# Business cycle and chaos in monetary economies

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**Abstract**

We consider a type of monetary economy inhabited by a representative agent with perfect foresight. Particular characteristic of our model is that the consumer receives utility directly from the money holdings. Following the framework of Matsuyama(1991) and Fukuda(1993), we give a sufficient condition for the existence of period two cycle of optimal real balances.

We also give an example which is capable of generating chaotic motion in the framework of our model.

The result suggests that we can explain volatile fluctuations of monetary economy without recourse to exogenous stochastic shocks.

**Key Words:** money-in-the- utility, cycles, bifurcation and chaos.

**JEL Classification:** E32.

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# 1 Introduction

Recent theories of deterministic economic dynamics have successfully shown that the chronic and seemingly volatile fluctuations of an economy may emerge from optimization behavior of economic agents. One of the most interesting branches of this field is the research into the effect of money on the dynamic movement of the economy. The primary purpose of this paper is to construct a deterministic model of dynamic monetary economy which exhibits cyclical or chaotic fluctuations.

In the literature of optimal dynamics of monetary economy, two different approaches coexist. One is the overlapping generations model and the other is the model with money in the utility function. The latter approach, initiated by Brock (1974), offers the simplest way to study the dynamic movement of the monetary economy and there are several outstanding papers on the subject along this line. The fundamental framework of this paper is that of Matsuyama (1991) and Fukuda (1993). Matsuyama showed that there exist optimal cyclical paths and chaotic paths of real balances under a specified utility function which leads to a logistic type return map. Fukuda has given a sufficient condition for the existence of a period two cycle if the utility is represented as the sum of utilities from consumption and money. He also gave several examples of utility functions which lead to cyclical or chaotic paths.

In this paper we give a general condition for the existence of the period two orbit without specifying utility functions. This is a generalization of the results of Matsuyama and Fukuda. We also show that our model is capable of generating chaotic paths by giving a numerical example.

The construction of this paper is as follows. After presenting a fundamental framework of the model in Section 2, we show that there exists a unique return map for optimal dynamic paths of real balances in Section 3. In Section 4, we state our main proposition on the existence of period two cycles. Section 5 is devoted to an example which exhibits cyclical and chaotic movement. Brief concluding remarks are given in Section 6.

## 2 The Model

Consider a representative consumer who receives utility from consumption and real money holdings. The money is deflated by the price level of consumption goods. The objective of the consumer is to maximize his utility over the infinite discrete time horizon given the price level of consumption goods and the initial endowment of goods and money. The goods are perishable and therefore not storable. To construct a concrete description of the model, we assume that the consumer has a constant utility function and maximizes the discounted sum of utilities over infinite time periods.

Let $c$ denote the quantity of consumption goods, the price of which is written as $p$. Let $M$ be the quantity of money and $m = M/p$ as real balances. We represent $c$ and $m$ with nonnegative reals. Utility function $u(c, m)$ of the consumer can take only nonnegative values and is bounded by a positive number.[1]

**Assumption 1** *$u : \mathbf{R}^2_+ \to \mathbf{R}$ is a continuous function. $u$ is twice continuously differentiable on $\mathbf{R}^2_{++}$ and satisfies the following conditions.*

$$u_1(c, m) > 0, \quad u_2(c, m) > 0, \tag{1}$$

$$u_{11}(c, m) < 0, \quad u_{22}(c, m) < 0, \tag{2}$$

$$u_{12}(c, m) = u_{21}(c, m) > 0, \tag{3}$$

*for $(c, m) \in \mathbf{R}^2_{++}$.*

Let $c_t$, $M_t$, and $p_t$ denote the consumption, money held and price level at

[1]Take any nonnegative utility function $v(c, m)$ and define bounded function $u(c, m)$ as $u(c, m) = v(c, m)/(1 + v(c, m)) \leq 1$. If $v$ satisfies Assumptions 1 and 3 below, then $u$ satisfies those Assumptions. Because $u$ is an increasing transformation of $v$, we can think of $u$ as another utility function which represents the same preference as $v$. This guarantees that we can restrict our attentions to bounded utility functions.

period $t$, respectively. Define undiscounted utility $u_t$ at period $t$ as

$$u_t = u(c_t, M_{t-1}/p_t).$$

This means that in period $t$ the consumer receives utility from the money holdings at the beginning of period $t$ deflated by current price level and from consumption of that period. This kind of construction is needed to obtain dynamics of money demand. Let $\rho$ be a constant discount rate and $0 \le \rho < 1$. The objective function of the consumer is

$$\sum_{t=1}^{\infty} \rho^{t-1} u(c_t, M_{t-1}/p_t).$$

This series converges absolutely as long as $u(c_t, M_{t-1}/p_t)$ is defined for $t = 1, 2, \ldots$, because $u$ is a bounded nonnegative function and $0 < \rho < 1$. Let $M_t^s$ denote the supply of money. The government controls nominal money supply through lump-sum tax or transfer denoted by $T_t$. The budget constraint of the representative consumer at period $t$ is given by

$$M_t + p_t c_t = M_{t-1} - p_t T_t + p_t y_t.$$

We assume $y_t > T_t$ for any $t$ so that the feasible set will not be empty. The problem of the consumer is as follows.

$$\max_{(c_t, M_t)} \sum_{t=1}^{\infty} \rho^{t-1} u(c_t, M_{t-1}/p_t) \tag{4}$$
$$\text{subject to } \; c_t + \frac{M_t}{p_t} + T_t = \frac{M_{t-1}}{p_t} + y_t.$$

We assume that the government pursues the policy of the balanced budget at each period. The budget constraint of the government is given by

$$p_t T_t = M_{t-1}^s - M_t^s.$$

The price of consumption goods at period $t$ is determined to clear the demands for goods and money. The equilibrium conditions for goods and money are given by

$$c_t = y_t \;\text{ and }\; M_t = M_t^s. \tag{5}$$

If the above conditions are satisfied, the budget constraint for the government is automatically satisfied.

