MAXIMAL PROGRAM OF A FOREST ABSORBING ATOMOSPHERIC CARBON DIOXIDE

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ABSTRACT. This paper considers optimal resource use of a forest which provides two types of beneficial goods, timber and service of removing carbon dioxide from the atmosphere. Using a variation of forest program model developed by Mitra and Wan (1985, 1986), some properties of maximal program of the forest are examined.

The results are the following: 1) There is some real positive number less than one such that when a discount factor greater than or equal to the number is used for utilization of the forest, there exists a maximal program starting from every initial forest stock. 2) In undiscounting case, for every maximal program starting from any initial forest, all trees are necessarily harvested in the long run. This implies that the forest is not utilized as a stock yard of carbon, but as an apparatus for absorbing atmospheric carbon dioxide. Also some turnpike properties are examined for both discounting and undiscounting cases.

1. Introduction

It is well known that the global climate is potentially affected by small change of atmospheric contents of some substances, so called greenhouse gases(GHGs). GHGs are observed to have been concentrating in the atmosphere since pre-industrial times (about 1750). Among the substances, carbon dioxide is assessed to be the most important cause of anthropogenic climate change(IPCC, 1996). Forests are the largest reservoir of carbon among terrestrial ecosystems. Forests also absorb the atmospheric carbon dioxide thorough the process of photo-synthesis accompanying growth in volume. Nowadays, in context of the issue of global warming, how to create and utilize forests is a great concern.

This paper considers optimal resource use of a forest which provides two types of socially beneficial goods, timbers and service of removing carbon dioxide from the atmosphere. Using a variation of forest program model developed by Mitra and Wan (1985, 1986), some properties of maximal forest program are examined.

Mitra-Wan model is a discrete time model and we also consider a discrete time model.For a continuous time model of forest program, see Heaps(1984).

This paper is constructed as the following: First, a single stand problem is established when a forest provides timbers as well as absorbs the atmospheric carbon as its growth. Then we show a forest program model and discuss some features of the model relating to multi-sector economic growth models. These two sections also introduce notation and assumption used in this paper. The existence of stationary maximal forest program is proved in the third section. The proof makes use of a technique developed by Mitra and Wan (1985, 1986), which exploits the relation between a single stand model(the Faustmann formula) and a forest program model and constructs relevant support prices. The forth section focuses undiscounting case. So called von Neumann facet of a stationary maximal forest program and turnpike property is examined. The fifth section turns to discounting case. It is shown that our model has the neighborhood turnpike property in generic.

2. A modified Faustmann formula

We start with the classic of optimization problem in forestry, which is initiated by Martin Faustmann, a German forester in the 19th century. Consider a even-aged forest such that the age of every tree in the forest is same and the area covered with the forest is one, measured by some unit. The problem is choice of harvesting ages of trees under the special program where every trees is cut at same time and every tree is planted at once. This is preliminary for studying more general model after this section. Notations and assumptions below are used throughout this paper.

Denote age of trees by $i = 0, 1, 2...$, where $i = 0$ means that the tree is a seedling just planted. Let F_i $(i = 1, 2, ...)$ stand for net value of timber harvested when all trees on the unit land are cut at age i. G_i $(i = 1, 2, ...)$ stands for total amount of carbon accumulated in trees whose ages are i and which stand on a unit forest land. Define g_i $(i = 1, 2, ...)$ with $g_1 = G_1, g_i = G_i - G_{i-1}$ for $i = 2, 3, ...$, that is, g_i denotes absorption rate of carbon by the forest. Let α represent a multiplier with which the value of removing one unit carbon from the atmosphere is translated into value of timber. It might be possible to interpret α as the Pigue subsidy.

Until harvesting is carried, growing trees continue to absorb atmospheric carbon dioxide. Suppose that when trees are harvested, timbers are supplied to society as well as a portion of carbon accumulated in the trees is released into the atmosphere. Assume the ratio of released carbon to so far accumulated carbon is exogenously fixed and denote the ratio by β. Since timber may be used as a durable good such as material of housing, it is natural to think that there is time lag between harvesting and release of carbon. But to introduce the time lag make the model too complicated to analyze. So we do not explicitly deal with the time lag. Interpret β as a parameter reflecting some time schedule of carbon release idiosyncratic to the society we consider. As final notation introduced in this section, ρ denotes the discount factor used in the forest program.

We use following assumptions:

[A.1] There is some positive integer N such that $F_i = F_N$, $G_i = G_N$ for all $i \geq N$.
[A.2] $F_N \geq 0$.
[A.3] $g_i \geq 0$, all i with at least one strict inequality.
[A.4] $\alpha \in (0, \infty)$.
[A.5] $\beta \in (0, 1)$.
[A.6] $\rho \in (0, 1]$.

[A.1] implies that the biological equilibrium exists about the forest and that within finite time, the forest reaches the natural steady state. By [A.2], we suppose that the forest in the biological equilibrium is profitable in terms of timber produce. [A.3] implies that trees can grow in some ages. The rate of carbon accumulation in a tree is approximately proportional to the growth rate of biomass of the tree(for example, FAO, 1995 regards half of the biomass as carbon). [A.4] means that to remove carbon dioxide from the atmosphere is socially beneficial. Regarding [A.5], if $\beta = 1$, it implies that all carbon fixed in trees is released into the atmosphere when the trees are harvested. It is the case where one completely burns down all of timbers. One might think that sooner or later timber becomes waste and waste is burned out or decomposed, so all accumulated carbon is virtually released into the atmosphere in the long run. But it seems plausible to think that some residual of carbon is left and it is reallocated to somewhere such as the soil or the bed of the sea. If not so, fossil fuel could not exist on earth. From this, it would be allowed that β is assumed to be less than one. On the other hand, if $\beta = 0$, none of carbon fixed in trees harvested is released into the atmosphere. Obviously the case seems unrealistic.[A.6] shows that we plan to consider discounting case as well as undiscounting case.

The problem we consider in this section is to choose the cutting ages which maximize land rent acquired from one unit bare land if harvesting and planting are repeated infinitely. When this problem is set up with infinite time horizon under stationary assumptions, the situation at the time when harvesting just has finished is same to the initial one no matter how many times harvesting and planting occur so far. This implies that the set of optimal cutting ages, if it exists, is same for every round. Therefore the problem can be written as the following.

$$\sup_i R(i;\alpha) = \left[\rho^i F_i + \alpha\left(\sum_{t=1}^{i} \rho^t g_t - \rho^i \beta G_i\right)\right] \frac{1-\rho}{1-\rho^i} \tag{2.1}$$

This problem was formulated by Martin Faustmann (Faustmann, 1849), so it is called the Faustmann formula (for the comprehensive economic interpretation, see Samuelson, 1976). In order that the problem (2.1) is meaningful, it is needed that there is some i such that $R(i) > 0$, which guarantees that planting is profitable so that planting and harvesting are necessarily repeated infinitely. From [A.2], [A.3], and [A.6],

$$R(N) > \left[\rho^N F_N + \alpha \sum_{t=1}^{N} (1-\beta) G_N\right] \frac{1-\rho}{1-\rho^N} > 0. \tag{2.2}$$

This confirms that the maximum value of $R(i)$ is positive.

Since the optimal cutting ages, the solutions of the problem (2.1), are needed to exists independently of values of α and to be less than or equal to N in this paper, we establish the sufficient condition.

Proposition 1. *There is some $\rho(\beta)$ such that for every $\rho \in [\rho(\beta), 1]$, the solutions of problem (2.1) exists and they are less than or equal to N.*

Proof. If $R(M) - R(M+1) > 0$ for every $M \geq N$, it is sufficient for the existence of solutions less than or equal to N.

