A Revealed Preference Theory with Preference Cycles

by

Takashi Oginuma

Waseda University

No.1002

A Revealed Preference Theory with Preference Cycles

Takashi Oginuma
Waseda University
Department of Economics

November 2010

E-mail: oginuma@waseda.jp

Abstract: In this paper, a theory of revealed preference that can be compatible with preference cycles is considered. The problem of preference cycles is treated in Schwartz(1972) and he advocated the notion of optimality. Deb(1977) showed that the notion of optimality can be captured by the maximal set based on transitive closure of a strict preference. By using this type of maximal sets (they are called quasi-maximal sets in this paper), we'd like to provide conditions for revealed preferences that rationalize the choice patterns of the decision-maker. Firstly, several properties of quasi-maximal sets are considered. The content of a quasi-maximal set is given by the union of (standard) maximal set and the maximal fully cyclic sets. Secondly, several conditions for revealed preferences that are compatible with preference cycles are considered. These conditions are WWARP, IIA and NBDC(or RWARP). Thirdly, as a special case of our results, a characterization of a top-cycle rule is considered. The conditions to characterize the top-cycle rule are different from the conditions to characterize it in Ehlers and Sprumont(2008). Fourthly, an alternative notion of a quasi-maximal set, i.e., an extended-maximal set, is considered. It can be seen as a procedural choice function, and it needs an extra-condition PNCA to characterize it.

1. Introduction

   The search for the conditions about choice functions that can be compatible with preference cycles is one interesting topic of choice theory. This problem is considered in Schwartz(1972) for individual choices. Deb(1977) characterized Schwartz's rule by using the transitive closure of a strict preference. Recently, Manzini and Mariotti(2008) give us a set of conditions for sequential choices that can be compatible with preference cycles. Lombardi(2009) considered the conditions to characterize reason-based choices and explained one kind of preference cycles.

   As for a related field, Ehlers and Sprumont(2008) provide us a characterization of top-cycle rules that is compatible with preference cycles, by using the conditions of Weakened Axiom of Revealed Preference(WWARP), Binary Dominance Consistency(BDC) and Weak Choice Consistency(WCC). Although their characterization might not be intended to provide us the conditions for individual choices, it is possible to interpret it as a characterization of individual choices.

   Considering a set of conditions about a choice function for revealed preference is considering how to rationalize the choice function. (As for the conditions, see Richter(1966), Sen(1971), Suzumura(1983) and Aleskerov and Monjardet(2002).)It is impossible to rationalize a choice function that contains a preference cycle by simple maximization of the corresponding revealed preference relation. However, if we use the notion of quasi-maximal sets, that are equivalent with maximal sets based on the transitive closure of strict preference, preference cycles are compatible with maximization of the revealed preference. The conditions considered in this paper are as the followings: WWARP, Independence of Irrelevant Alternatives(IIA) and Negative Binary Dominance Consistency(NBDC).

   In the next section, by using base preference relations, several properties of fully cyclic sets and quasi-maximal sets are considered. It is shown that the quasi-maximal set is equivalent with the union of the maximal set and the set of 'maximal' fully cyclic sets. Furthermore, it will be confirmed the equivalence of Schwartz's sets of optimal elements and the quasi-maximal sets. In the third section, the conditions about a choice function that can be rationalized by a quasi-maximal set are considered. They are WWARP, IIA and NBDC. It is shown that the revealed base preference is acyclic pseudo-transitive, i.e., pseudo-transitive if it is acyclic. Also, it is shown that the conditions about a choice function that induces an acyclic transitive base preference, i.e., transitive base preference if it is acyclic, are WWARP, IIA and Restricted Weak Axiom of Revealed Preference (RWARP). In the fourth section, as a special case, the conditions for a characterization of top-cycle rules are considered. This result means that the

conditions in Ehlers and Sprumont (2008), i.e., WWARP, BDC and WCC, are equivalent with our conditions, i.e., WWARP, IIA and NBDC if the choice function is resolute, i.e., one alternative must be chosen from the sets of two alternatives. In the fifth section, an extended-maximal set that is slightly different from quasi-maximal set is considered. The extended-maximal set does not include any cycles if the usual maximal set is non-empty. It is shown that if a choice function is rationalized by an extended-maximal set, then it satisfies the conditions: WWARP, IIA, NBDC and Priority of non-cyclic alternatives (PNCA).

2. Maximization and preference cycles

In this section, we would like to consider the maximization of preference on the set of choice alternatives when the preference permits preference cycles. As for the notion of maximization that can be compatible with preference cycles, the notion of the quasi-maximal set is introduced, and its characterization is considered.

To begin with, we would like to provide several notations in this paper. Let X be a finite set of alternatives with cardinality $|X| \geq 3$ and $\wp(X)$ the set of nonempty subsets of X. Let R be a *binary relation* defined on X, i.e., $R \subseteq X \times X$. R is considered as a weak preference relation. Thus, $(x, y) \in R$ means that x is weakly preferred to y. Denote $P(R)$ and $I(R)$ as $P(R) = \{(x, y) | (x, y) \in R$ and $(y, x) \notin R\}$ and $I(R) = \{(x, y) | (x, y) \in R$ and $(y, x) \in R\}$ respectively.

R is assumed *complete*, i.e., $(x, y) \in R$ or $(y, x) \in R$ for any $x, y \in X$, in this section. Since R is complete, R is *reflexive*, i.e., $\Delta \subseteq R$ where $\Delta = \{(x, x) | x \in X\}$. R is *acyclic* if there is no $x, \{x_i\}$ such that $(x, x_1), (x_i, x_{i+1}), (x_k, x) \in P(R)$ for $i = 1, 2, .., k-1$. R is *transitive*: if $(x, y), (y, z) \in R$, then $(x, z) \in R$.

For any $A \in \wp(X)$, $R|A$ is the restriction of R on A, i.e., $R|A = R \cap (A \times A)$. The *transitive closure* of $R|A$ is defined as follows: for all $x, y \in A$, $(x, y) \in T(R|A)$ if and only if there exists a positive integer n and $x_1, x_2, .., x_n \in A$ such that $x_1 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in R$ for $i = 1, 2, .., n-1$. Since R is complete, $T(R|A)$ is complete and transitive.

Define the *set of maximal elements* or *the maximal set* as $M(A, R) = \{x \in A | (y, x) \notin P(R)$ for all $y \in A\}$ for given $A \in \wp(X)$. The maximal set $M(A, R)$ is empty if R includes cycles. Thus, we need another notion of 'maximum.' Schwartz(1972) showed that the set of optimal elements, a substitute of the maximal set, is compatible with preference cycles. Later, Deb(1977) showed that the maximal set with the transitive closure of a strict preference is equivalent with the set of optimal elements.

In this section, the relations among the set of alternatives constituting preference

3

cycles are considered. By using the results about these sets, it is shown that Schwartz's set of optimal elements is equivalent with the union of the maximal set and the set of alternatives that constitutes a 'maximum' cycle in A.

Definition 1. Suppose a binary relation R is defined on X, and A is a nonempty subset of X, i.e., $A \in \wp(X)$.

(1) $R|A$ is called *fully cyclic* on A if $A=\{a_1, a_2, \ldots, a_n\}$ and (a_i, a_{i+1}), $(a_n, a_1) \in P(R)$ for i=1, 2, .., n-1 and $n \geq 3$. Alternatively, A is called *a fully cyclic set* (with respect to R) if $R|A$ is fully cyclic on A.

(2) Suppose there are k distinct sets $\{C_i\}$ such that $C_i \subseteq A$ and any $R|C_i$ is fully cyclic on C_i (i=1, 2, .., k). $C(A)=\cup_i C_i$ is *the set of all cyclic sets* in A if every $R|C_i$ is fully cyclic on C_i, and either $R|(A \setminus C(A))$ is acyclic or empty.

(3) Let fc(X) be the set of indices of all fully cyclic sets in X. The cardinality of fc(X) is finite, since X is a finite set. fc(A) is called *the set of indices of all fully cyclic sets in A*, and fc(A)=$\{i \in fc(X)| C_i \subseteq A\}$.