## 3 Maximization and Equilibrium

By solving problem (4) and putting $m_t = M_{t-1}/p_t$, we have the first order condition,

$$\frac{u_t(c_t, m_t)}{p_t} = \frac{\rho}{p_{t+1}} \{u_1(c_{t+1}, m_{t+1}) + u_2(c_{t+1}, m_{t+1})\}, \tag{6}$$

for internal solution.[2]

To simplify the analysis, we assume that the growth rate of the nominal money supply is fixed, i.e.,

**Assumption 2** $M_t^s = aM_{t-1}^s$, *where $a$ is positive and constant.*

For simplicity we assume that $y_t$ is constant over time and denote the quantity as $e$, that is $y_t = e$ for all $t$. In equilibrium, we have $M_t = M_t^s$. Thus,

$$p_t = \frac{M_{t-1}}{m_t},\ p_{t+1} = \frac{M_t}{m_{t+1}} = \frac{aM_{t-1}}{m_{t+1}}, \text{ and } c_t = e. \tag{7}$$

[2]Let $\{\lambda_t\}$ be sequence of Lagrangian multipliers. Lagrangian form of (4) is,

$$\mathcal{L} = \sum_{t=1}^{\infty} \rho^{t-1} u(c_t, M_{t-1}/p_t) + \sum_{t=1}^{\infty} \lambda_t (M_{t-1}/p_t + y_t - c_t - M_t/p_t - T_t).$$

Diffrentiating $\mathcal{L}$ with respect to $c_t$, $c_{t+1}$ and $M_t$, and letting them equal to zero, we have

$$\rho^{t-1} u_1(c_t, M_{t-1}/p_t) = \lambda_t,$$
$$\rho^t u_1(c_{t+1}, M_t/p_{t+1}) = \lambda_{t+1},$$
$$\frac{\rho^t}{p_{t+1}} u_2(c_{t+1}, M_t/p_{t+1}) - \frac{\lambda_t}{p_t} + \frac{\lambda_{t+1}}{p_{t+1}} = 0.$$

Substitution of the former two equality into the last one yields,

$$\frac{\rho^t}{p_{t+1}} u_2(c_{t+1}, M_t/p_{t+1}) - \frac{\rho^{t-1}}{p_t} u_1(c_t, M_{t-1}/p_t) + \frac{\rho^t}{p_{t+1}} u_1(c_{t+1}, M_t/p_{t+1}) = 0.$$

Putting $m_t = M_{t-1}/p_t$, we have (6).

Substitution of (7) into (6) yields,

$$m_t = \frac{\rho}{a} m_{t+1} \left\{ \frac{u_1(e, m_{t+1})}{u_1(e, m_t)} + \frac{u_2(e, m_{t+1})}{u_1(e, m_t)} \right\} \tag{8}$$

In equilibrium, real money sequence $\{m_t\}$ must satisfy the above equation.

The right hand side of eq. (8) is a function of $m_t$ and $m_{t+1}$. Thus eq. (8) describes implicitly a relationship between $m_t$ and $m_{t+1}$. Unfortunately, we cannot in general expect the existence of global implicit function $m_{t+1} = f(m_t)$ which solves eq. (8). Thus, as often done, we consider using the backward system $m_t = g(m_{t+1})$. To secure the existence of $g$, we need several technical Assumptions.

**Assumption 3** *We assume the following conditions:*

1. $u_2(e, m)$ *is bounded with respect to second argument* $m > 0$, *which assures the existence of* $+\infty > \lim_{m \downarrow 0} u_2(e, m) = s_2 > 0$.

2. $a > \rho$, *i.e., the growth rate of nominal money supply is larger than the discount rate.*

3. $\lim_{m \downarrow 0} u_1(e, m) = s_1 > 0$ *and* $\lim_{m \downarrow 0} \frac{u_2(e, m)}{u_1(e, m)} = \frac{s_2}{s_1} > \frac{a}{\rho} - 1\,( > 0)$.

4. *For any* $n > 0$, $\sup_{0 < m < n} u_{12}(e, m) < +\infty$, $\inf_{0 < m < n} u_{22}(e, m) > -\infty$.

Before stating the existence result, some comments on Assumption 3 are in order. Condition 1 means that the marginal utility of money at $c = e$ approaches its least upper bound as the real money holdings approach zero. This implies that when the price of money is high enough relative to that of consumption goods, the demand for money will vanish in a single period consumption program. Note that in our model the relative price of real money to consumption goods is always equal to unity and thus the budget line moves parallelly on $(c, m)$ semiplane as $p$ moves. This means if money is to be demanded by single period maximization with $c = e$, the marginal equality $u_2(e, m_0)/u_1(e, m_0) = 1$

must hold for some positive $m_0$. This immediately leads to the inequality

$$\frac{s_2}{s_1} = \lim_{m \downarrow 0} \frac{u_2(e,m)}{u_1(e,m)} > 1$$

because $u_{22}(e,m) < 0$ and $u_{12}(e,m) > 0$.