Set $\rho(\beta) = \beta^{1/N}$ (< 1) and let $\rho \in [\rho(\beta), 1]$. Then, from [A.1], [A.2], and [A.3],

$$\begin{aligned} R(M) - R(M+1) &= \frac{\rho^M [F_N + \alpha \sum_{t=1}^{N} (\rho^t - \beta) g_t]}{(1+\rho+\cdots+\rho^M)(1+\rho+\cdots+\rho^{M-1})} \\ &\geq \frac{\rho^M [F_N + \alpha \sum_{t=1}^{N} (\rho^N - \beta) g_t]}{(1+\rho+\cdots+\rho^N)(1+\rho+\cdots+\rho^{M-1})} \\ &\geq 0 \end{aligned} \tag{2.3}$$

The first inequality holds with strict inequality when $\rho \in [\rho(\beta), 1)$. When $\rho = 1$, the second inequality holds with strict inequality. So $R(M) - R(M+1) > 0$, all $M \geq N$ if $\rho \in [\rho(\beta), 1]$. □

Throughout the rest of paper, following assumption is used:

[A.6'] $\rho \in [\rho(\beta), 1]$

3. Maximal program of a forest

Although number of useful economic implications can be drawn from the Faustmann formula (see Johansson and Löfgren, 1985, for example), the forest program is quite special, i.e. the intermittent forest program. The proposition 4.1 of Mitra *et al.* (1991) shows that when a forest program is formulated as a linear control problem, the Faustmann formula can be applied to more general forest program where all trees are not necessary cut down at once. On the other hand, linearity is not so plausible assumption when one consider management of a forest across broad area such as water basin, state, region across different nations, continent, and the globe. For example, as issues ignored in the Faustmann formula or a linear forest program, Dasgupta (1982, Chapter 9) pointed out the employment issue and intermittent timber supply. So we turn to a non-linear forest program.

The model is a simple version developed by Mitra and Wan (1985, 1986) and further analyzed by Mitra *et al.* (1991), Wan (1989, 1993, 1994). The major difference is that they consider a forest as the source of timber, whereas this paper consider two functions of a forest, i.e. timber produce and carbon absorption. In spite of the difference, technique developed by Mitra and Wan is so powerful to be applicable to our model and the following consideration is strongly owed to tools they developed.

3.1. Additional notation and assumption. In order to show the model, we need to introduce additional notation and assumptions.

Let $x_i, i = 0, 1, ..., N$ denote the area of land occupied by the trees whose age are i. Here x_0 represents the area just planted and x_N represents the area covered by trees whose age are greater than or equal to N. Recall that by [A.1], we can identify the trees whose age is greater than N with the trees whose age is N. Suppose that available land to forest management is bounded above and standardize the total area as unit. Let us refer to $x = (x_0, x_1, ..., x_N) \in R_+^{N+1}$ as a forest resource. Denote by ν the $N+1$ dimensional vector with all the elements being one. Then the set of feasible forest resources is represented by

$$X = \{x | \ x \ \in R_+^{N+1} \text{ and } \nu x \leq 1\}. \tag{3.1}$$

Define $N+1 \times N+1$ matrix A and $N \times N+1$ matrix B by

$$A = \begin{bmatrix} 0 & \cdots\cdots\cdots & & & 0 \\ 1 & 0 & \cdots\cdots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.2}$$

Denote time by $t = 0, 1, ...$ We refer to time interval between $t-1$ and t as t-th period. The period is taken as trees become old by 1 by each period. Let input, $x \in X$, be a forest resource at the begin of a period and output, $y \in X$, be a forest resource at the end of same period. If $B(Ax - y) \geq 0$, then the pair (x, y) is feasible. Denote the set of feasible input-output pairs by D, i.e.,

$$D = \{(x, y) | \ x, y \in X \text{ and } B(Ax - y) \geq 0\}. \tag{3.3}$$

When sequence $< x_t >_{t \geq 0}$ satisfies $x_0 \in X$, and $(x_{t-1}, x_t) \in D$, all $t = 1, 2, ...$, we refer to $< x_t >$ as a forest program starting from x_0. In particular, for x such that (x, x) in D, we call x a sustainable forest and refer to the program $x_t = x$, all t as a stationary forest program. Define the set of sustainable forests, Δ, with

$$\Delta = \{x | \ (x.x) \in D\}. \tag{3.4}$$

Pick up an age m and consider a unique sustainable forest, $x \in \Delta$ such that

$$(3.5) \qquad x_i = \begin{cases} 1/m, & i = 0, 1, ..., m-1, \\ 0, & i = m, m+1, ..., N. \end{cases}$$

We refer to this forest resource as the sustainable forest with cutting age m.

Using the notation in previous section, we define $F = (F_1, F_2, ..., F_N) \in R^N$, $G = (G_1, G_2, ..., G_N) \in R_+^N$, and $g = (g_1, g_2, ..., g_N, 0) \in R_+^{N+1}$. Suppose that at the end of each period, forest practice is carried. Denote value of timber produce and amount of net fixed carbon in a period by l and c, respectively. Then l and c acquired from a feasible input-output pair (x, y) is expressed as

$$(3.6) \qquad l = FB(Ax - y),$$

$$(3.7) \qquad c = gx - \beta GB(Ax - y),$$

respectively.

Let $w(l, c)$ be a one period social welfare function. As for w, we assume that

[A.7] $w: \ R^2 \to R$ is a continuous, strict increase, and strict concave C^1 function.

As the reduced form of $w(l, c)$, we will also use the function $u: \ D \to R$ defined with

$$(3.8) \qquad u(x, y) = w[FB(Ax - y), gx - \beta GB(Ax - y)]$$

3.2. The model. We consider the following problem:

$$(3.9) \qquad V(x_0) = \sup(\liminf_{T \to \infty} \sum_{t=1}^{T} \rho^t [u(x_{t-1}, x_t) - u^*]),$$

$$\text{subject to } (x_{t-1}, x_t) \in D, \text{ all } t \text{ and } x_0 \in X \text{ is given,}$$

where supremum is taken over feasible forest programs starting from x_0 and u^* is some real number with which $V(x_0)$ comes to be bounded above and below for every x_0 in X. In the case of $\rho < 1$, choice of u^* is arbitrary. On the other hand, in undiscounting case($\rho = 1$), we need to verify the existence of such u^*. It will be done later, but here premising the existence, let us state some basic remarks about this problem.

Remarks:

1. *A forest program realizing $V(x_0)$ exists for every x_0 in X.*

Choose $x_0 \in X$ arbitrarily and take a sequence of forest programs, $< x_t^s >$, starting from same initial state, x_0, such that the sum of social welfare realized by $< x_t^s >$ converges to $V(x_0)$ as $s \to \infty$. Since the set of feasible input-output pairs, D, is compact, sequence $< x_1^s >$ has a limit point, x_1^*, such that $(x_0, x_1^*) \in D$, and we can choose the sub-sequence of $< x_t^{s'} >$ in which x_1^s converges x_1^*. By D's compactness, from this sub-sequence, we can also pick up the second sub-sequence

$< x_t^{s''} >$ where the sequence $x_2^{s''}$ is converged to some point, say x_2^* such that $(x_1^*, x_2^*) \in D$. Continuing this operation for $t = 3, 4, ...$ yields the sequence of limit points, $< x_t^* >$. Notice that $< x_t^* >$ generates $V(x_0)$ since when a sequence converges to some point($V(x_0)$), the every sub-sequence converges to same point. Also notice that every (x_{t-1}^*, x_t^*) is in D and $x_0^* = x_0$. Therefore $< x_t^* >$ is a forest program starting from x_0 and it realizes $V(x_0)$.

