{1,2\}) + v(\{1,3\}).$$

Namely, for any coalition S such that $1 \notin S$, the sum of marginal contributions of player 1 to coalition S and to coalition $N \backslash (S \cup \{1\})$ is the same. The same condition holds for players 2 and 3. This property is known as the PS property, introduced by Kar, Mitra and Mutuswami (2009). As they proved, under PS property, the Shapley value and the prenucleolus coincide.

Only If Part: Suppose not. Then, both conditions 1 and 2 do not hold. Assume, without loss of generality, that $v(\{1,2\}) \geq v(\{1,3\}) \geq v(\{2,3\})$. Since Condition 1 does not hold, we have $v(\{1,2\}) > v(\{1,3\}) \geq v(\{2,3\})$ or $v(\{1,2\}) = v(\{1,3\}) > v(\{2,3\})$.

First, suppose that $v(\{1,2\}) > v(\{1,3\}) \geq v(\{2,3\})$. Then, we have

$$-2v(\{2,3\}) \geq -2v(\{1,3\}) > -2v(\{1,2\}). \tag{22}$$

We also have

$$\begin{aligned} & v(\{1,2\}) + v(\{1,3\}) \geq v(\{1,2\}) + v(\{2,3\}) > v(\{1,3\}) + v(\{2,3\}), \\ & 2v(\{1,2\}) + 2v(\{1,3\}) + v(\{2,3\}) \\ \geq & 2v(\{1,2\}) + v(\{1,3\}) + 2v(\{2,3\}) \\ > & v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}). \end{aligned} \tag{23}$$

The addition of equations (22) and (23) implies

$$\begin{aligned} & 2v(\{1,2\}) + 2v(\{1,3\}) - v(\{2,3\}) \\ \geq & 2v(\{1,2\}) - v(\{1,3\}) + 2v(\{2,3\}) \\ > & -v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}). \end{aligned}$$

By dividing by 6, together with equations (9) to (11), we have

$$d_N(\{2,3\}) \geq d_N(\{1,3\}) > d_N(\{1,2\}).$$

[7] Note that both the Shapley value and the prenucleolus satisfy Equal Treatment Property, which requires symmetric players to obtain the same payoff.

It follows that

$$d_N(\{1,3\}) + d_N(\{2,3\}) > d_N(\{1,2\}) + d_N(\{2,3\}) \geq d_N(\{1,2\}) + d_N(\{1,3\}).$$

From equations (14) to (19), we have

$$\begin{aligned} v_N^{Sh}(\{1,2\}) &> v_N^{Sh}(\{1,3\}) \geq v_N^{Sh}(\{2,3\}), \\ v_N^{Sh}(\{3\}) &> v_N^{Sh}(\{2\}) \geq v_N^{Sh}(\{1\}). \end{aligned}$$

Since Condition 2 does not hold, $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \neq v(N)$. Suppose that $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) > v(N)$. Then, we have

$$-v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(N) < 0,$$

which implies, together with equation (7), $d_N(N, v) < 0$. In this case,

$$\{S \subseteq N, S \neq \emptyset : v_N^{Sh}(S) \geq v_N^{Sh}(\{1,2\})\} = \{1,2\},$$

which contradicts Theorem 6. If we assume that $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) < v(N)$, the same contradiction can be obtained.

Next, suppose that $v(\{1,2\}) = v(\{1,3\}) > v(\{2,3\})$. Then, by following the same calculation, we have

$$\begin{aligned} v_N^{Sh}(\{1,2\}) &= v_N^{Sh}(\{1,3\}) > v_N^{Sh}(\{2,3\}), \\ v_N^{Sh}(\{3\}) &= v_N^{Sh}(\{2\}) > v_N^{Sh}(\{1\}). \end{aligned}$$

If we assume $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) > v(N)$, we have

$$\{S \subseteq N, S \neq \emptyset : v_N^{Sh}(S) \geq v_N^{Sh}(\{1,2\})\} = \big\{\{1,2\}, \{1,3\}\big\},$$

If we assume $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) < v(N)$, we have

$$\{S \subseteq N, S \neq \emptyset : v_N^{Sh}(S) \geq v_N^{Sh}(\{2\})\} = \big\{\{2\}, \{3\}\big\}.$$

Both results are contradictions. □

Corollary 3 states that the coincidence region for 3-person game is exactly equal to the union of the set of all symmetric games, and the set of all games which satisfy PS property.