**Lemma 1** *Under Assumptions 1, 2, and 3, if eq. (8) has at least one solution $(m_t, m_{t+1}) \in \mathbf{R}^2_{++}$, then there exists one and only one continuously differentiable global implicit function $m_t = g(m_{t+1})$, $g : \mathbf{R}^*_+ \to \mathbf{R}^*_+$, which solves eq. (8). That is*

$$g(m) = \frac{\rho}{a} m \left\{ \frac{u_1(e,m)}{u_1(e,g(m))} + \frac{u_2(e,m)}{u_2(e,g(m))} \right\}$$

*for $m = m_{t+1}$ and $g(m) = m_t$. Moreover, every solution $(m_t, m_{t+1}) \in \mathbf{R}^2_{++}$ of eq. (8) is represented by this function.*[3]

**Proof.** See the Appendix.

# 4 Behavior of Equilibrium Paths

We first investigate the asymptotic behavior of the function $m_t = g(m_{t+1})$.

**Lemma 2** *Under Assumptions 1, 2, and 3,*

$$\lim_{m \downarrow 0} g(m) = 0, \tag{9}$$

$$\lim_{m \downarrow 0} g'(m) > 1. \tag{10}$$

**Proof.** See the Appendix.

Now consider stationary point $g(m^*) = m^*$. From eq. (8), this point is characterised by

$$m^* = \frac{\rho}{a} m^* \left\{ 1 + \frac{u_2(e,m^*)}{u_1(e,m^*)} \right\},$$

[3]Note that the usual implicit function theorem assures only locally the existence of an implicit function. To secure the result, we need a theorem on the extension of a function over to the closure of its domain.

or equivalently

$$\frac{u_2(e,m^*)}{u_1(e,m^*)} = \frac{a}{\rho} - 1\ (>0). \tag{11}$$

The stationary point $m^*$ is uniquely determined because $u_2(e,m)/u_1(e,m)$ is a decreasing function of $m$. Fig. 1 depicts the situation in two cases and Fig. 2 is its counterpart for forward dynamics. In case (i), the graph $G_1$ intersects the 45° line at point $m_1$, where the curvature of $G_1$ is positive. In this case any equilibrium path $\{m_t\}$ which starts from a point $m_0 \in ]0, m_1[$ converges to zero as $t \to \infty$ (Fig. 2). In case (ii), $G_2$ intersects the 45° line at point $m_2$, where the curvature of $G_2$ is negative and has large absolute value. In this case, cycles of period two or three may emerge. Using condition (11), we can derive a sufficient condition that there exists an equilibrium path with a period two cycle.

(Insert figures 1 and 2 about here.)

A general condition is as follows.

**Lemma 3** *Let $I \subset \mathbf{R}$ be a compact interval and $g : I \to I$ be a continuous mapping with fixed point $m^* = g(m^*)$. If $g$ is differentiable at $m^*$ and $g'(m^*) < -1$, then dynamical system $(I, g)$ has at least one periodic orbit of period two $\{x^*, y^*\}$ such that*

$$x^* < m^* < y^*, \quad or \quad y^* < m^* < x^*. \tag{12}$$

**Proof.** Without loss of generality, we can set $I = [0, \alpha]$, $\alpha > 0$. Divide the square $I \times I$ into four regions $A$, $B$, $C$, and $D$ by the diagonal line connecting $(0,0)$ and $(\alpha, \alpha)$, denoted by $L_1$, and the line through $(m^*, m^*)$ with gradient $-1$, denoted by $L_2$ (See Fig. 3). $L_2$ is a segment of the line defined by the equation

$$x + y = 2m^*, \quad \text{or} \quad y = 2m^* - x. \tag{13}$$

So,

$$\begin{aligned} A &= \{ (x,y) \in I \times I \mid x \geq y \text{ and } x + y \geq 2m^* \}, \\ B &= \{ (x,y) \in I \times I \mid x \leq y \text{ and } x + y \geq 2m^* \}, \end{aligned}$$

$$\begin{aligned} C &= \{ (x,y) \in I \times I \mid x \leq y \text{ and } x+y \leq 2m^* \}, \\ D &= \{ (x,y) \in I \times I \mid x \geq y \text{ and } x+y \leq 2m^* \}. \end{aligned}$$

Define continuous maps $G$ and $F$ from $I$ to $I \times I$ as

$$G(x) = (x, g(x)), \; F(y) = (g(y), y).$$

Because $g'(m) < -1$, there exists a neighbourhood $N$ of $m^*$ in $I$ such that

$$\begin{aligned} &x \in N, \; x < m^* \Longrightarrow (g(x) - m^*) + (x - m^*) > 0, \;\text{ and} \\ &x \in N, \; x > m^* \Longrightarrow (g(x) - m^*) + (x - m^*) < 0. \end{aligned}$$