2. *A forest program realizing $V(x_0)$ is maximal program.*

Let $< x_t^* >$ be a forest program realizing $V(x_0)$ and pick up arbitrarily a forest program starting from x_0, say $< x_t >$. Let $u_t = u(x_{t-1}^*, x_t^*)$ and $u_t' = u(x_{t-1}, x_t)$. If $V(x_0)$ is bounded, by construction of $V(x_0)$, the following inequality holds.

$$\begin{aligned} 0 &\geq \liminf \sum \rho^t (u_t' - u^*) - \liminf \sum \rho^t (u_t - u^*) \\ &= \liminf \sum \rho^t (u_t' - u^*) + \limsup \sum \rho^t (u_t - u^*) \\ &\geq \liminf \sum \rho^t (u_t' - u_t). \end{aligned}$$

This inequality implies that there is no forest program which starts from the same initial state and overtakes $< x_t^* >$. That is, $< x_t^* >$ is a maximal program. Particularly, in discounting case($\rho < 1$), by the boundedness of D, u is bounded and every $\sum_{t=1}^{T} \rho^t (u_t - u^*)$ converges. Hence the inequality is rewritten as $0 \geq \lim_{T\to\infty} \sum_{t=1}^{T} \rho^t (u_t' - u_t)$. This implies that $< x_t^* >$ is optimal in the sense of that the program catches up every forest program starting from the same initial state. (The terminology such as overtake, catch up, maximal, optimal follows McKenzie, 1986.)

3. *The forest model is similar to standard multi-sector optimal growth models but not same.*

The model is constructed on a compact convex production set D and an additive and concave one period felicity u. This set up is common with standard economic growth models. Fortunately, the model evades difficulty caused by non-convexity, which typically appears in one state variable bio-economics models(see Dechert and Nishimura, 1983 for the complete characterization), because we regard age distribution of trees as controllable. On the other hand, unfortunately, the model lacks an important assumption frequently used in economic growth models, that is free disposability. The reason why free disposability is violated in the forest model is simply that land available to forest management is bounded. Since free disposability is used almost everywhere in order to derive basic propositions on optimal economic growth, the lack requires a special approach to examine the forest model. The approach is to find support prices of stationary maximal forest program in heuristic way, which is developed by Mitra and Wan (1985, 1986). We will study our forest model along same line to Mitra and Wan.

4. Existence of a stationary maximal program and the supporting prices

In this section, we prove the following claim:

Proposition 2. *There is a stationary maximal forest program of the problem (3.9) for each ρ under [A.1] to [A.5], [A.6'], and [A.7].*

Proof. Throughout this proof, fix discount factor ρ at some value satisfying [A.6']. First, we derive dual prices supporting some sustainable forest.

Choose arbitrarily a sustainable forest x in Δ and consider a stationary forest program $< x_t >$ such that all $x_t = x$. The unique (l, c) in R^2 is given by this program, that is $l = FB(Ax - x)$, and $c = gx - \beta GB(Ax - x)$. Therefore forest program $< x >$ provides the unique $w_1(l, c)$ and $w_2(l, c)$, where $w_1(l, c)$ and $w_2(l, c)$ represents $\partial w(l, c)/\partial l$ and $\partial w(l, c)/\partial c$, respectively. By [A.7], both w_1 and w_2 are positive and bounded, so we can define α in R_{++} as

$$\alpha = w_2/w_1. \tag{4.1}$$

Also we can define a map $W: \Delta \to R_{++}$ which corresponds $x \in \Delta$ to $\alpha \in R_{++}$.

Given some positive real number α, we consider the Faustmann formula (2.1). Under [A.1] to [A.5] and [A.6'], there is the set of maximizers of (2.1) for each α. Denote the set by $M(\alpha)$ and let M be a correspondence which corresponds $\alpha \in R_{++}$ to $M(\alpha)$, a subset of $\{1, 2, ..., N\}$. Denote by $x(m)$ the sustainable forest with cutting age m and let λ_m be any non negative real number satisfying $\sum_{m \in M(\alpha)} \lambda_m \leq 1$. Denote by $\Delta(M(\alpha))$ a subset of Δ such that

$$\Delta(M(\alpha)) = \{x | \; x = \sum_{m \in M(\alpha)} \lambda_m x(m)\}. \tag{4.2}$$

Then we have a correspondence $\Phi: \{1, 2, ..., N\} \to \Delta$ which corresponds $M(\alpha)$ to $\Delta(M(\alpha))$.

Now consider the correspondence $\Phi \circ M \circ W: \Delta \to \Delta$. What we going to show is that this correspondence is upper hemi-continuous.

Let $\bar{x} \in \Delta$ be a sustainable forest arbitrarily chosen and suppose that $< x^s >$ is any sequence of sustainable forests converging to $\bar{x}$. Since the map W is continuous, $\alpha^s = W(x^s)$ converges to $W(\bar{x})$. Denote $W(x^s)$ and $W(\bar{x})$ by α^s, $\bar{\alpha}$, respectively. Then take $m^s \in M(\alpha^s)$ for each s and consider a sequence $< m^s >$. Since $M(\alpha^s)$ is a subset of $\{1, 2, ..., N\}$, there is a limit point, $\bar{m}$, of $< m^s >$. It holds with each s that $R(m^s, \alpha^s) \geq R(i, \alpha^s)$ for all $i \in \{1, 2, ..., N\}$. Therefore, $R(\bar{m}, \bar{\alpha}) \geq R(i, \bar{\alpha})$ for all $i \in \{1, 2, ..., N\}$ must hold. That is, $\bar{m} \in M(\bar{\alpha})$. This is valid no matter how to choose m^s and to choose the limit point. Therefore we have

$$\limsup_{s \to \infty} M(\alpha^s) \subset M(\bar{\alpha}).$$

Then choose $y^s \in \Phi \circ M \circ W(x^s)$ such that y^s converges to $\bar{y}$ as $s \to \infty$. By taking nonnegative numbers $\lambda^s_{m^s}$ such that $m^s \in M(\alpha^s)$ and $\sum_{m^s \in M(\alpha^s)} \lambda^s_{m^s} \leq 1$, y^s can be represented as

$$y^s = \sum_{m^s \in M(\alpha^s)} \lambda^s_{m^s} x(m^s).$$

Then taking enough large s', we have

$$y^s = \sum_{m \in \liminf_{s\to\infty} M(\alpha^s)} \lambda^s_m x(m), \ s \geq s'.$$

As a result, there are some $\bar{\lambda}_m \geq 0$ such that $\sum_{m \in \liminf M(\alpha^s)} \bar{\lambda}_m \leq 1$ and

$$\bar{y} = \sum_{m \in \liminf_{s\to\infty} M(\alpha^s)} \bar{\lambda}_m x(m).$$

Since $\liminf M(\alpha^s) \subset \limsup M(\alpha^s) \subset M(\bar{\alpha})$, we can rewrite $\bar{y}$ as

$$\bar{y} = \sum_{m \in M(\bar{\alpha})} \bar{\lambda}_m x(m).$$

This implies $\bar{y} \in \Phi \circ M \circ W(\bar{x})$. Therefore the correspondence $\Phi \circ M \circ W : \Delta \to \Delta$ is upper hemi-continuous.

By construction, Δ and $\Phi \circ M \circ W(x)$ $(x \in \Delta)$ are convex and compact. Also Δ contains $\Phi \circ M \circ W(x)$. Hence by the Kakutani's fixed point theorem, there is a fixed point, $z \in \Delta$, such that $z \in \Phi \circ M \circ W(z)$.

Using z (a fixed point of $\Phi \circ M \circ W : \Delta \to \Delta$), let us introduce the following notation:

$$\begin{aligned}
&l^* = FB(Az - z), \\
&c^* = gx - \beta GB(Az - z), \\
&\alpha^* = W(z), \\
&R^* = R(m^*, \alpha^*), \text{where } \alpha^* = W(z) \text{ and } m^* \in M(\alpha^*), \\
&p_0 = 0, \\
&p_i = \rho^{-1}[(1-\rho^i)(1-\rho)^{-1}R^* - \alpha^* \sum_{j=1}^{i} \rho^j g_j] \quad \text{for } i = 1, 2, ..., N, \\
&p = (p_0, p_1, ..., p_N).
\end{aligned}$$

Notice that m^* is a solution of the Faustmann formula (2.1) under $\alpha = \alpha^*$ and straightforward calculation yields

$$p_i \geq F_i - \alpha^* \beta G_i, \text{ all } i = 1, 2, ..., N, \tag{4.3}$$

where the equality holds if and only if $i \in M(\alpha^*)$.