Remark 1. For any $A \in \wp(X)$ and for any binary relation R on X, there exists the set of all cyclic sets $C(A) \subseteq A$ such that $R|(A \setminus C(A))$ is acyclic or $A \setminus C(A)$ is empty. Thus, $C(A)=\varnothing$ if A does not include any fully cyclic set.

(Proof) Since X is finite, there is only finite number of fully cyclic sets in X. Let the number of all fully cyclic sets in X be n. The number n might be 0 because there is a possibility of no fully cyclic set. Thus, there are distinct sets $\{C_i\}$ in X (i=1, 2, .., n) such that $C_i \subseteq X$ and any $R|C_i$ is fully cyclic on C_i. (Note that $C_i \supset C_j$ is possible for some i, j.) By the definition, C(X) be the union of all fully cyclic sets, i.e., $C(X)=\cup_i C_i$, and $C(X) \subseteq X$. If $X \setminus C(X)$ is not empty, then $R|(X \setminus C(X))$ should be acyclic. Similarly, for any $A \in \wp(X)$, there are at most n fully cyclic sets that are included in A. Remind that $C(A)=\cup_{i \in fc(A)} C_i$. Then, if $A \setminus C(A)$ is not empty, then $R|(A \setminus C(A))$ should be acyclic. $\square$

Definition 2. Suppose there is a partition of fc(A) where each cell I of the partition has the following property: if $i \in I$, then $\exists j \in I$ such that $C_i \cap C_j \neq \varnothing$. Denote the partition of fc(A) as $\{I_k(A)\}$, k=1, 2, .., κ, i.e., $I_i(A) \cap I_j(A)=\varnothing$ and $\cup I_k=fc(A)$. Denote the set of indices that constitutes the partition as fc*(A)=$\{1, 2, .., \kappa\}$. Define *a connected fully cyclic set* in A as $C^*_i(A)=\cup_{j \in I_i(A)} C_j$. $\{C^*_i(A)\}_{i \in fc^*(A)}$ is a partition of C(A). Denote the set of connected fully cyclic sets in A as $C^{con}(A)$.

Proposition 1. (1) Suppose C^*_j is a connected fully cyclic set in A, i.e., $C^*_j \in C^{con}(A)$.

Then, for given $x \in A \setminus C^*_j$, either $(x, y) \in R|A$ for all $y \in C^*$ or $(y, x) \in R|A$ for all $y \in C^*$.

(2) Let the set of all cyclic sets in A be $C(A) = \cup_{i \in fc(A)} C_i = \cup_{j \in fc^*(A)} C^*_j$. For given $A \in \wp(X)$, if $x \in A \setminus C(A)$ and $j \in fc^*(A)$, then $(x, y) \in R|A$ for all $y \in C^*_j$ or $(y, x) \in R|A$ for all $y \in C^*_j$.

(Proof). (1) Suppose $A \setminus C^*_j$ is nonempty and $x \in A \setminus C^*_j$. Then, either $(x, y) \in R|A$ for any $y \in C^*_j$ or $(y, x) \in R|A$ for any $y \in C^*_j$ holds. Suppose not, i.e., there are some y, $z \in C^*_j$ such that $(x, y), (z, x) \in P(R|A)$. Suppose $C^*_j = \{a_1, a_2, \ldots, a_k\}$, a connected fully cyclic set. Let $C_{j1} = \{b_1, b_2, \ldots, b_l\}$ and $C_{j2} = \{c_1, c_2, \ldots, c_m\}$ be such that $C_{ji} \subseteq C^*_j$ (i=1, 2), $C_{j1} \cap C_{j2} \neq \varnothing$ and $(b_i, b_{i+1}), (b_l, b_1), (c_j, c_{j+1}), (c_m, c_1) \in P(R|A)$, i=1, 2, .., l-1 and j=1, 2, .., m-1. Then, there are two cases: (a) both y and z belong to the same fully cyclic set, (b) y and z belongs to the different fully cyclic sets. Consider the case (a) first. Suppose y, $z \in C_{j1} \subseteq C^*_j$. Take $b_1 = y$ and $b_j = z$. Then, $\{x, y, b_2, .., b_{j-1}, z\}$ is another fully cyclic set in A and it has the nonempty intersection with C^*_j. It contradicts the assumption that C^*_j is a connected fully cyclic set in A. Consider the case (b) next. Suppose $y \in C_{j1} \subseteq C^*_j$ and $z \in C_{j2} \subseteq C^*_j$. Take $b_1 = y$ and $c_1 = z$. Suppose $b_i = c_j$. Then, $\{x, y, b_2, .., b_i, c_{j+1}, .., c_m, z\}$ is another fully cyclic set. It contradicts the assumption that C^*_j is a connected fully cyclic set in A. Hence, $(x, y) \in R|A$ for all $y \in C^*_j$ or $(y, x) \in R|A$ for all $y \in C^*_j$. (2) Suppose $A \setminus C(A)$ is nonempty and $x \in A \setminus C(A)$. Let C_i^* be a connected fully cyclic set in A. Then, $x \in A \setminus C^*_i$ since $C^*_i \subseteq C(A)$ and $x \in A \setminus C(A)$. Hence, by Proposition 1 (1), either $(x, y) \in R|A$ for any $y \in C^*_i$ or $(y, x) \in R|A$ for any $y \in C^*_i$ holds. The same argument holds for any $i \in fc^*(A)$.$\square$

Proposition 1 (1) shows that any alternative outside a connected fully cyclic set C^*_j in A can be compared with any alternative in the connected fully cyclic set C^*_j unilaterally. By using this property, Proposition 1 (2) states that any alternative that does not belong to any fully cyclic set in A can be compared with any alternative in any fully cyclic set in A unilaterally.

  The following Proposition 2 shows the relation among connected fully cyclic sets by using Proposition 1.

Proposition 2. Let $\{C^*_j\}_{j \in fc^*(A)}$ be the set of all connected fully cyclic sets in A. Then, for given $A \in \wp(X)$, if j, $k \in fc^*(A)$, then for given $x \in C^*_j$, either $(x, y) \in R|A$ for any $y \in C^*_k$ or $(y, x) \in R|A$ for any $y \in C^*_k$.

(Proof) Since $C^*_j \cap C^*_k = \varnothing$, for $\forall x \in C^*_j$, either $(x, y) \in R|A$ for $\forall y \in C^*_k$ or $(y, x) \in R|A$ for $\forall y \in C^*_k$, by Proposition 1 (1). $\square$

Proposition 2 states that any alternative in a connected fully cyclic set in A can be compared with any alternative in another connected fully cyclic set in A unilaterally, i.e., all disjoint connected fully cyclic sets in A can be ordered by using a preference relation R.

By using Proposition 2, we can consider the most preferred fully cyclic sets.

Definition 3. A connected fully cyclic set C^*_i in A is *one of the most preferred connected fully cyclic sets* in A if $\forall x \in C^*_i$ and $\forall y \in C(A) \setminus C^*_i$, $(y, x) \notin P(R)$. Denote the set of the most preferred connected fully cyclic sets in A by $C_M(A)$, i.e., $C_M(A)=\{x \in A| x \in C^*_i$ and C^*_i is any one of the most preferred fully cyclic sets in A}.

Since the number of the connected fully cyclic sets in A is finite and they can be ordered by Proposition 2, there exists the set of the most preferred fully cyclic sets in A.

Schwartz considered the optimization of choice alternatives O(A, P) by using the following notion. For any $A \in \wp(X)$,

$S(A, P)=\{B \in \wp(A)| ((A \setminus B) \times B) \cap P=\varnothing$ and $\forall C \subset B, ((A \setminus C) \times C) \cap P \neq \varnothing \}$.

$O(A, P)=\{x \in B| B \in S(A, P)\}$.

Since any element in B is not strictly preferred by any element in $A \setminus B$, S(A, P) is the set of all non-dominated sets like B in A. Thus, O(A, P) is the set of elements in the set of all non-dominated sets in A.