Using superscript $\circ$ to denote open kernel in $I \times I$, the above conditions can be written as

$$x \in N, \; x < m^* \Longrightarrow G(x) \in B^\circ \;\text{ and }\; F(x) \in A^\circ, \tag{14}$$

$$x \in N, \; x > m^* \Longrightarrow G(x) \in D^\circ \;\text{ and }\; F(x) \in C^\circ. \tag{15}$$

Let $K$ be the set of all fixed points of $g$. $K$ is closed in $I$, and since $I$ is compact, $K$ is compact. Take $x_0 \in N$, $x_0 < m^*$, and consider compact set $K_{x_0} = K \cap [0, x_0]$. If $K_{x_0} = \emptyset$, then $G(x)$ cuts diagonal $L_1$ for the first time at $x = m^*$ from (14) because $x_0 \in N$. Since $G$ is an injection, the set

$$J_1 = \{ (0,y) \mid 0 \leq y \leq g(0) \} \cup G(]0, m^*[) \cup \{ (x,x) \mid 0 \leq x \leq m^* \}$$

is a closed curve in $I \times I$ (See Fig. 4). From (14), $G(x_0) \in B^0$ and thus $G(]0, m^*[) \cap (A \cup D) = \emptyset$ because $K \cap [0, m^*[= \emptyset$. From (15), $F(y)$ connects the interior of $J_1$ with $F(\alpha)$ which lies on $J_1$ or in the exterior of $J_1$. In the latter case, by the Jordan curve theorem $F(]m^*, \alpha]) \cap J_1 = \emptyset$.[4] Since $K \cap [0, m^*[= \emptyset$,

[4]Let $H_1$, $H_2$ be the interior and exterior open connected regions of $\mathbb{R}^2$ defined by $J_1$. Then $J_1 \cap H_1 \cap H_2 = \mathbb{R}^2$ and $H_1 \cap H_2 = \emptyset$ by Jordan curve theorem. Let $F(]m^*, \alpha]) = F_1$ and $F_1 \cap H_1 \neq \emptyset$, $F_1 \cap H_2 \neq \emptyset$. $F_1$ is connected. If $F_1 \cap J_1 = \emptyset$, then $F_1 = F_1 \cap \mathbb{R}^2 = F_1 \cap (J_1 \cup H_1 \cup H_2) = (F_1 \cap J_1) \cup (F_1 \cap H_1) \cup (F_1 \cap H_2) = (F_1 \cap H_1) \cup (F_1 \cap H_2)$. Because $H_1 \cap H_2 = \emptyset$ and both are open, this contradicts the connectedness of $F_1$.

$F(y)$ must cross $J_1$ at its arc $G(]0, m^*[)$. If $K_{x_0} \neq \emptyset$, by compactness of $K_{x_0}$ we have $\bar{x} = \max K_{x_0}$. Then $K \cap ]\bar{x}, m^*[ = \emptyset$ and thus

$$J_2 = G(]\bar{x}, m^*[) \cup \{ (x, x) \mid \bar{x} \leq x \leq m^* \}$$

is a closed curve. Since $K \cap ]\bar{x}, m^*[ = \emptyset$, an analogous argument assures that $F(y)$ must cross $J_2$ at its arc $G(]\bar{x}, m^*[)$. So in any case there must exist a point $(x^*, y^*) \in B \cup C$ such that $x^* \neq y^*$ and $G(x^*) = F(y^*)$. This means $x^* < m^* < y^*$ and $x^* = g(y^*)$, $g(x^*) = y^*$ and thus $g(g(x^*)) = g(y^*) = x^*$, which was to be proved. (QED)

(Insert figure 3 and 4 about here.)

This geometric proof of Lemma 3 upholds our previous observation that if the curvature of the graph $(m_{t+1}, g(m_{t+1}))$ is negative with a large absolute value at $m^* = g(m^*)$, a cycle would emerge. To apply the result to our model, we prove the following corollary.

**Corollary 1** *Let $g$: $\mathbb{R}_+ \to \mathbb{R}_+$ be a continuous mapping. If $g$ is differentiable at $m^* = g(m^*) > 0$ and*

$$g'(m^*) < -1.$$

*then dynamical system $(\mathbb{R}_+, g)$ has at least one periodic orbit of period two.*

**Proof.** Because $g$ is continuous on compact interval $[0, m^*]$, it has the maximum value $M$ on $[0, m^*]$. Let $I = [0, M]$ and define $h : I \to I$ as

$$\begin{cases} h(x) = g(x) & \text{if } g(x) \leq M \\ h(x) = M & \text{if } g(x) > M \end{cases}$$

Then one can easily see that $h$ is continuous on $I$. Because $g'(m^*) < -1$, $g(m^*) < M$ and thus $h$ is diffrentiable at $m^*$ and

$$h'(m^*) = g'(m^*) < -1.$$

Thus $(I, h)$ has period two orbit $\{x^*, y^*\}$ which satisfies (12) by Lemma 3. We have to prove $g(x^*) \leq M$ and $g(y^*) \leq M$. If $g(x^*) > M$, then $h(x^*) = M =$

$y^*$. Since $y^* = M \neq x^*$, $x^* < M$. Thus by (12), $x^* < m^* < y^*$. But $M < g(x^*)$ means $m^* < x^*$ by definition of $M$, which is contradiction. Thus $g(x^*) \leq M$. The same holds for $g(y^*)$. So $h(x^*) = g(x^*)$ and $h(y^*) = g(y^*)$. (QED)

Now we can use this Corollary noting that our $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ has a continuous extension $\bar{g} : \mathbb{R}_+ \to \mathbb{R}_+$ by Lemma 2.