Remark 3 We show an interesting relationship between the Shapley value and the prenucleolus in the linear basis. The following lemma shows that the the Shapley value prescribes $\mathbf{0}$ to games $\bar{u}_S$, $S \subseteq N, |S| \geq 2$.

Lemma 4 (Lemma 5 of Yokote, Funaki and Kamijo (2013)) *Let N, $n \geq 2$ be a set of players. Then, we have $\phi(\bar{u}_S) = \mathbf{0}$ for all $S \subseteq N, |S| \geq 2$.*

The following result was proved by Chang and Tseng (2011).

Lemma 5 (Corollary 4 (iii) of Chang and Tseng (2011)) *Let $(N, v) \in \Gamma$ be a simple game. Then, $\phi(N, v) = \mathbf{0}$ implies that the family of coalitions $\{S \subseteq N : v(S) = 1\}$ is balanced.*

From Lemma 5, for any $\alpha \in \mathbb{R}$, $\{T \subseteq N, T \neq \emptyset : \bar{u}_S(T) \geq \alpha\} \neq \emptyset$ implies that the family of coalitions is balanced. Then, from Kohlberg's (1971) theorem, we have $PN(N, \bar{u}_S) = \mathbf{0}$ for all S such that $|S| \geq 2$. This result shows that the Shapley value coincides with the prenucleolus in $\bar{u}_S, S \subseteq N, S \neq \emptyset$. Namely, when we express a game by the linear combination of $\bar{u}_S$, the two values coincide in each games. The difference between them originates from the difference of the way to respond to the addition of games. ∎

Remark 4 Consider the following set of games $(\tilde{u}_S)_{S \subseteq N, S \neq \emptyset}$ introduced by Dragan (2012).

$$\begin{aligned} &\tilde{u}_S = u_S \text{ if } |S| = 1, \\ &\tilde{u}_S = u_S - \sum_{i \in S} \phi_j(u_S) u_{\{j\}} \text{ if } |S| \geq 2. \end{aligned}$$

Even if we use this set, the Shapley value appears in the coefficients and the coefficients can be calculated by using the dividend. However, the advantage of using our new linear basis is that we can deal with the relationship between the Shapley value and the prenucleolus more effectively, as we saw in the proof of Corollary 2. Since our linear basis consists of simple games, we can check balancedness of coalitions more easily. ∎

Remark 5 Although Theorem 6 gives necessary and sufficient condition, we can also give a sufficient condition which is easier to calculate. The condition clarifies a convex cone where the Shapley value coincides with the prenucleolus.

Proposition 2 *Let $\Gamma_{PS} \subset \Gamma$ denote the set of all games $(N, v) \in \Gamma, n \geq 2$ which satisfy the following two properties:*

1 $d_N(S, v) = d_N(T, v) := a$ *for all* $S, T \subseteq N$ *such that* $|S| = |T| = 2$.

2 $v_N^{Sh}(S) \leq (n-1)a$ *for all* $S \subseteq N, S \neq \emptyset$.

Then, we have $\phi(N,v) = PN(N,v)$ for all $v \in \Gamma^N_{PS}$. Moreover, Γ^N_{PS} is a convex cone.

We can prove this corollary by using the following proposition.

Proposition 3 (Suzuki and Nakayama (1976)) *Let (N,v) be a game. Then, $PN(N,v) = ENSC(N,v)$ if*

$$v(S) - \Big(\sum_{i\in S} m_i(N,v) + |S|b(N,v)\Big) \leq b(N,v),$$

for all $S \subseteq N, S \neq \emptyset$.

Since the proof is a bit cumbersome, we skip it. ∎

5 Concluding Remarks

In Theorem 2, we showed that the coefficients of $\bar{u}_S$ in a linear combination can be calculated by using the dividend of the Shapley value. However, there are two other ways to calculates the coefficients. First way uses the potential function introduced by Hart and Mas-Colell (1989). Second way uses the direct calculation from a game. Since the calculation by using the dividend is the easiest way, we listed the theorem only. For other ways, see Yokote, Funaki and Kamijo (2013).

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