Denote the maximal set as M(A, R), i.e., for any $A \in \wp(X)$, $M(A, R)=\{x \in A| (y, x) \notin P(R)$ for all $y \in A\}$. Since R is complete, $M(A, R)=\{x \in A| (x, y) \in R$ for all $y \in A\}$.

Proposition 3 (Schwartz(1972)). Suppose P=P(R). Then, for any $A \in \wp(X)$, M(A, R)$\subseteq$O(A, P). If A does not include any fully cyclic set, then M(A, R)=O(A, P).

(Proof) This proposition is a combination of Theorem 7 and Theorem 8 in Schwartz(1972). These relations are easily checked by using Theorem 1 and Proposition 5 in this paper. (For detail, see Remark 3 below.) □

To consider another characterization of the set of optimal elements, the following notion is introduced.

Definition 4. For $\forall A \in \wp(X)$, $T_q(R)$ is the *quasi-transitive closure* of R|A if and only if $T_q(R|A)=T(P(R|A)) \cup I(R|A)$.

This notion is different from the transitive closure of R|A, i.e., T(R|A). The difference is

shown in the following example.

Example 1.   The quasi-transitive closure of R|A is different from the transitive closure of R|A. Let A={x, y, z}.

(1) Suppose R|A={(x, x), (y, y), (z, z), (x, y), (y, z), (z, x)}. Then, T_q(R|A)=T(R|A)=R|A$\cup${(y, x), (z, y), (x, z)}.

(2) Suppose R|A={(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z)}. Then, T_q(R|A)=R|A. T(R|A)=R|A$\cup${(z, x)}.

(3) Suppose R|A={(x, x), (y, y), (z, z), (x, y), (y, z), (z, x), (x, z)}. Then, T_q(R|A)=R|A. T(R|A)=R|A$\cup${(y, x), (z, y)}.

Although T(R|A) is different from T_q(R|A), both operations replace cycles with indifferent relations. T_q(R|A) is different from R only when R includes cycles. Note that the difference between T(R|A) and T_q(R|A) is vanished if I(R|A) were transitive. The content of a quasi-transitive closure is given by the following formula.

Proposition 4.   (1) T_q(R|A∖(C(A)×C(A)))= R|A∖(C(A)×C(A)).

(2) There is a permutation function ρ from fc*(A) to fc*(A) such that for all x in C^*_i and for all y in C^*_j, (x, y)∈R|A if and only if $\rho(i) \geq \rho(j)$. For given $A \in \wp(X)$,

T_q(R|A)=R∖(C(A)×C(A)∪($\cup_{i, j \in fc^*(A), \rho(i) \geq \rho(j)}$($C^*_{\rho(i)}$×$C^*_{\rho(j)}$)).

(Proof) (1) Let A be A={x_1, x_2, .., x_{n+1}}. Consider the following three cases. (a) If (x_i, x_{i+1})∈P(R) for i=1, 2, .., n and (x_1, x_{n+1})∈P(R), then T_q(R|A)=R|A because P(T_q(R|A))=P(T(P(R|A)))=P(R|A). (b) If (x_i, x_{i+1})∈P(R) for i=1, 2, .., n and (x_{n+1}, x_1) ∈I(R), then T_q(R|A)=R|A because P(T_q(R|A))=P(T(P(R|A)))=P(R|A). (c) If (x_i, x_{i+1}) ∈P(R) for i=1, 2, .., n and (x_{n+1}, x_1)∈P(R), then T_q(R|A)≠R|A because P(T_q(R|A))≠P(R|A). This result can be extended to the following: T_q(R|A)≠R|A only when A include fully cyclic sets. Proposition 1 (2) showed that any alternative that does not belong to any fully cyclic set can be compared with any alternative in a cycle unilaterally. Such comparisons also do not belong to cycles. Thus, T_q(R∖(C(A)×C(A)))= R∖(C(A)×C(A)).

(2) Proposition 2 showed any two connected fully cyclic sets can be ordered unilaterally, i.e., if j, k∈fc*(A), then for $\forall x \in C^*_j$, either (x, y)∈R|A for $\forall y \in C^*_k$ or (y, x)∈R|A for $\forall y \in C^*_k$. Hence, there exists a permutation function ρ from fc*(A) to fc*(A) such that for all x in C^*_i and for all y in C^*_j, (x, y)∈R|A if and only if $\rho(i) \geq \rho(j)$. Since (1) in this Proposition holds, T_q(R|A∖(C(A)×C(A))) = R|A∖(C(A)×C(A)) for any $A \in \wp(X)$. By Proposition 2, $\cup_{i, j \in fc(A), \rho(i) \geq \rho(j)}$ ($C^*_{\rho(i)}$×$C^*_{\rho(j)}$)⊆R|C(A)=R∩(C(A)×C(A)). Since $\cup_{i, j \in fc^*(A),}$

$_{\rho(i)\geq\rho(j),\ \rho(i)\neq\rho(j)}(C^*_{\rho(i)}\times C^*_{\rho(j)})$ is acyclic, $\cup_{i,j\in fc^*(A),\ \rho(i)\geq\rho(j),\ \rho(i)\neq\rho(j)}(C^*_{\rho(i)}\times C^*_{\rho(j)})\subseteq T_q(R|C(A))$. By the definition of quasi-transitive closure, $T_q(C^*_i\times C^*_i)\subseteq I(T_q(R|C(A)))$ for any $i\in fc^*(A)$. Thus, $\cup_{i,j\in fc^*(A),\rho(i)\geq\rho(j)}(C^*_{\rho(i)}\times C^*_{\rho(j)})=T_q(R|C(A))$. $T_q(R|A)=R|(R|A \setminus (C(A)\times C(A))) \cup T_q(R|C(A)))=(R|A\setminus(C(A)\times C(A)))\cup(\cup_{i,\ j\in fc^*(A),\rho(i)\geq\rho(j)}(C^*_{\rho(i)}\times C^*_{\rho(j)}))$. $\square$

By using the notion of quasi-transitive closure, the following notion of 'maximum' is compatible with preference cycles.

Definition 5. For given $A\in\wp(X)$, $M(A, T_q(R|A))$ is the *quasi-maximal set* if $M(A, T_q(R|A))=\{x\in A|\ (y, x)\notin T_q(R|A)$ for all $y\in A\}$. Denote a quasi-maximal set by $M_q(A, R)$, i.e., $M_q(A, R)=M(A, T_q(R|A))$.

Remark 2. Generally, the quasi-maximal set is different from both the maximal set and the maximal set with the transitive closure of a weak preference.

  Consider the case of Example 1. Let $X=\{x, y, z\}$.
(1) Suppose $R=\{(x, x), (y, y), (z, z), (x, y), (y, z), (z, x)\}$. Then, $M(X, R)=\varnothing$, and $M(X, T_q(R))=\{x, y, z\}=M(X, T(R))$.
(2) Suppose $R=\{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z)\}$. Then, $M(X, R)=\{x, y\}=M(X, T_q(R))\neq M(X, T(R))=\{x, y, z\}$.

It is possible that the elements of the set of the most preferred fully cyclic sets are preferred to any other alternatives in A.

Definition 6. Define *the set of maximal connected fully cyclic sets (in* A*)* as $C^*_M(A)=\{x\in C_M(A)|\ (y, x)\notin T(P|A)$ for all $y\in A\}$.

  By the definition, $C^*_M(A)\subseteq M_q(A, R)$. But, $C^*_M(A)$ might be empty since the elements of the most preferred connected fully cyclic sets might be preferred by some other alternatives in A, i.e., $C_M(A)\cap M_q(A, R)=\varnothing$. By using the notion of the set of maximal fully cyclic sets, the content of a quasi-maximal set is given by the following formula.