Now let $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be the implicit function defined by Lemma 1 and let $g(m^*) = m^*$. The implicit function theorem tells

$$g'(m^*) = \frac{\frac{\rho}{av_1(m^*)}[v_1(m^*) + v_2(m^*) + m^*\{v_{12}(m^*) + v_{22}(m^*)\}]}{1 + \frac{\rho}{a}\frac{v_{12}(m^*)}{(v_1(m^*))^2}m^*\{v_1(m^*) + v_2(m^*)}$$

by eq. (8). Here $v_1(m) = u_1(e,m)$, $v_2(m) = u_2(e,m)$,etc.

By eq. (11) we have

$$g'(m^*) = \frac{1 + \frac{\rho}{av_1(m^*)}v_{12}(m^*) + v_{22}(m^*)}{1 + m^* \frac{v_{12}(m^*)}{v_1(m^*)}}$$

Thus if the inequality

$$u_{12}(e,m^*) + \frac{2a}{a+\rho}\frac{u_1(e,m^*)}{m^*} < -\frac{\rho}{a+\rho}u_{22}(e,m^*)$$

is satisfied, then $g'(m^*) < -1$. This is equivalent to

$$1 < \frac{a}{\rho} < -\frac{u_{12}(e,m^*) + u_{22}(e,m^*)}{u_{12}(e,m^*) + \frac{2u_1(e,m^*)}{m^*}}. \tag{16}$$

Thus, we have as our main proposition.

**Proposition 1** *Under Assumptions 1 to 3, if the inequality (16) holds for the stationary point $m^*$ of (8), then there exists an equilibrium real money path $\{m_t\}$ with period two orbit.*

Note that the stationary point $m^*$ depends on $a/\rho$ as well as on $e$ (See eq. (11)). In the next section, we will give an example which exhibits cyclical and chaotic movement in the framework of our model.

# 5 Numerical Example.

In this section, we specify a utility function as[5]

$$u(c,m) = b_1 \tanh b_2 c \tanh b_3(m+\alpha), \tag{17}$$

$+\infty > b_1,\, b_2,\, b_3,\, \alpha > 0$ for $c,\ m \leq 0$. One can easily see that (17) satisfies Assumption 1 and 3. For this utility function, (4) takes the form of

$$\max \sum_{t=1}^{\infty} \rho^{t-1} b_1 \tanh b_2 c_t \tanh b_3(m_t+\alpha) \tag{18}$$

$$\text{subject to } \ c_t + \frac{M_t}{p_t} + T_t = \frac{M_{t-1}}{p_t} + y_t \tag{19}$$

Solving this problem and letting $c_t = e$(constant) from the equilibrium condition of goods market, we have the following equation for an optimal dynamic path for money balances.

$$m_t = \frac{\rho}{a} m_{t+1} \frac{\tanh b_3(m_{t+1}+\alpha)}{\tanh b_3(m_t+\alpha)} + \frac{b_3 \tanh b_2 e \ \operatorname{sech}^2 b_3(m_{t+1}+\alpha)}{b_2 \operatorname{sech}^2 b_2 e \ \tanh b_3(m_t+\alpha)} \tag{20}$$

Hereafter we set $b_2 = 2$, $b_3 = 5, \alpha = 0.01$ and $e = 1.5$ as fixed and consider how the orbits of (20) change as parameter $\rho/a$ changes.

Taking account of 2 and 3 in Assumption 3, we set the range over which parameter $\rho/a$ moves as $0.01 \leq \rho/a < 1$. Rewrite (20) as

$$m_t = H(m_{t+1}; \rho/a, m_0),\ m_0 = \textit{initial value},$$

and write according to the values of parameter $\rho/a$,

$$\begin{aligned}
H_0 &= H(m_{t+1}; 0.01, m_0),\ H_1 = H(m_{t+1}; 0.1, m_0), \\
H_2 &= H(m_{t+1}; 0.2, m_0),\ H_3 = H(m_{t+1}; 0.3, m_0), \\
H_4 &= H(m_{t+1}; 0.4, m_0),\ H_5 = H(m_{t+1}; 0.5, m_0),
\end{aligned}$$

[5]The following example is provided by Masanori Yokoo. We thank for his cooperation.

$$
\begin{aligned}
H_6 &= H(m_{t+1}; 0.6, m_0),\ H_7 = H(m_{t+1}; 0.7, m_0), \\
H_8 &= H(m_{t+1}; 0.8, m_0),\ H_9 = H(m_{t+1}; 0.9, m_0).
\end{aligned}
$$

(Insert figure 5 about here.)

Fig.5 depicts those graphs. From this figure, we can see that the peak of the graph becomes higher as we increase the value of the parameter $\rho/a$.

In Section 4, we have shown that the periodic orbit of period two emerges if the curvature of the graph is less than -1 at the point at which the graph intersects 45° line. Then, at what values of parameter $\rho/a$ does it happen, i.e. at what values of the parameter is condition (16) satisfied ? Also, can a periodic orbit of period three, which leads to the emergence of the chaos in the sense of Li-Yorke, emerge as we change the parameter $\rho/a$?