Then, for any $(x, y) \in D$,

$$
\begin{aligned}
(4.4) \qquad & FB(Ax-y)+\alpha^*[gx-\beta GB(Ax-y)]+py-\rho^{-1}px \\
&=(F-\beta G)B(Ax-y)+\alpha^* gx+py-\rho^{-1}px \\
&\le p(Ax-y)+\alpha^* gx+py-\rho^{-1}px \\
&\quad \text{(equality holds iff } x_{i-1}-y_i=0 \text{ for all } i\notin M(\alpha^*)) \\
&=(pA++\alpha^* g-\rho^{-1}p)x \\
&\le \rho^{-1}R^*\nu x \quad \text{(equality holds iff } x_N=0) \\
&\le \rho^{-1}R^* \quad \text{(equality holds iff } \nu x=1) \\
&=FB(Az-z)+\alpha^*[gz-\beta GB(Az-z)]+pz-\rho^{-1}pz.
\end{aligned}
$$

Denoting $l=FB(Ax-y)$ and $c=\alpha^*[gx-\beta GB(Ax-y)]$, we have

$$
(4.5) \qquad l+\alpha^* c+py-\rho^{-1}px \le l^*+\alpha^* c^*+pz-\rho^{-1}pz.
$$

On the other hand, by strict concavity of w, we have

$$
\begin{aligned}
(4.6) \qquad u(x,y)-u(z,z) &= w(l,c)-w(l^*,c^*) \\
&\le w_1^*[(l+\alpha^* c)-(l^*+\alpha^* c^*)] \\
&\quad \text{(equality holds iff } (l,c)=(l^*,c^*)),
\end{aligned}
$$

where w_1^* denotes $\partial w(l^*,c^*)/\partial l$.

Define $q=w_1^* p$ and combine (4.6) with (4.5) and we have

$$
(4.7) \qquad u(x,y)+py-\rho^{-1}px \le u(z,z)+pz-\rho^{-1}pz \text{ for any } (x,y)\in D.
$$

Here the equality holds if and only if the following all conditions meet:
a. $(l,c)=(l^*,c^*)$,
b. $x_{i-1}-y_i=0$ for all $i\notin M[\alpha^*]$,
c. $x_N=0$,
d. $\nu x=1$.

Now we show the existence of a stationary maximal program using support prices q derived above.

1) Undiscounting case

Set $u^*=w(l^*,c^*)$. For any x_0 in X, any forest program $<x_t>$ starting from x_0, and every T,

$$
(4.8) \qquad \sum_{t=1}^{T}[u(x_{t-1},x_t)-u^*] \le q(x_T-x_0).
$$

Hence $V(x_0)$ is bounded above. Since the sustainable forest z is reachable within finite time from every initial forest x_0, $V(x_0)$ is also bounded below. Also we can claim that for every $x \in \Delta$, $u(x,x) \leq u^*$. Otherwise, there is some $y \in \Delta$ such that $u(y,y) > u^*$ and since y is reachable within finite time from every initial state $x \in X$, $V(x)$ diverges to infinity, contradicts (4.9). So we can write

$$u^* = \max_{x \in \Delta} u(x,x). \tag{4.9}$$

Also notice that (l^*, c^*) realizing $u^* = w(l^*, c^*)$ must be unique since w is strict concave.

The existence of a maximal program is proved along the line of Lemma 9.2 in McKenzie (1986). Let Δ^* represent the set of fixed points $z \in \Phi \circ M \circ W(z)$ when $\rho = 1$. As mentioned just above, every $z \in \Delta^*$ must satisfy $l^* = FB(Az - z)$ and $c^* = gz - \beta GB(Az - z)$ so that Δ^* is a compact subset of Δ. Then there is some z^* such that

$$pz^* = \min_{z \in \Delta^*} pz. \tag{4.10}$$

Suppose that a stationary forest program $< z^* >$ is not a maximal program. Then there is some forest program $< x_t >$ starting from z^* and some positive integer T^* such that

$$\sum_{t=1}^{T} [u(x_{t-1}, x_t) - u^*] > \epsilon > 0 \text{ for all } T > T^*. \tag{4.11}$$

Now define a sequence $< k_{T-1}, k_T > = (\sum_{t=1}^{T} x_{t-1}/T,\ \sum_{t=1}^{T} x_t/T)$. By convexity of D, $(k_{T-1}, k_T) \in D$ for all $T = 1, 2, \ldots$ and by compactness of D, there exists an limit point $(k^*, k^*) \in D$. Notice $k^* \in \Delta$, so that $u(k^*, k^*) - u^* \leq 0$. Since u is concave,

$$u(k_{T-1}, k_T) \geq \frac{1}{T} \sum_{t=1}^{T} u(x_{t-1}, x_t). \tag{4.12}$$

Therefore, we have

$$\begin{aligned} 0 &\geq u(k^*, k^*) - u^* \\ &\geq \liminf_{T \to \infty} u(k_{T-1}, k_T) - u^* \\ &\geq \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} [u(x_{t-1}, x_t) - u^*] \\ &\geq \lim_{T \to \infty} \epsilon / T \\ &= 0. \end{aligned} \tag{4.13}$$

Hence $k^* \in \Delta^*$.

On the other hand, from (4.9) and (4.11), for all $T > T^*$,

$$\epsilon < \sum_{t=1}^{T}[u(x_{t-1}, x_t) - u^*] \leq pz^* - px_T.$$

Therefore, for $T > T^*$,

$$\begin{aligned} \epsilon &< pz^* - \sum_{t=T^*+1}^{T} \frac{px_t}{T - T^*} \\ &= pz^* - \sum_{t=1}^{T}(\frac{px_t}{T})(\frac{T}{T - T^*}) + \sum_{t=1}^{T^*} \frac{px_T}{T - T^*}. \end{aligned} \tag{4.14}$$

Taking inferior limit of (4.14) with $T \to \infty$ and we have

$$\begin{aligned} \epsilon &\leq pz^* - \limsup_{T\to\infty} \sum_{t=1}^{T}(\frac{px_t}{T})(\frac{T}{T - T^*}) \\ &\leq pz^* - pk^*. \end{aligned} \tag{4.15}$$

That is, $pk^* < pz^*$. Since $k^* \in \Delta^*$ and z^* minimizes pz over $z \in \Delta^*$, a contradiction appears. Therefore, a stationary forest program $< z^* >$ is a maximal program.

2) Discounting case

As shown by Weitzman (1973), (4.7) is sufficient condition for the forest program $< z >$ to be a maximal program. For $u(z, z)$ and any program $< x_t >$ starting from z,

$$\sum_{t=1}^{\infty}[\rho^t u(x_{t-1}, x_t) - \rho^t u(z, z)] \leq \lim_{t\to\infty} \rho^t q = 0. \tag{4.16}$$

Therefore $< z >$ is a stationary maximal program. □

In the proof of Proposition 2, some characteristics of a stationary maximal forest program appear. We states them as corollary.

Corollary 1. *For every $x \in X$, there exists a maximal forest program starting from x.*

Proof. We have seen in the proof of Proposition 2 that by standardizing social welfare function u with $u^* = u(z^*, z^*)$, $V(x)$ is well defined for all $x \in X$. Therefore Remark 1 is applicable to every $x \in X$, so that a maximal program exists for all initial states in X. □

Corollary 2. *If $< z >$ is a stationary maximal forest program, then $z_N = 0$ and $\nu z = 1$.*

Proof. See the conditions a to d in the proof of Proposition 2 □

Corollary 2 says that in a stationary maximal program, all trees necessary fells and the land available to forest management is fully utilized. The former means that the forest is not used as stock yard, but as an apparatus absorbing the atmospheric carbon dioxide.