Theorem 1. For given $A\in\wp(X)$, $M_q(A, R)=M(A, R)\cup C^*_M(A)$.
(Proof) Remind that $C^*_M(A)\subseteq M_q(A, R)$. Thus, if $x\in C^*_M(A)$, then $x\in M_q(A, R)$. If $x\in M(A, R)$, then there is no $y\in A$ such that $(y, x)\in P(R)$. Thus, $(x, y)\in R$ for all $y\in A$. Then, $(x, y)\in T_q(R|A)$ for all $y\in A$, i.e., $x\in M_q(A, R)$. Hence, if $x\in M(A, R)\cup C^*_M(A)$, then $x\in M_q(A, R)$. Conversely, if $x\in M_q(A, R)$, there is no $y\in A$ such that $(y,$

x)∈P(T$_q$(R|A)). Then, (x, y)∈T$_q$(R|A) for all y∈A. If x∉C(A), then M$_q$(A, R)=M(A, R). Thus, x∈M(A, R). Since C*$_M$(A)⊆M$_q$(A, R), if x∈C(A), then x∈C*$_M$(A). Hence, x∈M(A, R)∪C*$_M$(A). □

  Theorem 1 shows that an element of the quasi-maximal set M$_q$(A, R) is either an element of the maximal set M(A, R) or an element of the set of the maximal connected fully cyclic set C*$_M$(A). This formula gives us a new view of 'maximum,' if A includes a fully cyclic set.
  Deb(1977) considered a maximal set with respect to the transitive closure of a strict preference restricted to a set A, i.e., M(A, T(P|A)), and he showed that for any A∈℘(X), O(A, P)=M(A, T(P|A)). The following is substantially the same result as Deb's theorem although the proof is different.

Proposition 5 (Deb(1977)). Let P=P(R). For any A∈℘(X), O(A, P)=M$_q$(A, R).
(Proof) Define B=M(A∖C(A), R). B is the maximal set of A∖C(A), i.e., the maximal set of the set in A that does not include any fully cyclic set. By Proposition 1 (2), for any x∈B and either (x, y)∈(R|A) for any y∈C$_M$(A) or (y, x)∈(R|A) for any y∈C$_M$(A) holds. Suppose (x, y)∈P(R|A) for any x∈B and any y∈C$_M$(A). Then, ((A∖B)×B)∩P(R|A)=∅ and ∀B′⊂B, ((A∖B′)×B′)∩P(R|A)≠∅. Hence B=O(A, P(R|A)). In this case, M(A, T$_q$(R|A))=B. Suppose that (y, x)∈P(R|A) for any x∈B and y∈C$_M$(A). ((A∖C$_M$(A))× C$_M$(A))∩P(R|A)=∅ and ∀B″⊂C$_M$(A), ((A∖B″)× B″)∩P(R|A)≠∅. Hence C$_M$(A)=O(A, P(R|A)). In this case, M(A, T$_q$(R|A))=C$_M$(A)=C*$_M$(A). Suppose (x, y)∈I(R|A) for any x∈B and for any y∈C$_M$(A). Then, ((A∖C$_M$(A)∪B)×C$_M$(A)∪B)∩P(R|A)=∅ and ∀C′⊂C$_M$(A)∪B, ((A∖C′)× C′)∩P(R|A)≠∅. Hence, C$_M$(A)∪B=O(A, P(R|A)). In this case M(A, T$_q$(R|A))=C$_M$(A)∪B=C*$_M$(A)∪B. Therefore, for any A∈℘(X), O(A, P(R|A))=M(A, T$_q$(R|A)). □

Remark 3. By using Theorem 1 and Proposition 5, Proposition 3 can be easily shown. By Proposition 5, for any A∈℘(X), O(A, P)=M$_q$(A, R). Then, for any A∈℘(X), M(A, R)⊆O(A, P) because M$_q$(A, R)=M(A, R)∪C*$_M$(A). If A does not include any fully cyclic set, then M$_q$(A, R)=M(A, R)=O(A, P) because C*$_M$(A)=∅. □

3.   The conditions for a revealed preference theory with preference cycles.
  At first, define a choice function as follows. A *choice function* f is a function from ℘(X) to ℘(X) such that f(A)≠∅ and f(A)⊆A for every A∈℘(X). This function can be

considered as the decision-maker's choice behavior. By using the notion of quasi-maximal set, we would like to consider the conditions of a choice function to induce a revealed preference relation satisfying several desirable properties. In other words, it is considered a form of rationalization of the choice function satisfying several appropriate conditions.

Define *a revealed (weak) preference* R^* by the following: For some $x, y \in X$ and some $A \in \wp(X)$, $(x, y) \in R^*$ if and only if $x \in f(A)$ and $y \in A$. The binary relations P^* and I^* are defined by $P^*=P(R^*)$ and $I^*=I(R^*)$. Also define *a revealed (weak) base preference* R_b by the following: For some $x, y \in X$ and some $A \in \wp(X)$, $(x, y) \in R_b$ if and only if $x \in f(\{x, y\})$. Similarly, P_b and I_b are defined by $P_b=P(R_b)$ and $I_b=I(R_b)$. By the definition, both R^* and R_b are complete. Although it is possible to consider incomplete preferences, we restrict our attention to the cases in which they are complete in this paper. (See Oginuma(2010) for the analysis of incomplete preference.)

The following properties in Sen (1971) are standard conditions to rationalize choice functions.

Property α: For $\forall A, B \in \wp(X)$ and for $\forall x, y \in A$, if $x \in f(B)$ and $A \subseteq B$, then $x \in f(A)$.

Property β: For $\forall A, B \in \wp(X)$ and for $\forall x, y \in f(A)$, If $A \subseteq B$, then $x \in f(B)$ if and only if $y \in f(B)$.

Property γ: Let M be a set of sets in X, let B be the union of all sets in M. If $x \in f(S)$ for all $S \subseteq M$, then $x \in f(B)$.

Property δ: For $\forall A, B \in \wp(X)$ and for $\forall x, y \in A \subseteq B$, if $x, y \in f(A)$, then $\{x\} \neq f(B)$.

Sen(1971) showed the statements in the following remark.

Remark 4 (Sen(1971)).

(1) $R^*=R_b$ if the choice function f satisfies Property α.

(2) A choice function f satisfies Property α and Property β if it is rationalizable by complete and transitive R^*, i.e., $f(A)=M(A, R^*)$ for any A in $\wp(X)$.

(3) A choice function f satisfies Property α and Property γ if it is rationalizable by complete and acyclic R^*, i.e., $f(A)=M(A, R^*)$ for any A in $\wp(X)$.

(4) A choice function f satisfies Property α, Property γ and property δ if it is rationalizable by complete and quasi-transitive R^*, i.e., $f(A)=M(A, R^*)$ for any A in $\wp(X)$.

If we permit the possibility of preference cycles, a choice function f might not satisfy Property α. If we don't suppose Property α, R^* should be distinguished from R_b because they are not necessarily the same.

Example 2. In general, R_b is different from R*. Let X be X={x, y, z}. Consider a choice function such that f({k})={k} for k∈{x, y, z}. f({x, y})={x}, f({y, z})={y}, f({x, z})={z} and f(X)=X. Suppose A={x, y} and B=X. Then, x, y∈f(B), x∈f(A), and y∉f(A). Thus, (y, x)∈R* but (y, x)∉R_b. Denote Δ as Δ={(x, x), (y, y), (z, z)}. Actually, R*={(x, y), (y, x), (y, z), (z, y), (x, z), (z, x)}∪Δ and R_b={(x, y), (y, z), (z, x)}∪Δ.

Consider the case in the former Remark 2 (1). If R_b includes a preference cycle, i.e., A={x, y, z} and (x, y), (y, z), (z, x)∈P(R_b), then the notion of the maximal set doesn't work , i.e., M(A, R_b)=∅. Because of this, the rationalization of the choice function f by the revealed base relation R_b , i.e., f(A)=M(A, R_b) is impossible. However, it is possible to consider a quasi-rationalization, i.e., f(A)=M_q(A, R_b). In other words, if we assume that the decision-maker adopts quasi-rational behavior, i.e., choose an alternative in M_q(A, R) if he or she has a preference R, then the choice function can been seen as a result of his or her quasi-rationalization.