A powerful tool for answering these questions is the bifurcation diagram (see Fig.6).

(Insert figure 6 about here.)

Fig. 6 reads as follows. (i) As we increase $\rho/a$ from 0.01, initially *period doubling bifurcation* occurs. A phenomenon that the stable fixed point becomes unstabilized and at the same time there emerges a stable period two, and if we increase $\rho/a$ still further, this periodic orbit of period two becomes unstabilized and there emerges stable period four etc. (ii) Beyond $\rho/a \simeq 0.45$, *period halving bifurcation* occurs. A phenomenon contrary to period doubling bifurcation, period four becomes period two and period two becomes fixed point etc. (iii) In the neighbourhoods of $\rho/a = 0.1$ or 0.7, there exist stable periodic orbit of period two. In fact, we can see from Fig. 7 that there exists periodic orbit of period two at $\rho/a = 0.7$. (iv) In the neighbourhoods of $\rho/a = 0.25$ or 0.5, there exist stable periodic orbits of period three. We can see from Fig. 8 that there exists periodic orbit of period three at $\rho/a = 0.5$.

(Insert figure 7 and 8 about here.)

# 6 Concluding Remarks

In this paper we have treated money according to classical definition, i.e., money is a good with the following three functions: unit of account, means of payment and store of value. Because money is used as a unit of account, the price of nominal money was always equal to unity in our model. It is a means of payment and thus one was able to exchange it for consumption goods. Money can store value and thus one could hold money over to the next period for prospective use. In view of our model, storable property is especially important for generating the dynamics of prices. This property is eventually the essence for any equilibrium dynamics model including that of overlapping generations.

We have seen that money with positive utility generates dynamic movement of prices. And we have investigated a condition under which there exists a cyclically oscillating equilibrium path of real balances. Finally, we have given an example in which equilibrium paths behave chaotically. This example suggests that nonlinearity of individual preference may give rise to a very complex or unstable movement of monetary economy. This also implies that we can explain irregular fluctuation of economic variables without recourse to exogenous stochastic shocks, which of course is a vigorously studied theme of business cycle theories nowadays.

# Appendix

For the proof of Lemmata 1 and 2, some notational convenience is of great help. Write, as in Section 4, $u_1(e,m) = v_1(e,m)$, $u_2(e,m) = v_2(e,m)$, etc. Setting $m_t = x, m_{t+1} = m$ in eq. (8), define $C^1$ function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ as

$$f(x,m) = x - \frac{\rho}{a} m \frac{v_1(m) + v_2(m)}{v_1(x)}. \qquad (\mathrm{A}_1)$$

Then eq. (8) is equivalent to

$$f(x,m)=0. \tag{A$_2$}$$

Write

$$f_1(x,m)=\frac{\partial f}{\partial x}(x,m), f_2(x,m)=\frac{\partial f}{\partial m}(x,m).$$

Simple calculation yields

$$f_1(x,m)=1+\frac{\rho}{a}\{v_1(m)+v_2(m)\}\frac{v_{12}(x)}{(v_1(x))^2},$$
$$f_2(x,m)=\frac{-\rho}{av_1(x)}[v_1(m)+v_2(m)+m\{v_{12}(m)+v_{22}(m)\}].$$

By Assumption 1,$f_1(x,m)>0$ for any $(x,m)\in \mathbb{R}^2_{++}$ and thus we can define continuous function

$$\phi(x,m)=\frac{-f_2(x,m)}{f_1(x,m)} \tag{A$_3$}$$

over $\mathbb{R}^2_{++}$.

Next note that

$$\begin{aligned} s_1 &= \lim_{m\downarrow 0} v_1(0)=\inf_{0<m} v_1(m)>0 \text{ and} \\ s_2 &= \lim_{m\downarrow 0} v_2(m)=\sup_{0<m} v_2(m)<+\infty, \end{aligned} \tag{A$_4$}$$

from Assumption 1 and 3.

**Proof of Lemma 1.** We first prove the uniqueness of the solution $(x_0,m_0)\in \mathbb{R}^2_{++}$ of eq.(A$_2$) with respect to $m_0$. Let $(x_1,m_0)$ be another solution of eq. (A$_2$) and let $x_0<x_1$. since $f_1(x,m)>0$ for any $(x,m)\in\mathbb{R}^2_{++}$, this means

$$0=f(x_0,m_0)<f(x_1,m_0)=0,$$

which is contradiction. Thus we have proved the uniqueness of solution and of implicit function.

Existence proof is divided into six steps, (a)–(f). Let $(x_0,m_0)\in\mathbb{R}^2_{++}$ be a solution of eq.(A$_2$) stated in the lemma. For $f_1(x_0,m_0)>0$, implicit function theorem guarantees the existence of local implicit function $g_0$ of class $C^1$ such that

$$g_0 : ]m_1,m_2[\to \mathbb{R}^*_+,\ m_1<m_0<m_2 \tag{i}$$

$$f(g_0(m), m) = 0,\ g_0(m_0) = x_0.$$

We make $C^1$ extension of this function as the unique implicit function of eq.($A_2$) over to $\mathbb{R}_+^*$.