5. Turnpike property in undiscounting case

5.1. Convergence to the von Neumann facet. In this section, we examine the turnpike property of maximal forest programs when discount factor is equal to one. Firstly, we define a value loss function and the von Neumann facet corresponding to a stationary maximal forest program $< z^* >$.

A value loss function $\delta(x, y) : D \to R_-$ is defined with

$$\delta(x, y) = u(x, y) + q(y - x) - u^*, \tag{5.1}$$

where q is support prices which was derived in the proof of Proposition 2. From (4.7), for all $(x, y) \in D$, $\delta(x, y)$ is nonpositive.

The von Neumann facet C corresponding to a stationary maximal program $< z^* >$ is a subset of D such as

$$C = \{(x, y) | \ \delta(x, y) = 0 \text{ and } (x, y) \in D\}. \tag{5.2}$$

Notice that C is not empty since it contains at least a point (z^*, z^*), where z^* is a stationary maximal forest defined in (4.10). Also notice that if $(x, y) \notin C$, then $\delta(x, y)$ is necessarily negative. The following result is immediate.

Proposition 3. *Every maximal forest program is attracted to the von Neumann facet in the long run when discount factor is equal to one.*

Proof. Define a distance d between any two sets A, B in $R^{N+1} \times R^{N+1}$ as

$$d(A, B) = \inf_{(x,y) \in A, (x', y') \in B} |(x, y) - (x', y')|, \tag{5.3}$$

where $| \cdot |$ denotes the Euclid norm. If $d((x, y), C) > \epsilon$, then there is some negative real number, δ, such that $\delta(x, y) < \delta$. Otherwise, we have a sequence $< (x^s, y^s) >$ such that $(x^s, y^s) \in D$, $d((x^s, y^s), C) > \epsilon$ for each s, and $\delta(x^s, y^s)$ converges to zero as $s \to \infty$. By compactness of D, there is an accumulation point (x^*, y^*) of $< (x^s, y^s) >$ such that $(x^*, y^*) \in D, (x^*, y^*) \notin C$, and $\delta(x^*, y^*) = 0$. A contradiction appears.

Suppose that a forest program $< x_t >$ such that

$$\limsup_{t \to \infty} d((x_{t-1}, x_t), C) = \epsilon > 0. \tag{5.4}$$

Then there is a positive real number ϵ' less than ϵ and there is a positive integer T such that $d((x_{t-1}, x_t), C) > \epsilon'$ for all $t > T$. This implies that there is a negative real number δ' such that $\delta(x_{t-1}, x_t) < \delta'$ for all $t > T$. Therefore, for this program $< x_t >$,

$$\sum_{t=1}^{T}[u(x_{t-1}, x_t) - u^*] = \sum_{t=1}^{T}\delta(x_{t-1}, x_t) + q(x_0 - x_T)$$

diverges to minus infinity as $T \to \infty$. On the other hand, $V(x_0)$ must be bounded below. Hence $< x_t >$ can not be a maximal program. In other words, if $< x_t >$ is a maximal program, then

$$\limsup_{t\to\infty} d((x_{t-1}, x_t), C) = 0$$

By construction of $d(\cdot)$, $\liminf_{t\to\infty} d((x_{t-1}, x_t), C) \geq 0$. That is,

$$0 \leq \liminf_{t\to\infty} d((x_{t-1}, x_t), C) \leq \limsup_{t\to\infty} d((x_{t-1}, x_t), C) = 0. \tag{5.5}$$

This shows $\lim_{t\to\infty} d((x_{t-1}, x_t), C) = 0$. □

A forest program attracted to the von Neumann facet C is called a good program (Brock, 1970). Notice that the set of maximal programs is a subset of the set of good programs. The following corollary states some characteristics of a good forest program.

Corollary 3. *If $< x_t >$ is a good program, then $x_{tN} \to 0$ and $\nu x_t \to 1$ as $t \to \infty$.*

Proof. It is immediate from Corollary 2 and Proposition 3. □

So a maximal forest program requires utilizing the forest not as a stock yard of carbon, but as an apparatus absorbing the atmospheric carbon dioxide in the long time.

5.2. Convergence to the stationary maximal forest program. Now we focus on the structure of the von Neumann facet C. First of all, we prove the following claim.

Proposition 4. *In general, the cutting ages adopted in a stationary maximal forest program are at most two for undiscounting case.*

Proof. Recall that if $< z^* >$ is a stationary maximal forest program, $u(z^*, z^*)$ must be maximized over Δ, which appears in (4.10). This can be paraphrased as the following:

When (l^*, c^*) is the outcome from stationary maximal forest programs, then $w(l^*, c^*)$ must be maximized over the set of outcome from a sustainable forests $x \in \Delta$.

Formally, the set , say O, is expressed as

$$O = \{(l, c) \in R^2 |\ l = FB(Ax - x), c = gx - \beta GB(Ax - x), x \in \Delta\}. \tag{5.6}$$

Denote by $x(0)$ $N+1$ dimensional vector $(0,...,0,1)$ and recall the definition of the sustainable forest with cutting age m in the expression (3.5). Then each $x(m), m=0,1,...,N$ is sustainable and linear independent of each other. Hence any $x \in \Delta \subset R^{N+1}$ is expressed as

$$x = \sum_{m=0}^{N} \lambda_m x(m), \tag{5.7}$$

where λ_m is a nonnegative real number such that $\sum_{m=0}^{N} \lambda_m \leq 1$.

Notice that the outcome from $(x(m), x(m))$ is

$$(l,c) = \begin{cases} (0,0) & \text{if } m = 0 \\ (F_m/m, (1-\beta)G_m/m) & \text{if } m = 1,2,...,N. \end{cases}$$

Hence the set O is rewritten as

$$\begin{aligned} O = \{(l,c) \in R^2 \mid l = \sum_{m=1}^{N} \lambda_m F_m/m, \\ c = (1-\beta)\sum_{m=1}^{N} \lambda_m G_m/m, \\ \lambda_m \geq 0 \text{ for all } m, \text{and } \sum_{m=1}^{N} \lambda_m \leq 1\}. \end{aligned} \tag{5.8}$$