The notions that are defined in the former section, can be used in the following by setting R=R_b. Thus, all fully cyclic sets {C_i}$_{i∈fc(X)}$ and all connected fully cyclic sets {C^*_j}$_{j∈fc^*(X)}$ can be treated as 'observable.' Therefore, all propositions in the section 2 are still valid in the following.

The following two conditions about choice functions do not contradict with quasi-maximal sets. These two conditions are sometimes supposed for pseudo-transitive rationalization. (For example, these two conditions are considered in Jamison and Lau(1973), and Fishburn(1975).)

Define (x, y)∈PIP if there exists some v, w∈X such that (x, v), (w, y)∈P and (v, w)∈I. A binary relation R is called *pseudo-transitive* if PIP⊆P where I=I(R) and P=P(R). Suppose R=P∪I. It is known that R is reflexive and pseudo-transitive if and only if P is *an interval order*, i.e., for ∀∈X, if (x, y), (v, w)∈P, then either (x, w)∈P or (v, y)∈P. (See, for example, Lemma 1 in Aleskerov, Bouyssou and Monjardet(2002), pp59-60.)

WWARP(Weakened Weak Axiom of Revealed Preference): If x, y∈X and there is A∈℘(X) such that x∈f(A), y∈A\f(A), then there is no B∈℘(X) such that y∈f(B), x∈B\f(B).

IIA(independence of irrelevant alternatives): For given A∈℘(X), if B⊆A\f(A), then f(A)=f(A\B).

Example 3. Let X be X={x, y, z} and f a choice function defined on ℘(X).

(1) Suppose $f(\{x, y\})=\{x\}$, and $f(\{y, z\})=\{y\}$. Then, WWARP and IIA requires that $f(\{x, y, z\})=\{x\}$ or $\{x, y, z\}$. If $f(\{x, z\})=\{x\}$, then $f(\{x, y, z\})=\{x\}$ or $\{x, y, z\}$. If $f(\{x, z\})=\{z\}$ or $\{x, z\}$, then $f(\{x, y, z\})=\{x, y, z\}$.

(2) Suppose $X=\{x, y, z\}$, and (x, y), $(y, z)\in P_b$. Then, if $(z, x)\in P_b$, $M_q(X, R_b)=\{x, y, z\}$. If $(x, z)\in P_b$, $M_q(X, R_b)=\{x\}$. Also, if (x, z), $(z, x)\in R_b$, $M_q(X, R_b)=\{x\}$.

Comparing (1) and (2) in the example, it is shown that the combination of WWARP and IIA is too weak to assure $f(A)=M_q(A, R_b)$ for any $A\in \wp(X)$. Consider the case: $f(\{x, y\})=\{x\}$, $f(\{y, z\})=\{y\}$ and $f(\{x, z\})=\{x, z\}$. Then, $f(\{x, y, z\})=\{x, y, z\}$ while $M_q(\{x, y, z\}, R_b)=\{x\}$.

To exclude the cases such that $f(A)\neq M_q(A, R_b)$ for some $A\in \wp(X)$, we'd like to introduce the following axiom.

NBDC(negative binary dominance consistency): For $\forall A\in \wp(X)$ and $\forall x\in A$, if there exists $y\in A$ such that $\{y\}=f(\{x, y\})$ and $\forall z\in A$, $z\in f(\{x, z\})$, then $x\notin f(A)$.

Since we do not exclude the preference cycles, we need the following definition.

Definition 5. Let R be a binary relation defined on X.
(1) $R|A$ is called *acyclic transitive* if and only if $\forall B\subseteq A$, $R|B$ is transitive when it is acyclic.
(2) $R|A$ is called *acyclic quasi-transitive* if and only if $\forall B\subseteq A$, $P(R|B)$ is transitive when it is acyclic.
(3) $R|A$ is also called *acyclic pseudo-transitive* if and only if $\forall B\subseteq A$, $P(R|A)$ is pseudo-transitive when it is acyclic.

Adding NBDC with WWARP and IIA, it assures that $R_b|A$ is acyclic pseudo-transitive.

Proposition 6. Suppose that f satisfies WWARP, NBDC and IIA.
(1) If $\forall A\in \wp(X)$, $R_b|A$ is acyclic, then $R_b|A$ is quasi-transitive.
(2) If $\forall A\in \wp(X)$, $R_b|A$ is acyclic, then $R_b|A$ is pseudo-transitive.

(Proof) By the definition of a choice function, $f(\{x\})=\{x\}$ for $x\in A$. Then, $R_b|A$ is reflexive. (1) Suppose x, y, z$\in A$ such that $f(\{x, y\})=\{x\}$, $f(\{y, z\})=\{y\}$. If f satisfies WWARP, IIA and NBDC, then $f(\{x, y, z\})=\{x\}$ or $\{x, y, z\}$. $f(\{x, y, z\})=\{x\}$ is compatible with $f(\{x, z\})=\{x\}$, and $f(\{x, y, z\})=\{x, y, z\}$ is compatible with $f(\{x, z\})=\{z\}$. Thus, $R_b|A$ is acyclic quasi-transitive. (2) Suppose w, x, y, z$\in A$ such that $f(\{x,$

y})={x}, f({y, z})={y, z} and f({w, z})={z}. Then, f({x, y, z})={x} or {x, z}, f({w, y, z})={y, z} and f({w, x, y, z})={x} or {x, z} by WWARP, IIA and NBDC. Suppose f({w, x})={w}. Then, f({w, x, z})={z} or {w, x, z}. If f({w, x, z})={z}, then f({x, z})={z}. By Proposition 6 (1), it means f({y, z})={z} or {x, y, z} is a fully cyclic set. If f({y, z})={z}, then it contradicts with the supposition: f({y, z})={y, z}. Thus, f({w, x})≠{w}. Next, suppose f({w, y})={w}. Then, by Proposition 6 (1), it means f({y, z})={z} or {w, y, z} is a fully cyclic set. If f({y, z})={z}, then it contradicts with the supposition: f({y, z})={y, z}. Thus, f({w, y})≠{w}. Suppose f({w, x})={w, x}. Then, f({w, x, y})={w, x}, and f({w, y, z})={y, z}. This contradicts with WWARP because y∈f({w, y, z}), w∉f({w, y, z}), w∈f({w, x, y}) and y∉f({w, y, z}). Hence, f({w, x})={x}. Thus, R_b|A is acyclic pseudo-transitive.□

Proposition 6 states that R_b|A is acyclic pseudo-transitive if a choice rule f satisfies WWARP, IIA and NBDC. Note that (1) of Proposition 6, is an implication of (2) in Proposition 6, because pseudo-transitivity implies transitivity, i.e., if (x, z)∈P, (z, z)∈I, and (z, y)∈P, then (x, y)∈P. (Also note that in the former proof of Proposition 6 (2), Proposition (1) is used.)

The following condition is one of the standard conditions to characterize choice functions.

BDC(binary dominance consistency): If {x}=f({x, y}) for all y in A, then {x}=f(A).

BDC says that if x is chosen from x and any other alternative in A, then x should be the chosen alternative from A.

Proposition 7. If a choice function f satisfies WWARP, IIA ad NBDC, then it satisfies BDC.
(Proof) If {x}=f({x, y}) for all y in A, then x∈f(A) by WWARP. Let y∈A be y≠x and {x}=f({x, y}). Then, y∉f(A). Because of IIA, y∈f(A) is possible when A=f(A). However, there exists z∈A such that there is no w∈A such that {z}=f({w, z}). Then, z∉f(A) by NBDC. Thus, y∉f(A) for any y≠x in A. Hence, f(A)={x}.□

Proposition 7 says that if a choice function satisfies WWARP, IIA, NBDC, then it satisfies BDC. By using these three propositions, we'd like to show the following theorem.

Theorem 2. A choice function f satisfies WWARP, NBDC and IIA if and only if $\forall A \in \wp(X)$, $f(A)=M_q(A, R_b)=M(A, R_b)\cup C^*{}_M(A)$, and $R_b|A$ is reflexive and acyclic pseudo-transitive.