By ($A_3$)

$$g_0'(m) = \phi(g_0(m), m). \tag{ii}$$

(a) For any $m_0 > 0$, $\phi$ is bounded on $\mathbb{R}_+^* \times ]0, m_0[$.

Proof. Let

$$\alpha(m) = \frac{\rho}{a}\left[v_1(m) + v_2(m) + m\left\{v_{12}(m) + v_{22}(m)\right\}\right].$$

Then $-f_2(x, m) = \alpha(m)/v_1(m)$. By (1) and (2) in Assumption 1 and 1 in Assumption 3, we have $0 < v_1(m) < v_1(m_0)$ and $0 < v_2(m) < s_2$ for $0 < m < m_0$. By (2), (3) and 4 in Assumption 3, there exist

$$\beta = \sup_{0<m\leq m_0} |v_{12}(m)|,\ \gamma = \sup_{0<m\leq m_0} |v_{22}(m)|.$$

Thus

$$|\alpha(m)| < \frac{\rho}{a}\left\{v_1(m_0) + s_2 + m_0(\beta + \gamma)\right\} = \delta$$

for $0 < m < m_0$. Then, for $0 < x$ and for $0 < m < m_0$, we have

$$|\phi(x, m)| < |f_2(x, m)| = \frac{|\alpha(m)|}{|v_1(x)|} < \frac{\delta}{s_1}.$$

Thus (a) holds.□

(b) There exist

$$\lim_{m\downarrow m_1} g_0(m) = x_1,\ \lim_{m\uparrow m_2} g_0(m) = x_2 \tag{iii}$$

in $\mathbb{R}$.

Proof. Let

$$L = \sup_{0<x, 0<m\leq m_2} |\phi(x, m)|$$

for $m_2$ defined above. We embed the final set $\mathbb{R}_+^*$ of $g_0$ naturally into $\mathbb{R}$ by inclusion and treat $g_0$ as a function from $]m_1, m_2[$ to $\mathbb{R}$. By (ii) and (a),

$$\sup_{m_1<m<m_2} |g_0'(m)| \leq L.$$

Thus by the theorem of mean value

$$\begin{aligned} |g_0(m) - g_0(m')| &\leq |m - m'| \sup_{m \leq n \leq m'} |g_0'(n)| \\ &\leq |m - m'|L, \end{aligned}$$

for all $m, m' \in ]m_1, m_2[$. So $g_0$ is Lipschitz continuous and thus uniformly continuous. Because $\mathbb{R}$ is a complete metric space, uniformly continuous mapping from a subset of metric space is extended uniform-continuously over to the closure of its domain. Thus (b) holds.□

(c) $x_1 = \bar{g}_0(m_1) > 0$ and $x_2 = \bar{g}_0(m_2) > 0$, where $\bar{g}_0 : [m_1, m_2] \longrightarrow \mathbb{R}$ is the extension referred in the proof of (b).

Proof. If not so for $x_1$, then $x_1 = 0$ by the continuity of $\bar{g}_0$. Then by the uniqueness of the limit and 3 of Assumption 3, we have

$$\lim_{m \downarrow m_1} v_1(g_0(m)) = s_1 > 0.$$

Thus by continuity of $v_1$ and $v_2$, we have from definition of $g_0$,

$$\begin{aligned} 0 = \lim_{m \downarrow m_1} g_0(m) = \lim_{m \downarrow m_1} \frac{\rho}{a} m \frac{v_1(m) + v_2(m)}{v_1(g_0(m))} \\ = \frac{\rho}{a} m_1 \frac{v_1(m_1) + v_2(m_1)}{s_1} > 0, \end{aligned}$$

which is contradiction. For $x_2$, the argument is completely analogous.□

(d) $f(x_1, m_1) = f(x_2, m_2) = 0$.

Proof. Immediate from continuity of $f$ and the definition of $x_1$ and $x_2$.□

(e) For any $m_3 \geq m_2$, there exists unique $C^1$ extension of $g_0$ over to $]0, m_3[$, denoted by $g^{m_3}$, unique in that it is only one implicit function of eq. ($A_2$) on $]0, m_3[$.

Proof. Let $P_1 \subset \mathbb{R}$ be defined as

$$P_1 = \{\, m \in [0, m_1] \mid g_0 \text{ is uniquely extended over } ]m, m_2[ \text{ as a } C^1 \text{ implicit function of eq. } (A_2). \,\}.$$

Because $m_1 \in P_1$, $P_1 \neq \emptyset$. Let $m, m' \in P_1, m < m'$ and let $g_m, g_{m'}$ be the functions corresponding to $m$ and $m'$. Then by uniqueness,

$$g_m = g_{m'} | ]m, m'[ \text{ (restriction)}. \tag{iv}$$

If not so, there exist two different extensions for $m$ and contradicts $m \in P_1$. For $\emptyset \neq P_1 \subset \mathbf{R}$ and $P_1$ is bounded, there exist $q = \inf P_1$. We prove $q \in P_1$. Apparently $0 \leq q \leq m_1$. Because $q = \inf P_1$, for any $q < n$ there exists $m \in P_1$ such that $m < n$. Let $g_m : ]m, m_2[ \rightarrow \mathbf{R}^*_+$ be the corresponding implicit function for $m$ and define $g_q : ]q, m_2[ \rightarrow \mathbf{R}^*_+$ as