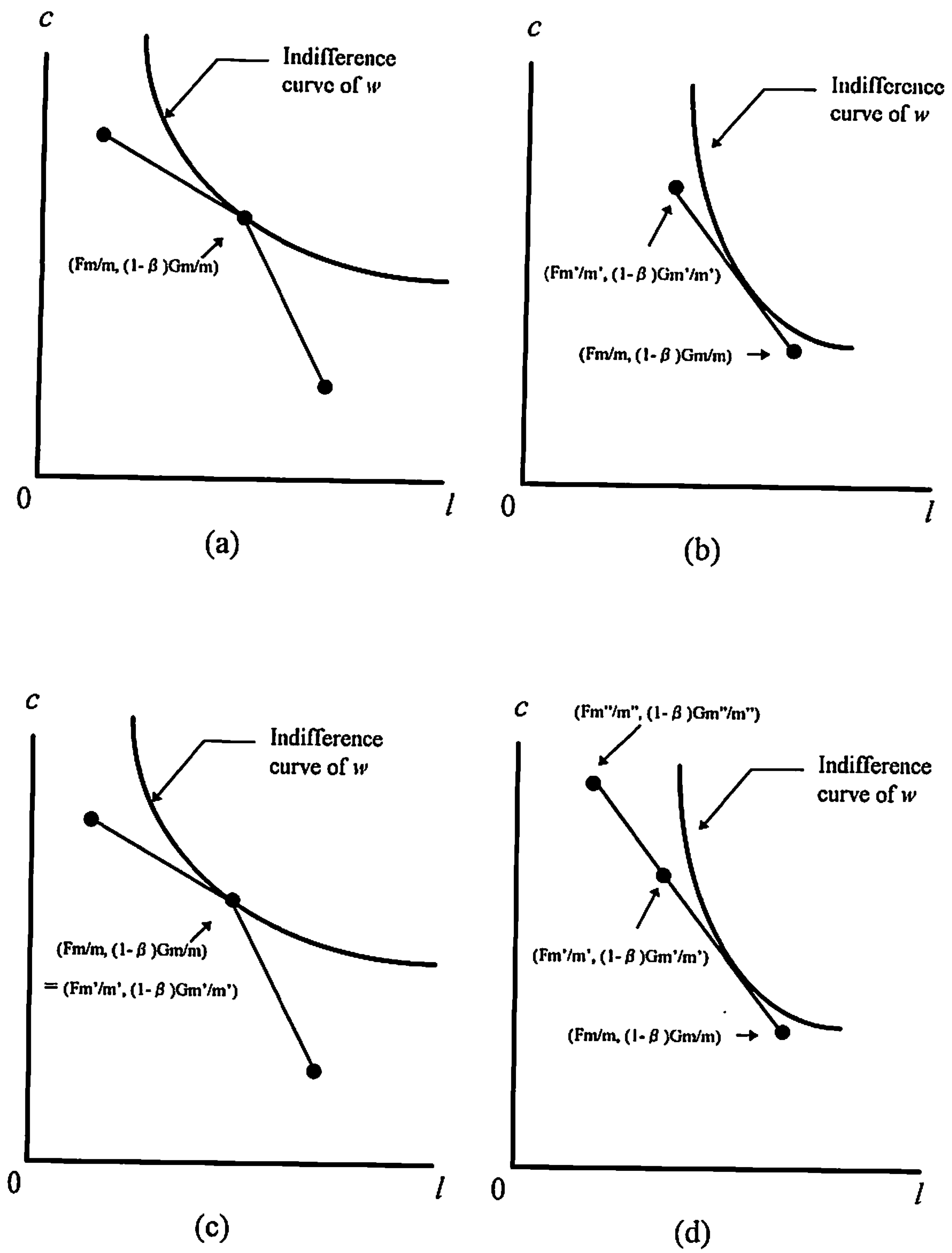
This expression and strict concavity of w imply that the outcome (l^*, c^*), which corresponds to stationary maximal forest programs, lies on some line segment spanned by two points, say $(F_m/m, (1-\beta)G_m/m)$ and $(F_{m'}/m', (1-\beta)G_{m'}/m')$. The situation is illustrated in Figure 1.

Figure 1(a) shows the case where the stationary maximal program is to sustain unique $x(m)$ and the corresponding cutting age is unique m. Figure 2(b) shows the case where the stationary maximal forest is also unique, expressed as $\lambda x(m) + (1-\lambda)x(m')$ with some $\lambda \in (0,1)$. But in this case, two different cutting ages are adopted in order to sustain the maximal program. Figure 1(c) shows the case where $x(m) = x(m')$ is a stationary maximal forest. In this case, every $\lambda x(m) + (1-\lambda)x(m'), \lambda \in [0,1]$ is a stationary maximal forest. In the case of Figure 1(d), stationary maximal forests lies on the convex combination of three different sustainable forests with cutting age m, m', m''. In the case, also stationary maximal forests are not unique.

Among these four cases, case (c) and (d) is obviously not generic in the sense that these situations disappear by small perturbation of F_i or G_i. So we conclude that the claim is valid. □

From above observation, we also have the following corollary.

Corollary 4. *In general, stationary maximal forest program is unique for undiscounting case.*



c
Indifference curve of w
(Fm/m, (1- β)Gm/m)
0
l
(a)
c
Indifference curve of w
(Fm'/m', (1- β)Gm'/m')
(Fm/m, (1- β)Gm/m)
0
l
(b)
c
Indifference curve of w
(Fm/m, (1- β)Gm/m)
= (Fm'/m', (1- β)Gm'/m')
0
l
(c)
c
(Fm"/m", (1- β)Gm"/m")
Indifference curve of w
(Fm'/m', (1- β)Gm'/m')
(Fm/m, (1- β)Gm/m)
0
l
(d)

Figure 1

Next we investigate relation between properties of the von Neumann facet and maximal forest programs in generic cases illustrated Figure 1 (a) and (b). We begin with showing the sufficient condition under which every maximal program converges to the stationary maximal program.

Proposition 5. *If there is no sequence* $< k_{t-1}, k_t >_{t \geq 1}$ *such that each* $(k_{t-1}, k_t) \in C$ *and* $\liminf |k_t - z^*| > 0$ *(z^* is the stationary maximal forest), every maximal forest program converges to the stationary maximal.*

Proof. Let $< x_t >$ be a maximal program starting from $x_0 \in X$. From Proposition 3, it is satisfied that $d((x_{t-1}, x_t), C) \to 0$ as $t \to \infty$. Take a sequence of the neighborhoods of C, $< U^s >$, such that

$$U^s = \{d((x,y),C) < \epsilon^s\},\ \epsilon^s > 0, \text{ and } \epsilon^s \to 0 (t \to 0).$$

Then we can consider a time sequence $< t^s >$ such that

$$(x_{t-1}, x_t) \in U^s, t \geq t^s$$

Define $x^s_\tau = x_{t^s+\tau}$ and make a sequence $< x^s_\tau >_{s \geq 0}$ for each τ. Then by compactness of D, there is some limit point x_τ for each τ such that

$$(x_{\tau-1}, x_\tau) \in C, \text{ all } \tau \geq 1.$$

If $< x_t >$ does not converge to the stationary maximal program, z^*, then there is some $\epsilon > 0$ such that

$$\liminf_{\tau \to \infty} |x^s_\tau - z^*| > \epsilon, \text{ all } s.$$

Therefore the sequence of limit points, $< x_\tau >$, must satisfies $\liminf_{\tau \to \infty} |x_\tau - z^*| \geq \epsilon > 0$. □

From this proposition, we immediately have the result for the case of Figure 1 (a).

Proposition 6. *If the stationary maximal forest has only one cutting age, i.e., if it is the sustainable forest with cutting age m, then every maximal forest program converges to the stationary maximal in undiscounting case.*

Proof. Let a forest program $< x_t >$ satisfies $(x_{t-1}, x_t) \in C$. In each period, the forest program requires to cut all trees whose ages are just m and to keep all other trees. Also it is required to amount of timber harvested is same in each period and the amount is equal to F_m/m. These conditions are met if and only if

$$x_t = (\underbrace{1/m, ..., 1/m}_{m}, 0, ..., 0) = z^*,$$

where z^* is the stationary maximal forest. Proof is completed by invoking Proposition 5. □

For the case of Figure 1 (b), we claim the following proposition.

Proposition 7. *When the stationary maximal forest z^* is represented as $\lambda x(m') + (1-\lambda)x(m)$ with some $\lambda \in (0,1)$ and $m', m \in \{1, 2, ..., N\}$, it is generic that every maximal forest program converges to the stationary maximal in undiscounting case.*

Proof. Let a forest program $< x_t >$ satisfy $(x_{t-1}, x_t) \in C$, all t. Then the elements of x_t, $x_{t,i}, i \geq m$ must be zero. Therefore we can restrict consideration in the sub-space R^m in R^{N+1}. Define $y_t \in R^m$ with

$$y_t = (x_{t,0} - z_0^*, ..., x_{t,m-1} - z_{m-1}^*). \tag{5.9}$$

Notice that $\nu y_t = 0$, all t since $\nu x_t = \nu z^* = 1$, all t.