(Proof) Firstly, consider the if-part of the theorem. Suppose $f(A)= M_q(A, R_b)$.

WWARP: Suppose $x\notin C(A)$. If $x\in M(A, R_b)\subseteq M_q(A, R_b)$ and $y\in A\setminus M_q(A, R_b)$, then $(x, y)\in R_b$. Alternatively suppose $x\in C(A)$. If $x\in M_q(A, R_b)$ and $y\in A\setminus M_q(A, R_b)$, then $x\in C^*{}_M(A)$ and $y\in A\setminus C^*{}_M(A)$. Then, $(x, y)\in R_b$. Thus, if $x\in M_q(A, R_b)$ and $y\in A\setminus M_q(A, R_b)$, then $(x, y)\in R_b$. Suppose $B\in \wp(X)$ and $x, y\in B$. If $y\in M_q(B, R_b)$ and $x, y\notin C(B)$, then $x\in M_q(B, R_b)$ since $(x, y)\in I(R_b)$. If $y\in M_q(B, R_b)$, $y\in C(B)$ and $x\notin C(B)$, then $x\in M_q(B, R_b)$ since $(x, y)\in I(R_b)$. If $y\in M_q(B, R_b)$, and $x, y\in C(B)$, then $x\in M_q(B, R_b)$ since $(x, y)\in I(T_q(R_b))$. Thus, there exists no $B\in \wp(X)$ such that $y\in M_q(B, R_b)$ and $x\in B\setminus M_q(B, R_b)$.

NBDC: For $\forall A\in \wp(X)$ and $\forall x\in A$, if there exists $y\in A$ such that $\{y\}=f(\{x, y\})$ and $\forall z\in A$, $z\in f(\{x, z\})$, i.e., $(y, x)\in P_b$ and $(z, x)\in R_b$ for all z in A, then by the definition of quasi-maximal sets, $x\notin M_q(A, R_b)$. Hence, $x\notin f(A)$.

IIA: If $B\subseteq A\setminus M_q(A, R_b)$, then there exists $y\in A\setminus B$ such that $(y, x)\in P(T_q(R_b|A))$ for any $x\in B$. Any fully cyclic set C_i in A is either $C_i\subseteq M_q(A, R_b)$ or $C_i\subseteq A\setminus M_q(A, R_b)$. Thus, if $x\in M_q(A, R_b)$, then $(x, y)\in T_q(R_b|A)$ for all $y\in A$. Thus, $(x, y)\in T_q(R_b|A)$ for all $y\in A\setminus B$. Then, $x\in M_q(A\setminus B, R_b)$. Conversely, if $x\in M_q(A\setminus B, R_b)$, then $(x, y)\in T_q(R_b|A)$ for all $y\in A\setminus B$. By the definition of B and $x\in A\setminus B$, $(x, y)\in T_q(R_b|A)$ for any $y\in B$. Then, $x\in M_q(A, R_b)$. Hence, $M_q(A, R_b)=M_q(A\setminus B, R_b)$.

Next, consider the only if-part of the theorem. Define R_b, P_b, and I_b as follows: $(x, y)\in R_b$ if $x\in f(\{x, y\})$, $(x, y)\in P_b$ if $\{x\}=f(\{x, y\})$, and $(x, y)\in I_b$ if $\{x, y\}=f(\{x, y\})$. Proposition 6 shows that $R_b|A$ is acyclic pseudo-transitive if f satisfies WWARP, IIA and NBDC. Let $C(A)$ be the set of fully cyclic sets in A. Suppose $x\in A\setminus C(A)$ for some $A\in \wp(X)$ and $x\in f(A)$. Then, there exists no $y\in A$ such that $\{y\}=f(\{x, y\})$ by WWARP and IIA. Remind that if $y\in f(A)$ contradicts with $\{y\}=f(\{x, y\})$ by IIA. Then, $x\in M(A, R_b)$. Let $C_M(A)$ be the set of most preferred connected fully cyclic sets in A. Suppose $x\in C(A)$ for some $A\in \wp(X)$ and $x\in f(A)$. Then, $x\in C_M(A)$ by Proposition 1 and Proposition 2. If $f(A)\cap C_M(A)\neq\varnothing$, then $C_M(A)=C^*{}_M(A)$. Thus, $x\in C^*{}_M(A)$. If $x\in f(A)$, then either $x\in f(A)\cap C(A)$ or $x\in f(A)\setminus C(A)$. Hence, if $x\in f(A)$, then $x\in M(A, R_b)\cup C^*{}_M(A)$. $\square$

By Theorem 2, $f(A)=M_q(A, R)=M(A, R)\cup C^*{}_M(A)$, and $R_b|A$ is acyclic psudo-transitive. If you would like to get acyclic transitive R_b, the conditions for the choice function must be strengthened. One candidate is the following.

RWARP(Restricted Weak Axiom of Revealed Preference): Suppose x, y∈X, and A∈℘(X). Suppose f(A)∩C(A)=∅. If x∈f(A) and y∈A∖f(A), then there is no B∈℘(X) such that y∈f(B) and x∈B.

   RWARP means that, if x, an element chosen from A, does not belong to a cycle in A and y is not chosen from A, then y is not chosen in any set B if x is an element of B.

Proposition 8. If a choice function f satisfies IIA and RWARP, then it satisfies NBDC.
(Proof) If x, y∈A and {y}=f({x, y}) and z∈f({x, z}) for any z∈A, then x does not belong to any fully cyclic set in A because there exists no w∈A such that {x}=f({x, w}). Then, for A′∈℘(X) such that x, y∈A′, x∉f(A′) by RWARP. Thus, x∉f(A).□

Theorem 3. A choice function f satisfies WWARP, RWARP and IIA if and only if ∀A∈℘(X), there exists a preference relation R_b such that $f(A)=M_q(A, R_b)$. For ∀A∈℘(X), $R_b|A$ is reflexive and acyclic transitive.
(Proof) Since Proposition 8 holds, Theorem 2 shows that ∀A∈℘(X), $f(A)=M_q(A, R_b)$ and $R_b|A$ is acyclic pseudo-transitive. If {x, y}=f({x, y}), then by RWARP, there is no A, B∈℘(X) such that A∩C(A)=∅, x∈f(A) and y∈A∖f(A) and y∈f(B) and x∈B. Thus, if {x, y}=f({x, y}), and {y, z}=f({y, z}), then {x, y, z}=f({x, y, z}). Thus, {x, z}=f({x, z}) by RWARP. Similarly, if {x}=f({x, y}), and {y, z}=f({y, z}), then {x}=f({x, y, z}). Thus, {x}=f({x, z}). If {x, y}=f({x, y}), and {y}=f({y, z}), then {x, y}=f({x, y, z}). Thus, {x}=f({x, z}). From Proposition 6 (1), $P_b|A$ is transitive when it is acyclic. Hence, $R_b|A$ is acyclic transitive.

   Since $R_b|A$ is acyclic transitive, $f(A)=M_q(A, R_b)$ satisfies WWARP, IIA and NBDC by Theorem 2. Furthermore, if x∈f(A) and y∉f(A) for A∩C(A)=∅, then (x, y)∈P(R_b) for all y∈B and B∈℘(X). Hence $f(A)=M_q(A, R_b)$ satisfies RWARP. □

Note that the transitivity of $R_b|A$ is not assured if {x, y, z}⊆B⊆A and B is a fully cyclic set in A. RWARP only requires that if x is at least as good as y and y is at least as good as z, then x is at least as good as z when x does not belong to any fully cyclic set in A.

4. Resolute choices and a top-cycle rule: a special case
   Top-cycle rules are choice functions that can be compatible with preference cycles. Thus, we'd like to consider a characterization of top-cycle rules as a special case.

A binary relation R is *tournament* if R is both complete and asymmetric, i.e., if (x, y)∈R, then (y, x)∉R.

Definition 5. Given tournament R on X and A∈℘(X), the *top-cycle* of R in A, denoted t(R|A) is the set of maximal elements of T(R|A) in A: x∈t(R|A) if and only if x∈M(A, T(R|A)), i.e., (x, y)∈T(R|A) for all y∈A∖{x}.