$$g_q(n) = g_m(n) \text{ for } q < m < n < m_2,\ m \in P_1.$$

By (iv), this value is determined without particular selection of $m < n$, $m \in P_1$ and thus $g_q$ is unique $C^1$ extension of $g_0$ to $]q, m_2[$ as an implicit function of eq. ($A_2$). Thus $q \in P_1$. Next we prove $q = 0$. If $q > 0$, considering $]q, m_2[$ for $]m_1, m_2[$ in (b),(c) and (d), there exists

$$0 < x_q = \lim_{m \downarrow q} g_q(m) \text{ such that } f(x_q, q) = 0.$$

Because $f(x_q, q) = 0$ and $f_1(x, m) > 0$ for any $(x, m) \in \mathbf{R}^2_{++}$, implicit function theorem tells that there exists an interval $]t, l[, 0 \leq t < q < l \leq m_2$ and corresponding implicit function $h : ]t, l[ \rightarrow \mathbf{R}^*_+$. But by the uniqueness solution $(x, m) \in \mathbf{R}^2_{++}$ with respect to $m$, which we have proved at first in a course of proofs,

$$h | ]q, l[ = g_q | ]q, l[$$

must hold. Thus defining $g_t : ]t, m_2[ \rightarrow \mathbf{R}^*_+$ as

$$g_t(n) = h(n) \text{ for } t < n < l,$$
$$g_t(n) = g_q(n) \text{ for } l \leq n < m_2,$$

we have extension over to $]t, l[$, which contradicts the definition of $q = \inf P_1$. Thus $q = 0$. So we have implicit function $g_q = \psi : ]0, m_2[ \rightarrow \mathbf{R}^*_+$.

For $m_3 > m_2$, define $P_2 \subset \mathbb{R}$ as

$$P_2 = \{\, m \in [m_2, m_3] \mid \psi \text{ is uniquely extended over } ]0, m[ \text{ as a } C^1 \text{ implicit function of eq. } (\mathrm{A}_2).\},$$

and make analogous argument for $\sup P_2$.□

(f) There exists unique $C^1$ implicit function $g : \mathbb{R}^*_+ \to \mathbb{R}^*_+$.

Proof. By (e), for any $0 < m$ there exists unique implicit function $g^m$ : $]0, m[\to \mathbb{R}^*_+$. So for $0 < m$ take $m' > n$ (e.g. $m' = [n] + 1$, $[\cdot]$ being Gauss symbol) and let $m = \max\{m', m_2\}$ and define

$$g(n) = g^m(n).$$

By the uniqueness of solution of eq.($\mathrm{A}_2$), this is well-defined.

This completes the proof of Lemma 1.□ (QED)

**Proof of Lemmma 2.**

1) Take $m_0$. Then for $0 < m < m_0$, we have

$$0 < g(m) = \frac{\rho}{a} m \left( \frac{v_1(m) + v_2(m)}{v_1(g(m))} \right) < \frac{\rho}{a} m \left( \frac{v_1(m_0) + s_2}{s_1} \right) < +\infty.$$

Letting $m \downarrow 0$, we have 1).□

2) We first prove

$$\lim_{m \downarrow 0} f_1(g(m), m) = 1. \tag{v}$$

By 1), $\lim_{m\downarrow 0} g(m) = 0$. Thus for an $r > 0$ there exist $m_1 > 0$ such that for $m_1 > m > 0$,

$$0 < g(m) < r.$$

From 4, for this $r$, there exists

$$\sup_{0<m<r} v_{12}(m) = s_{12}\ (> 0),$$

and for $m_1 > m > 0$,

$$0 < v_{12}(g(m)) \leq s_{12},$$

because $0 < m < m_1$ implies $0 < g(m) < r$. Then

$$\begin{aligned} 1 \leq f_1(g(m),m) &= 1 + \frac{\rho}{a}\frac{v_{12}(g(m))}{(v_1(g(m)))^2} \\ &\leq f_1(g(m),m) = 1 + \frac{\rho}{a}\frac{s_{12}}{(s_1)^2} \end{aligned}$$

for $0 < m < m_1$. Letting $m \downarrow 0$, we have (v). We next show

$$\lim_{m\downarrow 0} -f_2(g(m),m) > 1. \tag{vi}$$

From 4 of Assumption 3, for the same $r > 0$ as above, there exists

$$\inf_{0<m<r} v_{22}(m) = s_{22}\ (> 0).$$

Thus for $0 < m < r$

$$ms_{22} < mv_{12}(m) + v_{22}(m) < ms_{12}.$$

Letting $m \downarrow 0$ we have $mv_{12}(m) + v_{22}(m) \to 0$ and thus,

$$\begin{aligned} \lim_{m\downarrow 0} -f_2(g(m),m) &= \lim_{m\downarrow 0} \frac{\rho}{av_1(g(m))}[v_1(m) + v_2(m) + \{v_{12}(m) + v_{22}(m)\}] \\ &= \frac{\rho}{a}\frac{s_1 + s_2}{s_1} = \frac{\rho}{a}(1 + \frac{s_2}{s_1}) > 1 \end{aligned}$$

by 3 of Assumption 3. Because

$$g'(m) = \phi(g(m),m) = \frac{-f_2(g(m),m)}{f_1(g(m),m)},$$

we have 2).□ (QED)

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