In each period, $< x_t >$ and $< z^* >$ must yield same amount of timber and the ages of trees harvested are just m' and/or m. Therefore

$$F_{m'}(y_{t-1,m'-1} - y_{t,m'}) + F_m(y_{t-1,m}) = 0$$

This implies that the sequence $< y_t >$ can be described by a linear system $y_t = Ly_{t-1}$, where L is defined with

$$L = \begin{bmatrix} 0 & \cdots\cdots\cdots\cdots\cdots\cdots & 1 - F_m/F_{m'} \\ 1 \quad 0 & \cdots\cdots\cdots\cdots\cdots & 0 \\ 0 \quad 1 \quad 0 & \cdots\cdots\cdots\cdots & 0 \\ \cdots\cdots \quad 1 & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots\cdots \quad 1 & \cdots\cdots\cdots & F_m/F_{m'} \\ \cdots\cdots\cdots\cdots \quad 1 & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots & \ddots \quad \cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots & 1 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow m\text{+1st line.} \\ \\ \\ \\ \end{array}$$

Since $|x_{t,i} - z_i^*| = |y_{t,i}| \leq 1$, all t, i, y_t must be in direct sum of the eigen spaces corresponding to the eigen value whose absolute value is less than or equal to one. If L has an eigen value whose absolute value is equal to one, we can get $< y_t >$ such that y_t does not converges to zero and $|y_{t,i}| \leq 1$, all t, i. This implies the existence of $< x_{t-1}, x_t > \in C$ such that $\liminf_{t\to\infty} |x_t - z^*| > 0$. Then we check the existence of eigen value, μ, such as $|\mu| = 1$. The characteristic polynomial of L is

$$\mu^m + \mu^{m'}(F_m/F_{m'}) - (1 + F_m/F_{m'}) = 0.$$

Taking μ as $\cos\theta + i\sin\theta$, where " i " is the imaginary number, yields

$$\mu^m = [1 + F_m/F_{m'} - (F_m/F_{m'})\cos(m'\theta)] - i[(F_m/F_{m'})\sin(m'\theta)],$$

and

(5.10)

$$|\mu^m|^2 = 1 = 1 + 2(F_m/F_{m'})^2 + 2(F_m/F_{m'}) - 2(1 + F_m/F_{m'})(F_m/F_{m'})\cos(m'\theta).$$

Arranging (5.10), we have

$$(F_m/F_{m'} + 1)(1 - \cos(m'\theta)) = 0.$$

Therefore $\sin(m'\theta) = 0$ and $\mu^{m'} = 1$. Substitute these into (5.9) and we have $\mu^m = 1$. As a result, an eigen vector belonging to μ such as $|\mu| = 1$, say $y(\mu)$, must satisfy

(5.11) $$y(\mu) = L^m y(\mu) = L^{m'} y(\mu).$$

In other words, the linear system L has m'- and m-period when it starts at $y(\mu)$. Hence possible period is common measure of m' and m.

Period-1 is always a common measure and this implies $\mu = 1$. Simple calculation yields a corresponding eigen vector,

$$y(1) = \begin{pmatrix} 1 + F_m/F_{m'} \\ \vdots \\ 1 + F_m/F_{m'} \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

But $y(1)$ also satisfies $\nu y(1) = 0$ since $y(1)$ must satisfy (5.9). Therefore $y(1) = 0$. This implies that when m' and m are mutually prime, every maximal path converges to the stationary maximal.

What we have to examine next is the case where m' and m have a common measure greater than one. Suppose that the existence of such a common measure and denote it by n. Consider an eigen vector, y, corresponding to an eigen value, μ, such that $\mu^n = 1$ and $\mu \neq 1$. Decompose y to two vectors, $y_{m'}$ and y_m such that $y_{m'} + y_m = y$, every tree on $y_{m'}$ is cut at age m', and every tree on y_m is cut at age m. Using n dimensional vectors $r' = (r'_0, r'_1, ..., r'_{n-1})\prime$ and $r = (r_0, r_1, ..., r_{n-1})\prime$, then $y_{m'}$ and y_m are represented as

$$y'_{m'} = (\underbrace{r', \dots, r'}_{m'/n \text{ times}}, 0, \dots\dots, 0)', \quad y_m = (\underbrace{r, \dots\dots, r}_{m/n \text{ times}}, 0, \dots, 0)',$$

where prime "$'$" expresses transportation.

If $(x_{t-1}, x_t) \in C$, then

$$gx_{t-1} - \beta GB(Ax_{t-1} - x_t) = gz^* - \beta GB(Az^* - z^*).$$

(See the condition a in Proposition 2.)

From this condition about carbon, the following equation must hold:

$$(g_1, \dots, g_{m'-1}, g_{m'} - \beta G_{m'}) \overbrace{\begin{bmatrix} r'_0 & r'_{n-1} & \cdots & r'_1 \\ r'_1 & r'_0 & \cdots & r'_2 \\ \vdots & \vdots & & \vdots \\ r'_{n-1} & r'_{n-2} & \cdots & r'_0 \\ r'_0 & r'_{n-1} & \cdots & r'_1 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ r'_{n-1} & r'_{n-2} & \cdots & r'_0 \end{bmatrix}}^{m' \times n \text{ matrix}}$$

$$+ (g_1, \dots, g_{m-1}, g_m - \beta G_m) \overbrace{\begin{bmatrix} r_0 & r_{n-1} & \cdots & r_1 \\ r_1 & r_0 & \cdots & r_2 \\ \vdots & \vdots & & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_0 \\ r_0 & r_{n-1} & \cdots & r_1 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_0 \end{bmatrix}}^{m \times n \text{ matrix}}$$

$$= 0.$$

This equation can be rewritten as

$$G'_{n\times m'}\begin{pmatrix} r'_0 \\ r'_1 \\ \vdots \\ r'_{n-1} \\ \vdots \\ r'_{n-1} \end{pmatrix} + G_{n\times m}\begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \\ \vdots \\ r_{n-1} \end{pmatrix} = 0,$$

where $G'_{n\times m'}$ and $G_{n\times m}$ are defined as

$$G'_{n\times m'} = \overbrace{\begin{bmatrix} g_1 & \cdots\cdots\cdots\cdots\cdots\cdots & & g_{m'-1} & g_{m'} - \beta G_{m'} \\ g_2 & \cdots\cdots\cdots\cdots\cdots\cdots & & g_{m'} - \beta G_{m'} & g_1 \\ g_3 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & g_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ g_{n-1} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & g_{n-2} \\ g_n & \cdots \quad g_{m'} - \beta G_{m'} & g_1 & \cdots & g_{n-1} \end{bmatrix}}^{n\times m' \text{ matrix}},$$

$$G_{n\times m} = \overbrace{\begin{bmatrix} g_1 & \cdots\cdots\cdots\cdots\cdots\cdots & & g_{m-1} & g_m - \beta G_m \\ g_2 & \cdots\cdots\cdots\cdots\cdots\cdots & & g_m - \beta G_m & g_1 \\ g_3 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & g_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ g_{n-1} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & g_{n-2} \\ g_n & \cdots \quad g_m - \beta G_m & g_1 & \cdots & g_{n-1} \end{bmatrix}}^{n\times m \text{ matrix}}.$$

This equation is collapsible as the following:

$$G'_{n\times n}r' + G_{n\times n}r = 0, \tag{5.12}$$

where $G'_{n\times n}$ and $G_{n\times n}$ are defined as

$$G'_{n\times n} = \overbrace{\begin{bmatrix} \sum_{i=1}^{m'/n} g_{(i-1)n+1} & \cdots & \sum_{i=1}^{m'/n} g_{(i-1)n+n} - \beta G_{m'} \\ \sum_{i=1}^{m'/n} g_{(i-1)n+2} & \cdots & \sum_{i=1}^{m'/n} g_{(i-1)n+1} \\ \vdots & & \vdots \\ \sum_{i=1}^{m'/n} g_{(i-1)n+(n-1)} & \cdots & \sum_{i=1}^{m'/n} g_{(i-1)n+(n-2)} \\ \sum_{i=1}^{m'/n} g_{(i-1)n+n} - \beta G_{m'} & \cdots & \sum_{i=1}^{m'/n} g_{(i-1)n+(n-1)} \end{bmatrix}}^{n\times n \text{ matrix}},$$

$$G_{n\times n} = \overbrace{\begin{bmatrix} \sum_{i=1}^{m/n} g_{(i-1)n+1} & \cdots & \sum_{i=1}^{m/n} g_{(i-1)n+n} - \beta G_{m} \\ \sum_{i=1}^{m/n} g_{(i-1)n+2} & \cdots & \sum_{i=1}^{m/n} g_{(i-1)n+1} \\ \vdots & & \vdots \\ \sum_{i=1}^{m/n} g_{(i-1)n+(n-1)} & \cdots & \sum_{i=1}^{m/n} g_{(i-1)n+(n-2)} \\ \sum_{i=1}^{m/n} g_{(i-1)n+n} - \beta G_{m} & \cdots & \sum_{i=1}^{m/n} g_{(i-1)n+(n-1)} \end{bmatrix}}^{n\times n \text{ matrix}}.$$