Definition 6. A choice function f is a *top-cycle rule* if there is a tournament R on X such that f(A)=t(R|A) for all A∈℘(X).

Ehlers and Sprumont(2008) give us a characterization of a top-cycle rule by using the following conditions.

WWARP(Weakened Weak Axiom of Revealed Preference): If x, y∈X and there is A∈℘(X) such that x∈f(A), y∈A∖f(A), then there is no B∈℘(X) such that y∈f(B), x∈B∖f(B).
BDC(Binary Dominance Consistency): If A∈℘(X), x∈A, and f({x, y})={x} for all y∈A∖{x}, then f(A)={x}.
WCC(Weak Contraction Consistency): If A∈℘(X), and |A|≥2, then f(A)⊆∪$_{x∈A}$f(A∖{x})

Note that NBDC is independent with BDC and WCC.

Remark 5. (1) NBDC is different from BDC. (2) NBDC is different from WCC.
(Proof) (1) Suppose that f({x, y})={y}, f({y, z})={z} and f({x, z})={x, z}. Then, NBDC implies x∉f({x, y, z}), while BDC is compatible with x∈f({x, y, z}). (2) Suppose that f({x, y})={y}, f({y, z})={z} and f({x, z})={x, z}. Then, NBDC implies x∉f({x, y, z}), while WCC is compatible with x∈f({x, y, z}). □

Denote the cardinality of the set A as |A.| A choice function f is called *resolute* if |f(A)|=1 if |A|=2.

Proposition 9. If a choice function f is resolute and satisfies WWARP, then f satisfies IIA and NBDC if and only if it satisfies BDC and WCC.
(Proof) By Proposition 7, if f satisfies WWARP, IIA and NBDC, then f satisfies BDC. By Theorem 2, if f satisfies WWARP, IIA and NBDC, then f(A)=M(A, R$_b$)∪C*$_M$(A).

Since f is resolute, i.e., $|f(A)|=1$ if $|A|=2$, either $f(A)=M(A, R_b)$ or $f(A)=C^*_M(A)$. For given A, $|f(A)|=1$ when $f(A)=M(A, R_b)$. Suppose $|f(A)|=1$. Then, $\cup_{x\in A}f(A\setminus\{x\})=f(A\setminus f(A))\cup f(A)$. Hence, WCC holds. If $f(A)=C^*_M(A)$, then $\cup_{x\in A}f(A\setminus\{x\})=C^*_M(A)$. Hence, WCC holds.

Conversely, suppose f is resolute and satisfies WWARP, BDC and WCC. If $x, y\in A$ and $f(\{x, y\})=\{x\}$, then $y\notin f(A)$. Because of WWARP, if $y\in f(A)$, then $x\in f(A)$. Since f is resolute, $|f(A)|=1$ for $f(A)\cap C(A)=\varnothing$. If there exists $y\in A$ such that $\{y\}=f(\{x, y\})$ and $\forall z\in A$, $z\in f(\{x, z\})$, then $f(A)\cap C(A)=\varnothing$. Thus, $x\notin f(A)$, i.e., NBDC holds. If $|f(A)|=1$ and $\{x\}=f(A)$, then $\{x\}=f(A\setminus\{y\})$ for $y\neq x$. By using this relation repeatedly, it is shown that $\{x\}=f(A\setminus B)$ for $B\subseteq A\setminus f(A)$. If $x\in f(A\setminus B)$ for $B\subseteq A\setminus f(A)$, then $f(A)\cap B=\varnothing$. Suppose $x\notin f(A)$ and $x\in A\setminus B$. Then, there exists $y\in B$ such that $f(\{x, y\})=\{y\}$, contradiction. Thus, IIA holds.□

Theorem 4. Suppose a choice function f is resolute. Then, the following three statements are equivalent.
 (1) The choice function f satisfies WWARP, IIA and NBDC.
 (2) The choice function f satisfies WWARP, BDC and WCC.
 (3) The choice function f is a top-cycle rule.
(Proof) Since Proposition 9, the statement (1) holds if and only if the statement (2). By Theorem 2, $f(A)=M(A, R_b)\cup C^*_M(A)$ for all $A\in \wp(X)$. Since f is resolute, R_b is a tournament. Thus, for all $A\in \wp(X)$, $f(A)=M(A, R_b)$ with $|M(A, R_b)|=1$ or $f(A)=C^*_M(A)$. Hence, the statement (1) holds if and only if the statement (3) holds. □

The equivalence between the statement (2) and the statement (3) is already shown by Ehlers and Sprument (2008). But the proof is different from ours. Since Theorem 3 can be seen as a special case of Theorem 2, $f(A)=M_q(A, R_b)$ can be seen as a generalization of a top-cycle rule.

5. An alternative model of choice and discussions.
  In this paper, we considered the conditions for choice functions inducing revealed preferences that are compatible with preference cycles. To "rationalize" such choice functions, the notion of extended-maximal sets is introduced. There is an alternative notion of maximal sets that "rationalizes" cyclic choices.

Definition 13. For given $A\in \wp(X)$, $M_e(A, R_b)$ is an *extended-maximal set* if and only if

$M_E(A, R_b)=M(A, R_b)$ if $M(A, R_b)\neq\varnothing$

$\qquad\qquad=C^*_M(A)$ if $M(A, R_b)=\varnothing$

The extended-maximal set is a natural extension of the (standard) maximal set because $M_E(A, R_b)$ equals to $M(A, R_b)$ if $M(A, R_b)$ is nonempty.

Example 4. The extended-maximal set is different from the quasi-maximal set. Suppose $X=\{w, x, y, z\}$, and $R=\{(x, y), (y, z), (z, x), (x, w), (w, x), (y, w), (w, y), (w, z), (z, w), (x, x), (y, y), (z, z)\}$. Then, $T_q(R)=\{ (x, y), (y, x), (y, z), (z, y), (z, x), (x, z), (x, w), (w, x), (y, w), (w, y), (z, w), (w, z), (x, x), (y, y), (z, z)\}$.

  Thus, $M_q(X, R)=M(X, T_q(R))=\{w, x, y, z\}$ while $M_E(X, R)=M(X, R) =\{w\}$.

  Example 4 shows that the cycles are excluded from any extended-maximal set. Since quasi-maximal set doesn't have this property, quasi-maximal sets are different from extended-maximal sets. This difference caused by the respective choice procedures.

  The quasi-maximal sets can be seen as the following procedure: (1) Taking quasi-transitive closure of the preference relation conditioned by A, i.e., treating the elements of the same cycle in A as "indifferent". (2) Based on the quasi-transitive closure of preference relation, a choice is made by selecting one element in the maximal set. Similarly, the extended-maximal set can be seen as the following procedure: (1) Computing the maximal set. (2) If the maximal set is nonempty, then a choice is made by selecting one element in the quasi-maximal set. (3) If the maximal set is empty, then taking quasi-transitive closure of the preference relation conditioned by A, and selects an element from the quasi-maximal set.

  It is difficult to judge which procedure is realistic. Probably experiments or appropriate observations are needed to answer the question. As for a formal difference, it can be captured by the following condition.

PNCA(priority of non-cyclic alternatives):If $f(A)\setminus C(A)\neq\varnothing$, then $f(A)\cap C(A)=\varnothing$

Theorem 5: A choice function f satisfies WWARP, NBDC, IIA and PNCA if and only if $\forall A\in\wp(X)$, $f(A)=M_E(A, R_b)$.

(Proof) Only-if part: Since $M_E(A, R_b)$ is either $M(A, R_b)$ or $C^*_M(A)$, $\forall A\in\wp(X)$, $f(A)=M_E(A, R_b)$ satisfies WWARP, NBDC, IIA. PNCA is a direct implication of the extended-maximal set $M_E(A, R_b)$.