On the other hand, from the condition about timber, i.e. $FB(Ax_{t-1} - x_t) = FB(Az^* - z^*)$ for $(x_{t-1}, x_t) \in C$, we have for each i,

$$F'_m r'_i + F_m r_i = 0 \tag{5.13}$$

Combine (5.13) with (5.12) and we have

$$[G_{n\times n} - (F_m/F_{m'})G_{n\times n}]r = 0. \tag{5.14}$$

$[G_{n\times n} - (F_m/F_{m'})G_{n\times n}]$ is a circulation matrix and the elements are

$$\sum_{i=1}^{m/n} g_{(i-1)n+j} - (F_m/F_{m'}) \sum_{i=1}^{m/n} g_{(i-1)n+j}, j \in \{1, ..., n-1\}$$

$$\text{and } (\sum_{i=1}^{m/n} g_{in} - \beta G_m) - (F_m/F_{m'})(\sum_{i=1}^{m/n} g_{(in)} - \beta G_{m'}).$$

Obviously the determinant of $[G_{n\times n} - (F_m/F_{m'})G_{n\times n}]$ depends on F, G, and β. In other words, if $|G_{n\times n} - (F_m/F_{m'})G_{n\times n}| = 0$, small perturbation of G or F breaks the equality. Therefore we can conclude that in generic, there is no r which satisfies the equation (5.12).

By gathering the results when the stationary maximal program has just one cutting age and just two cutting ages, we complete proof. □

6. Turnpike property in discounting case

Now we turn to discounting case. Since it is known that the forest program model does not have the global asymptotic stability for every discount factor less than one(Mitra *et al.*, 1992 and Wan, 1989, 1993, 1994), we restrict consideration to the neighborhood turnpike property in this section.

Proposition 8. *Our forest program model generically has the neighborhood turnpike property. That is, for any* $\epsilon > 0$*, there is some* ρ^* *such that if* $\rho \in (\rho^*, 1)$*, every maximal program starting,* $< x_t(\rho) >$*, from any* $x \in X$ *satisfies*

$$\limsup_{t \to \infty} |z(\rho) - x_t(\rho)| < \epsilon,$$

where $z(\rho)$ *represents a stationary maximal program when discount factor is* ρ*.*

The proof has three steps.

Lemma 1. *Denote by* z^* *a stationary maximal program in undiscounting case. If* z^* *is unique, then for any* $\epsilon > 0$*, there is some* $\rho' < 1$ *such that if* $\rho > \rho'$*,*

$$|z(\rho) - z^*| < \epsilon.$$

Proof. Suppose that the claim does not hold. Then there is a sequence $< z(\rho^s) >$ converging to some point $\bar{z}$ such that $|\bar{z} - z^*| > 0$ as $\rho^s \to 1$. As seen in the proof of Proposition 2, if a fixed point of the correspondence $\Phi \circ M \circ W$ is given, then the support prices satisfying (4.7) are uniquely determined. Denote by $q(\rho^s)$ the vector of prices supporting the stationary maximal forest $z(\rho^s)$. Since the support prices are bounded as shown in the proof of Proposition 2, we can take a sub-sequence of $< \rho^s >$, $< \rho^{s'} >$, such that $z(\rho^{s'})$ converges to $\bar{z}$ and $q(\rho^{s'})$ converges to some $\bar{q}$ as $\rho^{s'} \to 1$. Since every $z(\rho^{s'})$ and $q(\rho^{s'})$ satisfies the dual equation (4.7), $\bar{z}$ and $\bar{q}$ also must satisfies (4.7). Therefore $\bar{z}$ is a candidate of a stationary maximal forest in undiscounting case. But the stationary maximal forest is unique so that $\bar{z} = z^*$. A contradiction appears. □

Lemma 2. *Let* $< x_t(\rho) >$ *denote a maximal program starting from* x *under the discount factor* ρ*. Then for any* $\epsilon > 0$*, there is some* $\rho" < 1$ *independently of* x *such that if* $\rho > \rho"$*, for every* $x \in X$*,*

$$\sup_t |x_t(\rho) - x_t(1)| < \epsilon.$$

Proof. Pick up $x \in X$ arbitrarily. Suppose that, for this initial state, x, there is no $\rho < 1$ satisfying above claim. Then there is some sequence, $< x_t(\rho^s) >$, of maximal programs starting from same initial state, x, such that $< x_t(\rho^s) > \to < \bar{x}_t >$ as $\rho^s \to 1$ and $< \bar{x}_t >$ is not a maximal program in undiscounting case. On the other hand, letting $V(x, \rho^s)$ be sum of social welfare acquired from the program, $< x_t(\rho^s) >$, it holds that

$$V(x,\rho^s) \geq \liminf_{T\to\infty} \sum_{t=1}^{T} (\rho^s)^t [u(x_{t-1}(1), x_t(1)) - u^*].$$

Therefore we have

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [u(\bar{x}_{t-1}, \bar{x}_t) - u^*] \geq \liminf_{T\to\infty} \sum_{t=1}^{T} [u(x_{t-1}(1), x_t(1)) - u^*].$$

But $< x_t(1) >$ is a maximal program, so that,

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [u(\bar{x}_{t-1}, \bar{x}_t) - u^*] = \liminf_{T\to\infty} \sum_{t=1}^{T} [u(x_{t-1}(1), x_t(1)) - u^*].$$

This implies that $< \bar{x}_t >$ is a maximal program. A contradiction appears. As a result, for each $x \in X$, there is some $\rho(x)$ such that if $\rho > \rho(x)$, then $\sup_t |x_t(\rho) - x_t(1)| < \epsilon$.

Next consider the problem

$$\sup_{x\in X} \rho(x).$$

Since X is compact, there is a maximizer of this problem, x^*, and $\rho(x^*) < 1$. Take $\rho(x^*)$ as ρ" and proof is finished. □

Proof. (Proposition 7)

As seen in proof of Proposition 6, it is generic that the stationary maximal program is unique in undiscounting case. Therefore in the generic case, by Lemma 1, for any $\epsilon > 0$, we can choose ρ' such as

(6.1) $$|z(\rho) - z^*| < \epsilon/3 \text{ ,all } \rho > \rho'.$$

Also as seen in Proposition 6, in undiscounting case, every maximal program starting from any initial state generically converges to the stationary maximal forest. Therefore there is some $T^* > 0$ such that if $t > T^*$, then

(6.2) $$|x_t(1) - z^*| < \epsilon/3 \text{ for every } x \in X \text{ .}$$

Further more, by Lemma 2, we can choose ρ" such that for every $x \in X$ and all $\rho > \rho$",

(6.3) $$\sup_t |x_{(}\rho) - x_t(1)| < \epsilon/3.$$

Define ρ^* with

$$\rho^* = \max[\rho', \rho", \rho(\beta)].$$

Then we have for $\rho > \rho^*$ and $t > T^*$

$$
\begin{aligned}
|x_t(\rho) - z(\rho)| &\leq |x_t(\rho) - x_t(1)| + |x_t(1) - z^*| + |z^* - z(\rho)| \\
&< \epsilon.
\end{aligned}
$$

This implies that for any ϵ, in generic there is some $\rho^* < 1$ such that when discount factor ρ is greater than ρ^*, for every maximal program, $x_t(\rho)$, starting from any $x \in X$, $\limsup_{t \to \infty} |z(\rho) - x_t(\rho)| < \epsilon$. □

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