If-part: If a choice function f satisfies WWARP, NBDC, and IIA, then $\forall A\in\wp(X)$,

$f(A)=M_q(A, R_b)$ by Theorem 2. Since f satisfies PNCA, $M_q(A, R_b)=M(A, R_b)$ if $M(A, R_b)\neq\varnothing$. It means $M_q(A, R_b)=C^*_M(A)$ if $M(A, R_b)=\varnothing$. Hence, $\forall A \in \wp(X)$, $f(A)=M_E(A, R_b)$. □

PNCA requires as follows: if an alternative that does not belong to any cyclic set in A exists, then any alternative in a cyclic set in A cannot be chosen in A.

An extended maximal set can be seen as a model of procedural choice. Although the conditions for Manzini and Mariotti's model of "Rational Shortlist Method" have some similarity with our model, they are rather different procedures.

Manzini and Mariotti(2007) formulated the following procedural decision method.

RSM(Rational Shortlist Method): $M(M(A, R_1), R_2)=\{x\in A|\ (y, x)\notin P(R_2)$ for all $y\in\{z\in A|\ (w, z)\notin P(R_1)$ for all $w\in A\}$ for all $A\in\wp(X)$.

Consider the relation between Manzini and Mariotti's RSM and our choice based on extended-maximal sets. The following two conditions are needed to characterize RSM.

WARP*: For all A, B$\in \wp(X)$, If $\{x, y\}\subseteq A\subseteq B$, and $\{x\}=f(\{x, y\})=f(B)$, then $y\notin f(A)$.
EXPANSION: For all A, B$\in \wp(X)$, if $\{x\}=f(A)=f(B)$, then $\{x\}=f(A\cup B)$.

Remark 6. (1) WARP* can be derived from IIA. Suppose IIA, then if $\{x\}=f(B)$ and $\{x, y\}\subseteq A\subseteq B$, then $f(A)=f(\{x, y\})=\{x\}$. So, WARP* is a weaker condition of IIA.
(2) $y\neq x$ and $y\in A\cup B$, then $\{x\}=f(\{x, y\})$ by IIA. Hence, by NBDC implies EXPANSION. Thus, EXPANSION is a weaker condition of NBDC.

Unique choice: Let f be a choice function defined on F. For all $A\in \wp(X)$, $|f(A)=1|$.

Although two conditions in Manzini and Mariotti(2007) are weakenings of our conditions but they supposed that the range of f is a singleton, i.e., $f(A)\in X$ for any $A\in \wp(X)$. The assumption of unique choice might be too strong in some situations since such choice functions also exclude the cases where multiple alternatives are chosen.

Finally, we'd like to consider the examples of cyclic revealed preferences.

Remark 7. Suppose the agent has multiple preferences on X, i.e., there are n valuations that correspond to preferences for the same decision-maker. Denote R^i as a binary relation that corresponds to i-th type of valuation. For any i=1, 2, ., n, $R^i\subseteq X\times X$. One

interpretation of our model of revealed preference is as follows: Suppose that $D=(D^1,$ $D^2, .., D^n)$ is *an assignment of choice domains (or a system of rights))* on X, i.e., for any i, $D^i \subseteq X \times X$ and $\cup_{i \in \{1, 2,.., n\}} D^i = X \times X$. Let $R = \cup_{i \in \{1, 2,.., n\}} R^i \cap D^i$. Even if all R^i are complete and transitive, then R is complete but not necessarily transitive. Such multiplicity of preferences can be considered one source of cyclic preferences. The other source of cyclic preferences is indecisiveness originated from shortage of information about alternatives. This type of problem is considered in Oginuma(2010).

Appendix:　Money pump and preference cycles

  One strong theoretical critique of preference cycles is the money pump argument. If the decision-maker has a preference with cycles, then she suffers money loss by a sequence of trade offers by some clever agent. Consider the following example.

Example A1: Let X be the set of alternatives such that X={x, y, z}. Suppose there are two agents. One is the DM and another is the opponent. Let M be the set of money quantity such that M={q∈ℜ|q≥0}. Suppose DM has the initial wealth (x, 10) at first. Also suppose the DM has the following binary relation defined on X×M: R={((x, k), (x, k−η)) ((x, k−δ)), (y, k)), ((y, k−δ)), (z, k)), ((z, k−δ)), (x, k)) for any η such that k≥η≥0, and for given δ′>0 and for any δ such that δ′≥δ>0}. The opponent sequentially offers the following. Assume ε>0 is a sufficiently small number, i.e., δ′≥ε. (1) Offer (z, 0) in exchange of (x, ε). (2) If the DM has (z, 10−ε), then offer (y, 0) in exchange of (z, ε). (3) If the DM has (y, 10−2ε), then offer (x, 0) in exchange of (y, ε). The DM can respond to the offers by "yes" or "no". If the response is "yes", the offer by the opponent is executed while it is "no", the offer by the opponent is not executed and keep the status of that time.

  We can formalize this idea as the following description of sequential choice. As for the last game,
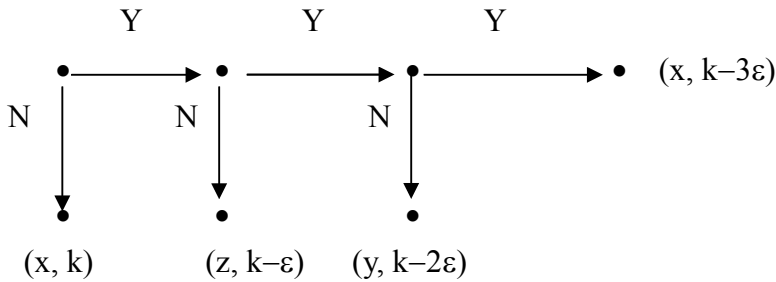
In the last decision node x the decision node N must be dominated by Y, i.e., ((x, k−3ε), (y, k−2ε)). Similarly, if the previous node is at the second, then the choice between G and N depends on the magnitude of 2ε. Suppose Y is chosen. Then, N must be chosen because (x, k) is preferred to (x, k−3ε). Thus, the DM is not vulnerable to the sequential offers by the opponent. Suppose N is chosen. Then, the DM's wealth is (z, k−ε). Thus, either (N, Y, Y), (Y, Y, N), or (Y, N, N) can be the subgame perfect equilibrium. (It is also the solution of backward induction.) This result cannot be seen as one kind of "money pump." Hence, even if the decision-maker has a preference with a cycle, she

might not suffer by "money pump" trade offers by the opponent. If the DM recognizes the total sequence of the trade offers, she does not accept all of the offers by the opponent.

Repetition of "money pump" trades might awake the DM and she recognizes the sequence of trades. If she adopts the backward induction reasoning, then she is difficult to be applied money pumping. Thus, it might be possible that the DM who has a preference with cycles can survive for not a few moments.

References.

Aleskerov, F., Monjardet, B. 2002. *Utility Maximization, Choice and Preference*.

Deb, R. 1977. On Schwartz's rule. *Journal of Economic Theory* 16, 103-110.

Ehlers, L., Sprumont, Y. 2008. Weakened WARP and Top-cycle choice functions. *Journal of Mathematical Economics* 44, 87-94.

Fishburn, P. C. 1975. Semiorders and choice functions. Econometrica 43, 975-976.

Jamison, D. T., Lau, L. J. 1973. Semiorders and the Theory of Choice. *Econometrica* 41, 901-912.

Lombardi, M. 2009. Reason-based choice correspondences. *Mathematical Social Sciences* 57, 58-66.

Manzini, P., M. Mariotti, 2007. Sequentially rationalizable choice. *American Economic Review* 97, 1824-1839.

Oginuma, T. 2010. Choice and Incomplete Preference. Mimeo.

Richter, M. K. 1966. Revealed Preference Theory. Econometerica, 34, 635-645.

Schwartz, T., 1972. Rationality and the Myth of the Maximum. *Noûs*, 97-117.

Sen, A. K., 1971. Choice Functions and Revealed Preference. *Review of Economic Studies*, 38, 307-317.

Suzumura, K., 1983. *Rational choice, collective decisions, and social welfare*. Cambridge University Press.